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On uniformly approximate convex vector-valued function*

by

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Abstract: Let X, Y be real Banach spaces. Let Z be a Banach space partially ordered by a pointed closed convex cone K. Let $f(\cdot)$ be a locally uniformly approximate convex function mapping an open subset $\Omega_Y \subset Y$ into Z. Let $\Omega_X \subset X$ be an open subset. Let $\sigma(\cdot)$ be a differentiable mapping of Ω_X into Ω_Y such that the differentials of $\sigma\Big|_x$ are locally uniformly continuous function of x. Then $f(\sigma(\cdot))$ mapping X into Z is also a locally uniformly approximate convex function. Therefore, in the case of $Z = R^n$ the composed function $f(\sigma(\cdot))$ is Fréchet differentiable on a dense G_δ-set, provided X is an Asplund space, and Gateaux differentiable on a dense G_δ-set, provided X is separable. As a consequence, we obtain that in the case of $Z = R^n$ a locally uniformly approximate convex function defined on a $C^{1,u}_{\mathbf{E}}$-manifold is Fréchet differentiable on a dense G_δ-set, provided $\mathbf{E}$ is an Asplund space, and Gateaux differentiable on a dense G_δ-set, provided $\mathbf{E}$ is separable.

Keywords: vector valued functions, strongly $\alpha(\cdot)$-K-paraconvexity, differentiable manifolds, Gateaux and Fréchet differentiability.

1. Introduction

Let $(X, \|.\|)$ be a real Banach space. Let Z be a Banach space partially ordered by a pointed closed convex cone K. Let $f(\cdot)$ be a continuous function defined on an open convex subset $\Omega \subset X$. We say that the function $f(\cdot)$ is *K-convex* if, for $0 \le t \le 1$,

$$f\big(tx + (1-t)y\big) \le_K tf(x) + (1-t)f(y)$$

(in other words

$$tf(x) + (1-t)f(y) \in f\big(tx + (1-t)y\big) + K)$$

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for all $x, y \in \Omega$ and for t, $0 \leq t \leq 1$ (see, for example, Jahn, 1986, 2004, Pallaschke and Rolewicz, 1997).

In the case of $Z = R$ and $K = \{z \in R : z \geq 0\}$ we obtain the classical definition of a convex real-valued function.

We recall that a set $B \subset \Omega$ of second Baire category is called *residual* if its complement $\Omega \setminus B$ is of the first Baire category (i.e. it is a countable union of nowhere dense sets). Mazur (1933) proved that for each continuous convex real-valued function $f(\cdot)$ there is a residual subset A_G such that on the set A_G the function f is Gateaux differentiable. Asplund (1968) showed that if in the dual space X^* there exists an equivalent locally uniformly rotund norm, then for each continuous convex real-valued function $f(\cdot)$ there is a residual subset A_F such that on the set A_F the function f is Fréchet differentiable. The spaces X such that for the dual space X^* there exists an equivalent locally uniformly rotund norm are now called *Asplund spaces*. It can be shown that each reflexive space and spaces having separable duals are Asplund spaces. Even more, a space X is an Asplund space if and only if each its separable subspace $X_0 \subset X$ has a separable dual (Phelps, 1989).

Basing on a uniformization of the notion of approximate subgradient introduced and developed by Ioffe and Mordukhovich (see Ioffe, 1984, 1986, 1989, 1990, 2000; Mordukhovich, 1976, 1980, 1988, 2005a, 2005b) and adapting the method of Preiss and Zajíček (1984) the author extended the Mazur and Asplund results on larger (than convex) classes of function called strongly $\alpha(\cdot)$-paraconvex functions (Rolewicz, 1999, 2001a, 2001b, 2002, 2005a, 2005b, 2006). We say that a function $f(\cdot)$ is uniformly approximate convex if there is a function $\alpha(\cdot)$ (satisfying certain conditions) such that $f(\cdot)$ is a strongly $\alpha(\cdot)$-paraconvex function.

In the papers by Rolewicz (2007, 2009) it was shown that if σ is a mapping of a convex open set into a convex open set, such that the differentials * of σ, $\partial\sigma\big|_x$, are locally uniformly continuous in the norm topology, then the composition of a locally uniformly approximate real-valued convex function $f(\cdot)$ with $\sigma(\cdot)$, $f(\sigma(\cdot))$, is also a locally uniformly approximate real-valued convex function. As a consequence we get that $f(\sigma(\cdot))$ is Fréchet differentiable on a residual set, provided X is an Asplund space, and it is Gateaux differentiable on a residual set, provided X is a separable space. As a consequence we obtain that a locally uniformly approximate convex real-valued functions defined on $C^{1,u}_{\mathbf{E}}$-manifolds over a real Banach space $\mathbf{E}$ are Fréchet differentiable on a dense G_δ-set, provided $\mathbf{E}$ is an Asplund space, and are Gateaux differentiable on a dense G_δ-set, provided $\mathbf{E}$ is separable.

In this paper those results are extended on vector-valued functions having values in R^n.

*We shall say briefly *differentials*, since under assumptions of continuity each Gateaux differential is also a Fréchet differential.

2. Uniformly approximate convex vector-valued functions

Let k belong to the relative interior of K, $k \in Int_r K$.

Let $\alpha(\cdot)$ be a nondecreasing function mapping the interval $[0, +\infty)$ into the interval $[0, +\infty]$ such that

$$\lim_{t \downarrow 0} \frac{\alpha(t)}{t} = 0. \tag{2.1}$$

Let a continuous function $f(\cdot)$ be defined on an open convex subset $\Omega \subset X$ and having values in Y. We say that the function $f(\cdot)$ is *strongly* $\alpha(\cdot)$-k-*paraconvex* if there is $C \geq 0$ and such that for all $x, y \in \Omega$ and $0 \leq t \leq 1$ we have

$$f\big(tx+(1-t)y\big) \leq_K tf(x)+(1-t)f(y)+C\min[t,(1-t)]\alpha(\|x-y\|_X)k. \tag{2.2}$$

We say that a continuous function $f(\cdot)$ defined on an open convex subset $\Omega \subset X$ and having values in Z is *strongly* $\alpha(\cdot)$-K-*paraconvex* if it is *strongly* $\alpha(\cdot)$-k-*paraconvex* for all $k \in Int_r K$.

The set of all strongly $\alpha(\cdot)$-K-paraconvex functions is denoted $\alpha PC_K(\Omega)$.

