

Equivalence of control systems
on the Euclidean group $SE(2)$ *

by

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Abstract: We classify, under (local) state space equivalence, all full-rank left-invariant control affine systems evolving on the Euclidean group $SE(2)$.

Keywords: Left-invariant control system, state space equivalence, Euclidean group.

1. Introduction

Invariant control systems are smooth, nonlinear control systems evolving on (real, finite dimensional) Lie groups with dynamics invariant under translations (see Jurdjevic and Sussmann, 1979; Jurdjevic, 1997; Agrachev and Sachkov, 2004; Biggs and Remsing, 2012). In order to understand the local geometry of (nonlinear) control systems, one needs to introduce natural equivalence relations. The most natural equivalence relation for (smooth, nonlinear) control systems is equivalence up to coordinate changes in the state space. This is called state space equivalence (see Biggs and Remsing, no date; Jakubczyk, 1990). Two control systems are state space equivalent if they are related by a diffeomorphism (in which case their trajectories, corresponding to the same controls, are also related by that diffeomorphism).

In this paper we consider only left-invariant control affine systems, evolving on the Euclidean group $SE(2)$. We classify, under state space equivalence, all such full-rank control systems. This classification is obtained by making use of an algebraic characterization of state space equivalence. A representative is identified for each equivalence class, in a systematic manner.

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2. Invariant control systems and equivalence

Left-invariant control affine systems

A left-invariant control affine system Σ is a control system of the form

$$\dot{g} = g \Xi(\mathbf{1}, u) = g(A + u_1 B_1 + \cdots + u_\ell B_\ell), \quad g \in \mathbf{G}, u \in \mathbb{R}^\ell.$$

Here \mathbf{G} is a (real, finite-dimensional) *matrix* Lie group and the *parametrization map* $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \rightarrow \mathfrak{g}$ is an affine injection (i.e., B_1, \dots, B_ℓ are linearly independent). The admissible controls are piecewise continuous maps $u(\cdot) : [0, T] \rightarrow \mathbb{R}^\ell$ and the *trace* of the system $\Gamma = A + \Gamma^0 = A + \langle B_1, \dots, B_\ell \rangle$ is an affine subspace of (the Lie algebra) \mathfrak{g} . A system Σ is called *homogeneous* if $A \in \Gamma^0$, and *inhomogeneous* otherwise. Furthermore, Σ has *full rank* provided the Lie algebra generated by its trace equals the whole Lie algebra \mathfrak{g} . Note that Σ is completely determined by the specification of its state space \mathbf{G} and its parametrization map $\Xi(\mathbf{1}, \cdot)$. Hence, we shall specify a system Σ (on \mathbf{G}) by simply writing

$$\Sigma : A + u_1 B_1 + \cdots + u_\ell B_\ell.$$

State-space equivalence

State space equivalence is well understood (see Agrachev and Sachkov, 2004; Jakubczyk, 1990); it establishes a one-to-one correspondence between the trajectories of equivalent systems. This equivalence relation is very strong. Consequently, there are so many equivalence classes that any general classification appears to be very difficult, if not impossible. However, there is a chance for some reasonable classification in low dimensions.

Now, let \mathbf{G} be a fixed matrix Lie group and let Σ and Σ' be two (left-invariant control affine) systems on \mathbf{G} . We say that Σ and Σ' are *locally state space equivalent* at $g \in \mathbf{G}$ and $g' \in \mathbf{G}$ if there exist open neighbourhoods N and N' of g and g' , respectively, and a (local) diffeomorphism $\phi : N \rightarrow N'$ (mapping g to g') such that $T_g \phi \cdot \Xi(g, u) = \Xi'(\phi(g), u)$ for all $g \in N$ and $u \in \mathbb{R}^\ell$ (i.e., the diagram

$$\begin{array}{ccc} N \times \mathbb{R}^\ell & \xrightarrow{\phi \times \text{id}_{\mathbb{R}^\ell}} & N' \times \mathbb{R}^\ell \\ \Xi \downarrow & & \downarrow \Xi' \\ TN & \xrightarrow{T\phi} & TN' \end{array}$$

commutes). Σ and Σ' are called *globally state space equivalent* if this happens globally (i.e., $N = \mathbf{G}$ and $N' = \mathbf{G}'$). In this paper we consider only local state space equivalence, which will be referred to, simply, as *equivalence*. Any equivalence between two control systems can be reduced to an equivalence between neighbourhoods of the identity. More precisely,

PROPOSITION 1 (*Biggs and Remsing, no date*) Σ and Σ' are equivalent at $g \in \mathbf{G}$ and $g' \in \mathbf{G}$ if and only if they are equivalent at $g = \mathbf{1} \in \mathbf{G}$ and $g' = \mathbf{1} \in \mathbf{G}$.

