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Some aggregating operators based on the Choquet integral with fuzzy number intuitionistic fuzzy information and their applications to multiple attribute decision making*

by

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Abstract: With respect to multiple attribute decision making (MADM) problems in which attribute values take the form of fuzzy number intuitionistic fuzzy values, a new decision making analysis method is developed. First, some operational laws of fuzzy number intuitionistic fuzzy values, score function and accuracy function of fuzzy number intuitionistic fuzzy values are introduced. Then, we have developed two fuzzy number intuitionistic fuzzy Choquet integral aggregation operators: fuzzy number intuitionistic fuzzy Choquet ordered averaging (FNIFCOA) operator and fuzzy number intuitionistic fuzzy Choquet ordered geometric mean (FNIFCOGM) operator. The prominent characteristic of the operators is that they can not only consider the importance of the elements or their ordered positions, but also reflect the correlation among the elements or their ordered positions. We have studied some desirable properties of the FNIFCOA and FNIFCOGM operators, such as commutativity, idempotency and monotonicity, and applied the FNIFCOA and FNIFCOGM operators to multiple attribute decision making with fuzzy number intuitionistic fuzzy information. Finally an illustrative example has been given to show the developed method.

Keywords: fuzzy number intuitionistic fuzzy values; operational laws; fuzzy number intuitionistic fuzzy Choquet ordered averaging (FNIFCOA) operator; fuzzy number intuitionistic fuzzy Choquet ordered geometric mean (FNIFCOGM) operator

1. Introduction

Atanassov (1986) introduced the concept of intuitionistic fuzzy set (IFS), which is a generalization of the concept of fuzzy set (Zadeh, 1965). The intuitionistic

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fuzzy sets have been gaining increasing attention. Gau and Buehrer (1993) introduced the concept of vague set. But Bustince and Burillo (1996) showed that vague sets are intuitionistic fuzzy sets. Xu and Yager (2006) developed some geometric aggregation operators. Xu (2007a) developed the intuitionistic fuzzy weighted averaging (IFWA) operator, the intuitionistic fuzzy ordered weighted averaging (IFOWA) operator, and the intuitionistic fuzzy hybrid aggregation (IFHA) operator. Wei (2008) utilized the maximizing deviation method for intuitionistic fuzzy multiple attribute decision making with incomplete weight information. Wei and Zhao (2011) established the minimum deviation models for multiple attribute decision making in intuitionistic fuzzy setting. Wei (2011b) developed the grey relational analysis method for intuitionistic fuzzy multiple attribute decision making with preference information on alternatives. Wei and Zhao (2012) proposed some induced correlated aggregating operators with intuitionistic fuzzy information and applied these operators to multiple attribute group decision making. Atanassov and Gargov (1989) introduced the concept of interval-valued intuitionistic fuzzy sets (IVIFSs) as a further generalization of IFSs, as well as of interval valued fuzzy sets. Atanassov (1994) defined some operational laws of the IVIFSs. Xu (2007) developed the interval-valued intuitionistic fuzzy weighted geometric (IIFWG) operator and the interval-valued intuitionistic fuzzy geometric (IIFG) operator and gave an application of the IIFWG and IIFG operators to multiple attribute group decision making. Xu and Chen (2007) developed some arithmetic aggregation operators. Ye (2009) proposed multicriteria fuzzy decision-making method based on a novel accuracy function under interval-valued intuitionistic fuzzy environment. Wei (2009) proposed some geometric aggregation functions for dynamic multiple attribute decision making in intuitionistic fuzzy setting. Wei (2010) developed some induced geometric aggregation operators with intuitionistic fuzzy information. Li (2010) proposed linear programming method for MADM with interval-valued intuitionistic fuzzy sets. Liu and Yuan (2007) introduced the concept of fuzzy number intuitionistic fuzzy set (FNIFS). The fundamental characteristic of the FNIFS is that the values of its membership function and non-membership function are triangular fuzzy numbers rather than exact numbers. Wang (2008a) developed the fuzzy number intuitionistic fuzzy weighted averaging (FNIFWA) operator, fuzzy number intuitionistic fuzzy ordered weighted averaging (FNIFOWA) operator and fuzzy number intuitionistic fuzzy hybrid aggregation (FNIFHA) operator. Wang (2008b) proposed some aggregation operators, including fuzzy number intuitionistic fuzzy weighted geometric (FNIFWG) operator, fuzzy number intuitionistic fuzzy ordered weighted geometric (FNIFOWG) operator and fuzzy number intuitionistic fuzzy hybrid geometric (FNIFHG) operator. All of the existing fuzzy number intuitionistic fuzzy aggregation operators only consider situations where all the elements in the fuzzy number intuitionistic fuzzy set are independent. However, in many practical situations, the elements in the fuzzy number intuitionistic fuzzy set are correlated. Therefore, we need to find some new ways to deal with these situations in which the decision data in question are correlated. The Choquet integral is a very useful way of measuring the expected utility of an uncertain event, and can be utilized to depict the correlation of the decision data under consideration.

Motivated by the correlation properties of the Choquet integral, in this paper we propose some fuzzy number intuitionistic fuzzy aggregation operators, whose prominent characteristic is that they can not only consider the importance of the elements or their ordered positions, but also reflect the correlation of the elements or their ordered positions. To do so, the remainder of this paper is set out as follows. In the next section, we introduce some basic concepts related to fuzzy number intuitionistic fuzzy sets and some operational laws of fuzzy number intuitionistic fuzzy numbers. In Section 3 we have developed two fuzzy number intuitionistic fuzzy Choquet integral aggregation operators: fuzzy number intuitionistic fuzzy Choquet ordered averaging (FNIFCOA) operator and fuzzy number intuitionistic fuzzy Choquet ordered geometric mean (FNIFCOGM) operator. In Section 4, we have developed an approach to multiple attribute decision making based on FNIFCOA operator and FNIFCOGM operator with fuzzy number intuitionistic fuzzy information. In Section 5, an illustrative example is presented. In Section 6, we conclude the paper and give some remarks.

2. Preliminaries

Atanassov (1986) extended the fuzzy set to the IFS as follows:

Definition 1 *Given a fixed set $X = \{x_1, x_2, \cdots, x_n\}$, an intuitionistic fuzzy set (IFS) is defined as*

$$A = (\langle x_i, t_A(x_i), f_A(x_i)\rangle \,|x_i \in X) \tag{1}$$

which assigns to each element x_i a membership degree $t_A(x_i)$ and a non-membership degree $f_A(x_i)$ under the condition
$0 \le t_A(x_i) + f_A(x_i) \le 1$, *for all $x_i \in X$.*

Atanassov and Gargov (1989) further introduced the interval-valued intuitionistic fuzzy set (IVIFS), a generalization of the IFS. The fundamental characteristic of the IVIFS is that the values of its membership function and non-membership function are intervals rather than exact numbers.

Definition 2 *Given a fixed set $X = \{x_1, x_2, \cdots, x_n\}$, an IVIFSs $\tilde{A}$ over X is an object having the form:*

$$\tilde{A} = \left\{\left\langle x_i, \tilde{t}_A(x_i), \tilde{f}_A(x_i)\right\rangle |x_i \in X\right\}, \tag{2}$$

where $\tilde{t}_A(x_i) \subset [0,1]$ and $\tilde{f}_A(x_i) \subset [0,1]$ are interval numbers, and

$$0 \le \sup \tilde{t}_A(x_i) + \sup \tilde{f}_A(x_i) \le 1, , \forall\, x_i \in X.$$

Liu and Yuan (2007) introduced the concept of fuzzy number intuitionistic fuzzy set (FNIFS), and the fundamental characteristic of the FNIFS is that the values of its membership function and non-membership function are triangular fuzzy numbers.