PROPOSITION 2.1 *(Rolewicz, 2010). Let X, Z be Banach spaces. Let $K \subset Z$ be a convex pointed cone. Let $k_0 \in Int_r K$. Then each strongly $\alpha(\cdot)$-k_0-paraconvex function $f(\cdot)$ mapping a convex set $Q \subset X$ into Z is strongly $\alpha(\cdot)$-K-paraconvex.*

The following Proposition is obvious

PROPOSITION 2.2 *(Rolewicz, 2010). Let X be a real Banach space. Let $K \subset \mathbb{R}^n$ be a closed convex pointed cone. Let a function $f(\cdot)$ mapping a convex set $Q \subset X$ into $\mathbb{R}^n$ be strongly $\alpha(\cdot)$-K-paraconvex . Then there are n linearly independent functionals $\{\ell_1, \ell_2, ..., \ell_n\}$ defined on $\mathbb{R}^n$ such that the functions $\{\ell_1(f(\cdot)), \ell_2(f(\cdot)), ..., \ell_n(f(\cdot)\}$ are strongly $\alpha(\cdot)$-K-paraconvex.*

Using Proposition 2.2 and results about differentiability of uniformly approximate convex real-valued functions (Rolewicz, 1999, 2002, 2005a, 2005b, 2006) we can obtain

THEOREM 2.1 *(Rolewicz, 2010). Let Ω_X be an open convex set in a real Banach space $(X, \|\cdot\|_X)$. Let K be a convex closed pointed cone in $\mathbb{R}^n$ with any norm $\|\cdot\|$. Let $f(\cdot)$ be a strongly $\alpha(\cdot)$-K-paraconvex function defined on Ω_X with values in $\mathbb{R}^n$. Then the function $f(\cdot)$ is:*

(a) *Fréchet differentiable on a dense G_δ-set provided X is an Asplund space,*

(b) *Gateaux differentiable on dense G_δ-set provided X is separable.*

A vector-valued function $f(\cdot)$ defined on a convex set $\Omega \subset X$ with values in the space Z is called *uniformly approximate* K-*paraconvex* if for arbitrary

$k \in Int_r K$ and arbitrary $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, k)$ such that if $x, y \in \Omega$ and $\|x - y\| < \delta$ and $0 \leq t \leq 1$, then

$$f\big(tx + (1-t)y\big) \leq_K tf(x) + (1-t)f(y) + \varepsilon \min[t, (1-t)]\|x - y\|k. \tag{2.6}$$

The class of all uniformly approximate K-paraconvex functions defined on Ω with values in the space Z shall be denoted $UAC_K(\Omega)$.

PROPOSITION 2.3 *Let $(X, \|.\|)$ be a real Banach space. Let $\Omega \subset X$ be an open convex subset. Then $UAC_K(\Omega)$ is a convex cone.*

Proof. Take any $f \in UAC_K(\Omega)$ and any $\lambda > 0$. Since $f \in UAC_K(\Omega)$ for every $\varepsilon > 0$ and $k \in Int_r K$ there is $\delta > 0$ such that

$$f\big(tx + (1-t)y\big) \leq_K tf(x) + (1-t)f(y) + \frac{\varepsilon}{\lambda} \min[t, (1-t)]\|x - y\|k, \quad (2.6)_{\lambda,k}$$

provided $\|x - y\| < \delta$.

Multiplying $(2.6)_{\lambda,k}$ by λ we get

$$\lambda f\big(tx + (1-t)y\big) \leq_K t\lambda f(x) + (1-t)\lambda f(y) + \varepsilon \min[t, (1-t)]\|x - y\|k, \quad (2.7)_{\lambda,k}$$

i.e. $\lambda f \in UAC_K(\Omega)$.

Now, take arbitrary $f, g \in UAC_K(\Omega)$. Since $f \in UAC_K(\Omega)$, (respectively $g \in UAC_K(\Omega)$) for every $\varepsilon > 0$ and $k \in Int_r K$ there is $\delta_f > 0$ (resp. $\delta_g > 0$) such that

$$f\big(tx + (1-t)y\big) \leq_K tf(x) + (1-t)f(y) + \frac{\varepsilon}{2} \min[t, (1-t)]\|x - y\|k, \quad (2.6)_f$$

(respectively

$$g\big(tx + (1-t)y\big) \leq_K tg(x) + (1-t)g(y) + \frac{\varepsilon}{2} \min[t, (1-t)]\|x - y\|k,) \quad (2.6)_g$$

provided $\|x - y\| < \delta_f$ (resp. $\|x - y\| < \delta_g$).

Let $\delta = \min[\delta_f, \delta_g]$. Take $x, y \in \Omega$ such that $\|x - y\| < \delta$. Then, by adding $(2.6)_f$ and $(2.6)_g$ we get

$$(f + g)\big(tx + (1-t)y\big) = f\big(tx + (1-t)y\big) + g\big(tx + (1-t)y\big)$$

$$\leq_K tf(x) + (1-t)f(y) + tg(x) + (1-t)g(y) + \varepsilon \min[t, (1-t)]\|x - y\|k. \quad (2.6)_{f+g}$$

Thus $f + g \in UAC_K(\Omega)$. ∎

Recall that the set of all strongly $\alpha(\cdot)$-K-paraconvex functions defined on Ω with values in the space Z is denoted $\alpha PC_K(\Omega)$. In a similar way as in Proposition 2.3 we can demonstrate

PROPOSITION 2.4 *Let $\alpha(\cdot)$ be a nondecreasing function mapping the interval $[0,+\infty)$ into the interval $[0,+\infty]$ such that*

$$\lim_{t\downarrow 0} \frac{\alpha(t)}{t} = 0. \tag{2.1}$$

Let $(X, \|\cdot\|)$ be a real Banach space. Let Ω be an open convex subset of X. Then $\alpha PC_K(\Omega)$ is a convex cone.

It is trivial that $\alpha PC_K(\Omega) \subset UAC_K(\Omega)$. The following can be shown:

PROPOSITION 2.5 *(compare Rolewicz, 2001b), Let $(X, \|.\|)$ be a real Banach space. Let $\Omega \subset X$ be an open convex subset. Then*

$$\bigcup_{\alpha} \alpha PC_K(\Omega) = UAC_K(\Omega), \tag{2.7}$$

where the union is taken over all nondecreasing functions $\alpha(\cdot)$ mapping the interval $[0,+\infty)$ into the interval $[0,+\infty)$ satisfying (2.1). In other words, a function $f(\cdot)$ is uniformly approximate K-paraconvex if and only if there is $\alpha(\cdot)$ satisfying (2.1) such that the function $f(\cdot)$ is strongly $\alpha(\cdot)$-K-paraconvex.

