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Comparing two ways of measuring the power of party members in simple majority voting games*

by

Maria Ekes

Szkoła Główna Handlowa
Al. Niepodległości 162, 02-554 Warszawa, Poland
e-mail: maria.ekes@sgh.waw.pl

Abstract: We examine two approaches to the problem of measuring the voting power of individuals in a voting body with an a priori coalition structure by means of the Shapley-Shubik index. In order to analyze this question we apply games with a priori unions and composite games. We compare these two approaches basing on voting games with 100 voters and different coalition structures and present our conclusions.

Keywords: simple majority voting games, games with a priori unions, composite games, Shapley-Shubik index, Owen index

1. Introduction

In this paper we investigate the way to measure the power of individuals in a voting body possibly divided into some blocks (parties). We are modeling such a situation in two different ways – by applying the framework of games with a priori unions (see Owen, 1977) and by applying composite games (Felsenthal, Machover, 1998). In both cases we measure the power of individual voters using the Shapley-Shubik index. We make simulations for a specific voting body composed of 100 members and compare both approaches.

Our model is to some extent similar to the model considered by Shapley and Milnor in the appendix of the paper on oceanic games (see Shapley and Milnor, 1978). The basic difference is that we consider game with a finite number of players and we are interested in the power of individual players, while Shapley and Milnor are describing the model with a continuum of players, therefore the power of individuals cannot be the issue in their (continuous) case. Shapley and Milnor are interested in the power of two large shareholders, which can be treated as counterparts of our parties, and the total power of the ocean of small shareholders, corresponding to individual voters in our model.

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We begin with describing the formal models. In the sequel we present examples of applications for a voting body composed of 100 members with various divisions into parties (blocs).

2. The model

Let $N = \{1, 2 \ldots n\}$ denote the set of voters (or seats). We consider a decision-making situation in which the voting body is supposed to make a decision (to pass or to reject a proposal) by means of a voting rule. We assume that voters, who do not vote for a proposal (do not vote "yes"), vote against it, and there is no possibility of abstention. The voting rule specifies whether the set of voters, who accepted the proposal, forms a winning coalition or not. Formally we have 2^n possible coalitions (vote configurations) $S \subseteq N$. The voting rule is then defined by the set of winning coalitions W. Usually it is assumed that

- $\emptyset \notin W$,
- $N \in W$,
- if $S \in W$ then $N \backslash S \notin W$,
- if $S \in W$ and $S \subset T$ then $T \in W$.

The voting rule is equivalently given by a simple voting game v_W as follows

$$v_W(S) = \begin{cases} 1 & \text{if } S \in W \\ 0 & \text{if } S \notin W \end{cases}$$

for each $S \subseteq N$.

In our former research (Ekes, 2006) we applied the Banzhaf-Coleman index (in fact we should call it Penrose-Banzhaf-Coleman index, but as many other authors we will use only two last names) as a measure of power of voters. We say that a voter i is *critical* for a coalition S if $v_W(S) = 0$ and $v_W(S \cup \{i\}) = 1$, or $v_W(S) = 1$ and $v_W(S \backslash \{i\}) = 0$.

The Banzhaf-Coleman index of a voter j is the probability that this voter is critical assuming that all voting configurations are equally probable, that is

$$\beta_j(W) = \frac{\#\{S \subset N : (j \in S \in W \wedge S - \{j\} \notin W) \vee (j \notin S \notin W \wedge S \cup \{j\} \in W)\}}{2^n} = \frac{1}{2^{n-1}} \sum_{\substack{S \subset N \\ j \in S}} (v_W(S) - v_W(S - \{j\})).$$

Here we will be dealing with the Shapley-Shubik index, which is given (for a player j) by the formula

$$Sh_j(W) = \sum_{\substack{S \subset N \\ j \in S}} \frac{(s-1)!(n-s)!}{n!} \left(\nu_W(S) - \nu_W(S - \{j\})\right).$$

The Shapley-Shubik index has also a probabilistic interpretation – if we assume that all orderings of voters are equally probable, then the Shapley-Shubik index of a voter j is the probability that this voter is pivotal (i.e. changes the already existing coalition from the losing one to a winning one, regardless of what happens after his accession). Most important characterizations of the Banzhaf-Coleman and Shapley-Shubik indices are given in Penrose (1946), Shapley (1953), Shapley and Shubik (1954), Coleman (1964), Banzhaf (1965), Dubey (1975), Owen (1978), and Young (1985).

In the real world voting bodies the situation is more intricate, since voters are divided into some blocs (parties) ex ante, which may constrain their actual voting behavior. This partition may be the consequence of the political party membership, which is an obvious reason of some constraints in voting in bodies like parliaments. It might also reflect different national interests of citizens of various members of international communities like EU or IMF. This situation can be described by games with a priori unions (precoalitions) introduced in 1977 by Owen. Let $T = (T_1, T_2, ..., T_m)$ be a partition of the set N into subsets which are nonempty, pairwise disjoint and $\bigcup_{i=1}^{m} T_i = N$. The sets T_i are called precoalitions (a priori unions) and they can be interpreted as parties occupying seats in the voting body (note that some T_i can be singletons). Let M denote the set of all precoalitions, that is $M = \{1, 2, ..., m\}$. Owen proposed a modification of the Shapley value, taking into account the a priori division of the set of players. This new value is called Owen value or Owen index in case of simple games. Therefore the Owen index for a player j belonging to the set T_i is given by the formula

$$O_j(W,T) = \sum_{\substack{H \subset M \\ i \notin H}} \sum_{\substack{S \subset T_i \\ j \notin S}} \frac{h!(m-h-1)!s!(t_i-s-1)!}{m!t_i!} (\nu_W(H \cup S \cup \{j\}) - \nu_W(H \cup S)),$$

where h denotes the cardinality of the set H, t_i – the cardinality of the party T_i, and s – the cardinality of the coalition S.