Henceforth, we will assume that any equivalence is between neighbourhoods of identity. We recall an algebraic characterization of this equivalence. Let Σ and Σ' be two full-rank systems.

PROPOSITION 2 (*Biggs and Remsing, no date*) Σ and Σ' are equivalent if and only if there exists a Lie algebra automorphism $\psi \in \text{Aut}(\mathfrak{g})$ such that $\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, u)$ for all $u \in \mathbb{R}^\ell$.

3. Classification

Let (the fixed state space) \mathbf{G} be the Euclidean group $SE(2)$. The group

$$SE(2) = \left\{ \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R \end{bmatrix} : \mathbf{v} \in \mathbb{R}^{2 \times 1}, R \in SO(2) \right\}$$

is a (real, three-dimensional) connected matrix Lie group. The associated Lie algebra

$$\mathfrak{se}(2) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}$$

has standard basis

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(The bracket operation is given by $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$ and $[E_1, E_2] = 0$.) With respect to this basis, the group $\text{Aut}(\mathfrak{se}(2))$ of Lie algebra automorphisms of $\mathfrak{se}(2)$ is given by

$$\left\{ \begin{bmatrix} x & y & v \\ -\varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} : x, y, v, w \in \mathbb{R}, x^2 + y^2 \neq 0, \varsigma = \pm 1 \right\}.$$

Note that $\langle E_1, E_2 \rangle$ is an invariant subspace of every such automorphism.

We now proceed to classify (under local state space equivalence) all full-rank left-invariant control affine systems on $SE(2)$. This reduces (by Proposition 2) to the algebraic classification of the corresponding affine parametrization maps. We outline the approach to be used in classifying these systems. First, we distinguish between the number of controls involved and the homogeneity of the systems; this yields four types of systems.

REMARK Let $A = a_1E_1 + a_2E_2 + a_3E_3$. The condition $a_3 = 0$ is invariant. More precisely, for any automorphism ψ , the coefficient of E_3 in A is zero if and only if the coefficient of E_3 in $\psi \cdot A$ is zero.

The trace of any full-rank system must admit a vector with a nonzero E_3 term. This property, together with the invariance of the coefficient of E_3 , allows us to further distinguish between various families of equivalence classes. For each of these families, we simplify an arbitrary system by successively applying automorphisms. Finally, we verify that all the candidates for class representatives are distinct and not equivalent. Families of these representatives are typically parametrized by some constants $\alpha > 0$, $\beta \neq 0$ and a vector $\gamma = (\gamma_1, \dots, \gamma_k)$.

When convenient, a system specified by

$$\sum_{i=1}^3 a_i E_i + u_1 \sum_{i=1}^3 b_i E_i + u_2 \sum_{i=1}^3 c_i E_i + u_3 \sum_{i=1}^3 d_i E_i$$

will be represented as

$$\left[\begin{array}{c|ccc} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array} \right].$$

As any automorphism $\psi : \mathfrak{se}(2) \rightarrow \mathfrak{se}(2)$ is identified with its matrix (with respect to the standard basis), the evaluation $\psi \cdot \Xi(\mathbf{1}, u)$ becomes a matrix multiplication.

We start with single-input systems. (Only the inhomogeneous case need be considered as the homogeneous systems do not have full rank.)

PROPOSITION 3 *Every single-input (inhomogeneous) system is equivalent to exactly one of the following systems*

$$\begin{aligned} \Sigma_{1,\alpha}^{(1,1)} &: \alpha E_3 + u E_2 \\ \Sigma_{2,\alpha\gamma}^{(1,1)} &: E_2 + \gamma_1 E_3 + u(\alpha E_3). \end{aligned}$$

Here $\alpha > 0$ and $\gamma_1 \in \mathbb{R}$, with different values of these parameters yielding distinct (non-equivalent) class representatives.