PROPOSITION 2.6 *Let Ω be an open convex set in a real Banach space X. Let $f(\cdot)$ be a function defined on Ω with values in the Banach space Z ordered by a convex pointed cone K with non-empty interior. Suppose that $f(\cdot)$ is differentiable on Ω and that the differentials of $f\big|_x$ are uniformly continuous functions of x in the norm topology. Then the function $f(\cdot)$ is uniformly approximate K-paraconvex.*

Proof. Since the differentials of $\partial f\big|_x$ are uniformly continuous function of x in the norm topology, there is a function β_0 mapping the interval $[0,+\infty)$ into the interval $[0,+\infty)$ such that

$$\lim_{t\downarrow 0} \beta_0(t) = 0, \tag{2.8}$$

and

$$\|\partial f\big|_x - \partial f\big|_y\| \le \beta_0(\|x-y\|). \tag{2.9}$$

We define

$$F(t) = f\big(tx + (1-t)y\big) - tf(x) + (1-t)f(y).$$

It is easy to observe that $F(0) = F(1) = 0$.

Let ϕ be an arbitrary linear continuous functional of norm 1. Let $F_\phi = \phi(F)$. Now we shall calculate its derivative

$$\frac{dF_\phi}{dt}\Big|_t = \partial\phi(f)\Big|_{(tx+(1-t)y)}(x-y) - \phi(f(x)) + \phi(f(y)). \tag{2.10}$$

Since F_ϕ is real-valued and $F_\phi(0) = F_\phi(1) = 0$, by the Rolle theorem there is $t_0, 0 \leq t_0 \leq 1$, such that $\frac{dF_\phi}{dt}\Big|_{t_0} = 0$. Thus for arbitrary $t, 0 \leq t \leq 1$

$$|\frac{dF_\phi}{dt}\Big|_t| = |\frac{dF_\phi}{dt}\Big|_t - \frac{dF_\phi}{dt}\Big|_{t_0}| \leq \|\partial f\Big|_{(tx+(1-t)y)} - \partial f\Big|_{(t_0x+(1-t_0)y)}(x-y)\|$$

$$\leq \beta_0\Big(\|(tx+(1-t)y) - (t_0x+(1-t_0)y)\|\Big)\|x-y\| \leq \beta_0\Big(\|x-y\|\Big)\|x-y\|$$

$$= \beta\Big(\|x-y\|\Big), \tag{2.11}$$

where the function $\beta(t) = t\beta_0(t)$ satisfies (2.1).

Since $F_\phi(0) = F_\phi(1) = 0$, for $0 \leq t \leq \frac{1}{2}$ by (1.8) we have

$$F_\phi(t) = \int_0^t \frac{dF_\phi}{ds}\Big|_s ds \leq t\beta\Big(\|x-y\|\Big).$$

Similarly, for $\frac{1}{2} \leq t \leq 1$ by (2.11) we have

$$F_\phi(t) = \int_t^1 \frac{dF_\phi}{ds}\Big|_s ds \leq (1-t)\beta\Big(\|x-y\|\Big).$$

Finally,

$$F_\phi(t) \leq \min[t,(1-t)]\beta(\|x-y\|). \tag{2.12}$$

Since ϕ is arbitrary linear functional of norm one this implies that

$$\|F(t)\| \leq \min[t,(1-t)]\beta(\|x-y\|). \tag{2.13}$$

Thus, by definition of $F(t)$

$$\|f\big(tx+(1-t)y\big) - tf(x) + (1-t)f(y)\| \leq \min[t,(1-t)]\beta(\|x-y\|). \tag{2.14}$$

Since the cone K has non-empty interior for each $k \in K$, there is $C > 0$ such that the ball of radius r, $B(r,0) = \{z : \|z\| = r$ is contained in $K - Crk$, $B(r,0) \subset K - Crk$. Thus, from (2.14) it follows that

$$tf(x) + (1-t)f(y) - f\big(tx+(1-t)y\big) \geq_K -C\min[t,(1-t)]\beta(\|x-y\|)k, \tag{2.15}$$

i.e.

$$f\big(tx+(1-t)y\big) \leq_K tf(x) + (1-t)f(y) + C\min[t,(1-t)]\beta(\|x-y\|)k, \tag{2.16}$$

i.e. the function $f(\cdot)$ is strongly $\beta(\cdot)$-paraconvex. Therefore it is uniformly approximate K-paraconvex. ■

As a consequence of Propositions 2.5 and 2.6 we get

EXAMPLE 2.1 *Let Ω be an open convex set in a real Banach space X. Let $f(\cdot)$ be a convex function defined on Ω with values in the Banach space Z ordered by a convex pointed cone K with non-empty interior. Suppose that $f(\cdot)$ is differentiable on Ω and that the differentials of $f\big|_x$ are uniformly continuous function of x in the norm topology. Let $g(\cdot)$ be a differentiable function defined on Ω with values in the space Z. Suppose that the differentials of $g\big|_x$ are uniformly continuous function of x in the norm topology. Then the sum of the functions $f(\cdot)$ and $g(\cdot)$, $f(\cdot)+g(\cdot)$, is uniformly approximate K-paraconvex.*

There is a natural question whether every uniformly approximate K-paraconvex function is a sum of a convex and uniformly differentiable functions. It is not so. We shall present another class of uniformly approximate K-paraconvex functions, based on the following Theorem.

THEOREM 2.2 *Let Ω be an open convex set in a real Banach space Y. Let $f(\cdot)$ be a Lipschitz uniformly approximate K-paraconvex function defined on Ω_Y with values in the Banach space Z ordered by a convex pointed cone K with non-empty interior. Let Ω_X be an open convex set in a real Banach space X. Let σ be a mapping of a Ω_X into Ω_Y such that the differentials of $\sigma\big|_x$ are uniformly continuous function of x in the norm topology. Then the composed function $f(\sigma(\cdot))$ is uniformly approximate K-paraconvex.*

The proof is based on the following

LEMMA 2.1 *(Rolewicz, 2007, 2009) Let Ω_X (Ω_Y) be an open convex set in a real Banach space X (respectively Y). Let σ be a mapping of a Ω_X into Ω_Y such that the differentials of $\partial\sigma\big|_x$ are uniformly continuous functions of x in the norm topology. Then there is a function $\beta(\cdot)$ mapping the interval $[0,+\infty)$ into the interval $[0,+\infty)$ such that*

$$\lim_{t\downarrow 0}\frac{\beta(t)}{t}=0, \tag*{$(2.1)_\beta$}$$

and such that for all $x,y\in\Omega_X$ and $0\le t\le 1$

$$\|\sigma\big(tx+(1-t)y\big)-[t\sigma(x)+(1-t)\sigma(y)]\|\le \min[t,(1-t)]\beta(\|x-y\|). \tag{2.17}$$

