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Unique fuzzy optimal value of fully fuzzy linear programming problems*

by

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Abstract: Kumar, Kaur and Singh (2011), proposed a new method to find the exact fuzzy optimal solution of fully fuzzy linear programming (FFLP) problems with equality constraints. In this paper, an FFLP problem is chosen to show that the fuzzy optimal value, obtained by using the existing method, is not necessarily a unique fuzzy number i.e., the fuzzy optimal value of the FFLP problem, obtained by the existing method, does not conform to the uniqueness property of fuzzy optimal value. To overcome this shortcoming of the existing method, a new method is proposed for solving FFLP problems with equality constraints. To show the advantage of the proposed method the results of the chosen FFLP problem, obtained by using the existing and the proposed methods, are compared.

Keywords: fully fuzzy linear programming, unique fuzzy optimal value, fuzzy number.

1. Introduction

Any linear programming model representing real world situations involves a lot of parameters whose values are assigned by experts. However, both experts and decision makers frequently do not precisely know the value of those parameters. Therefore it is useful to consider the knowledge of experts about the parameters as fuzzy data (Zadeh, 1965). Bellman and Zadeh (1970) proposed the concept of decision making in fuzzy environment. Many authors (Tanaka, Okuda and Asai, 1973; Zimmermann, 1978; Campos and Verdegay, 1989; Maleki, Tata and Mashinchi, 2000; Buckley and Feuring, 2000; Maleki, 2002; Hashemi, Modarres, Nasrabadi and Nasrabadi, 2006; Ganesan and Veeramani, 2006; Allahviranloo, Lotfi, Kiasary, Kiani and Alizadeh, 2008; Ebrahimnejad, Nasseri, Lotfi

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and Soltanifar, 2010) adopted this concept for solving fuzzy linear programming problems. Lotfi, Allahviranloo, Jondabeha and Alizadeh (2009) proposed a method to find the approximate solution of FFLP problems with equality constraints. Kumar, Kaur and Singh (2011) pointed out that there is no method to find the exact fuzzy optimal solution of FFLP problems with equality constraints and proposed a method to find the exact fuzzy optimal solution of such FFLP problems. In this paper, the shortcoming of the existing method (Kumar, Kaur and Singh, 2011) is pointed out and to overcome this shortcoming a new method is proposed to find the fuzzy optimal solution of FFLP problems with equality constraints. This paper is organized as follows: In Section 2, some basic definitions and arithmetic operations between two trapezoidal fuzzy numbers are presented. In Section 3, a brief review of the existing method (Kumar, Kaur and Singh, 2011) is presented. In Section 4, shortcoming of this method is pointed out. In Section 5, a new method is proposed for solving FFLP problems and the advantages of the proposed method over the existing method are discussed. In Section 6, the results of some FFLP problems, obtained by using the existing and the proposed methods, are compared. Conclusions are discussed in Section 7.

2. Preliminaries

In this section, the necessary background and some notions of fuzzy set theory are presented (Kumar, Kaur and Singh, 2011).

2.1. Basic definitions

DEFINITION 2.1 *A fuzzy number $\tilde{A}$ defined on the universal set of real numbers $\mathbb{R}$, denoted as $\tilde{A} = (a, b, c, d)$, is said to be a trapezoidal fuzzy number if its membership function, $\mu_{\tilde{A}}(x)$, is given by*

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)} & , \quad a \leq x < b \\ 1 & , \quad b \leq x \leq c \\ \frac{(x-d)}{(c-d)} & , \quad c < x \leq d \\ 0 & , \quad \textit{otherwise.} \end{cases}$$

DEFINITION 2.2 *A trapezoidal fuzzy number (a, b, c, d) is said to be non-negative fuzzy number if and only if $a \geq 0$.*

DEFINITION 2.3 *Two trapezoidal fuzzy numbers $\tilde{A} = (a, b, c, d)$ and $\tilde{B} = (e, f, g, h)$ are said to be equal if and only if $a = e, b = f, c = g$ and $d = h$.*