Let Σ be an arbitrary system represented as

$$\left[\begin{array}{c|c} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{array} \right].$$

First, we consider the case of $b_3 = 0$. Then, as Σ has full rank, $a_3 \neq 0$. Let ψ be the automorphism specified by $\varsigma = 1$, $x = 1$, $y = 0$, $v = -\frac{a_1}{a_3}$ and $w = -\frac{a_2}{a_3}$. Then

$$\begin{bmatrix} 1 & 0 & -\frac{a_1}{a_3} \\ 0 & 1 & -\frac{a_2}{a_3} \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{c|c} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & 0 \end{array} \right] = \left[\begin{array}{c|c} 0 & b_1 \\ 0 & b_2 \\ a_3 & 0 \end{array} \right].$$

Now, by applying the automorphism specified by $\varsigma = \text{sgn}(a_3)$, $x = b_2$, $y = -b_1$ and $v = w = 0$, we get

$$\begin{bmatrix} b_2 & -b_1 & 0 \\ \text{sgn}(a_3)b_1 & \text{sgn}(a_3)b_2 & 0 \\ 0 & 0 & \text{sgn}(a_3) \end{bmatrix} \begin{bmatrix} 0 & b_1 \\ 0 & b_2 \\ a_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \text{sgn}(a_3)(b_1^2 + b_2^2) \\ \alpha & 0 \end{bmatrix}$$

where $\alpha = \text{sgn}(a_3)a_3 > 0$. Lastly, we apply the automorphism specified by $\varsigma = 1$, $x = \frac{\text{sgn}(a_3)}{b_1^2 + b_2^2}$ and $y = v = w = 0$ to obtain

$$\begin{bmatrix} \frac{\text{sgn}(a_3)}{b_1^2 + b_2^2} & 0 & 0 \\ 0 & \frac{\text{sgn}(a_3)}{b_1^2 + b_2^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \text{sgn}(a_3)(b_1^2 + b_2^2) \\ \alpha & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ \alpha & 0 \end{bmatrix}.$$

Thus, Σ is equivalent to $\Sigma_{1,\alpha}^{(1,1)}$.

Next, we assume $b_3 \neq 0$. Let ψ be the automorphism specified by $\varsigma = 1$, $x = 1$, $y = 0$, $v = -\frac{b_1}{b_3}$ and $w = -\frac{b_2}{b_3}$. Then

$$\begin{bmatrix} 1 & 0 & -\frac{b_1}{b_3} \\ 0 & 1 & -\frac{b_2}{b_3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = \begin{bmatrix} a_1 - \frac{a_3 b_1}{b_3} & 0 \\ a_2 - \frac{a_3 b_2}{b_3} & 0 \\ a_3 & b_3 \end{bmatrix} = \begin{bmatrix} c_1 & 0 \\ c_2 & 0 \\ a_3 & b_3 \end{bmatrix}$$

for some corresponding constants c_1 and c_2 . (Note that if $c_1 = c_2 = 0$, then Σ is not inhomogeneous). Now, by applying the automorphism specified by $\varsigma = \text{sgn}(b_3)$, $x = c_2$, $y = -c_1$ and $v = w = 0$, we get

$$\begin{bmatrix} c_2 & -c_1 & 0 \\ \text{sgn}(b_3)c_1 & \text{sgn}(b_3)c_2 & 0 \\ 0 & 0 & \text{sgn}(b_3) \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ c_2 & 0 \\ a_3 & b_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \text{sgn}(b_3)(c_1^2 + c_2^2) & 0 \\ \text{sgn}(b_3)a_3 & \alpha \end{bmatrix}$$

where $\alpha = \text{sgn}(b_3)b_3 > 0$. Lastly, we apply the automorphism specified by $\varsigma = 1$, $x = \frac{\text{sgn}(b_3)}{c_1^2 + c_2^2}$ and $y = v = w = 0$ to obtain

$$\begin{bmatrix} \frac{\text{sgn}(b_3)}{c_1^2 + c_2^2} & 0 & 0 \\ 0 & \frac{\text{sgn}(b_3)}{c_1^2 + c_2^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \text{sgn}(b_3)(c_1^2 + c_2^2) & 0 \\ \text{sgn}(b_3)a_3 & \alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ \gamma_1 & \alpha \end{bmatrix}$$

where $\gamma_1 = \text{sgn}(b_3)a_3$. Thus, Σ is equivalent to $\Sigma_{2,\alpha\gamma}^{(1,1)}$. (Here $\gamma = \gamma_1$.)