Proof. **of Theorem 2.2.** Let k be an arbitrary element of the interior of K, $k\in Int\ K$. Since $f(\cdot)$ is a uniformly approximate K-paraconvex function, there are a nondecreasing function $\alpha(\cdot)$ mapping the interval $[0,+\infty)$ into the interval $[0,+\infty)$ such that

$$\lim_{t\downarrow 0}\frac{\alpha(t)}{t}=0 \tag{2.1}$$

and $C_k>0$ such that for all $y_1,y_2\in\Omega_Y$ and $0\le t\le 1$

$$f\big(ty_1+(1-t)y_2\big) \leq_K tf(y_1)+(1-t)f(y_2)+C_k \min[t,(1-t)]\alpha(\|y_1-y_2\|)k. \quad (2.18)$$

Let $x_1, x_2 \in \Omega_X$. We put $y_1 = \sigma(x_1)$ and $y_2 = \sigma(x_2)$. Then by (2.18)

$$f\big(t\sigma(x_1) + (1-t)\sigma(x_2)\big) \leq_K tf(\sigma(x_1)) + (1-t)f(\sigma(x_2)) + C_k \min[t,(1-t)]\alpha(\|y_1-y_2\|)k. \quad (2.18')$$

Recall that $f(\cdot)$ is a Lipschitz function. We shall denote the Lipschitz constant by M. Thus by Lemma 2.1

$$\|f\Big(\sigma\big(tx_1+(1-t)x_2\big)\Big) - f\Big(t\sigma(x_1)) + (1-t)\sigma(x_2)\Big)\|$$
$$\leq M\|\big(\sigma(tx_1+(1-t)x_2)\big)-t\sigma(x_1)+(1-t)\sigma(x_2)\| \leq M\min[t,(1-t)]\beta(\|x_1-x_2\|). \quad (2.19)$$

Since the cone K has non-empty interior and $k \in Int\ K$, there is $C'_k > 0$ such that for each element z of norm r, $\|z\| = r$, the element z belongs to $K - C'_k rk$. Thus,

$$f(\sigma(tx_1+(1-t)x_2)) \leq_K f\big(t\sigma(x_1) + (1-t)\sigma(x_2))\big) + C'_k M \min[t,(1-t)]\beta(\|x_1-x_2\|)k. \quad (2.20)$$

Since $\sigma(\cdot)$ is also a Lipschitz function, denoting its Lipschitz constant by L, by (2.18') we get

$$f(\sigma(tx_1+(1-t)x_2)) \leq_K tf(\sigma(x_1))+(1-t)f(\sigma(x_2))+C_k\min[t,(1-t)]\alpha(\|y_1-y_2\|)k$$
$$+C'_kM\min[t,(1-t)]\beta(\|x_1-x_2\|)k \leq_K tf(\sigma(x_1)) + (1-t)f(\sigma(x_2))$$
$$+\big(C_k\min[t,(1-t)]L\alpha(\|x_1-x_2\|)+C'_kM\min[t,(1-t)]\beta(\|x_1-x_2\|)\big)k. \quad (2.21)$$

It is easy to see that the function $\alpha_1(u) = C_k L\alpha(u) + C'_k M\beta(u)$ satisfies (2.1) and that by (2.20) the function $f(\sigma(\cdot))$ is strongly $\alpha_1(\cdot)$-paraconvex. Thus, it is a uniformly approximate K-paraconvex function. ■

As a consequence of Proposition 2.6, Example 2.1 and Theorem 2.2 we get

EXAMPLE 2.2 *Let Y be a real Banach space. Let Ω_Y be an open convex set in a Y. Let $f(\cdot)$ be a Lipschitz convex function defined on Ω_Y with values in the Banach space Z ordered by a convex pointed cone K with non-empty interior. Let Ω_X be an open convex set in a real Banach space X. Let $g(\cdot)$ be a differentiable function defined on Ω_X with values in Z. Suppose that the differentials of $g\big|_x$ are uniformly continuous function of x in the norm topology.*

Let $h(\cdot)$ be a real-valued Lipschitz convex function defined on Ω_Y with values in Z. Let σ be a mapping of a Ω_X into Ω_Y such that the differentials of $\sigma\big|_x$ are uniformly continuous function of x in the norm topology.

Then the sum

$$u(\cdot) = f(\cdot) + g(\cdot) + h(\sigma(\cdot)) \tag{2.22}$$

is uniformly approximate K-paraconvex.

Let $(X, \|\cdot\|)$ be a normed space. Let $f(\cdot)$ be a function defined on a subset $\Omega_X \subset X$ with values in the Banach space Z ordered by a convex pointed cone K. We say that the function is *vector bounded* (*vector upper bounded*, *vector bounded from below*) if there is $k \in K$ (respectively $k_u \in K$, $k_b \in K$) such that

$$-k \leq_K f(x) \leq_K k \tag{2.23}$$

(respectively

$$f(x) \leq_K k_u \tag{2.23_u}$$

$$-k_b \leq_K f(x).) \tag{2.23_b}$$

PROPOSITION 2.7 *Let X be a real Banach space X. Let $f(\cdot)$ be a bounded function defined on $\Omega_X \subset X$ with values in the Banach space Z ordered by a convex pointed cone K with non-empty interior. Then $f(\cdot)$ is a vector bounded function.*

Proof. Since the function $f(\cdot)$ is bounded, there is $M > 0$ such that for all $x \in \Omega_X$

$$\|f(x)\| \leq M. \tag{2.24}$$

The cone K has non-empty interior and $k \in Int\ K$. Thus, there is $C'_k > 0$ such that for each element z of norm less than M, $\|z\| \leq_K M$, the element z belongs to $(K - C'_k Mk) \cap -K + C'_k Mk)$. Hence

$$-C'_k Mk \leq_K f(x) \leq_K C'_k Mk \tag{2.25}$$

i.e. $f(\cdot)$ is a vector bounded function. ∎

PROPOSITION 2.8 *Let X be a real Banach space X. Let $f(\cdot)$ be a vector bounded function defined on $\Omega_X \subset X$ with values in the Banach space Z ordered by a convex pointed cone K with bounded basis. Then $f(\cdot)$ is a bounded function.*

Proof. Since the function $f(\cdot)$ is vector bounded, there is $k \in K$ such that for all $x \in \Omega_X$

$$-k \leq_K f(x) \leq_K k. \tag{2.23}$$

The cone K has bounded basis. Thus, there is $M > 0$ such that

$$\big[(K - k) \cap (-K + k)\big] \subset \{z \in Z : \|z\|\} \leq M.$$

So for all $x \in \Omega_X$
$$\|f(x)\| \leq M, \tag{2.24}$$
i.e. $f(\cdot)$ is a bounded function. ■

Let $(X, \|\cdot\|)$ be a real Banach space. Let $f(\cdot)$ be a function defined on a subset $\Omega_X \subset X$ with values in the Banach space Z ordered by a convex pointed cone K. We say that the function is *vector Lipschitz* if there is $k \in K$ (respectively $k_u \in K$, $k_b \in K$) such that

$$-\|x - x'\|k \leq_K f(x) - f(x') \leq_K \|x - x'\|k \tag{2.26}$$
for arbitrary $x, x' \in \Omega$.