Definition 3 *Given a fixed set* $X = \{x_1, x_2, \cdots, x_n\}$*, a FNIFS* $\tilde{A}$ *over* X *is an object having the form:*

$$\tilde{A} = \left\{ \left\langle x_i, \tilde{t}_A(x_i), \tilde{f}_A(x_i) \right\rangle | x_i \in X \right\}, \tag{3}$$

where $\tilde{t}_A(x_i) \subset [0,1]$ *and* $\tilde{f}_A(x_i) \subset [0,1]$ *are triangular fuzzy numbers, and*

$$\tilde{t}_A(x_i) = (a(x_i), b(x_i), c(x_i)), X \to [0,1],$$
$$\tilde{f}_A(x) = (l(x_i), m(x_i), p(x_i)), X \to [0,1], 0 \leq c(x_i) + p(x_i) \leq 1, \forall\, x \in X$$

For convenience, let $\tilde{t}_A(x_i) = (a(x_i), b(x_i), c(x_i)), \tilde{f}_A(x) = (l(x_i), m(x_i), p(x_i))$*, so* $\tilde{a}(x_i) = \langle (a(x_i), b(x_i), c(x_i)), (l(x_i), m(x_i), p(x_i)) \rangle$ *and we call* $\tilde{a}(x_i)$ *a fuzzy number intuitionistic fuzzy value (FNIFV).*

Definition 4 *Let* $\tilde{a}(x_i) = \langle (a(x_i), b(x_i), c(x_i)), (l(x_i), m(x_i), p(x_i)) \rangle$ *and* $\tilde{a}(x_j) = \langle (a(x_j), b(x_j), c(x_j)), (l(x_j), m(x_j), p(x_j)) \rangle$ *be two FNIFVs, then*
(1) $\tilde{a}(x_i) \oplus \tilde{a}(x_j)$

$$= \langle (a(x_i) + a(x_j) - a(x_i)a(x_j), b(x_i) + b(x_j) - b(x_i)b(x_j), c(x_i) + c(x_j) - c(x_i)c(x_j)), (l(x_i)l(x_j), m(x_i)m(x_j), p(x_i)p(x_j)) \rangle;$$

(2) $\tilde{a}(x_i) \otimes \tilde{a}(x_j) = \langle (a(x_i)a(x_j), b(x_i)b(x_j), c(x_i)c(x_j)),$

$$(l(x_i) + l(x_j) - l(x_i)l(x_j), m(x_i) + m(x_j) - m(x_i)m(x_j), p(x_i) + p(x_j) - p(x_i)p(x_j)) \rangle;$$

(3) $\lambda\tilde{a}(x_i) = \left\langle \left(1 - (1 - a(x_i))^\lambda, 1 - (1 - b(x_i))^\lambda, 1 - (1 - c(x_i))^\lambda\right), \left((l(x_i))^\lambda, (m(x_i))^\lambda, (p(x_i))^\lambda\right) \right\rangle, \ \lambda \geq 0;$

(4) $(\tilde{a}(x_i))^\lambda = \left\langle \left((a(x_i))^\lambda, (b(x_i))^\lambda, (c(x_i))^\lambda\right), \left(1 - (1 - l(x_i))^\lambda, 1 - (1 - m(x_i))^\lambda, 1 - (1 - p(x_i))^\lambda\right) \right\rangle, \lambda \geq 0;$

Definition 5 *Let* $\tilde{a}(x_i) = \langle (a(x_i), b(x_i), c(x_i)), (l(x_i), m(x_i), p(x_i)) \rangle$ *be a FNIFV, a score function* S *of an FNIFV* $\tilde{a}(x_i)$*can be represented as follows (Wang, 2008a-b):*

$$S(\tilde{a}(x_i)) = \tfrac{a(x_i) + 2b(x_i) + c(x_i)}{4} - \tfrac{l(x_i) + 2m(x_i) + p(x_i)}{4}$$
$$S(\tilde{a}(x_i)) \in [-1, 1]. \tag{4}$$

Definition 6 *Let $\tilde{a}(x_i) = \langle (a(x_i), b(x_i), c(x_i)), (l(x_i), m(x_i), p(x_i)) \rangle$ be a FNIFV, an accuracy function H of an FNIFV $\tilde{a}(x_i)$ can be represented as follows:*

$$H(\tilde{a}(x_i)) = \frac{(a(x_i)+2b(x_i)+c(x_i))+(l(x_i)+2m(x_i)+p(x_i))}{4} \text{čň}$$

$$H(\tilde{a}(x_i)) \in [0,1]. \tag{5}$$

It is meant to evaluate the degree of accuracy of the FNIFV $\tilde{a}(x_i) = \langle (a(x_i), b(x_i), c(x_i)), (l(x_i), m(x_i), p(x_i)) \rangle$. The larger the value of $H(\tilde{a}(x_i))$, the higher the degree of accuracy of the FNIFV $\tilde{a}(x_i)$.

Based on the score function $S$and the accuracy function$H$, in the following, we shall give an order relation between two fuzzy number intuitionistic fuzzy values, defined as follows:

Definition 7 *Let $\tilde{a}(x_i)$ and $\tilde{a}(x_j)$ be two FNIFVs, $s(\tilde{a}(x_i))$ and $s(\tilde{a}(x_j))$ be the scores of $\tilde{a}(x_i)$ and $\tilde{a}(x_j)$, respectively, and let$H(\tilde{a}(x_i))$ and $H(\tilde{a}(x_j))$ be the accuracy degrees of $\tilde{a}(x_i)$ and $\tilde{a}(x_j)$, respectively. Then, if $s(\tilde{a}(x_i)) < s(\tilde{a}(x_j))$, then $\tilde{a}(x_i)$ is smaller than $\tilde{a}(x_j)$, denoted by $\tilde{a}(x_i) < \tilde{a}(x_j)$; if $s(\tilde{a}(x_i)) = s(\tilde{a}(x_j))$, then*

1. *if $H(\tilde{a}(x_i)) = H(\tilde{a}(x_j))$, then $\tilde{a}(x_i)$ and $\tilde{a}(x_j)$ represent the same information, denoted by $\tilde{a}(x_i) = \tilde{a}(x_j)$;*
2. *if $H(\tilde{a}(x_i)) < H(\tilde{a}(x_j))$, $\tilde{a}(x_i)$ is smaller than $\tilde{a}(x_j)$, denoted by $\tilde{a}(x_i) < \tilde{a}(x_j)$.*

However, the above aggregation operators with fuzzy number intuitionistic fuzzy information are based on the assumption that the attributes of decision makers are independent, as characterized by the independence axiom (Keeney & Raiffa, 1976; Wakker, 1999), that is, these operators are based on the implicit assumption that attributes of decision makers are independent one of another; their effects are viewed as additive. For real decision making problems, there is always some degree of inter-dependence between attributes. Usually, there is interaction among attributes of decision makers. The independence axiom generally can not be satisfied. Thus, it is necessary to consider this issue.