2.2. Arithmetic operations

In this section, arithmetic operations between two trapezoidal fuzzy numbers, defined on the universal set of real numbers $\mathbb{R}$, are presented. Let $\tilde{A} = (a, b, c, d)$

and $\tilde{B} = (e, f, g, h)$ be two trapezoidal fuzzy numbers, then

(i) $\tilde{A} \oplus \tilde{B} = (a, b, c, d) \oplus (e, f, g, h) = (a + e, b + f, c + g, d + h)$
(ii) $\ominus\tilde{A} = \ominus(a, b, c, d) = (-d, -c, -b, -a)$
(iii) $\tilde{A} \ominus \tilde{B} = (a, b, c, d) \ominus (e, f, g, h) = (a - h, b - g, c - f, d - e)$
(iv) Let $\tilde{A} = (a, b, c, d)$ be any trapezoidal fuzzy number and $\tilde{B} = (x, y, z, w)$ be a non-negative trapezoidal fuzzy number, then

$$\tilde{A} \otimes \tilde{B} \simeq \begin{cases} (ax, by, cz, dw), & a \geq 0 \\ (aw, by, cz, dw), & a < 0, b \geq 0 \\ (aw, bz, cz, dw), & b < 0, c \geq 0 \\ (aw, bz, cy, dw), & c < 0, d \geq 0 \\ (aw, bz, cy, dx), & d < 0. \end{cases}$$

3. The existing method

Kumar, Kaur and Singh (2011) proposed a method to find the exact fuzzy optimal solution of FFLP problems with equality constraints (P_1):

$$\begin{aligned} &\text{Maximize (or Minimize) } \sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j \\ &\text{subject to} \qquad\qquad\qquad\qquad (P_1): \\ &\sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i \quad \forall\ i = 1, 2, ..., m \end{aligned}$$

where, $\tilde{a}_{ij}$, $\tilde{c}_j, \tilde{b}_i$ are any fuzzy numbers and $\tilde{x}_j$ is a non-negative fuzzy number.

The steps of the proposed method are as follows:

Step 1 Assuming $\tilde{c}_j = (p_j, q_j, r_j, s_j), \tilde{x}_j = (x_j, y_j, z_j, w_j), \tilde{a}_{ij} = (a_{ij}, b_{ij}, c_{ij}, d_{ij})$ and $\tilde{b}_i = (b_i, g_i, h_i, k_i)$, the FFLP problem (P_1) can be written as:

$$\begin{aligned} &\text{Maximize (or Minimize) } \sum_{j=1}^{n} (p_j, q_j, r_j, s_j) \otimes (x_j, y_j, z_j, w_j) \\ &\text{subject to} \qquad\qquad\qquad\qquad (P_2) \\ &\sum_{j=1}^{n} (a_{ij}, b_{ij}, c_{ij}, d_{ij}) \otimes (x_j, y_j, z_j, w_j) = (b_i, g_i, h_i, k_i) \quad \forall\ i = 1, 2, ..., m \end{aligned}$$

where (x_j, y_j, z_j, w_j) is a non-negative trapezoidal fuzzy number.

Step 2 Assuming $(a_{ij}, b_{ij}, c_{ij}, d_{ij}) \otimes (x_j, y_j, z_j, w_j) = (m_{ij}, n_{ij}, o_{ij}, t_{ij})$ the FFLP problem (P_2) can be written as:

$$\begin{aligned} &\text{Maximize (or Minimize) } \sum_{j=1}^{n} (p_j, q_j, r_j, s_j) \otimes (x_j, y_j, z_j, w_j) \\ &\text{subject to} \qquad\qquad\qquad\qquad (P_3) \\ &\sum_{j=1}^{n} (m_{ij}, n_{ij}, o_{ij}, t_{ij}) = (b_i, g_i, h_i, k_i) \quad \forall\ i = 1, 2, ..., m \end{aligned}$$

where (x_j, y_j, z_j, w_j) is a non-negative trapezoidal fuzzy number.