PROPOSITION 2.9 *Let X be a Banach space X. Let $f(\cdot)$ be a Lipschitz function defined on $\Omega_X \subset X$ with values in the Banach space Z ordered by a convex pointed cone K with non-empty interior. Then $f(\cdot)$ is a vector Lipschitz function.*

Proof. Since the function $f(\cdot)$ is Lipschitz, there is $M > 0$ such that for all $x, x' \in \Omega_X$.

$$\|f(x) - f(x')\| \leq M\|x - x'\| \tag{2.27}$$

The cone K has non-empty interior and $k \in Int\ K$. Thus there is $C'_k > 0$ such that for each element z of norm less than M, $\|z\| \leq_K M$, the element z belongs to $(K - C'_k Mk) \cap -K + C'_k Mk)$. Hence
$$-C'_k M\|x - x'\|k \leq_K f(x) - f(x') \leq_K C'_k M\|x - x'\|k \tag{2.28}$$
i.e. $f(\cdot)$ is a vector Lipschitz function. ■

PROPOSITION 2.10 *Let Ω be a set in a real Banach space X. Let $f(\cdot)$ be a vector Lipschitz function defined on $\Omega_X \subset X$ with values in the Banach space Z ordered by a convex pointed cone K with bounded basis. Then $f(\cdot)$ is a Lipschitz function.*

Proof. Since the function $f(\cdot)$ is vector Lipschitz, there is $k \in K$ such that for all $x, x' \in \Omega_X$

$$-k \leq_K f(x) - f(x') \leq_K k. \tag{2.29}$$

The cone K has bounded basis. Thus
$$\big[(K - Mk) \cap -K + Mk)\big] \subset \{z \in Z : \|z\| \leq M\}.$$
So, for all $x \in \Omega_X$
$$\|f(x)\| \leq M, \tag{2.24}$$
i.e. $f(\cdot)$ is a bounded function. ■

3. Localization

In this section we shall investigate localization of the notions of uniformly approximate K-paraconvex functions and strongly $\alpha(\cdot)$-K-paraconvex functions.

Let X be a real Banach space. Let $f(\cdot)$ be a mapping defined on an open subset $\Omega \subset X$ with values in the Banach space Z ordered by a convex pointed cone K. We say that $f(\cdot)$ is *locally uniformly approximate K-paraconvex* if for all $x_0 \in \Omega$ there is a convex open neighbourhood U_{x_0} of x_0 such that the function $f(\cdot)$ restricted to U_{x_0}, $f\Big|_{U_{x_0}}(\cdot)$, is uniformly approximate K-paraconvex. In other words, a function $f(\cdot)$ is locally uniformly approximate K-paraconvex if for all $x_0 \in \Omega$ there is a convex open neighbourhood U_{x_0} of x_0 such that for arbitrary $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, U_{x_0})$ such that for $x, z \in U_{x_0}$ such that $\|x - x_0\| < \delta$ and $\|z - x_0\| < \delta$ and $0 \leq t \leq 1$ and every k belonging to the relative interior of K, $k \in K$ there is C_k such that

$$f\big(tx + (1-t)z\big) \leq_K tf(x) + (1-t)f(z) + C_k \varepsilon \min[t, (1-t)]\|x - z\|k. \tag{2.6}$$

The class of all locally uniformly approximate K-paraconvex functions defined on Ω shall be denoted $UAC_K^{Loc}(\Omega)$.

Basing on the definition of locally uniformly approximate K-paraconvex function and Proposition 2.3 we can easily demonstrate

PROPOSITION 3.1 *Let $(X, \|.\|)$ be a real Banach space. Let Ω be an open subset of X, $\Omega \subset X$. Then, $UAC_K^{Loc}(\Omega)$ is a convex cone.*

Proof. Take any $f \in UAC_K^{Loc}(\Omega)$ and any $\lambda > 0$. Take arbitrary $x_0 \in \Omega$. By definition there is a convex open neighbourhood U_{x_0} of x_0 such that the function $f(\cdot)$ restricted to U_{x_0}, $f\Big|_{U_{x_0}}(\cdot)$, is uniformly approximate K-paraconvex. Thus, by Proposition 2.4. $\lambda f\Big|_{U_{x_0}}(\cdot)$ is uniformly approximate K-paraconvex, too. Therefore $\lambda f \in UAC_K^{Loc}(\Omega)$.

Take any $f, g \in UAC_K^{Loc}(\Omega)$. Take arbitrary $x_0 \in \Omega$. By definition there are convex open neighbourhoods $U_{x_0}^f$ of x_0 and $U_{x_0}^g$ of x_0 such that the function $f(\cdot)$ restricted to $U_{x_0}^f$, $f\Big|_{U_{x_0}^f}(\cdot)$ and the function $g(\cdot)$ restricted to $U_{x_0}^g$, $g\Big|_{U_{x_0}^g}(\cdot)$ are uniformly approximate K-paraconvex. Let $U_{x_0} = U_{x_0}^f \cap U_{x_0}^g$. Thus by Proposition 2.4. $(f+g)\Big|_{U_{x_0}}(\cdot)$ is uniformly approximate K-paraconvex. Therefore, $(f + g) \in UAC_K^{Loc}(\Omega)$. ∎

Let X be a real Banach space. Let $\alpha(\cdot)$ be a nondecreasing function mapping the interval $[0, +\infty)$ into the interval $[0, +\infty)$, satisfying (2.1). Let $f(\cdot)$ be a mapping defined on an open subset $\Omega \subset X$ with values in the Banach space Z

ordered by a convex pointed cone K. We say that $f(\cdot)$ is *locally strongly* $\alpha(\cdot)$*-*K*-paraconvex* if for all $x_0 \in \Omega$ there is a convex open neighbourhood U_{x_0} of x_0 such that the function $f(\cdot)$ restricted to U_{x_0}, $f\Big|_{U_{x_0}}(\cdot)$, is strongly $\alpha(\cdot)$-Kparaconvex.