Let $\mu(x_i)\,(i = 1, 2, \cdots, n)$ be the weight of the elements $x_i \in X\,(i = 1, 2, \cdots, n)$, where μ is a fuzzy measure, defined as follows:

Definition 8 *(Wang et al., 2000). A fuzzy measure μ on the set $X$is a set function $\mu : \theta(x) \to [0,1]$satisfying the following axioms:*

(1)$\mu(\varphi) = 0 \quad \mu(X) = 1$;

(2) $A \subseteq B$ implies *$\mu(A) \le \mu(B)$, for all $A, B \subseteq X$;*

(3) $\mu(A \cup B) = \mu(A) + \mu(B) + \rho\mu(A)\mu(B)$, for all $A, B \subseteq X$and $A \cap B = \varphi$, where $\rho \in (-1, \infty)$.

Especially, if $\rho = 0$, then the condition (3) reduces to the axiom of additive measure:

$\mu(A \cup B) = \mu(A) + \mu(B)$, for all $A, B \subseteq X$and$A \cap B = \varphi$.
If all the elements in X are independent, we have
$\mu(A) = \sum_{x_i \in A} \mu(\{x_i\})$, for all$A \subseteq X$.

DEFINITION 9 *(Grabisch et al., 2000). Let* f*be a positive real-valued function on*X*, and* μ *be a fuzzy measure on*X*. The discrete Choquet integral of* f *with respective to* μ *is defined by*

$$C_\mu(f) = \sum_{i=1}^{n} f_{\sigma(i)} \left[\mu\left(A_{\sigma(i)}\right) - \mu\left(A_{\sigma(i-1)}\right)\right]$$

where$(\sigma(1), \sigma(2), \cdots, \sigma(n))$ *is a permutation of* $(1, 2, \cdots, n)$*, such that* $f_{\sigma(i-1)} \geq f_{\sigma(i)}$ *for all* $j = 2, \cdots, n$*,* $A_{\sigma(k)} = \left\{x_{\sigma(j)} \,|j \leq k\right\}$*, for* $k \geq 1$*, and* $A_{\sigma(0)} = \varphi$.

So, the discrete Choquet integral is a linear expression up to a reordering of the elements.

3. Some aggregating operators based on the Choquet integral with fuzzy number intuitionistic fuzzy information

Wang (2008a) proposed the fuzzy number intuitionistic fuzzy weighted averaging (FNIFWA) operator and fuzzy number intuitionistic fuzzy ordered weighted averaging (FNIFOWA) operator.

DEFINITION 10 *Let* $\tilde{a}(x_i) = \langle (a(x_i), b(x_i), c(x_i)), (l(x_i), m(x_i), p(x_i)) \rangle$ $4\,(i = 1, 2, \cdots, n)$*be a collection of FNIFVs, and let FNIFWA:*$Q^n \to Q$*, if*

$$\begin{aligned} &FNIFWA_\omega(\tilde{a}(x_1), \tilde{a}(x_2), \cdots, \tilde{a}(x_n)) = \sum_{i=1}^{n} \omega_i \tilde{a}(x_i) \\ &= \left\langle \left(1 - \prod_{i=1}^{n} (1 - a(x_i))^{\omega_i}, 1 - \prod_{i=1}^{n} (1 - b(x_i))^{\omega_i}, 1 - \prod_{i=1}^{n} (1 - c(x_i))^{\omega_i}\right), \right. \\ &\left. \left(\prod_{i=1}^{n} l(x_i)^{\omega_i}, \prod_{i=1}^{n} m(x_i)^{\omega_i}, \prod_{i=1}^{n} p(x_i)^{\omega_i}\right) \right\rangle \end{aligned} \tag{6}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ *is the weight vector of* $\tilde{a}(x_i)\,(i = 1, 2, \cdots, n)$*, and* $\omega_j > 0$*,* $\sum_{i=1}^{n} \omega_i = 1$*, then FNIFWA is called the fuzzy number intuitionistic fuzzy weighted averaging (FNIFWA) operator.*

DEFINITION 11 *Let* $\tilde{a}(x_i) = \langle (a(x_i), b(x_i), c(x_i)), (l(x_i), m(x_i), p(x_i)) \rangle$ $(i = 1, 2, \cdots, n)$ *be a collection of FNIFVs. A fuzzy number intuitionistic fuzzy ordered weighted averaging (FNIFOWA) operator of dimension* n *is a mapping*

FNIFOWA:$Q^n \rightarrow Q$, *that has an associated weight vector* $w = (w_1, w_2, \cdots, w_n)^T$ *such that* $w_j > 0$ *and* $\sum_{j=1}^{n} w_j = 1$. *Furthermore,*

$$\begin{aligned} & FNIFOWA_w\left(\tilde{a}\left(x_1\right), \tilde{a}\left(x_2\right), \cdots, \tilde{a}\left(x_n\right)\right) \\ & = \sum_{i=1}^{n} w_i \tilde{a}\left(x_{\sigma(i)}\right) \\ & = \left\langle \left(1 - \prod_{i=1}^{n}\left(1 - a\left(x_{\sigma(i)}\right)\right)^{w_i}, 1 - \prod_{i=1}^{n}\left(1 - b\left(x_{\sigma(i)}\right)\right)^{w_i}, \right.\right. \\ & \left. 1 - \prod_{i=1}^{n}\left(1 - c\left(x_{\sigma(i)}\right)\right)^{w_i}\right), \\ & \left.\left(\prod_{i=1}^{n} l\left(x_{\sigma(i)}\right)^{w_i}, \prod_{i=1}^{n} m\left(x_{\sigma(i)}\right)^{w_i}, \prod_{i=1}^{n} p\left(x_{\sigma(i)}\right)^{w_i}\right)\right\rangle \end{aligned} \tag{7}$$

where$(\sigma(1), \sigma(2), \cdots, \sigma(n))$ *is a permutation of* $(1, 2, \cdots, n)$, *such that* $\tilde{a}\left(x_{\sigma(i-1)}\right) \geq \tilde{a}\left(x_{\sigma(i)}\right)$ *for all* $i = 2, \cdots, n$.

Based on Definition 8, in what follows, we shall develop the fuzzy number intuitionistic fuzzy Choquet ordered averaging (FNIFCOA) operator based on the well-known Choquet integral (Choquet, 1953).

Definition 12 *Let* $\tilde{a}(x_i) = \langle (a(x_i), b(x_i), c(x_i)), (l(x_i), m(x_i), p(x_i)) \rangle$ $(i = 1, 2, \cdots, n)$ *be a collection of FNIFVs on*X, *and* μ*be a fuzzy measure on* X, *then we call*

$$\begin{aligned} & FNIFCOA_\mu\left(\tilde{a}\left(x_1\right), \tilde{a}\left(x_2\right), \cdots, \tilde{a}\left(x_n\right)\right) \\ & = \tilde{a}\left(x_{\sigma(1)}\right)\left(\mu\left(A_{\sigma(1)}\right) - \mu\left(A_{\sigma(0)}\right)\right) \oplus a\left(\tilde{x}_{\sigma(2)}\right)\left(\mu\left(A_{\sigma(2)}\right)\right. \\ & \left. -\mu\left(A_{\sigma(1)}\right)\right) \oplus \cdots \oplus a\left(\tilde{x}_{\sigma(n)}\right)\left(\mu\left(A_{\sigma(n)}\right) - \mu\left(A_{\sigma(n-1)}\right)\right) \end{aligned} \tag{8}$$

a fuzzy number intuitionistic fuzzy Choquet ordered averaging (FNIFCOA) operator, where $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ *is a permutation of* $(1, 2, \cdots, n)$, *such that* $\tilde{\alpha}\left(x_{\sigma(j-1)}\right) \geq \tilde{\alpha}\left(x_{\sigma(j)}\right)$ *for all* $j = 2, \cdots, n$, $A_{\sigma(k)} = \left\{x_{\sigma(j)} \,|\, j \leq k\right\}$, *for* $k \geq 1$, *and*$A_{\sigma(0)} = \varphi$.