Note that when calculating the Owen index of a player $j \in T_i$ we restrict the number of possible permutations of the set of players. We take into account only those permutations in which all players from each precoalition appear together. In order to find all such permutations, we need to order the parties first and then to order the players in each party. Owen index can therefore be interpreted as a probability of a player $j \in T_i$ being pivotal, provided that all permutations of the set of players, respecting the coalition structure, are equally probable. Owen index was also axiomatized in various ways, see e.g. Owen (1977), Hart and Kurz (1983). In 1981 Owen proposed also a modification of the Banzhaf-Coleman index for games with a priori unions, and the axiomatic characterization of this modification is given in Albizuri (2001).

Another way of measuring the power of individual voters, when voters are a priori divided into some parties, is the application of the model of a composite game. Assume that any bill is decided in a voting body in a two-stage process of voting. First, the proposal is voted inside each party – we will consider the situation where voting rule is the simple majority. Then the bill is voted by the whole voting body and we assume that all parties vote as blocks, according to the decision taken in the inside party voting. Such a scenario of voting is modeled by the composite game (see Owen, 1964, where it is called compound game, or Felsenthal and Machover, 1998). We define the composite game in the following way: let (N_i, W_i), for $i = 1, 2, \ldots, m$, be simple games with disjoint sets of voters and let (M, W) be a simple game with the cardinality of the set M equal to m. The game $(N, W[W_1, ..., W_m])$, where $N = \bigcup_{i=1}^{m} N_i$ and

$$W[W_1, ..., W_m] = \{S \subset N : \{i \in \{1, 2, ..., m\} | S \cap N_i \in W_i\} \in W\},$$

is called *composite game* with *components* (N_i, W_i) and the *top* (M, W).

The calculation of the Banzhaf-Coleman index of a voter in a composite game is quite simple, since it is the product of the Banzhaf-Coleman index of a voter in the inside-party voting (component game) and the party index in the weighted voting game of parties (top game). The Shapley-Shubik index in a composite game (we shall denote it by $Sh^c(W, T)$) does not have such a property, therefore, it has to be calculated directly from its definition.

In the former paper of the present author, Ekes (2006) we argued that the Banzhaf-Coleman index, modified by Owen for games with a priori unions, can be interpreted as a measure of power of a voter belonging to the party in which there is no party discipline, provided that all other parties obey party discipline. Such interpretation in case of the Owen index is not valid any more.

In the sequel we present an example of a voting body composed of 100 voters, who are divided into two parties or vote independently. We calculate the power of voters for both approaches (game with precoalitions and composite game) for all possible configurations of sizes of both parties.

3. Presentation of results

We consider the situation of the voting body composed of 100 voters, who are members of one of two parties or who are voting as independent voters. The coalition structure is therefore the following: $T = (T_1, T_2, \{j_1\}, ..., \{j_l\})$, where $2 \leq t_1, t_2$ and $t_1 + t_2 + l = 100$. We assume that the voting rule in our example is the simple majority, which means that any proposal is accepted if it has at least 51 votes for. We are not interested in case where a single party constitutes the winning majority, therefore we assume that $t_1, t_2 \leq 50$. We have calculated values of both indices $O(T)$ and $Sh^c(T)$ for all such t_1 and t_2 and for all voters. We omit the symbol W in the notation of indices since the simple majority rule defines the set of winning coalitions in the game with a priori unions as well as in the composite game. We also note that, due to the symmetry of Shapley-Shubik index, the power of all voters in the same party is equal and the power of all independent voters is the same. Therefore, we will use the notation $Sh^c_{T_i}(T)$, $O_{T_i}(T)$ for $i = 1, 2$ and $Sh^c_{j_k}(T)$, $O_{j_k}(T)$ for $k = 1, 2, \ldots, l$. Parties are symmetric in our case – if we consider the coalition structures of the form $T^1 = \{T^1_1, T^1_2, \{j^1_1\}, ..., \{j^1_l\}\}$, $T^2 = \{T^2_1, T^2_2, \{j^2_1\}, ..., \{j^2_l\}\}$ such that $t^2_1 = t^1_2 \wedge t^1_1 = t^2_2$, then $Sh^c_{T^1_i}(T^1) = Sh^c_{T^2_j}(T^2)$, $O_{T^1_i}(T^1) = O_{T^2_j}(T^2)$ for $i, j = 1, 2$, $i \neq j$ and $Sh_{j^1_k}c(T^1) = Sh^c_{j^2_k}(T^2)$, $O_{j^1_k}(T^1) = O_{j^2_k}(T^2)$ for $k = 1, 2, \ldots, l$. This observation allows us to consider only the value of both indices for members of the first party and for independent voters. The number of the elements of our coalition structure is then equal to $100 - t_1 - t_2 + 2 = l + 2$.