The set of all locally strongly $\alpha(\cdot)$-paraconvex functions defined on Ω shall be denoted $\alpha PC^{Loc}(\Omega)$. In a similar way as in Proposition 3.1 we can demonstrate

PROPOSITION 3.2 *Let* $\alpha(\cdot)$ *be a nondecreasing function mapping the interval* $[0, +\infty)$ *into the interval* $[0, +\infty)$ *satisfying (1.2). Let* $(X, \|.\|)$ *be a real Banach space. Let* Ω *be an open convex subset of* X*. Then* $\alpha PC^{Loc}(\Omega)$ *is a convex cone.*

Let $(X, \|\cdot\|)$ be a normed space. Let $f(\cdot)$ be a function defined on an open subset $\Omega \subset X$ with values in the Banach space Z ordered by a convex pointed cone K.

We say that a function $f(\cdot)$ is *locally bounded* if for any $x_0 \in \Omega$, there is a convex neighbourhood U_{x_0} of the point x_0 such that the restriction of the function $f(\cdot)$ to the set U_{x_0}, $f|_{U_{x_0}}(\cdot)$ is bounded.

We say that the function $f(\cdot)$ is *locally vector Lipschitz* if for any $x_0 \in \Omega$, there is a convex neighbourhood U_{x_0} of the point x_0 such that the restriction of the function $f(\cdot)$ to the set U_{x_0}, $f|_{U_{x_0}}(\cdot)$ is vector Lipschitz.

Repeating the considerations of Jourani (1996) we shall prove

PROPOSITION 3.3 *Let* $(X, \|\cdot\|)$ *be a normed space. Let a function* $f(\cdot)$ *defined on an open subset* $\Omega \subset X$ *with values in the Banach space* Z *ordered by a convex pointed cone* K *be locally strongly* $\alpha(\cdot)$*-*K*-paraconvex and locally vector bounded. Then it is locally vector Lipschitz.*

Proof. Let $x_0 \in \Omega$ be arbitrary. Since f is locally bounded, there are $k \in Int_r\, K$ and $r > 0$ such that for any $y \in \Omega$ such that $\|y - x_0\| < r$ we have

$$-k \leq_K f(y) \leq_K k. \tag{3.1}$$

Let x, u be two arbitrary elements of Ω such that $\|x - x_0\| < \frac{r}{2}$, $\|u - x_0\| < \frac{r}{2}$. Let ε be an arbitrary positive number, let $\beta = \varepsilon + \|x - u\|$ and let

$$v = u + \frac{r}{2\beta}(u - x). \tag{3.2}$$

Observe that

$$\|v - x_0\| < \|u - x_0\| + \frac{r}{2\beta}\|u - x\| < \frac{r}{2} + \frac{r}{2}\frac{\|x - u\|}{\varepsilon + \|x - u\|} < r$$

and so

$$-k \leq_K f(v) \leq_K k. \tag{3.1_v}$$

Let $\lambda = \frac{2\beta}{r+2\beta}$. Observe that $u = \lambda v + (1-\lambda)x$.

Since the function $f(\cdot)$ is strongly $\alpha(\cdot)$-K-paraconvex, there is a constant $C > 0$, such that

$$f(u) = f\big(\lambda v + (1-\lambda)x\big) \leq_K \lambda f(v) + (1-\lambda)f(x) + C\lambda\alpha(\|x-v\|)k. \tag{3.2}$$

Thus,

$$f(u) - f(x) \leq_K \lambda(f(v) - f(x)) + C\lambda\alpha(\|x-v\|)k. \tag{3.3}$$

Since $\lambda\|v-x\| = \|u-x\|$, we get

$$f(u) - f(x) \leq_K \lambda(f(v) - f(x)) + C\lambda\alpha(\frac{\|u-x\|}{\lambda})k. \tag{3.3}$$

Recall that $0 < \lambda < 1$ and thus

$$f(u) - f(x) \leq_K \lambda(f(v) - f(x)) + C\lambda\alpha(\|x-v\|)k \leq_K \lambda\big(2 + C\alpha(2r)\big)k$$

$$\leq_K \frac{2\beta}{r}\big(2a + C\alpha(2r)\big)k \leq L(\varepsilon + \|u-x\|)k, \tag{3.4}$$

where $L = \frac{2}{r}\big(2a + C\alpha(2r)\big)$.

By exchanging the roles of x and u we get

$$f(x) - f(u) \leq_K L(\varepsilon + \|u-x\|)k. \tag{3.6}$$

Thus

$$-L(\varepsilon + \|u-x\|)k \leq_K f(u) - f(x). \tag{3.7}$$

By (3.4) and (3.7) and the arbitrariness of ε we obtain

$$-L\|u-x\|k \leq_K f(u) - f(x) \leq_K L\|u-x\|k. \tag{3.8}$$

■

PROPOSITION 3.4 *Let $(X, \|\cdot\|)$ be a normed space. Let a function $f(\cdot)$ defined on an open subset $\Omega \subset X$ with values in the Banach space Z ordered by a convex pointed cone K with non-empty interior be continuous. Then it is locally vector bounded.*

Proof. Let $x_0 \in \Omega$. Since the function $f(\cdot)$ is continuous for every $\varepsilon > 0$, there is a neighbourhood U_{x_0} of the point x_0 such that for any $x \in U_{x_0}$

$$\|f(x) - f(x_0)\| < \varepsilon. \tag{3.9}$$

The cone K has non-empty interior for any $k \in Int\ K$. Therefore there is $C'_k > 0$ such that

$$-C'_k \varepsilon k \leq_K f(x) - f(x_0) \leq_K C'_k \varepsilon k \tag{3.10}$$

and simultaneuosly

$$-C'_k \|f(x_0)\| k \leq_K f(x) - f(x_0) \leq_K C'_k \|f(x_0)\| k. \tag{3.11}$$

Finally,

$$-C'_k(\|f(x_0)\| + \varepsilon)\|k \leq_K f(x) \leq_K C'_k(\|f(x_0)\| + \varepsilon)k. \tag{3.12}$$

■

Without the assumption that K is open, Proposition 3.5 does not hold as shown by the following simple example.