Step 3 Using Definitions 2.2 and 2.3, the FFLP problem (P_3) can be written as:

$$\begin{aligned}
&\text{Maximize (or Minimize) } (\sum_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j,y_j,z_j,w_j))\\
&\text{subject to}\\
&\quad \sum_{j=1}^{n} m_{ij}=b_i \quad \forall\ i=1,2,...,m\\
&\quad \sum_{j=1}^{n} n_{ij}=g_i \quad \forall\ i=1,2,...,m\\
&\quad \sum_{j=1}^{n} o_{ij}=h_i \quad \forall\ i=1,2,...,m\\
&\quad \sum_{j=1}^{n} t_{ij}=k_i \quad \forall\ i=1,2,...,m\\
&\quad x_j\geq 0, y_j-x_j\geq 0, z_j-y_j\geq 0, w_j-z_j\geq 0.
\end{aligned} \qquad (P_4)$$

Step 4 Suppose the fuzzy linear programming problem (P_4) has l feasible solutions and $(x_j^t,y_j^t,z_j^t,w_j^t)$ is the t^{th} feasible solution; then the aim is to find that feasible solution out of all l feasible solutions corresponding to which the value of objective function is maximum (or minimum), i.e., the aim is to find $\underset{1\leq t\leq l}{\text{maximum (or minimum)}}\{\sum_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j^t,y_j^t,z_j^t,w_j^t)\}$. Kumar, Kaur and Singh (2011) have used the concept that if $\underset{1\leq t\leq l}{\text{maximum (or minimum)}}\{\text{Rank}(\sum_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j^t,y_j^t,z_j^t,w_j^t))\}$ is $\text{Rank}(\sum_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j^\theta,y_j^\theta,z_j^\theta,w_j^\theta))$ then $\underset{1\leq t\leq l}{\text{maximum (or minimum)}}\{\sum_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j^t,y_j^t,z_j^t,w_j^t)\}$ will also be $\sum_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j^\theta,y_j^\theta,z_j^\theta,w_j^\theta)$, where $\text{Rank}(a,b,c,d)=\frac{1}{4}(a+b+c+d)$, i.e. according to the existing method (Kumar, Kaur and Singh, 2011) the fuzzy optimal solution of (P_4) can be obtained by solving the following crisp linear programming problem:

$$\begin{aligned}
&\text{Maximize (or Minimize) } (\text{Rank}(\sum_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j,y_j,z_j,w_j)))\\
&\text{subject to}\\
&\quad \sum_{j=1}^{n} m_{ij}=b_i \quad \forall\ i=1,2,...,m\\
&\quad \sum_{j=1}^{n} n_{ij}=g_i \quad \forall\ i=1,2,...,m\\
&\quad \sum_{j=1}^{n} o_{ij}=h_i \quad \forall\ i=1,2,...,m\\
&\quad \sum_{j=1}^{n} t_{ij}=k_i \quad \forall\ i=1,2,...,m\\
&\quad x_j\geq 0, y_j-x_j\geq 0, z_j-y_j\geq 0, w_j-z_j\geq 0.
\end{aligned} \qquad (P_5)$$

Step 5 Use an appropriate existing method (Taha, 2003) for solving the crisp linear programming problem (P_5) and find the optimal solution x_j, y_j, z_j and w_j.

Step 6 Find the fuzzy optimal solution $\{\tilde{x}_j\}$ by putting the values of x_j, y_j, z_j and w_j, obtained from Step 5, in $\tilde{x}_j = (x_j, y_j, z_j, w_j)$ and find the fuzzy optimal value $\sum_{j=1}^{n} (\tilde{c}_j \otimes \tilde{x}_j)$ by putting the values of $\tilde{x}_j$.

4. Shortcoming of the existing method

Let $\{x_j\}$ and A be the optimal solution and optimal value of a linear programming problem, respectively. If there exists any feasible solution $\{y_j\}$ of the same linear programming problem such that the value of the objective function of the linear programming problem corresponding to $\{y_j\}$ is also A, then $\{y_j\}$ is said to be an alternative optimal solution of the same linear programming problem, i.e. corresponding to all alternative optimal solutions the values of objective function should be same.

In this section, an FFLP problem, chosen in Example 4.1, is solved to show that the results of the FFLP problem from Example 4.1, obtained by using the existing method (Kumar, Kaur and Singh, 2011), do not conform to the property of alternative optimal solutions.