With the operation of fuzzy number intuitionistic fuzzy values, the FNIFCOA operator can be transformed into the following form by induction over n:

$$\begin{aligned} & FNIFCOA_\mu\left(\tilde{a}\left(x_1\right), \tilde{a}\left(x_2\right), \cdots, \tilde{a}\left(x_n\right)\right) \\ & = \left\langle \left(1 - \prod_{i=1}^{n}\left(1 - a\left(x_{\sigma(i)}\right)\right)^{\mu\left(A_{\sigma(i)}\right) - \mu\left(A_{\sigma(i-1)}\right)}, \right.\right. \\ & 1 - \prod_{i=1}^{n}\left(1 - a\left(x_{\sigma(i)}\right)\right)^{\mu\left(A_{\sigma(i)}\right) - \mu\left(A_{\sigma(i-1)}\right)} \\ & \left. 1 - \prod_{i=1}^{n}\left(1 - a\left(x_{\sigma(i)}\right)\right)^{\mu\left(A_{\sigma(i)}\right) - \mu\left(A_{\sigma(i-1)}\right)}\right), \\ & \left(\prod_{j=1}^{n}\left(l\left(x_{\sigma(i)}\right)\right)^{\mu\left(A_{\sigma(i)}\right) - \mu\left(A_{\sigma(i-1)}\right)}, \prod_{j=1}^{n}\left(m\left(x_{\sigma(i)}\right)\right)^{\mu\left(A_{\sigma(i)}\right) - \mu\left(A_{\sigma(i-1)}\right)}, \right. \\ & \left.\left. \prod_{j=1}^{n}\left(p\left(x_{\sigma(i)}\right)\right)^{\mu\left(A_{\sigma(i)}\right) - \mu\left(A_{\sigma(i-1)}\right)}\right)\right\rangle \end{aligned} \tag{9}$$

whose aggregated value is also a fuzzy number intuitionistic fuzzy values.

Especially, if $\mu\left(\left\{x_{\sigma(i)}\right\}\right) = \mu\left(A_{\sigma(i)}\right) - \mu\left(A_{\sigma(i-1)}\right)$, $i = 1, 2, \cdots, n$, *then FNIFCOA operator reduces to the FNIFWA operator. If* $\mu(A) = \sum\limits_{x_i \in A} \mu(\{x_i\})$, *for all* $A \subseteq X$, *where* $|A|$ *is the number of elements in the set* A, *then* $w_i = \mu\left(A_{\sigma(i)}\right) - \mu\left(A_{\sigma(i-1)}\right)$, $i = 1, 2, \cdots, n$, *where* $w = (w_1, w_2, \cdots, w_n)^T$, $w_i \geq 0$, $i = 1, 2, \cdots, n$, *and* $\sum\limits_{i=1}^{n} w_i = 1$, *then FNIFCOA operator reduces to the FNIFOWA operator.*

It is easy to prove that the FNIFCOA operator has the following properties.

THEOREM 1 *(Commutativity).*

$$\begin{aligned}&FNIFCOA_\mu\left(\tilde{a}(x_1), \tilde{a}(x_2), \cdots, \tilde{a}(x_n)\right)\\&= FNIFCOA_\mu\left(\tilde{a}'(x_1), \tilde{a}'(x_2), \cdots, \tilde{a}'(x_n)\right)\end{aligned}$$

where $\left(\tilde{a}'(x_1), \tilde{a}'(x_2), \cdots, \tilde{a}'(x_n)\right)$ *is any permutation of* $\left(\tilde{a}(x_1), \tilde{a}(x_2), \cdots, \tilde{a}(x_n)\right)$

THEOREM 2 *(Idempotency) If* $\tilde{a}(x_j) = \tilde{a}(x)$ *for allj, then*

$$FNIFCOA_\mu\left(\tilde{a}(x_1), \tilde{a}(x_2), \cdots, \tilde{a}(x_n)\right) = \tilde{a}(x).$$

THEOREM 3 *(Monotonicity) If* $\tilde{a}(x_j) \leq \tilde{a}'(x_j)$ *for allj, then*

$$\begin{aligned}&FNIFCOA_\mu\left(\tilde{a}(x_1), \tilde{a}(x_2), \cdots, \tilde{a}(x_n)\right)\\&\leq FNIFCOA_\mu\left(\tilde{a}'(x_1), \tilde{a}'(x_2), \cdots, \tilde{a}'(x_n)\right).\end{aligned}$$

EXAMPLE 1 *Let* $\tilde{a}_1 = \langle(0.3, 0.4, 0.5), (0.1, 0.2, 0.3)\rangle$,$\tilde{a}_2 = \langle(0.4, 0.5, 0.5), (0.1, 0.1, 0.2)\rangle$, $\tilde{a}_3 = \langle(0.2, 0.3, 0.3), (0.3, 0.4, 0.5)\rangle$, *and* $\tilde{a}_4 = \langle(0.2, 0.3, 0.4), (0.4, 0.5, 0.6)\rangle$ *be four FNIFVs, by (4), we calculate the scores of* $\tilde{a}_j\,(j = 1, 2, 3, 4)$*:*

$$S(\tilde{a}_1) = 0.200, S(\tilde{a}_2) = 0.350, S(\tilde{a}_3) = -0.125, S(\tilde{a}_4) = -0.200$$

Since

$$S(\tilde{a}_2) > S(\tilde{a}_1) > S(\tilde{a}_3) > S(\tilde{a}_4),$$

then

$$\begin{aligned}\tilde{a}_{\sigma(1)} &= \langle(0.4, 0.5, 0.5), (0.1, 0.1, 0.2)\rangle,\\\tilde{a}_{\sigma(2)} &= \langle(0.3, 0.4, 0.5), (0.1, 0.2, 0.3)\rangle,\\\tilde{a}_{\sigma(3)} &= \langle(0.2, 0.3, 0.3), (0.3, 0.4, 0.5)\rangle,\\\tilde{a}_{\sigma(4)} &= \langle(0.2, 0.3, 0.4), (0.4, 0.5, 0.6)\rangle.\end{aligned}$$