We calculate the value of the Owen index for the voter in the party T_1 according to the following formula:

$$O_{T_1}(T) = P_1 + P_2,$$

where

$$P_1 = \sum_{s=51-t_1}^{\min(50,l)} \binom{l}{s} \binom{t_1 - 1}{50 - s} \frac{s!(l + 1 - s)!(50 - s)!(t_1 - (50 - s) - 1)!}{t_1!(l + 2)!},$$

if $t_1 + l \geq 51$, otherwise $P_1 = 0$ and

$$P_2 = \sum_{s=\max(51-t_1-t_2,0)}^{50-t_2} \binom{l}{s} \binom{t_1 - 1}{50 - t_2 - s} \times \frac{(1 + s)!(l - s)!(50 - t_2 - s)!(t_1 - (50 - t_2 - s) - 1)!}{t_1!(l + 2)!}.$$

For independent voters we have the following:

$$O_{j_k}(T) = S_1 + S_2 + S_3 + S_4,$$

where

$$S_1 = \binom{l - 1}{50} \frac{50!(l + 1 - 50)!}{(l + 2)!}, \text{ if } l > 50,$$

otherwise $S_1 = 0$

$$S_2 = \binom{l - 1}{50 - t_1} \frac{(50 - t_1 + 1)!(l + 1 - (50 - t_1 + 1))!}{(l + 2)!}, \text{ if } t_2 \neq 50,$$

otherwise $S_2 = 0$

$$S_3 = \binom{l - 1}{50 - t_2} \frac{(50 - t_2 + 1)!(l + 1 - (50 - t_2 + 1))!}{(l + 2)!}, \text{ if } t_1 \neq 50,$$

otherwise $S_3 = 0$, and

$$S_4 = \binom{l - 1}{50 - t_1 - t_2} \frac{(50 - t_1 - t_2 + 2)!(l + 1 - (50 - t_1 - t_1 + 2))!}{(l + 2)!},$$

if $t_1 + t_2 \leq 50$, otherwise $S_4 = 0$.

Note that Owen index has a product property which is similar to the property of Banzhaf-Coleman index in a composite game. After simplification of the formula for $O_{T_1}(T)$ we obtain:

$$O_{T_1}(T) = \frac{1}{t_1}(\tilde{P}_1 + \tilde{P}_2),$$

where

$$\tilde{P}_1 = \sum_{s=51-t_1}^{\min(50,l)} \binom{l}{s} \frac{s!(l-s+1)!}{(l+2)!}$$

if $t_1 + l \geq 51$, otherwise $\tilde{P}_1 = 0$ and

$$\tilde{P}_2 = \sum_{s=\max(51-t_1-t_2,0)}^{50-t_2} \binom{l}{s} \frac{(1+s)!(l-s)!}{(l+2)!}.$$

This new formula has an interesting interpretation – it is the product of the Shapley-Shubik index of a member of a party T_1 in an (arbitrary) majority voting game inside this party and the Shapley-Shubik index of this party in a top game among parties, which is the weighted majority voting game with the quota 51.

The calculation of the Shapley-Shubik index in a composite game has to be done in two steps. First, we have to find the set of all coalitions, for which the first party is decisive in the weighted voting game of parties (by parties we mean the two "large" parties and all independent voters). We denote this set of coalitions by $Dec(T_1)$. For a coalition $C \in Dec(T_1)$ we find the number of independent players in this coalition and we denote it $l(C)$. Moreover, we use the notation $\tau_1 = \left[\frac{t_1}{2}\right]$, $\tau_2 = \left[\frac{t_2}{2}\right]$, where the symbol $[x]$ denotes the largest integer not greater than x for any real x. In all formulae below we will assume that $\binom{n}{k} = 0$ for $n < k$. The value of the Shapley-Shubik index in a composite game for a player from the party T_1 is given by the formula:

$$Sh^c_{T_1}(T) = \bar{P}_1 + \bar{P}_2,$$

where

$$\bar{P}_1 = \frac{1}{100!} \sum_{\substack{C \in Dec(T_1) \\ T_2 \notin C}} \sum_{p_2=0}^{\tau_2} \binom{t_2}{p_2} \binom{t_1-1}{\tau_1} \binom{l}{l(C)} \times$$
$$(p_2 + l(C) + \tau_1)!\,(100 - (p_2 + l(C) + \tau_1) - 1)!$$

$$\bar{P}_2 = \frac{1}{100!} \sum_{\substack{C \in Dec(T_1) \\ T_2 \in C}} \sum_{p_2=\tau_2+1}^{t_2} \binom{t_2}{p_2} \binom{t_1-1}{\tau_1} \binom{l}{l(C)} \times$$
$$(p_2 + l(C) + \tau_1)!\,(100 - (p_2 + l(C) + \tau_1) - 1)!$$