EXAMPLE 4.1 *Maximize* $((0,2,2,4)\otimes\tilde{x}_1\oplus(5,7,12,14)\otimes\tilde{x}_2\oplus(5,7,8,10)\otimes\tilde{x}_3)$
subject to

$$\tilde{x}_1 \oplus \tilde{x}_2 = (1,1,1,1)$$
$$\tilde{x}_1 = \tilde{x}_3$$
$$\tilde{x}_2 \oplus \tilde{x}_3 = (1,1,1,1)$$

where, $\tilde{x}_1, \tilde{x}_2$ and $\tilde{x}_3$ are non-negative trapezoidal fuzzy numbers.

Solution: On solving the chosen FFLP problem by using the existing method (Kumar, Kaur and Singh, 2011) it is found that all the fuzzy feasible solutions $\tilde{x}_1 = \tilde{x}_3 = (a,a,a,a)$ and $\tilde{x}_2 = (1-a, 1-a, 1-a, 1-a)$, $0 \le a \le 1$ are fuzzy optimal solutions of the chosen FFLP problem. Putting $\tilde{x}_1 = \tilde{x}_3 = (a,a,a,a)$ and $\tilde{x}_2 = (1-a,1-a,1-a,1-a)$ in the objective function we obtain the fuzzy optimal value equal $(5, 7+2a, 12-2a, 14)$. Since the fuzzy optimal value depends upon a, so for the chosen FFLP problem infinite fuzzy numbers, representing the fuzzy optimal values, can be obtained. Since the fuzzy optimal values of the FFLP problem, chosen in Example 4.1, corresponding to alternative fuzzy optimal solutions, are not equal, this contradicts the uniqueness property of fuzzy optimal value of an FFLP problem. So, it is not appropriate to apply the existing method (Kumar, Kaur and Singh, 2011) for solving FFLP problems.

5. The proposed method based on RMDS approach

Kumar, Singh, Kaur and Kaur (2010) used four parameters Rank, Mode, Divergence and Left spread for comparing trapezoidal fuzzy numbers. It can be easily seen that if $\tilde{A}$ and $\tilde{B}$ are two trapezoidal fuzzy numbers such that Rank($\tilde{A}$) = Rank($\tilde{B}$), Mode($\tilde{A}$) = Mode($\tilde{B}$), Divergence($\tilde{A}$) = Divergence($\tilde{B}$) and Left spread($\tilde{A}$) = Left spead($\tilde{B}$) then $\tilde{A} = \tilde{B}$. In this section, to overcome the shortcomings of the existing method, a new method, based on RMDS approach, is proposed for solving FFLP problems.

5.1. The RMDS approach

In this section, the existing ranking approach of Kumar, Singh, Kaur and Kaur (2010) is presented. Let $\tilde{A} = (a_1, b_1, c_1, d_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers, then use the following steps to compare $\tilde{A}$ and $\tilde{B}$:

Step 1 Find Rank($\tilde{A}$) $= \frac{1}{4}(a_1 + b_1 + c_1 + d_1)$ and Rank($\tilde{B}$) $= \frac{1}{4}(a_2 + b_2 + c_2 + d_2)$

Case (i) If Rank($\tilde{A}$) > Rank($\tilde{B}$) then $\tilde{A} \succ \tilde{B}$
Case (ii) If Rank($\tilde{A}$) < Rank($\tilde{B}$) then $\tilde{A} \prec \tilde{B}$
Case (iii) If Rank($\tilde{A}$) = Rank($\tilde{B}$) then go to Step 2.

Step 2 Find Mode($\tilde{A}$) $= \frac{1}{2}(b_1 + c_1)$ and Mode($\tilde{B}$) $= \frac{1}{2}(b_2 + c_2)$

Case (i) If Mode($\tilde{A}$) > Mode($\tilde{B}$) then $\tilde{A} \succ \tilde{B}$
Case (ii) If Mode($\tilde{A}$) < Mode($\tilde{B}$) then $\tilde{A} \prec \tilde{B}$
Case (iii) If Mode($\tilde{A}$) = Mode($\tilde{B}$) then go to Step 3.