EXAMPLE 3.1 *Let $X = [0,1]$. Let $Z = \mathbb{R}^2$ and $K = \{(0,t), t \geq 0\}$. Then the function $f(t) = (t,0)$ is continuous but it is not locally vector bounded.*

By Propositions 3.3 and 3.4 we get

COROLLARY 3.1 *Let $(X, \|\cdot\|)$ be a normed space. Let a function $f(\cdot)$ defined on an open subset $\Omega \subset X$ with values in the Banach space Z ordered by a convex pointed cone K with non-empty interior be continuous and locally strongly $\alpha(\cdot)$-K-paraconvex. Then it is locally vector Lipschitz.*

Basing on Theorem 2.2 and Proposition 3.3 we can prove

PROPOSITION 3.5 *Let Ω_X (Ω_Y) be an open set in a real Banach space X (respectively Y). Let $f(\cdot)$ be a locally Lipschitz uniformly approximate K-paraconvex function defined on Ω_Y with values in the Banach space Z ordered by a convex pointed cone K with non-empty interior. Let Ω_X be an open convex set in a real Banach space X. Let σ be a mapping of a Ω_X into Ω_Y such that the differentials of $\sigma\big|_x$ are uniformly continuous function of x in the norm topology. Then the composed function $f(\sigma(\cdot))$ is locally uniformly approximate K-paraconvex.*

Proof. Take arbitrary $x \in \Omega_X$. Since $f(\cdot)$ is a locally uniformly approximate K-paraconvex function, there is a convex neighbourhood $\hat{U}_{\sigma(x)}$ of $\sigma(x)$ such that the restriction of $f(\cdot)$ to the set $\hat{U}_{\sigma(x)}$, $f\Big|_{\hat{U}_{\sigma(x)}}(\cdot)$, is uniformly approximate K-paraconvex. Without loss of generality we may assume that that the restriction of $f(\cdot)$ to the set $\hat{U}_{\sigma(x)}$, $f\Big|_{\hat{U}_{\sigma(x)}}(\cdot)$ is a Lipschitz function, since each uniformly approximate K-paraconvex function is locally Lipschitz.

By our assumptions the differentials of $\sigma\big|_x$ are locally uniformly continuous function of x. Thus, there is a convex neighbourhood U_x of x such that the differentials of $\sigma\big|_{U_x}$ are uniformly continuous function of its argument on U_x. Without loss of generality we may assume that

$$\sigma(U_x) \subset \hat{U}_{\sigma(x)}. \tag{3.13}$$

Thus, by Theorem 2.2 $f\Big|_{U_x}(\sigma(\cdot))$ is uniformly approximate K-paraconvex function. Therefore, by definition, $f(\sigma(\cdot))$ is a locally uniformly approximate K-paraconvex function defined on Ω_X. ∎

Using Theorem 2.1 we can obtain

THEOREM 3.1 *Let Ω_Y be an open convex set in a real Banach space Y. Let Z be n-dimensional Banach space ordered by a convex pointed cone K with non-empty interior. Let $f(\cdot)$ be a locally uniformly approximate K-paraconvex function function defined on Ω_Y with values in Z. Let Ω_X be an open convex set in a real Banach space X. Let σ be a mapping of a Ω_X into Ω_Y such that the differentials of $\sigma\Big|_x$ are uniformly continuous function of x in the norm topology. Then the composed function $f(\sigma(\cdot))$ is:*

(a). *Fréchet differentiable on a dense G_δ-set provided X is an Asplund space,*

(b). *Gateaux differentiable on dense G_δ-set provided X is separable.*

Proof. We denote by D the set of points of Ω_X for which the composed function $f(\sigma(\cdot))$ is Fréchet differentiable in case (a) and Gateaux differentiable in case (b).

By Proposition 3.5 the composed function $f(\sigma(\cdot))$ is locally uniformly approximate K-paraconvex. Recall that by Proposition 3.3 each locally uniformly approximate K-paraconvex functions is also locally Lipschitz.

Therefore, there is an open covering $\mathfrak{U}=\{U_\gamma\}, \gamma \in \Gamma$ of Ω_X such that for each $\gamma \in \Gamma$, the restricted function $f(\sigma\Big|_{U_\gamma}(\cdot))$ is uniformly approximate K-paraconvex and vector Lipschitz.

Therefore, as a simple consequence of Theorem 2.1, we get that the set $D \cap U_\gamma$ for which the composed function $f(\sigma(\cdot))$ is Fréchet differentiable in case (a) (see Rolewicz, 1999, 2002, 2005) and Gateaux differentiable in case (b) (see Rolewicz, 2006) is a G_δ-set.

This means that D is a local G_δ-set.

Hence, by the Michael theorem (Michael, 1954) D is a G_δ-set. ∎

4. Differentiability of locally uniformly approximate K-paraconvex functions with values in finite dimensional spaces on $C^{1,u}_{\mathbf{E}}$-manifolds

As an application of Theorem 3.1 we get a result concerning differentiability of locally uniformly approximate K-paraconvex functions on manifolds with values in finite dimensional spaces.

Let $\mathbf{E}$, $\mathbf{F}$, be real Banach spaces. We say that a function $\psi : \mathbf{E} \to \mathbf{F}$ is of the class $C^{1,u}_{\mathbf{E},\mathbf{F}}$ if it is continuously differentiable and, moreover, that differential $\partial\psi\Big|_x$ is locally uniformly continuous as a function of x in the norm topology. Of course, if $\psi \in C^{1,u}_{\mathbf{E},\mathbf{F}}$, then ψ belongs to the class of continuously differentiable functions, $\psi \in C^{1}_{\mathbf{E},\mathbf{F}}$.

If $\mathbf{E} = \mathbf{F}$ we denote briefly $C^{1,u}_{\mathbf{E},\mathbf{E}} = C^{1,u}_{\mathbf{E}}$.

Now we shall determine $C^{1,u}_{\mathbf{E}}$-manifold in the classical way (compare Lang, 1962).

Let X be a set. An $C^{1,u}_{\mathbf{E}}$-*atlas* is a collections of pairs (U_i, ϕ_i) (i ranging in some indexing set) satisfying the following conditions:

AT 1. Each U_i is a subset of X and $\{U_i\}$ covers X,

AT 2. Each ϕ_i is a bijection of U_i onto an open subset $\phi_i(U_i)$ of the space $\mathbf{E}$, and for all i, j, $\phi_i(U_i \cap U_j)$ is an open subset of the space $\mathbf{E}$,

AT 3. The map $\phi_j\phi_i^{-1}$ mapping $\phi_i(U_i \cap U_j)$ onto $\phi_j(U_i \cap U_j)$ is of the class $C^{1,u}_{\mathbf{E}}$ for all i, j.

Each pair (U_i, ϕ_i) is called a *chart*. If $x \in U_i$, then the pair (U_i, ϕ_i) is called a *chart at* x.