Assume the fuzzy measure of attribute of $\tilde{a}_i\,(i=1,2,3,4)$ *and attribute sets of* $\tilde{a}_i\,(i=1,2,3,4)$ *as follows:*

$$\begin{aligned}&\mu(\varphi)=0, \mu(\tilde{a}_1)=0.40, \mu(\tilde{a}_2)=0.25, \mu(\tilde{a}_3)=0.38, \mu(\tilde{a}_4)=0.25,\\ &\mu(\tilde{a}_1,\tilde{a}_2)=0.56, \mu(\tilde{a}_1,\tilde{a}_3)=0.65, \mu(\tilde{a}_1,\tilde{a}_4)=0.50, \mu(\tilde{a}_2,\tilde{a}_3)=0.45,\\ &\mu(\tilde{a}_2,\tilde{a}_4)=0.39, \mu(\tilde{a}_3,\tilde{a}_4)=0.40, \mu(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3)=0.80, \mu(\tilde{a}_1,\tilde{a}_2,\tilde{a}_4)=0.75,\\ &\mu(\tilde{a}_1,\tilde{a}_3,\tilde{a}_4)=0.72, \mu(\tilde{a}_2,\tilde{a}_3,\tilde{a}_4)=0.60, \mu(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,\tilde{a}_4)=1.00.\end{aligned}$$

Then, by (9), it follows that

$$\begin{aligned}&FNIFCOA_{\mu}(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,\tilde{a}_4)=\\ &\Big\langle\Big(1-(1-0.4)^{0.25}\times(1-0.3)^{0.56\text{-}0.25}\times(1-0.2)^{0.80\text{-}0.56}\times(1-0.1)^{1\text{-}0.80},\\ &1-(1-0.5)^{0.25}\times(1-0.4)^{0.56\text{-}0.25}\times(1-0.3)^{0.80\text{-}0.56}\times(1-0.2)^{1\text{-}0.80},\\ &1-(1-0.5)^{0.25}\times(1-0.5)^{0.56\text{-}0.25}\times(1-0.3)^{0.80\text{-}0.56}\times(1-0.3)^{1\text{-}0.80}\Big),\\ &\Big(0.1^{0.25}\times0.1^{0.56\text{-}0.25}\times0.3^{0.80\text{-}0.56}\times0.3^{1\text{-}0.80},\\ &0.1^{0.25}\times0.2^{0.56\text{-}0.25}\times0.4^{0.80\text{-}0.56}\times0.4^{1\text{-}0.80},\\ &0.2^{0.25}\times0.3^{0.56\text{-}0.25}\times0.5^{0.80\text{-}0.56}\times0.5^{1\text{-}0.80}\Big)\Big\rangle=\\ &\langle(0.269,0.370,0.420),(0.162,0.228,0.339)\rangle.\end{aligned}$$

Wang (2008b) proposed the fuzzy number intuitionistic fuzzy weighted geometric (FNIFWG) operator and fuzzy number intuitionistic fuzzy ordered weighted geometric (FNIFOWG) operator.

Definition 13 *Let* $\tilde{a}(x_i)=\langle(a(x_i),b(x_i),c(x_i)),(l(x_i),m(x_i),p(x_i))\rangle$ $(i=1,2,\cdots,n)$ *be a collection of FNIFVs, and let FNIFWG:*$Q^n\to Q$*; if*

$$\begin{aligned}&FNIFWG_{\omega}(\tilde{a}(x_1),\tilde{a}(x_2),\cdots,\tilde{a}(x_n))=\prod_{i=1}^{n}\tilde{a}(x_i)^{\omega_i}\\ &=\left\langle\left(\prod_{i=1}^{n}a(x_i)^{\omega_i},\prod_{i=1}^{n}b(x_i)^{\omega_i},\prod_{i=1}^{n}c(x_i)^{\omega_i}\right),\right.\\ &\left.\left(1-\prod_{i=1}^{n}(1-l(x_i))^{\omega_i},1-\prod_{i=1}^{n}(1-m(x_i))^{\omega_i},1-\prod_{i=1}^{n}(1-p(x_i))^{\omega_i}\right)\right\rangle\end{aligned}\tag{10}$$

where $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ *is the weight vector of* $\tilde{a}(x_i)\,(i=1,2,\cdots,n)$, *and* $\omega_i>0$, $\sum_{i=1}^{n}\omega_i=1$, *then FIFWG is called the fuzzy number intuitionistic fuzzy weighted geometric (FIFWG) operator.*

Definition 14 *Let* $\tilde{a}(x_i)=\langle(a(x_i),b(x_i),c(x_i)),(l(x_i),m(x_i),p(x_i))\rangle$ $(i=1,2,\cdots,n)$ *be a collection of FNIFVs. A fuzzy number intuitionistic fuzzy ordered weighted geometric (FNIFOWG) operator of dimension* n *is a mapping FNIFOWG:*$Q^n\to Q$*, that has an associated weight vector* $w=(w_1,w_2,\cdots,$

$w_n)^T$ *such that* $w_i > 0$ *and* $\sum_{i=1}^{n} w_i = 1$. *Furthermore,*

$$FNIFOWG_w\left(\tilde{a}(x_1), \tilde{a}(x_2), \cdots, \tilde{a}(x_n)\right) = \prod_{i=1}^{n} \tilde{a}\left(x_{\sigma(i)}\right)^{w_i}$$
$$= \left\langle \left(\prod_{i=1}^{n} a\left(x_{\sigma(i)}\right)^{w_i}, \prod_{i=1}^{n} b\left(x_{\sigma(i)}\right)^{w_i}, \prod_{i=1}^{n} c\left(x_{\sigma(i)}\right)^{w_i} \right), \right.$$
$$\left. \left(1 - \prod_{i=1}^{n} \left(1 - l\left(x_{\sigma(i)}\right)\right)^{w_i}, 1 - \prod_{i=1}^{n} \left(1 - m\left(x_{\sigma(i)}\right)\right)^{w_i}, 1 - \prod_{i=1}^{n} \left(1 - p\left(x_{\sigma(i)}\right)\right)^{w_i} \right) \right\rangle \tag{11}$$

where $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ *is a permutation of* $(1, 2, \cdots, n)$, *such that* $\tilde{a}\left(x_{\sigma(i-1)}\right) \geq \tilde{a}\left(x_{\sigma(i)}\right)$ *for all* $i = 2, \cdots, n$.

In the following, we shall develop the fuzzy number intuitionistic fuzzy Choquet ordered geometric mean (FNIFCOGM) operator based on the Choquet integral (Choquet, 1953).

Definition 15 *Let* $\tilde{a}(x_i) = \langle (a(x_i), b(x_i), c(x_i)), (l(x_i), m(x_i), p(x_i)) \rangle$ $(i = 1, 2, \cdots, n)$ *be a collection of FNIFVs on* X, *and* μ *be a fuzzy measure on* X, *then we call*

$$\begin{aligned} & FNIFCOGM_\mu\left(\tilde{a}(x_1), \tilde{a}(x_2), \cdots, \tilde{a}(x_n)\right) \\ & = \tilde{a}\left(x_{\sigma(1)}\right)^{\mu\left(A_{\sigma(1)}\right) - \mu\left(A_{\sigma(0)}\right)} \oplus a\left(\tilde{x}_{\sigma(2)}\right)^{\mu\left(A_{\sigma(2)}\right) - \mu\left(A_{\sigma(1)}\right)} \oplus \cdots \oplus \\ & a\left(\tilde{x}_{\sigma(n)}\right)^{\mu\left(A_{\sigma(n)}\right) - \mu\left(A_{\sigma(n-1)}\right)} \end{aligned} \tag{12}$$

a fuzzy number intuitionistic fuzzy Choquet ordered geometric mean (FNIFCOGM) operator, where $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ *is a permutation of* $(1, 2, \cdots, n)$, *such that* $\tilde{\alpha}\left(x_{\sigma(j-1)}\right) \geq \tilde{\alpha}\left(x_{\sigma(j)}\right)$ *for all* $j = 2, \cdots, n$, $A_{\sigma(k)} = \left\{x_{\sigma(j)} \,|\, j \leq k\right\}$, *for* $k \geq 1$, *and* $A_{\sigma(0)} = \varphi$.