Step 3 Find Divergence($\tilde{A}$) $= (d_1 - a_1)$ and Divergence($\tilde{B}$) $= (d_2 - a_2)$

Case (i) If Divergence($\tilde{A}$) > Divergence($\tilde{B}$) then $\tilde{A} \succ \tilde{B}$
Case (ii) If Divergence($\tilde{A}$) < Divergence($\tilde{B}$) then $\tilde{A} \prec \tilde{B}$
Case (iii) If Divergence($\tilde{A}$) = Divergence($\tilde{B}$) then go to Step 4.

Step 4 Find Left spread($\tilde{A}$) $= (b_1 - a_1)$ and Left spread($\tilde{B}$) $= (b_2 - a_2)$

Case (i) If Left spread($\tilde{A}$) > Left spread($\tilde{B}$) then $\tilde{A} \succ \tilde{B}$
Case (ii) If Left spread($\tilde{A}$) < Left spread($\tilde{B}$) then $\tilde{A} \prec \tilde{B}$
Case (iii) If Left spread($\tilde{A}$) = Left spread($\tilde{B}$) then $\tilde{A} = \tilde{B}$.

5.2. The proposed method

In this section, to overcome the shortcomings of the method by Kumar, Kaur and Singh (2011), a new method is proposed for solving FFLP problems (P_1).

Step 1 Use Steps 1 to 4 of the existing method and check whether an alternative optimal solution of crisp linear programming problem (P_5) exists or not.

Case (i) If an alternative optimal solution does not exist then the fuzzy optimal solution, obtained by using the existing method, is exact fuzzy optimal solution of the FFLP problem.
Case (ii) If alternative optimal solution existts, then Go to Step 2.

Step 2 Let the optimal value of crisp linear programming problem (P_5) be a and let it occur for p feasible solutions $\{x_j^k, y_j^k, z_j^k, w_j^k\}$ where

$k=1,...,p$. Now, the aim is to find $\underset{1\le k\le p}{\text{maximum (or minimum)}}\{\sum_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes (x_j^k,y_j^k,z_j^k,w_j^k)\}$. Since $\text{Rank}(\sum_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j^k,y_j^k,z_j^k,w_j^k)) = a\forall\ k = 1,...,p$, so using Step 2 of the existing method, discussed in Section 5.1 if $\underset{1\le t\le l}{\text{maximum (or minimum)}}\{\text{Mode}(\sum_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j^k,y_j^k,z_j^k,w_j^k))\}$ is Mode $(\sum_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j^{\phi},y_j^{\phi},z_j^{\phi},w_j^{\phi}))$ then $\underset{1\le t\le l}{\text{maximum (or minimum)}}\{\sum_{j=1}^{n}(p_j,q_j, r_j,s_j)\otimes(x_j^k,y_j^k,z_j^k,w_j^k)\}$ will also be $\sum_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j^{\phi},y_j^{\phi},z_j^{\phi},w_j^{\phi})$, i.e., the fuzzy optimal solution of (P_4) can be obtained by solving the crisp linear programming problem (P_6):

$$
\begin{aligned}
&\text{Maximize (or Minimize) } (\text{Mode}(\sum_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j,y_j,z_j,w_j)))\\
&\text{subject to}\\
&\qquad \sum_{j=1}^{n} m_{ij}=b_i \quad \forall\ i=1,2,...,m\\
&\qquad \sum_{j=1}^{n} n_{ij}=g_i \quad \forall\ i=1,2,...,m\\
&\qquad \sum_{j=1}^{n} o_{ij}=h_i \quad \forall\ i=1,2,...,m \qquad\qquad (P_6)\\
&\qquad \sum_{j=1}^{n} t_{ij}=k_i \quad \forall\ i=1,2,...,m\\
&\qquad \text{Rank}(\sum_{j=1}^{n}\tilde{c}_j\otimes\tilde{x}_j)=a\\
&\qquad x_j\ge 0, y_j-x_j\ge 0, z_j-y_j\ge 0, w_j-z_j\ge 0.
\end{aligned}
$$

Case (i) If there does not exist any alternative optimal solution, then put the values of x_j, y_j, z_j and w_j in $\tilde{x}_j=(x_j,y_j,z_j,w_j)$ to find the fuzzy optimal solution $\{\tilde{x}_j\}$ and find the fuzzy optimal value $\sum_{j=1}^{n}(\tilde{c}_j\otimes\tilde{x}_j)$ by putting the values of $\tilde{x}_j$.