For independent players we have the following:

$$Sh^c_{j_k}(T) = \bar{S}_1 + \bar{S}_2 + \bar{S}_3 + \bar{S}_4,$$

where

$$\bar{S}_1 = \frac{1}{100!} \binom{l-1}{50} \sum_{p_1=0}^{\tau_1} \sum_{p_2=0}^{\tau_2} \binom{t_1}{p_1} \binom{t_2}{p_2} \times$$
$$(50 + p_1 + p_2)!(100 - (50 + p_1 + p_2) - 1)!,$$
$$\text{if } l > 50, \text{ otherwise } \bar{S}_1 = 0,$$

$$\bar{S}_2 = \frac{1}{100!} \binom{l-1}{50-t_1} \sum_{p_1=\tau_1+1}^{t_1} \sum_{p_2=0}^{\tau_2} \binom{t_1}{p_1} \binom{t_2}{p_2} \times$$
$$(50 - t_1 + p_1 + p_2)!(100 - (50 - t_1 + p_1 + p_2) - 1)!,$$
$$\text{if } t_2 \neq 50, \text{ otherwise } \bar{S}_2 = 0$$

$$\bar{S}_3 = \frac{1}{100!} \binom{l-1}{50-t_2} \sum_{p_1=0}^{\tau_1} \sum_{p_2=\tau_2+1}^{t_2} \binom{t_1}{p_1} \binom{t_2}{p_2} \times$$
$$(50 - t_2 + p_1 + p_2)!(100 - (50 - t_2 + p_1 + p_2) - 1)!,$$
$$\text{if } t_1 \neq 50, \text{ otherwise } \bar{S}_3 = 0,$$

$$\bar{S}_4 = \frac{1}{100!} \binom{l-1}{50-t_1-t_2} \sum_{p_1=\tau_1+1}^{t_1} \sum_{p_2=\tau_2+1}^{t_2} \binom{t_1}{p_1} \binom{t_2}{p_2} \times$$
$$(50 - t_1 - t_2 + p_1 + p_2)!(100 - (50 - t_1 - t_2 + p_1 + p_2) - 1)!,$$
$$\text{if } t_1 + t_2 \leq 50, \text{ otherwise } \bar{S}_4 = 0.$$

Now we present figures showing values of both indices for the members of the first party. We can interpret those charts as the presentation of the power of members of the first party in terms of the function of the size of the second party.

Values of both indices decrease monotonically with the increasing size of the second party. It means that the power of the first party member falls, while the size of the opponent grows in the voting body. We can also notice that the indices considered behave in different ways. The range of the Owen index is less than the range of the Shapley-Shubik index in a composite game. Owen index is almost constant for the small sizes of the opponent party and then it decreases rather slowly. Shapley-Shubik index in a composite game is considerably greater

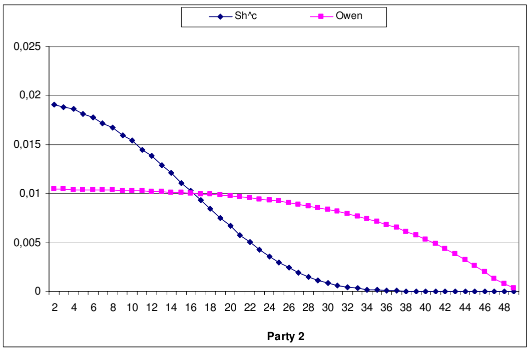

Figure 1. The power of the first party member as a function of t_2 at $t_1 = 5$

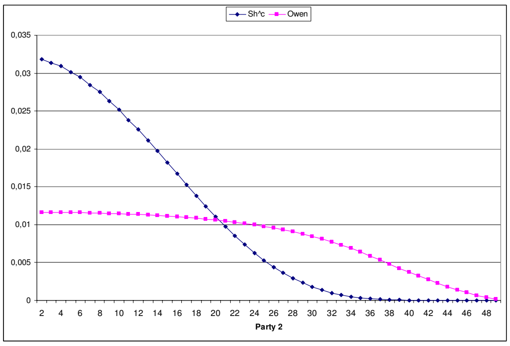

Figure 2. The power of the first party member as a function of t_2 at $t_1 = 15$

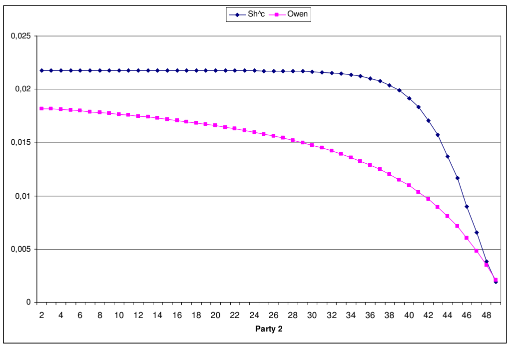

Figure 3. The power of the first party member as a function of t_2 at $t_1 = 46$

than Owen index for small sizes of the opponent party (for t_1 greater than 46 the Shapley-Shubik index in a composite game is greater than Owen index for all possible sizes of the second party). For small t_1 the Shapley-Shubik index decreases quickly with the increase of the size of the second party, it reaches the level of the Owen index, then it has an inflection point and decreases slowly to the values close to zero. For larger t_1 the behavior of the Shapley-Shubik index is different. For small sizes of the opponent party it is almost constant and when the size of the second party is quite large, then the value of the considered index starts to decrease. The point of intersection with the Owen index moves to the right (to the larger sizes of the second party), when the size of the first party grows and eventually the Shapley-Shubik index in a composite game is larger than Owen index for all possible values of t_2.

Owen index of a voter from the party T_1 attains its maximum in the situation where this party has the maximal possible number of members equal to 50 and the opponent has minimal possible number of members equal to 2. Moreover, if we fix the number t_2, then the maximal value of the Owen index for the voter from the party T_1 is attained in the situation where the size of the party T_1 is maximal (equal to 50). Relations among values of the Owen index for various configurations of sizes of both parties are presented in Fig. 4.