Finally, we verify that all these systems are not equivalent. Assume two systems $\Sigma_{1,\alpha}^{(1,1)}$ and $\Sigma_{1,\alpha'}$ are equivalent. Then there exists an automorphism such that

$$\begin{bmatrix} x & y & v \\ -\varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ \alpha & 0 \end{bmatrix} = \begin{bmatrix} v\alpha & y \\ w\alpha & \varsigma x \\ \varsigma\alpha & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ \alpha' & 0 \end{bmatrix}$$

which implies that $\alpha = \alpha'$. Similarly, two systems $\Sigma_{2,\alpha\gamma}^{(1,1)}$ and $\Sigma_{2,\alpha'\gamma'}^{(1,1)}$ are equivalent only if $\alpha = \alpha'$ and $\gamma = \gamma'$. Lastly, any automorphism leaves $\langle E_1, E_2 \rangle$ invariant. Thus, as $uE_2 \in \langle E_1, E_2 \rangle$ and $u(\alpha E_3) \notin \langle E_1, E_2 \rangle$, it follows that $\Sigma_{1,\alpha}^{(1,1)}$ and $\Sigma_{2,\alpha'\gamma'}^{(1,1)}$ cannot be equivalent.

We now proceed to two-input systems. First, we consider the homogeneous case and then the inhomogeneous case.

PROPOSITION 4 *Every two-input homogeneous system is equivalent to exactly one of the following systems*

$$\begin{aligned} \Sigma_{1,\alpha\gamma}^{(2,0)} &: \gamma_1 E_2 + \gamma_2 E_3 + u_1(\alpha E_3) + u_2 E_2 \\ \Sigma_{2,\alpha\gamma}^{(2,0)} &: \gamma_1 E_2 + \gamma_2 E_3 + u_1(E_2 + \gamma_3 E_3) + u_2(\alpha E_3). \end{aligned}$$

Here $\alpha > 0$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, with different values of these parameters yielding distinct (non-equivalent) class representatives.

Let Σ be an arbitrary system represented as

$$\left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array} \right].$$

For the case of $c_3 = 0$, an argument similar to that of Proposition 3 shows that Σ is equivalent to $\Sigma_{1,\alpha\gamma}^{(2,0)}$ for some $\gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2$ and $\alpha > 0$. Now assume that $c_3 \neq 0$. Let ψ be the automorphism specified by $\varsigma = 1$, $x = 1$, $y = 0$, $v = -\frac{c_1}{c_3}$ and $w = -\frac{c_2}{c_3}$. Then

$$\psi \cdot \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array} \right] = \left[\begin{array}{c|cc} a_1 - \frac{a_3 c_1}{c_3} & b_1 - \frac{b_3 c_1}{c_3} & 0 \\ a_2 - \frac{a_3 c_2}{c_3} & b_2 - \frac{b_3 c_2}{c_3} & 0 \\ a_3 & b_3 & c_3 \end{array} \right] = \left[\begin{array}{c|cc} a'_1 & b'_1 & 0 \\ a'_2 & b'_2 & 0 \\ a_3 & b_3 & c_3 \end{array} \right]$$

for some corresponding constants a'_1, a'_2, b'_1 and b'_2 . By applying an automorphism ψ' specified by $\varsigma = \text{sgn}(c_3)$, $x = \frac{\text{sgn}(c_3)b'_2}{b_1'^2 + b_2'^2}$, $y = -\frac{\text{sgn}(c_3)b'_1}{b_1'^2 + b_2'^2}$ and $v = w = 0$, we get

$$\psi' \cdot \left[\begin{array}{c|cc} a'_1 & b'_1 & 0 \\ a'_2 & b'_2 & 0 \\ a_3 & b_3 & c_3 \end{array} \right] = \left[\begin{array}{c|cc} \frac{a'_1 b'_2 - a'_2 b'_1}{b_1'^2 + b_2'^2} & 0 & 0 \\ \gamma_1 & 1 & 0 \\ \gamma_2 & \gamma_3 & \alpha \end{array} \right]$$

for some $\gamma = (\gamma_1, \gamma_2, \gamma_3)$. Here $\alpha = \text{sgn}(c_3)c_3 > 0$ and $a'_1 b'_2 - a'_2 b'_1 = 0$ as the system is homogeneous. Thus, Σ is equivalent to $\Sigma_{2,\alpha\gamma}^{(2,0)}$.