Case (ii) If alternative solution exists then go to Step 3.

Step 3 In the same direction, as discussed in Step 2, solve the crisp linear programming problem (P_7) and check whether an alternative optimal solution exists or not:

$$
\begin{aligned}
&\text{Maximize (or Minimize) } (\text{Divergence}(\sum_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j,y_j,z_j,w_j)))\\
&\text{subject to}\\
&\qquad \sum_{j=1}^{n} m_{ij}=b_i \quad \forall\ i=1,2,...,m
\end{aligned}
$$

$$\begin{aligned}&\sum_{j=1}^{n} n_{ij} = g_i \quad \forall\ i = 1,2,...,m\\&\sum_{j=1}^{n} o_{ij} = h_i \quad \forall\ i = 1,2,...,m\\&\sum_{j=1}^{n} t_{ij} = k_i \quad \forall\ i = 1,2,...,m\\&\text{Rank}(\sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j) = a\\&\text{Mode}(\sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j) = b\\&x_j \geq 0, y_j - x_j \geq 0, z_j - y_j \geq 0, w_j - z_j \geq 0\end{aligned} \qquad (P_7)$$

where, b is the optimal value of the crisp linear programming problem (P_6).
Case (i) If there does not exist any alternative optimal solution then put the values of x_j, y_j, z_j and w_j in $\tilde{x}_j = (x_j, y_j, z_j, w_j)$ to find the fuzzy optimal solution $\{\tilde{x}_j\}$ and find the fuzzy optimal value $\sum_{j=1}^{n}(\tilde{c}_j \otimes \tilde{x}_j)$ by putting the values of $\tilde{x}_j$.
Case (ii) If alternative solution exists then go to Step 4.

Step 4 In the same direction, as discussed in Step 2, solve the crisp linear programming problem (P_8):

Maximize (or Minimize) (Left spread($\sum_{j=1}^{n}(p_j, q_j, r_j, s_j) \otimes (x_j, y_j, z_j, w_j)$))

subject to

$$\begin{aligned}&\sum_{j=1}^{n} m_{ij} = b_i \quad \forall\ i = 1,2,...,m\\&\sum_{j=1}^{n} n_{ij} = g_i \quad \forall\ i = 1,2,...,m\\&\sum_{j=1}^{n} o_{ij} = h_i \quad \forall\ i = 1,2,...,m\\&\sum_{j=1}^{n} t_{ij} = k_i \quad \forall\ i = 1,2,...,m\\&\text{Rank}(\sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j) = a\\&\text{Mode}(\sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j) = b\\&\text{Divergence}(\sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j) = c\\&x_j \geq 0, y_j - x_j \geq 0, z_j - y_j \geq 0, w_j - z_j \geq 0\end{aligned} \qquad (P_8)$$

where c is the optimal value of the crisp linear programming problem (P_7). Find the fuzzy optimal solution $\{\tilde{x}_j\}$ by putting the values of x_j, y_j, z_j and w_j in $\tilde{x}_j = (x_j, y_j, z_j, w_j)$ and the fuzzy optimal value $\sum_{j=1}^{n}(\tilde{c}_j \otimes \tilde{x}_j)$ by putting the values of $\tilde{x}_j$.

5.3. Advantages of the proposed method

As discussed in Section 4, by using the method of Kumar, Kaur and Singh (2011) a unique fuzzy number, representing the fuzzy optimal value, is not obtained while by using the proposed method always a unique fuzzy number, representing the fuzzy optimal value, will be obtained. To show the advantage of the proposed method, the FFLP problem from Example 4.1, on solving which by using the existing method infinite fuzzy numbers representing the fuzzy optimal value are obtained, is solved by using the proposed method and it is found that a unique fuzzy number, representing the fuzzy optimal value, is obtained.