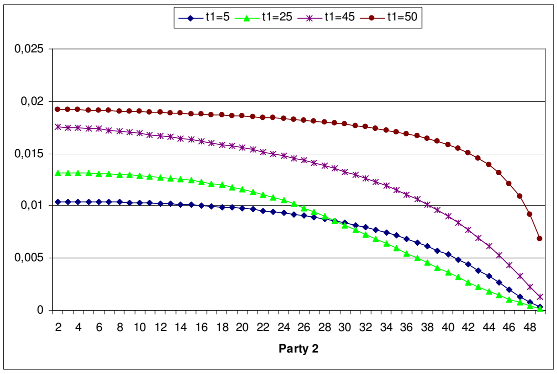

Figure 4. Relations among values of the Owen index of the member of T_1 (as a function of t_2) for various configurations of sizes of both parties

Index Sh^c has different properties. The global maximum of the value of this index (for the member of the party T_1 is attained in the configuration $t_1 = 22$, $t_2 = 2$. If we increase the size of the first party, the starting value of the index Sh^c grows up until t_1 reaches the value equal to 22 and then the starting point goes down. For large values of t_1 the power of the member of the first party is almost constant as a function of t_2 – it starts decreasing slightly only for large, almost maximal, sizes of the second party. Fig. 5 presents the behavior of the considered index for some chosen values of t_1.

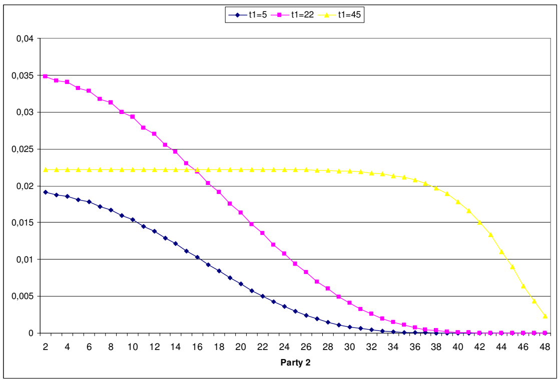

Figure 5. Relations among values of the Shapley-Shubik index in a composite game of the member of T_1 (as a function of t_2) for various configurations of sizes of both parties

It is interesting to see in what configurations the power of a member of the first party is maximal if we fix the size of the second party. Both indices have different properties also in this case. Owen index of the member of the first party is maximal if the size of the first party is maximal (equal to 50) for each fixed size of the second party. The point at which the maximal value of the Shapley-Shubik index in a composite game is attained depends on the fixed size of the second party. For $t_2 = 2$ the maximal power (measured by the Shapley-Shubik index in a composite game) of the member of the first party is attained for $t_1 = 22$. If we increase the fixed t_2, then the value of t_1, at which the maximum is achieved, also increases. For $t_2 > 43$ maximum of power of the party T_1 members is reached when the first party is of maximal size.

In the sequel we present charts showing the behavior of both indices treated as functions of the own party size for fixed values of t_2.

The most striking observation is that the power of the member of T_1, measured by the index Sh^c is not an increasing function of the own party size for most values of t_2. If we fix the size of the opponent, then the power of the member of the first party increases, attains the maximum and finally decreases with an increasing size of own party. Only for large sizes of the opponent party, the power is an increasing function of the own size. When we consider the Owen index, the picture changes – see Fig. 7.

In this case we observe that the power of the member of the first party is an (almost) monotonic function of the own party size. For large sizes of the opponent we notice a slight decrease of the power of a member of T_1, but then the power increases monotonically and achieves maximum always for $t_1 = 50$.

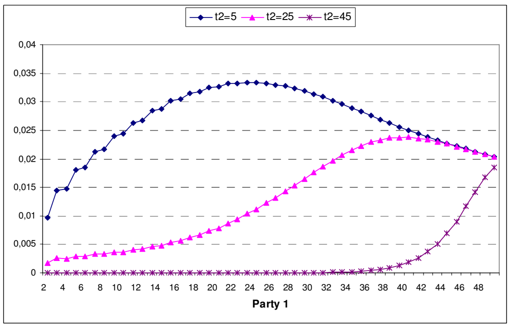

Figure 6. Relations among values of the Shapley-Shubik index of the member of T_1 in a composite game (as a function of t_1) for various configurations of sizes of both parties

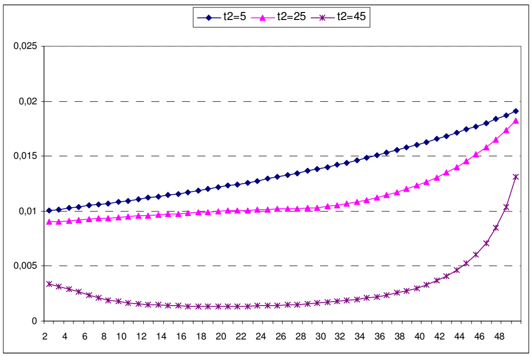

Figure 7. Relations among values of the Owen index of the member of T_1 (as a function of t_1) for various configurations of sizes of both parties

Up to this point we have only considered migration from one party to the set of independent voters – we have kept the size of one party fixed and increased the size of another party. Now we take a look at behavior of both indices when the migration between parties occurs – we assume that members of the second party are joining the first party. The following figures illustrate the influence of this kind of changes in the configuration of sizes on both indices.