We verify that all these systems are not equivalent. Assume two systems $\Sigma_{1,\alpha\gamma}^{(2,0)}$ and $\Sigma_{1,\alpha'\gamma'}^{(2,0)}$ are equivalent. Then there exists an automorphism such that

$$\left[\begin{array}{ccc} x & y & v \\ -\varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{array} \right] \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \gamma_1 & 0 & 1 \\ \gamma_2 & \alpha & 0 \end{array} \right] = \left[\begin{array}{c|cc} y\gamma_1 + v\gamma_2 & v\alpha & y \\ \varsigma x\gamma_1 + w\gamma_2 & w\alpha & \varsigma x \\ \varsigma\gamma_2 & \varsigma\alpha & 0 \end{array} \right] = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \gamma'_1 & 0 & 1 \\ \gamma'_2 & \alpha' & 0 \end{array} \right]$$

which implies that $\alpha = \alpha'$ and $\gamma = \gamma'$. Similarly, $\Sigma_{2,\alpha\gamma}^{(2,0)}$ and $\Sigma_{2,\alpha'\gamma'}^{(2,0)}$ are equivalent only if $\alpha = \alpha'$ and $\gamma = \gamma'$. Lastly, $u_2 E_2 \in \langle E_1, E_2 \rangle$ and $u_2(\alpha E_3) \notin \langle E_1, E_2 \rangle$, and so $\Sigma_{1,\alpha\gamma}^{(2,0)}$ cannot be equivalent to $\Sigma_{2,\alpha'\gamma'}^{(2,0)}$.

PROPOSITION 5 *Every two-input inhomogeneous system is equivalent to exactly one of the following systems*

$$\begin{aligned}\Sigma_{1,\alpha\beta\gamma}^{(2,1)} &: \alpha E_3 + u_1(E_1 + \gamma_1 E_2) + u_2(\beta E_2) \\ \Sigma_{2,\alpha\beta\gamma}^{(2,1)} &: \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(\alpha E_3) + u_2 E_2 \\ \Sigma_{3,\alpha\beta\gamma}^{(2,1)} &: \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(E_2 + \gamma_3 E_3) + u_2(\alpha E_3).\end{aligned}$$

Here $\alpha > 0$, $\beta \neq 0$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, with different values of these parameters yielding distinct (non-equivalent) class representatives.

Consider an arbitrary system Σ represented as

$$\left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array} \right].$$

First, we assume that $b_3 = c_3 = 0$ (in this case $a_3 \neq 0$). Let ψ be the automorphism specified by $\varsigma = 1$, $x = 1$, $y = 0$, $v = -\frac{a_1}{a_3}$ and $w = -\frac{a_2}{a_3}$. Then

$$\psi \cdot \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & 0 & 0 \end{array} \right] = \left[\begin{array}{c|cc} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ a_3 & 0 & 0 \end{array} \right].$$

Now, by applying an automorphism ψ' specified by $\varsigma = \text{sgn}(a_3)$, $x = \frac{c_2}{b_1 c_2 - b_2 c_1}$, $y = -\frac{c_1}{b_1 c_2 - b_2 c_1}$ and $v = w = 0$, we get

$$\psi' \cdot \left[\begin{array}{c|cc} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ a_3 & 0 & 0 \end{array} \right] = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & \gamma_1 & \beta \\ \alpha & 0 & 0 \end{array} \right]$$

for some $\gamma = \gamma_1$. Here $\alpha = \text{sgn}(a_3)a_3 > 0$ and $\beta = \frac{\text{sgn}(a_3)(c_1^2 + c_2^2)}{b_1 c_2 - b_2 c_1} \neq 0$. Thus, Σ is equivalent to $\Sigma_{1,\alpha\beta\gamma}^{(2,1)}$. When $b_3 \neq 0$ and $c_3 = 0$, a very similar argument shows that Σ is equivalent to $\Sigma_{2,\alpha\beta\gamma}^{(2,1)}$ for some $\gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2$, $\alpha > 0$ and $\beta \neq 0$.

Next, we assume that $c_3 \neq 0$. Let ψ be the automorphism specified by $\varsigma = 1$, $x = 1$, $y = 0$, $v = -\frac{c_1}{c_3}$ and $w = -\frac{c_2}{c_3}$. Then

$$\psi \cdot \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array} \right] = \left[\begin{array}{c|cc} a_1 - \frac{a_3 c_1}{c_3} & b_1 - \frac{b_3 c_1}{c_3} & 0 \\ a_2 - \frac{a_3 c_2}{c_3} & b_2 - \frac{b_3 c_2}{c_3} & 0 \\ a_3 & b_3 & c_3 \end{array} \right] = \left[\begin{array}{c|cc} a'_1 & b'_1 & 0 \\ a'_2 & b'_2 & 0 \\ a_3 & b_3 & c_3 \end{array} \right]$$

for some corresponding constants a'_1, a'_2, b'_1 and b'_2 . Now, by applying the automorphism ψ' specified by $\varsigma = \text{sgn}(c_3)$, $x = \frac{\text{sgn}(c_3)b'_2}{b_1'^2 + b_2'^2}$, $y = -\frac{\text{sgn}(c_3)b'_1}{b_1'^2 + b_2'^2}$ and $v = w = 0$, we get