Step 1 Assuming $\tilde{x}_1 = (x_1, y_1, z_1, w_1)$, $\tilde{x}_2 = (x_2, y_2, z_2, w_2)$ and $\tilde{x}_3 = (x_3, y_3, z_3, w_3)$ and using Steps 1 to 4 of the existing method, the FFLP problem, chosen in Example 4.1, can be written as:
Maximize $\frac{1}{4}(5x_2+5x_3+2y_1+7y_2+7y_3+2z_1+12z_2+8z_3+4w_1+14w_2+10w_3)$
subject to

$$\begin{aligned}
&x_1 + x_2 = 1,\ y_1 + y_2 = 1,\ z_1 + z_2 = 1,\ w_1 + w_2 = 1\\
&x_1 = x_3,\ y_1 = y_3,\ z_1 = z_3,\ w_1 = w_3\\
&x_2 + x_3 = 1,\ y_2 + y_3 = 1,\ z_2 + z_3 = 1,\ w_2 + w_3 = 1\\
&x_1 \geq 0, y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, w_1 - z_1 \geq 0\\
&x_2 \geq 0, y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, w_2 - z_2 \geq 0\\
&x_3 \geq 0, y_3 - x_3 \geq 0, z_3 - y_3 \geq 0, w_3 - z_3 \geq 0.
\end{aligned}$$

Step 2 Since, on solving the crisp linear programming problem, obtained in Step 1, alternative optimal solution is obtained and the optimal value of the crisp linear programming problem is $\frac{19}{2}$ so, using Step 2 of the proposed method the solution of the chosen problem can be obtained by solving the following crisp linear programming problem:

$$\text{Maximize } (y_1 + \frac{7}{2}y_2 + \frac{7}{2}y_3 + z_1 + 6z_2 + 4z_3)$$

subject to

$$\begin{aligned}
&x_1 + x_2 = 1, y_1 + y_2 = 1, z_1 + z_2 = 1, w_1 + w_2 = 1\\
&x_1 = x_3, y_1 = y_3, z_1 = z_3, w_1 = w_3\\
&x_2 + x_3 = 1, y_2 + y_3 = 1, z_2 + z_3 = 1, w_2 + w_3 = 1\\
&5x_2 + 5x_3 + 2y_1 + 7y_2 + 7y_3 + 2z_1 + 12z_2+\\
&8z_3 + 4w_1 + 14w_2 + 10w_3 = 38\\
&x_1 \geq 0, y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, w_1 - z_1 \geq 0\\
&x_2 \geq 0, y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, w_2 - z_2 \geq 0\\
&x_3 \geq 0, y_3 - x_3 \geq 0, z_3 - y_3 \geq 0, w_3 - z_3 \geq 0.
\end{aligned}$$

Step 3 Since, on solving the crisp linear programming problem, obtained in Step 2, alternative optimal solution is obtained and the optimal value of the crisp linear programming problem is $\frac{19}{2}$ so, using Step 3 of the proposed method the solution of the chosen problem can be obtained by solving the following crisp linear programming problem:

$$\begin{aligned}
&\text{Maximize } (4w_1 + 14w_2 + 10w_3 - 5x_2 - 5x_3)\\
&\text{subject to}\\
&\qquad x_1 + x_2 = 1, y_1 + y_2 = 1, z_1 + z_2 = 1, w_1 + w_2 = 1\\
&\qquad x_1 = x_3, y_1 = y_3, z_1 = z_3, w_1 = w_3\\
&\qquad x_2 + x_3 = 1, y_2 + y_3 = 1, z_2 + z_3 = 1, w_2 + w_3 = 1\\
&\qquad 5x_2 + 5x_3 + 2y_1 + 7y_2 + 7y_3 + 2z_1 + 12z_2+\\
&\qquad 8z_3 + 4w_1 + 14w_2 + 10w_3 = 38\\
&\qquad y_1 + \frac{7}{2}y_2 + \frac{7}{2}y_3 + z_1 + 6z_2 + 4z_3 = \frac{19}{2}\\
&\qquad x_1 \geq 0, y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, w_1 - z_1 \geq 0\\
&\qquad x_2 \geq 0, y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, w_2 - z_2 \geq 0\\
&\qquad x_3 \geq 0, y_3 - x_3 \geq 0, z_3 - y_3 \geq 0, w_3 - z_3 \geq 0.
\end{aligned}$$