Figs. 8-10 show that the Owen index is monotonic when considering migrations from one party to another – it increases with the increase of the own party size. The Shapley-Shubik index in a composite game does not show such monotonicity – it grows up to some point and then starts to decrease (this phenomenon occurs not in all cases but in most of them).

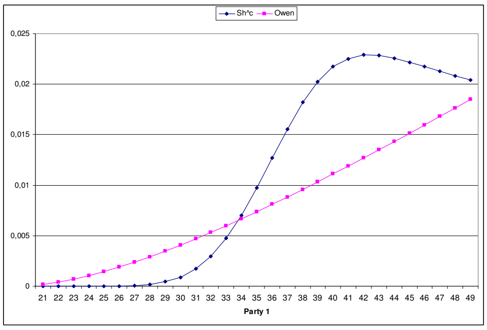

Figure 8. The power of a first party member – comparison of both indices ($l = 30$, migration from T_2 to T_1)

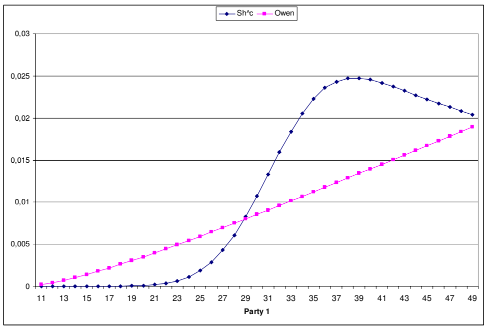

Figure 9. The power of a first party member – comparison of both indices ($l = 40$, migration from T_2 to T_1)

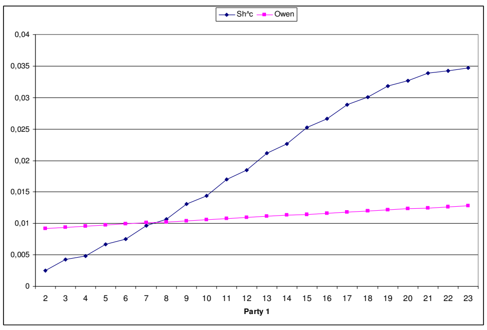

Figure 10. The power of a first party member – comparison of both indices ($l = 75$, migration from T_2 to T_1)

We have not analyzed yet the behavior of both indices for independent voters. Now we take a look at this question. We will treat the power of an independent voter as a function of the size of the first party with the number of all independent voters fixed (if we fix l, then by choosing t_1 we determine also t_2). Observe that if we fix l, then we sometimes have to restrict the range of t_1 and t_2, e.g., if $l = 40$, then $10 \leq t_1,\ t_2 \leq 50$. The subsequent figures present the behavior of both indices for different configurations of sizes of the voting body.

When observing Figs. 11-13 we conclude, first, that the power of an independent voter measured by the Owen index is always greater than the power measured by the Shapley-Shubik index in a composite game. The second observation is that both functions are symmetric, which is obvious, because if the size of one party grows, then evidently the size of the second drops. What is more interesting is that the maximum power of an independent voter is obtained in the situation where both parties are of the same size, or the difference between their sizes is equal to 1 – in this case we have two points with the same maximum value of power. The global maximum of the power of an independent voter is attained in case where there is only one such voter.

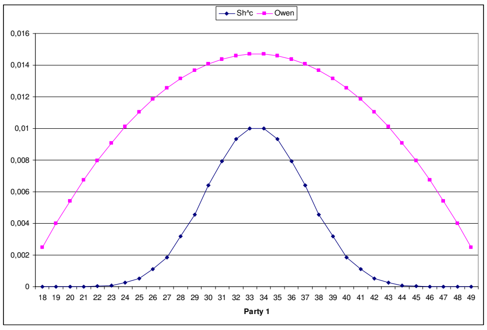

Figure 11. The power of an independent voter as a function of t_1 for $l = 33$

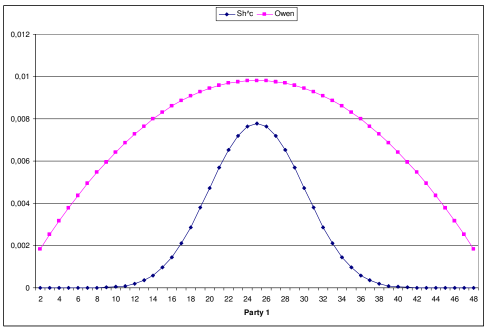

Figure 12. The power of an independent voter as a function of t_1 for $l = 50$

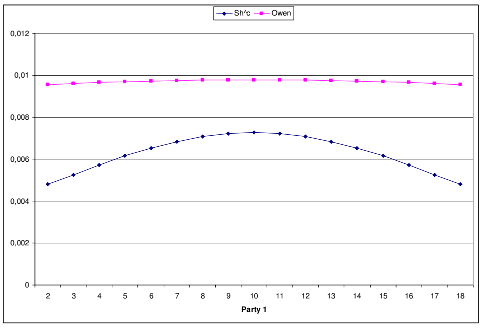

Figure 13. The power of an independent voter as a function of t_1 for $l = 80$

An interesting issue is also the question when it is profitable for an independent voter to join a party. In case where there are not too many individual voters, they are better off being independent than joining any party (especially in the case of the Owen index). When their number grows, it becomes better for them to join the bigger party. If the number of individual voters is very large (about 70 and more), then it is profitable for them to join any party. Figs. 14-19 illustrate this question.