$$\psi' \cdot \left[\begin{array}{c|cc} a'_1 & b'_1 & 0 \\ a'_2 & b'_2 & 0 \\ a_3 & b_3 & c_3 \end{array} \right] = \left[\begin{array}{c|cc} \beta & 0 & 0 \\ \gamma_1 & 1 & 0 \\ \gamma_2 & \gamma_3 & \alpha \end{array} \right]$$

for some $\gamma = (\gamma_1, \gamma_2, \gamma_3)$. Here $\alpha = \text{sgn}(c_3)c_3 > 0$ and $\beta \neq 0$. Thus, Σ is equivalent to $\Sigma_{3,\alpha\beta\gamma}^{(2,1)}$.

Finally, we verify that all these systems are not equivalent. Assume two systems $\Sigma_{1,\alpha\beta\gamma}^{(2,1)}$ and $\Sigma_{1,\alpha'\beta'\gamma'}^{(2,1)}$ are equivalent. Then there exists an automorphism ψ such that $\psi \cdot \Xi_{1,\alpha\beta\gamma}^{(2,1)}(\mathbf{1}, \cdot) = \Xi_{1,\alpha'\beta'\gamma'}^{(2,1)}(\mathbf{1}, \cdot)$, i.e.,

$$\left[\begin{array}{c|cc} \alpha v & x + \gamma_1 y & \beta y \\ \alpha w & \varsigma \gamma_1 x - \varsigma y & \varsigma \beta x \\ \varsigma \alpha & 0 & 0 \end{array} \right] = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & \gamma'_1 & \beta' \\ \alpha' & 0 & 0 \end{array} \right].$$

This implies that $\alpha = \alpha'$, $\beta = \beta'$ and $\gamma = \gamma'$. Similarly, $\Sigma_{2,\alpha\beta\gamma}^{(2,1)}$ and $\Sigma_{3,\alpha\beta\gamma}^{(2,1)}$ are equivalent to $\Sigma_{2,\alpha'\beta'\gamma'}^{(2,1)}$ and $\Sigma_{3,\alpha'\beta'\gamma'}^{(2,1)}$, respectively, only if $\alpha = \alpha'$, $\beta = \beta'$ and $\gamma = \gamma'$. Lastly, $u_2 E_2, u_2(\beta E_2) \in \langle E_1, E_2 \rangle$ and $u_2(\alpha E_3) \notin \langle E_1, E_2 \rangle$, and so neither $\Sigma_{1,\alpha\beta\gamma}^{(2,1)}$ nor $\Sigma_{2,\alpha\beta\gamma}^{(2,1)}$ can be equivalent to $\Sigma_{3,\alpha'\beta'\gamma'}^{(2,1)}$. Likewise, $\Sigma_{1,\alpha\beta\gamma}^{(2,1)}$ cannot be equivalent to $\Sigma_{2,\alpha'\beta'\gamma'}^{(2,1)}$.

REMARK Note that $\Sigma_{2,\alpha\beta\gamma}^{(2,1)}$ and $\Sigma_{3,\alpha\beta\gamma}^{(2,1)}$ differ only from $\Sigma_{1,\alpha\gamma}^{(2,0)}$ and $\Sigma_{2,\alpha\gamma}^{(2,0)}$, respectively, by the βE_1 term. Specifically, we have

$$\begin{aligned} \Xi_{2,\alpha\beta\gamma}^{(2,1)}(\mathbf{1}, u) &= \beta E_1 + \Xi_{1,\alpha\gamma}^{(2,0)}(\mathbf{1}, u) \\ \Xi_{3,\alpha\beta\gamma}^{(2,1)}(\mathbf{1}, u) &= \beta E_1 + \Xi_{2,\alpha\gamma}^{(2,0)}(\mathbf{1}, u). \end{aligned}$$

We are left to deal with three-input homogeneous systems (as there are clearly no inhomogeneous systems of this type).

PROPOSITION 6 Every three-input (homogeneous) system is equivalent to exactly one of the following systems

$$\begin{aligned} \Sigma_{1,\alpha\beta\gamma}^{(3,0)} &: \gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 \\ &\quad + u_1(\alpha E_3) + u_2(E_1 + \gamma_4 E_2) + u_3(\beta E_2) \\ \Sigma_{2,\alpha\beta\gamma}^{(3,0)} &: \gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 \\ &\quad + u_1(E_1 + \gamma_4 E_2 + \gamma_5 E_3) + u_2(\alpha E_3) + u_3(\beta E_2) \\ \Sigma_{3,\alpha\beta\gamma}^{(3,0)} &: \gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 \\ &\quad + u_1(E_1 + \gamma_4 E_2 + \gamma_5 E_3) + u_2(\beta E_2 + \gamma_6 E_3) + u_3(\alpha E_3). \end{aligned}$$

Here $\alpha > 0$, $\beta \neq 0$ and $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6 \in \mathbb{R}$, with different values of these parameters yielding distinct (non-equivalent) class representatives.