Step 4 Since, on solving the crisp linear programming problem, obtained in Step 3, alternative optimal solution is obtained and the optimal value of the crisp linear programming problem is 9 so, using Step 4 of the proposed method the solution of the chosen problem can be obtained by solving the following crisp linear programming problem:

$$\begin{aligned}
&\text{Maximize } (2y_1 + 7y_2 + 7y_3 - 5x_2 - 5x_3)\\
&\text{subject to}\\
&\qquad x_1 + x_2 = 1, y_1 + y_2 = 1, z_1 + z_2 = 1, w_1 + w_2 = 1\\
&\qquad x_1 = x_3, y_1 = y_3, z_1 = z_3, w_1 = w_3\\
&\qquad x_2 + x_3 = 1, y_2 + y_3 = 1, z_2 + z_3 = 1, w_2 + w_3 = 1\\
&\qquad 5x_2 + 5x_3 + 2y_1 + 7y_2 + 7y_3 + 2z_1 + 12z_2+\\
&\qquad 8z_3 + 4w_1 + 14w_2 + 10w_3 = 38\\
&\qquad y_1 + \frac{7}{2}y_2 + \frac{7}{2}y_3 + z_1 + 6z_2 + 4z_3 = \frac{19}{2}\\
&\qquad 4w_1 + 14w_2 + 10w_3 - 5x_2 - 5x_3 = 9\\
&\qquad x_1 \geq 0, y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, w_1 - z_1 \geq 0\\
&\qquad x_2 \geq 0, y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, w_2 - z_2 \geq 0\\
&\qquad x_3 \geq 0, y_3 - x_3 \geq 0, z_3 - y_3 \geq 0, w_3 - z_3 \geq 0.
\end{aligned}$$

The obtained optimal solution is $x_1 = 1, y_1 = 1, z_1 = 1, w_1 = 1, x_2 = 0, y_2 = 0, z_2 = 0, w_2 = 0, x_3 = 1, y_3 = 1, z_3 = 1$ and $w_3 = 1$. Using Step 4 of the proposed method the fuzzy optimal solution is $\tilde{x}_1 = (1, 1, 1, 1), \tilde{x}_2 = (0, 0, 0, 0)$, $\tilde{x}_3 = (1, 1, 1, 1)$ and the fuzzy optimal value is $(5, 9, 10, 14)$.

6. Comparative study

To show the advantages of the proposed method over the one of Kumar, Kaur and Singh (2011) the results of the FFLP problem from Example 4.1, obtained by using the existing and the proposed method are shown in Table 1.

Table 1: Results of the FFLP problems

Example	Fuzzy optimal value	
	Existing method (Kumar, Kaur and Singh, 2011)	Proposed method
6.1 (Kumar, Kaur and Singh, 2011)	$(9, 27, 75)$	$(9, 27, 75)$
6.2 (Kumar, Kaur and Singh, 2011)	$(5, 16, 33)$	$(5, 16, 33)$
4.1	$(5, 7+2a, 12-2a, 14), 0 \leq a \leq 1$	$(5, 9, 10, 14)$

It is obvious from the results, shown in Table 1, that on solving the existing FFLP problems (Kumar, Kaur and Singh, 2011, Example 6.1 and 6.2) by using the method of Kumar, Kaur and Singh (2011) a unique fuzzy number, representing the fuzzy optimal value, is obtained, but on solving the FFLP problem, chosen in Example 4.1, by using the existing method infinite fuzzy numbers, representing the fuzzy optimal value of the same problem, are obtained, contrary to the uniqueness property of fuzzy optimal value of FFLP problems. However, on solving all the existing and chosen FFLP problems by using the proposed method a unique fuzzy number, representing the fuzzy optimum, is obtained. Hence, it can be concluded that it is better to use the proposed method as compared to the existing method of Kumar, Kaur and Singh (2011).

7. Conclusions

In this paper it is shown that by using the proposed method all the shortcomings of the existing method (Kumar, Kaur and Singh, 2011) are removed. Hence, it can be concluded that it is better to use the proposed method as compared to the existing method (Kumar, Kaur and Singh, 2011).

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