With the operation of fuzzy number intuitionistic fuzzy values, the FNIFCOGM operator (12) can be transformed into the following form by induction over n:

$$\begin{aligned} & FNIFCOGM_\mu\left(\tilde{a}(x_1), \tilde{a}(x_2), \cdots, \tilde{a}(x_n)\right) \\ & = \left\langle \left(\prod_{j=1}^{n} \left(a\left(x_{\sigma(i)}\right)\right)^{\mu\left(A_{\sigma(i)}\right) - \mu\left(A_{\sigma(i-1)}\right)}, \right. \right. \\ & \left. \prod_{j=1}^{n} \left(b\left(x_{\sigma(i)}\right)\right)^{\mu\left(A_{\sigma(i)}\right) - \mu\left(A_{\sigma(i-1)}\right)}, \prod_{j=1}^{n} \left(c\left(x_{\sigma(i)}\right)\right)^{\mu\left(A_{\sigma(i)}\right) - \mu\left(A_{\sigma(i-1)}\right)} \right), \\ & \left(1 - \prod_{i} = 1^n \left(1 - l\left(x_{\sigma(i)}\right)\right)^{\mu\left(A_{\sigma(i)}\right) - \mu\left(A_{\sigma(i-1)}\right)}, \right. \\ & 1 - \prod_{i=1}^{n} \left(1 - m\left(x_{\sigma(i)}\right)\right)^{\mu\left(A_{\sigma(i)}\right) - \mu\left(A_{\sigma(i-1)}\right)}, \\ & \left. \left. 1 - \prod_{i=1}^{n} \left(1 - p\left(x_{\sigma(i)}\right)\right)^{\mu\left(A_{\sigma(i)}\right) - \mu\left(A_{\sigma(i-1)}\right)} \right) \right\rangle \end{aligned} \tag{13}$$

whose aggregated value is also a fuzzy number intuitionistic fuzzy values.

Especially, if $\mu\left(\left\{x_{\sigma(i)}\right\}\right)=\mu\left(A_{\sigma(i)}\right)-\mu\left(A_{\sigma(i-1)}\right)$, $i=1,2,\cdots,n$, *then FNIFCOGM operator reduces to the FNIFWG operator. If* $\mu(A)=\sum\limits_{x_i\in A}\mu(\{x_i\})$, *for all* $A\subseteq X$, *where* $|A|$ *is the number of elements in the set* A, *then* $w_i=\mu\left(A_{\sigma(i)}\right)-\mu\left(A_{\sigma(i-1)}\right)$, $i=1,2,\cdots,n$, *where* $w=(w_1,w_2,\cdots,w_n)^T$, $w_i\geq 0$, $i=1,2,\cdots,n$, *and* $\sum\limits_{i=1}^{n}w_i=1$, *then the FNIFCOGM operator reduces to the FNIFOWG operator.*

It is easy to prove that the FNIFCOGM operator has the following properties.

THEOREM 4 *(Commutativity).*

$$\begin{aligned}&FNIFCOGM_\mu\left(\tilde{a}(x_1),\tilde{a}(x_2),\cdots,\tilde{a}(x_n)\right)\\&=FNIFCOGM_\mu\left(\tilde{a}'(x_1),\tilde{a}'(x_2),\cdots,\tilde{a}'(x_n)\right)\end{aligned}$$

where $\left(\tilde{a}'(x_1),\tilde{a}'(x_2),\cdots,\tilde{a}'(x_n)\right)$ *is any permutation of* $\left(\tilde{a}(x_1),\tilde{a}(x_2),\cdots,\tilde{a}(x_n)\right)$.

THEOREM 5 *(Idempotency) If* $\tilde{a}(x_j)=\tilde{a}(x)$ *for all* j, *then*

$$FNIFCOGM_\mu\left(\tilde{a}(x_1),\tilde{a}(x_2),\cdots,\tilde{a}(x_n)\right)=\tilde{a}(x).$$

THEOREM 6 *(Monotonicity) If* $\tilde{a}(x_j)\leq\tilde{a}'(x_j)$ *for all* j, *then*

$$\begin{aligned}&FNIFCOGM_\mu\left(\tilde{a}(x_1),\tilde{a}(x_2),\cdots,\tilde{a}(x_n)\right)\\&\leq FNIFCOGM_\mu\left(\tilde{a}'(x_1),\tilde{a}'(x_2),\cdots,\tilde{a}'(x_n)\right).\end{aligned}$$

EXAMPLE 2 *Let* $\tilde{a}_1=\langle(0.3,0.4,0.5),(0.1,0.2,0.3)\rangle$,$\tilde{a}_2=\langle(0.4,0.5,0.5),(0.1,0.1,0.2)\rangle$, $\tilde{a}_3=\langle(0.2,0.3,0.3),(0.3,0.4,0.5)\rangle$, *and* $\tilde{a}_4=\langle(0.2,0.3,0.4),(0.4,0.5,0.6)\rangle$ *be four FNIFVs, by (4), we calculate the scores of* $\tilde{a}_j\,(j=1,2,3,4)$:

$$S(\tilde{a}_1)=0.200,S(\tilde{a}_2)=0.350,S(\tilde{a}_3)=-0.125,S(\tilde{a}_4)=-0.200.$$

Since

$$S(\tilde{a}_2)>S(\tilde{a}_1)>S(\tilde{a}_3)>S\left(\tilde{a}_4\right)$$

then

$$\begin{aligned}\tilde{a}_{\sigma(1)}&=\langle(0.4,0.5,0.5),(0.1,0.1,0.2)\rangle,\\\tilde{a}_{\sigma(2)}&=\langle(0.3,0.4,0.5),(0.1,0.2,0.3)\rangle,\\\tilde{a}_{\sigma(3)}&=\langle(0.2,0.3,0.3),(0.3,0.4,0.5)\rangle,\\\tilde{a}_{\sigma(4)}&=\langle(0.2,0.3,0.4),(0.4,0.5,0.6)\rangle\end{aligned}$$

Assume the fuzzy measure of attribute of $\tilde{a}_i\,(i=1,2,3,4)$ *and attribute sets of* $\tilde{a}_i\,(i=1,2,3,4)$ *as follows:*

$$\begin{aligned}&\mu(\varphi)=0,\mu(\tilde{a}_1)=0.40,\mu(\tilde{a}_2)=0.25,\mu(\tilde{a}_3)=0.38,\mu(\tilde{a}_4)=0.25,\\&\mu(\tilde{a}_1,\tilde{a}_2)=0.56,\mu(\tilde{a}_1,\tilde{a}_3)=0.65,\mu(\tilde{a}_1,\tilde{a}_4)=0.50,\mu(\tilde{a}_2,\tilde{a}_3)=0.45,\\&\mu(\tilde{a}_2,\tilde{a}_4)=0.39,\mu(\tilde{a}_3,\tilde{a}_4)=0.40,\mu(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3)=0.80,\mu(\tilde{a}_1,\tilde{a}_2,\tilde{a}_4)=0.75,\\&\mu(\tilde{a}_1,\tilde{a}_3,\tilde{a}_4)=0.72,\mu(\tilde{a}_2,\tilde{a}_3,\tilde{a}_4)=0.60,\mu(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,\tilde{a}_4)=1.00.\end{aligned}$$