We again consider an arbitrary system Σ represented as

$$\left[\begin{array}{c|ccc} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array} \right].$$

First, we assume $c_3 = d_3 = 0$ (in this case $b_3 \neq 0$). Let ψ be the automorphism specified by $\varsigma = 1$, $x = 1$, $y = 0$, $v = -\frac{b_1}{b_3}$ and $w = -\frac{b_2}{b_3}$. Then

$$\psi \cdot \left[\begin{array}{c|ccc} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & 0 & 0 \end{array} \right] = \left[\begin{array}{c|ccc} a_1 - \frac{b_1 a_3}{b_3} & 0 & c_1 & d_1 \\ a_2 - \frac{b_2 a_3}{b_3} & 0 & c_2 & d_2 \\ a_3 & b_3 & 0 & 0 \end{array} \right] = \left[\begin{array}{c|ccc} a'_1 & 0 & c_1 & d_1 \\ a'_2 & 0 & c_2 & d_2 \\ a_3 & b_3 & 0 & 0 \end{array} \right]$$

for some corresponding constants a'_1 and a'_2 . Now, by applying an automorphism ψ' specified by $\varsigma = \text{sgn}(b_3)$, $x = \frac{d_2}{c_1 d_2 - c_2 d_1}$, $y = -\frac{d_1}{c_1 d_2 - c_2 d_1}$ and $v = w = 0$, we get

$$\psi' \cdot \left[\begin{array}{c|ccc} a'_1 & 0 & c_1 & d_1 \\ a'_2 & 0 & c_2 & d_2 \\ a_3 & b_3 & 0 & 0 \end{array} \right] = \left[\begin{array}{c|ccc} \gamma_1 & 0 & 1 & 0 \\ \gamma_2 & 0 & \gamma_4 & \beta \\ \gamma_3 & \alpha & 0 & 0 \end{array} \right]$$

for some $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$. Here $\alpha = \text{sgn}(b_3)b_3 > 0$ and $\beta = \frac{\text{sgn}(b_3)(d_1^2 + d_2^2)}{c_1 d_2 - c_2 d_1} \neq 0$. Thus, Σ is equivalent to $\Sigma_{1, \alpha \beta \gamma}^{(3,0)}$. For $c_3 \neq 0$ and $d_3 = 0$, a very similar argument shows that Σ is equivalent to $\Sigma_{2, \alpha \beta \gamma}^{(3,0)}$ for some $\gamma = (\gamma_1, \dots, \gamma_5) \in \mathbb{R}^5$, $\alpha > 0$ and $\beta \neq 0$.

Now we assume $d_3 \neq 0$. Let ψ be the automorphism specified by $\varsigma = 1$, $x = 1$, $y = 0$, $v = -\frac{d_1}{d_3}$ and $w = -\frac{d_2}{d_3}$. Then

$$\begin{aligned} \psi \cdot \left[\begin{array}{c|ccc} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array} \right] &= \left[\begin{array}{c|ccc} a_1 - \frac{a_3 d_1}{d_3} & b_1 - \frac{b_3 d_1}{d_3} & c_1 - \frac{c_3 d_1}{d_3} & 0 \\ a_2 - \frac{a_3 d_2}{d_3} & b_2 - \frac{b_3 d_2}{d_3} & c_2 - \frac{c_3 d_2}{d_3} & 0 \\ a_3 & b_3 & c_3 & d_3 \end{array} \right] \\ &= \left[\begin{array}{c|ccc} a'_1 & b'_1 & c'_1 & 0 \\ a'_2 & b'_2 & c'_2 & 0 \\ a_3 & b_3 & c_3 & d_3 \end{array} \right] \end{aligned}$$

for some corresponding constants a'_i, b'_i, c'_i , $i = 1, 2$. By applying an automorphism ψ' specified by $\varsigma = \text{sgn}(d_3)$, $x = \frac{c_2}{b'_1 c'_2 - b'_2 c'_1}$, $y = -\frac{c'_1}{b'_1 c'_2 - b'_2 c'_1}$ and $v = w = 0$, we get