Note that if we do not consider a coalition structure in the voting body, then all voters have the same voting power (considering any majority voting rule and any symmetric index). In case of the Shapley-Shubik index the power of each individual voter in the concerned voting body is equal 0.01. We can ask the following questions: when the party membership increases the power of a voter or for which coalition structures the power of an independent voter is greater than in the situation where the coalition structure does not exist? The answers to those questions for both indices – Sh^c and O – are given in the subsequent four figures, Figs. 20-23. In each of these figures the values of respective index of a member of T_1 or of an independent voter are shown for all possible configurations of sizes of both parties (rows correspond to sizes of the first party, and columns – to sizes of the second party; the left upper corner corresponds to the case $t_1 = t_2 = 2$, while the right bottom corner describes the case $t_1 = t_2 = 50$). The cells are shaded if the value of respective index is greater than 0.01.

From the figures shown we can conclude that for both indices accounting for the coalition structure in most cases increases the power of a party member (compared to the case without any a priori coalition structure). If we look at the index Sh^c, then for small t_1 the power of a party member is less than 0.01 in case where t_1 is substantially less than t_2. For larger values of t_1 the power of T_1's member becomes less than 0.01 when the size of the second party is greater than the size of the first one. Independent voters are better off when considering the party structure only in cases where both parties are approximately of the same size and both are rather large.

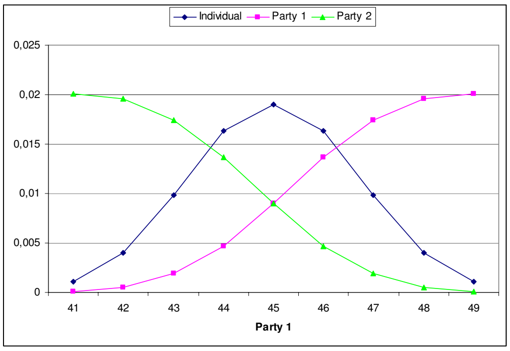

Figure 14. Comparison of the Shapley-Shubik index of individual voters and party members in the composite game, $l = 10$

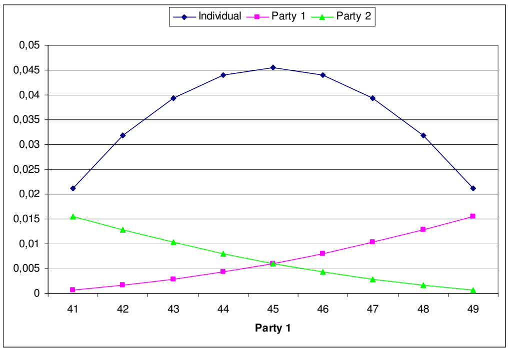

Figure 15. Comparison of the Owen index of individual voters and party members, $l = 10$

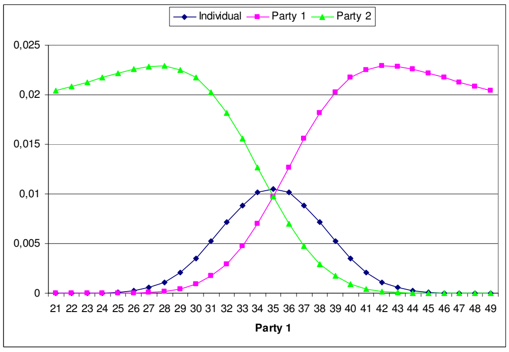

Figure 16. Comparison of the power of individual voters and party members in the composite game, $l = 30$

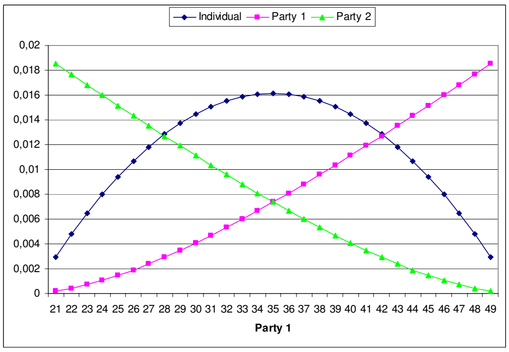

Figure 17. Comparison of the Owen index of individual voters and party members, $l = 30$

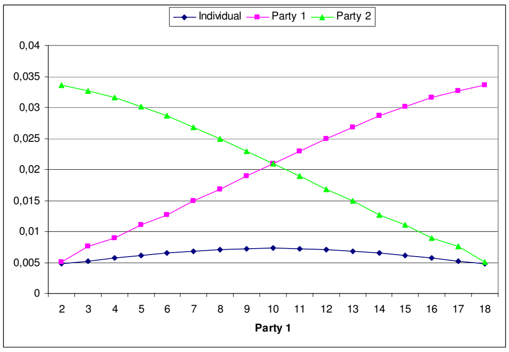

Figure 18. Comparison of the power of individual voters and party members in the composite game, $l = 80$

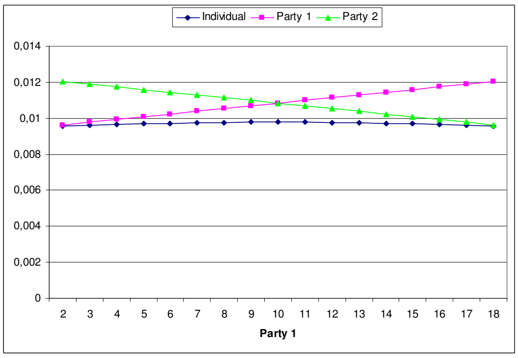

Figure 19. Comparison of the Owen index of individual voters and party members, $l = 80$

The last observation is such that the Owen index in general promotes independent voters while the Shapley-Shubik index in a composite game gives more power to the party members.