Then, by (13), it follows that

$$\begin{aligned}&FNIFCOGM_\mu(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,\tilde{a}_4)\\&=\big\langle\big(0.4^{0.25}\times0.3^{0.56-0.25}\times0.2^{0.80-0.56}\times0.1^{1-0.80},\\&\quad0.5^{0.25}\times0.4^{0.56-0.25}\times0.3^{0.80-0.56}\times0.2^{1-0.80},\\&\quad0.5^{0.25}\times0.5^{0.56-0.25}\times0.3^{0.80-0.56}\times0.3^{1-0.80}\big),\\&\quad\big(1-(1-0.1)^{0.25}\times(1-0.1)^{0.56-0.25}\times(1-0.3)^{0.80-0.56}\times(1-0.3)^{1-0.80},\\&\quad1-(1-0.1)^{0.25}\times(1-0.2)^{0.56-0.25}\times(1-0.4)^{0.80-0.56}\times(1-0.4)^{1-0.80},\\&\quad1-(1-0.2)^{0.25}\times(1-0.3)^{0.56-0.25}\times(1-0.5)^{0.80-0.56}\times(1-0.5)^{1-0.80}\big)\big\rangle\\&=\langle(0.292,0.344,0.399),(0.194,0.274,0.661)\rangle.\end{aligned}$$

4. An approach to multiple attribute decision making with fuzzy number intuitionistic fuzzy information

In this section, we shall develop an approach to multiple attribute decision making with fuzzy number intuitionistic fuzzy information as follows.

Let $A=\{A_1,A_2,\cdots,A_m\}$ be a discrete set of alternatives, $G=\{G_1,G_2,\cdots,G_n\}$ be the set of attributes, and $\omega=(\omega_1,\omega_2,\cdots,\omega_n)$ the weighting vector of the attributes $G_j\,(j=1,2,\cdots,n)$, where$\omega_j\in[0,1],\sum\limits_{j=1}^{n}\omega_j=1$.Suppose that $\tilde{R}=(\tilde{r}_{ij})_{m\times n}=\langle(a_{ij},b_{ij},c_{ij}),(l_{ij},m_{ij},p_{ij})\rangle_{m\times n}$ is the fuzzy number intuitionistic fuzzy decision matrix, where (a_{ij},b_{ij},c_{ij}) indicates the degree to which the alternative A_i satisfies the attribute G_j, (l_{ij},m_{ij},p_{ij}) indicates the degree to which the alternative A_i does not satisfy the attribute G_j $\quad(a_{ij},b_{ij},c_{ij})\subset[0,1]$, $(l_{ij},m_{ij},p_{ij})\subset[0,1]$, $c_{ij}+p_{ij}\le1$, $i=1,2,\cdots,m$, $j=1,2,\cdots,n$.

In the following, we apply the FNIFCOA and FNIFCOGM operator to multiple attribute decision making with fuzzy number intuitionistic fuzzy information.

Step 1. If we emphasize the group's influence, we utilize the decision information given in matrix$\tilde{R}$, and the FNIFCOA operator

$$\begin{aligned}&\tilde{r}_i=\langle(a_j,b_j,c_j),(l_j,m_j,p_j)\rangle=\text{FNIFCOA}(\tilde{r}_{i1},\tilde{r}_{i2},\cdots,\tilde{r}_{in}),\\&i=1,2,\cdots,m,j=1,2,\cdots,n,\end{aligned}\tag{14}$$

to derive the overall preference values $\tilde{r}_i\,(i=1,2,\cdots,m)$of the alternative A_i. Otherwise we utilize the decision information given in matrix$\tilde{R}$, and the

FNIFCOGM operator

$$\tilde{r}_i = \langle (a_j, b_j, c_j), (l_j, m_j, p_j) \rangle = \text{FNIFCOGM}(\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{in}),$$
$$i = 1, 2, \cdots, m, j = 1, 2, \cdots, n, \tag{15}$$

to derive the overall preference values $\tilde{r}_i\,(i = 1, 2, \cdots, m)$of the alternative A_i.

Step 2. Calculate the scores $S(\tilde{r}_i)$ $(i = 1, 2, \cdots, m)$of the overall fuzzy number intuitionistic fuzzy preference values $\tilde{r}_i$ $(i = 1, 2, \cdots, m)$to rank all the alternatives $A_i\,(i = 1, 2, \cdots, m)$ and then to select the best one(s). If there is no difference between two scores $S(\tilde{r}_i)$ and $S(\tilde{r}_j)$, then we need to calculate the accuracy degrees $H(\tilde{r}_i)$ and $H(\tilde{r}_j)$ of the overall fuzzy number intuitionistic fuzzy preference values $\tilde{r}_i$ and $\tilde{r}_j$, respectively, and then rank the alternatives A_i and A_j in accordance with the accuracy degrees $H(\tilde{r}_i)$ and $H(\tilde{r}_j)$.

Step 3. Rank all the alternatives $A_i\,(i = 1, 2, \cdots, m)$ and select the best one(s) in accordance with $S(\tilde{r}_i)$ and $H(\tilde{r}_i)\,(i = 1, 2, \cdots, m)$.

Step 4. End.

5. Illustrative example

In this section we shall present a numerical example to show potential evaluation of emerging technology commercialization with hesitant fuzzy information in order to illustrate the method proposed in this paper. The experts select four attributes to evaluate the possible emerging technology enterprises: ①G_1 is the technical advancement; ②G_2 is the potential market and market risk; ③G_3 is the industrialization infrastructure, human resources and financial conditions; ④G_4 is the employment creation and the development of science and technology. The five possible emerging technology enterprises $A_i\,(i = 1, 2, \cdots, 5)$ are to be evaluated using the fuzzy number intuitionistic fuzzy values by the decision maker under the above four attributes, and the decision matrices $\tilde{R}_k = \left(\tilde{r}_{ij}^{(k)}\right)_{5\times 4}(k = 1, 2, 3)$ as follows:

$$\tilde{R} = \begin{bmatrix} \langle (0.2, 0.3, 0.4), (0.3, 0.4, 0.4) \rangle & \langle (0.5, 0.6, 0.7), (0.1, 0.1, 0.1) \rangle & \langle (0.5, 0.5, 0.6), (0.1, 0.1, 0.2) \rangle & \langle (0.4, 0.5, 0.6), (0.1, 0.1, 0.1) \rangle \\ \langle (0.3, 0.4, 0.5), (0.1, 0.2, 0.3) \rangle & \langle (0.4, 0.5, 0.5), (0.1, 0.1, 0.2) \rangle & \langle (0.3, 0.4, 0.5), (0.1, 0.2, 0.3) \rangle & \langle (0.1, 0.2, 0.3), (0.3, 0.4, 0.5) \rangle \\ \langle (0.1, 0.2, 0.3), (0.3, 0.4, 0.5) \rangle & \langle (0.3, 0.4, 0.5), (0.2, 0.2, 0.3) \rangle & \langle (0.6, 0.7, 0.8), (0.1, 0.1, 0.1) \rangle & \langle (0.1, 0.1, 0.2), (0.4, 0.5, 0.6) \rangle \\ \langle (0.4, 0.5, 0.6), (0.1, 0.1, 0.1) \rangle & \langle (0.7, 0.7, 0.7), (0.1, 0.1, 0.1) \rangle & \langle (0.4, 0.5, 0.5), (0.1, 0.2, 0.3) \rangle & \langle (0.3, 0.4, 0.5), (0.2, 0.3, 0.3) \rangle \\ \langle (0.6, 0.6, 0.7), (0.1, 0.1, 0.1) \rangle & \langle (0.4, 0.4, 0.4), (0.1, 0.2, 0.3) \rangle & \langle (0.6, 0.6, 0.6), (0.1, 0.1, 0.1) \rangle & \langle (0.2, 0.3, 0.3), (0.3, 0.4, 0.5) \rangle \end{bmatrix}.$$

Then, we utilize the approach developed to get the most desirable emerging technology enterprise(s).