Observe that AT 3 implies that $\left(\phi_j\phi_i^{-1}\right)^{-1} = \phi_i\phi_j^{-1} \in C^{1,u}_{\mathbf{E}}$.

Suppose now that X is a topological space and let U be an open set in X. Suppose that there is a topological isomorphism ϕ mapping U onto an open set $U' \in \mathbf{E}$. We say that (U, ϕ) is *compatible* with the $C^{1,u}_{\mathbf{E}}$-atlas (U_i, ϕ_i) if for all i the maps $\phi_i\phi^{-1}$ and $\phi\phi_i^{-1}$ belong to $C^{1,u}_{\mathbf{E}}$. We say that two $C^{1,u}_{\mathbf{E}}$-atlases are *compatible* if each chart of one is compatible with the other $C^{1,u}_{\mathbf{E}}$-atlas.

A topological space X equipped with $C^{1,u}_{\mathbf{E}}$-atlas (U_i, ϕ_i) shall be called $C^{1,u}_{\mathbf{E}}$-*manifold*.

DEFINITION 4.1 *We say that a function $f(\cdot)$ defined on a $C^{1,u}_{\mathbf{E}}$-manifold X with values in the Banach space Z ordered by a convex pointed cone K with non-empty interior is locally uniformly approximate K-paraconvex on X if there is a $C^{1,u}_{\mathbf{E}}$-atlas (U_i, ϕ_i) such that for all i the function $f(\phi_i^{-1}(\cdot))$ is locally uniformly approximate K-paraconvex on the set $\phi_i(U_i) \subset \mathbf{E}$.*

As an immediate consequence of AT 3 and Proposition 2.2 we obtain

PROPOSITION 4.1 *Let X be topological space. Let $C^{1,u}_{\mathbf{E}}$-atlas (U_i, ϕ_i) on X. If a function $f(\phi_i^{-1}(\cdot))$ is locally uniformly approximate K-paraconvex on $U_i \cap U_j$ then the function $f(\phi_j^{-1}(\cdot))$ is also locally uniformly approximate K-paraconvex on $U_i \cap U_j$.*

As a consequence of definition of compatibility of $C^{1,u}_{\mathbf{E}}$-atlases and Proposition 3.5 we obtain

PROPOSITION 4.2 *Let X be topological space. Let (U_i, ϕ_i) and (V_j, μ_j) be two compatible $C^{1,u}_{\mathbf{E}}$-atlases on X. Let $f(\cdot)$ be a function defined on X with values in the Banach space Z ordered by a convex pointed cone K with non-empty interior. If a function $f(\cdot)$ is locally uniformly approximate K-paraconvex with respect to the $C^{1,u}_{\mathbf{E}}$-atlas (U_i, ϕ_i), then it is also locally uniformly approximate K-paraconvex with respect to the $C^{1,u}_{\mathbf{E}}$-atlas (V_j, μ_j).*

Let X be a $C^{1,u}_{\mathbf{E}}$-manifold. Let (U_i, ϕ_i) be a $C^{1,u}_{\mathbf{E}}$-atlas on X. Let $f(\cdot)$ be a function defined X with values in the Banach space Z ordered by a convex pointed cone K with non-empty interior. We say that the function $f(\cdot)$ is *Fréchet (Gateaux) differentiable at $x_0 \in U_i$* if the function $f(\phi_i^{-1}(\cdot))$ is Fréchet (respectively Gateaux) differentiable at $\phi_i(x_0)$.

Basing on this definition and Theorem 3.1 we get

THEOREM 4.1 *Let X be a $C^{1,u}_{\mathbf{E}}$-manifold. Let $f(\cdot)$ be a function defined X with values in the finite dimensional Banach space Z ordered by a convex pointed cone K with non-empty interior. Suppose that the function $f(\cdot)$ is locally uniformly approximate K-paraconvex function defined on X. Then it is:* (a). *Fréchet differentiable on a dense G_δ-set provided $\mathbf{E}$ is an Asplund space,* (b). *Gateaux differentiable on a dense G_δ-set provided $\mathbf{E}$ is separable.*

Now we shall determine $C^{1,u}_{\mathbf{E}}$-submanifold in the classical way (compare Lang, 1962).

Let X be a $C^{1,u}_{\mathbf{E}}$-manifold. Let Y be a subset of X. We assume that for each point $y \in Y$ there exists a chart (V, ψ) in X such that $V_1 = \psi(V \cap Y)$ is an open set in some Banach subspace $\mathbf{E}_1 \subset \mathbf{E}$. The map ψ induces a bijection

$$\psi_1 : Y \cap V \to V_1 \tag{4.1}$$

and, moreover, $\psi_1 \in C^{1,u}_{\mathbf{E}_1}$

The collection of pairs $(Y \cap V, \psi_1)$ obtained in the above manner constitute the atlas for Y. We shall call Y $C^{1,u}_{\mathbf{E}_1}$*-submanifold* of X.

THEOREM 4.2 *Let X be a $C^{1,u}_{\mathbf{E}}$-manifold. Let Y be an its $C^{1,u}_{\mathbf{E}_1}$-submanifold. Let $f(\cdot)$ be a function defined on X with values in the finite dimensional Banach space Z ordered by a convex pointed cone K with non-empty interior. Suppose that the function $f(\cdot)$ is locally uniformly approximate K-paraconvex function defined on X. Then the restriction $f\Big|_Y$ is locally uniformly approximate K-paraconvex function defined on Y.*

Proof. By our assumption the function $f(\psi^{-1})$ is a locally uniformly approximate K-paraconvex function on $\psi(V)$. Thus, its restriction $f(\psi^{-1})\Big|_Y(\cdot) = f(\psi_1^{-1}(\cdot))$ to V_1 is also a locally uniformly approximate K-paraconvex function and by definition $f\Big|_Y$ is locally uniformly approximate K-paraconvex function defined on Y. ∎

As an obvious consequence of Theorems 4.1 and 4.2 we obtain

THEOREM 4.3 *Let X be a $C^{1,u}_{\mathbf{E}}$-manifold. Let Y be an its $C^{1,u}_{\mathbf{E}_1}$-submanifold. Let $f(\cdot)$ be a function defined on X with values in the finite dimensional Banach space Z ordered by a convex pointed cone K with non-empty interior. Suppose that the function $f(\cdot)$ is locally uniformly approximate K-paraconvex function. Then the restriction $f\Big|_Y$ is:*
(a). *Fréchet differentiable on a dense G_δ-set provided $\mathbf{E}_1$ is an Asplund space,*
(b). *Gateaux differentiable on a dense G_δ-set provided $\mathbf{E}_1$ is separable.*

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