From the figures shown we can conclude that for both indices accounting for the coalition structure in most cases increases the power of a party member (compared to the case without any a priori coalition structure). If we look at the index Sh^c, then for small t_1 the power of a party member is less than 0.01 in case where t_1 is substantially less than t_2. For larger values of t_1 the power of T_1's member becomes less than 0.01 when the size of the second party is greater than the size of the first one. Independent voters are better off when considering the party structure only in cases where both parties are approximately of the same size and both are rather large.

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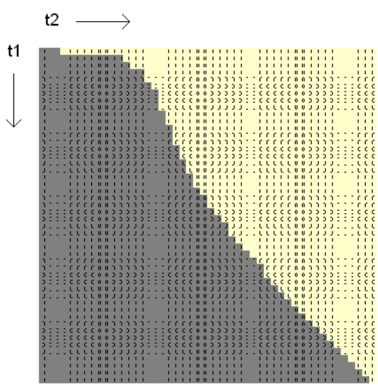

Figure 20. The Shapley-Shubik index of a member of T_1 in a composite game – comparison with the symmetric case

4. Conclusions

Comparison of the two approaches in the case of a voting body considered here leads to some conclusions concerning the properties of both methods of measuring the voting power of individuals in a voting body with a coalition structure. First of all, we observe that the Shapley-Shubik index in a composite game is more respondent to changes of coalition structure and it has a larger range of values. The Owen index is in most cases monotonic with respect to the size of the own party and the size of the opponent. If the size of the opponent is arbitrarily settled, then the maximal power is always achieved when the own party size is maximal (= 50). If the own party has an arbitrarily settled size,

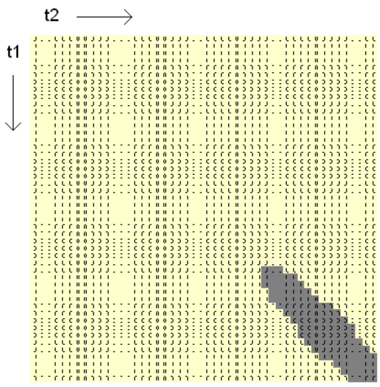

Figure 21. The Shapley-Shubik index of an independent voter in a composite game – comparison with the symmetric case

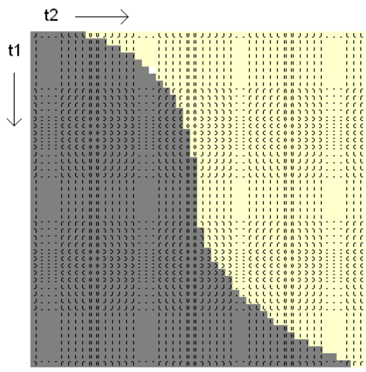

Figure 22. The Owen index of a member of T_1 – comparison with the symmetric case

then the power of its member is a decreasing function of the opponent's size. Moreover – the larger the own party size, the larger the maximal possible power of its member, which means that the global maximum is attained when the own party has 50 members and the opponent has two members. The index Sh^c does not reveal such monotonicity. As long as the own party size is arbitrarily settled, the power of its member is also a decreasing function of the opponent's size. However, with the arbitrarily settled size of the opponent party, the maximum of power depends on the size of the opponent. The global maximum is achieved in the situation where the own party has 22 members and the opponent has 2 members.

Independent voters are always better off if we measure their power by means of the Owen index than when applying the Shapley-Shubik index in a composite game.

In our opinion it was also interesting to compare the properties of the

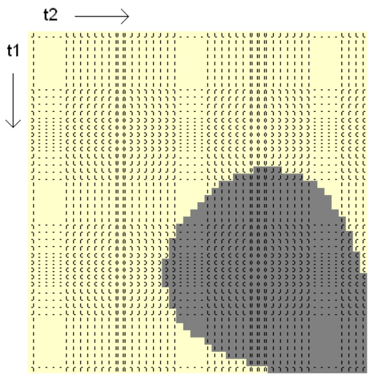

Figure 23. Owen index of an independent voter – comparison with the symmetric case

Shapley-Shubik index in a composite game with the properties of the Banzhaf-Coleman index in a composite game which we have examined in Ekes (2006) (a comparison of properties of various indices in games with precoalitions is also given in Sosnowska, 2003 and Malawski, 2004). Hence, the two indices behave in a very similar way in case of the examined voting body. Those similarities appear in spite of the fact that indices considered here have different structure, especially the Shapley-Shubik index is normalized while the Banzhaf-Coleman index is not normalized. It follows from our analysis that behavior of the Shapley-Shubik index in a composite game is much closer to the behavior of the Banzhaf-Coleman index in a composite game than to the behavior of the Owen index in game with precoalitions. It can be concluded from these observations that behavior of those indices in the concerned case depends more on the structure of the composite game than on the structure of the index itself.

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