Step 1. Suppose the fuzzy measure of attribute of $G_j\,(j = 1, 2, \cdots, n)$and attribute sets of Gas follows:

$$\mu(G_1) = 0.30, \mu(G_2) = 0.35, \mu(G_3) = 0.30, \mu(G_4) = 0.22, \mu(G_1, G_2) = 0.70,$$
$$\mu(G_1, G_3) = 0.60, \mu(G_1, G_4) = 0.55, \mu(G_2, G_3) = 0.50, \mu(G_2, G_4) = 0.45,$$
$$\mu(G_3, G_4) = 0.40, \mu(G_1, G_2, G_3) = 0.82, \mu(G_1, G_2, G_4) = 0.87,$$
$$\mu(G_1, G_3, G_4) = 0.75, \mu(G_2, G_3, G_4) = 0.60, \mu(G_1, G_2, G_3, G_4) = 1.00.$$

Step 2. If we emphasize the group's influence, we utilize the decision information given in matrix $\tilde{R}$, and the FNIFCOA operator to obtain the overall preference values $\tilde{r}_i$ of the emerging technology enterprises $A_i\,(i = 1, 2, \cdots, 5)$.

$$\tilde{r}_1 = \langle (0.385, 0.471, 0.575), (0.155, 0.174, 0.193) \rangle$$
$$\tilde{r}_2 = \langle (0.306, 0.407, 0.469), (0.122, 0.178, 0.285) \rangle$$
$$\tilde{r}_3 = \langle (0.329, 0.425, 0.540), (0.210, 0.239, 0.288) \rangle$$
$$\tilde{r}_4 = \langle (0.516, 0.568, 0.613), (0.113, 0.132, 0.139) \rangle$$
$$\tilde{r}_5 = \langle (0.505, 0.516, 0.556), (0.122, 0.149, 0.170) \rangle .$$

Step 3. Calculate the scores $S(\tilde{r}_i)\;\;(i = 1, 2, \cdots, 5)$of the overall fuzzy number intuitionistic fuzzy preference values $\tilde{r}_i\;\;(i = 1, 2, \cdots, 5)$

$$S(\tilde{r}_1) = 0.603, S(\tilde{r}_2) = 0.413, S(\tilde{r}_3) = 0.372$$
$$S(\tilde{r}_4) = 0.874, S(\tilde{r}_5) = 0.751.$$

Step 4. Rank all the emerging technology enterprises$A_i\,(i = 1, 2, 3, 4, 5)$ in accordance with the scores$S(\tilde{r}_i)\;\;(i = 1, 2, \cdots, 5)$ of the overall fuzzy number intuitionistic fuzzy preference values$\tilde{r}_i\;\;(i = 1, 2, \cdots, 5)$: $A_4 \succ A_5 \succ A_1 \succ A_2 \succ A_3$, and thus the most desirable emerging technology enterprise is A_4.

If we emphasize the individual influence, we utilize the decision information given in matrix$\tilde{R}$ and the FNIFCOGM operator to obtain the overall preference values $\tilde{r}_i$ of the emerging technology enterprises $A_i\,(i = 1, 2, \cdots, 5)$:

$$\tilde{r}_1 = \langle (0.339, 0.434, 0.538), (0.186, 0.235, 0.248) \rangle$$
$$\tilde{r}_2 = \langle (0.272, 0.382, 0.456), (0.140, 0.208, 0.310) \rangle$$
$$\tilde{r}_3 = \langle (0.213, 0.295, 0.415), (0.246, 0.305, 0.387) \rangle$$
$$\tilde{r}_4 = \langle (0.462, 0.540, 0.600), (0.119, 0.152, 0.165) \rangle$$
$$\tilde{r}_5 = \langle (0.450, 0.484, 0.507), (0.140, 0.185, 0.234) \rangle .$$

Then, by applying (4) to calculate the scores $S(\tilde{r}_i)\;\;(i = 1, 2, \cdots, 5)$of the collective overall fuzzy number intuitionistic fuzzy preference values $\tilde{r}_i\;\;(i = 1, 2, \cdots, 5)$ we obtain

$$S(\tilde{r}_1) = 0.421, S(\tilde{r}_2) = 0.313, S(\tilde{r}_3) = -0.013$$
$$S(\tilde{r}_4) = 0.777, S(\tilde{r}_5) = 0.592.$$

Therefore, the ranking order is$A_4 \succ A_5 \succ A_1 \succ A_2 \succ A_3$. Thus, we can see that the most desirable emerging technology enterprise is stillA_4. Especially, if the triangular fuzzy numbers $(a_j, b_j, c_j)$and (l_j, m_j, p_j) are reduced to the interval numbers $[a_j, b_j]$ and$[l_j, m_j]$, then, the FNIFCOA or FNIFCOGM operator is reduced to the interval-valued intuitionistic fuzzy Choquet ordered averaging (IVIFCOA) operator or interval-valued intuitionistic fuzzy Choquet ordered geometric mean (IVIFCOGM) operator (Xu,2010); if $a_j = b_j = c_j = \mu_j$, $l_j = m_j = p_j = \nu_j$, then the FNIFCOA or FNIFCOGM operator is reduced to the intuitionistic fuzzy Choquet ordered averaging (IFCOA) operator or intuitionistic fuzzy Choquet ordered geometric mean (IFCOGM) operator (Xu,2010).

6. Conclusion

The traditional Choquet integral aggregation operators are generally suitable for aggregating the information taking the form of numerical values, and yet they will fail in dealing with fuzzy number intuitionistic fuzzy information. In this paper, we have developed two fuzzy number intuitionistic fuzzy Choquet integral aggregation operators: fuzzy number intuitionistic fuzzy Choquet ordered averaging (FNIFCOA) operator and fuzzy number intuitionistic fuzzy Choquet ordered geometric mean (FNIFCOGM) operator. The prominent characteristic of the operators is that they can not only account for the importance of the elements or their their ordered positions, but also reflect the correlation among the elements or their ordered positions. We have studied some desirable properties of the FNIFCOA and FNIFCOGM operators, such as commutativity, idempotency and monotonicity, and applied the FNIFCOA and FNIFCOGM operators to multiple attribute decision making with fuzzy number intuitionistic fuzzy information. Finally an illustrative example has been given to show the working of the developed method. In the future, we shall continue working in the application of the fuzzy number intuitionistic fuzzy multiple attribute decision making to other domains.

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