$$\psi' \cdot \left[\begin{array}{c|ccc} a'_1 & b'_1 & c'_1 & 0 \\ a'_2 & b'_2 & c'_2 & 0 \\ a_3 & b_3 & c_3 & d_3 \end{array} \right] = \left[\begin{array}{c|ccc} \gamma_1 & 1 & 0 & 0 \\ \gamma_2 & \gamma_4 & \beta & 0 \\ \gamma_3 & \gamma_5 & \gamma_6 & \alpha \end{array} \right]$$

for some $\gamma = (\gamma_1, \dots, \gamma_6)$. Here $\alpha = \text{sgn}(d_3)d_3 > 0$ and $\beta \neq 0$. Thus, Σ is equivalent to $\Sigma_{3, \alpha \beta \gamma}^{(3,0)}$.

Finally, we verify that all these systems are not equivalent. Assume two systems $\Sigma_{1,\alpha\beta\gamma}^{(3,0)}$ and $\Sigma_{1,\alpha'\beta'\gamma'}^{(3,0)}$ are equivalent. Then there exists an automorphism such that $\psi \cdot \Xi_{1,\alpha\beta\gamma}^{(3,0)}(\mathbf{1}, \cdot) = \Xi_{1,\alpha'\beta'\gamma'}^{(3,0)}(\mathbf{1}, \cdot)$, i.e.,

$$\left[\begin{array}{c|ccc} x\gamma_1 + y\gamma_2 + v\gamma_3 & \alpha v & x + \gamma_4 y & \beta y \\ -\varsigma y\gamma_1 + \varsigma x\gamma_2 + w\gamma_3 & \alpha w & \varsigma x\gamma_4 - \varsigma y & \varsigma\beta x \\ \hline & \varsigma\alpha & 0 & 0 \end{array} \right] = \left[\begin{array}{c|ccc} \gamma'_1 & 0 & 1 & 0 \\ \gamma'_2 & 0 & \gamma'_4 & \beta \\ \hline & \alpha' & 0 & 0 \end{array} \right].$$

This implies that $\alpha = \alpha'$, $\beta = \beta'$ and $\gamma = \gamma'$. Similarly, $\Sigma_{2,\alpha\beta\gamma}^{(3,0)}$ and $\Sigma_{3,\alpha\beta\gamma}^{(3,0)}$ are equivalent to $\Sigma_{2,\alpha'\beta'\gamma'}^{(3,0)}$ and $\Sigma_{3,\alpha'\beta'\gamma'}^{(3,0)}$, respectively, only if $\alpha = \alpha'$, $\beta = \beta'$ and $\gamma = \gamma'$. Lastly, $u_3(\beta E_2) \in \langle E_1, E_2 \rangle$ and $u_3(\alpha E_3) \notin \langle E_1, E_2 \rangle$, and so neither $\Sigma_{1,\alpha\beta\gamma}^{(3,0)}$ nor $\Sigma_{2,\alpha\beta\gamma}^{(3,0)}$ can be equivalent to $\Sigma_{3,\alpha'\beta'\gamma'}^{(3,0)}$. Likewise, $\Sigma_{1,\alpha\beta\gamma}^{(3,0)}$ cannot be equivalent to $\Sigma_{2,\alpha'\beta'\gamma'}^{(3,0)}$.

We collect the results in a theorem. In addition, invariant classifying conditions are now included.

THEOREM *Let*

$$\Sigma : \sum_{i=1}^3 a_i E_i + u_1 \sum_{i=1}^3 b_i E_i + u_2 \sum_{i=1}^3 c_i E_i + u_3 \sum_{i=1}^3 d_i E_i$$

be a full-rank left-invariant control affine system on SE (2). (For a single-input system, $u_2 = u_3 = 0$, whereas for a two-input system, $u_3 = 0$.)

(i) *Every single-input system Σ is equivalent to exactly one of the following systems*

$$\begin{aligned} \Sigma_{1,\alpha}^{(1,1)} : \alpha E_3 + u E_2 & & b_3 = 0 \\ \Sigma_{2,\alpha\gamma}^{(1,1)} : E_2 + \gamma_1 E_3 + u(\alpha E_3) & & b_3 \neq 0. \end{aligned}$$

(ii) *Every two-input homogeneous system Σ is equivalent to exactly one of the following systems*

$$\begin{aligned} \Sigma_{1,\alpha\gamma}^{(2,0)} : \gamma_1 E_2 + \gamma_2 E_3 + u_1(\alpha E_3) + u_2 E_2 & & c_3 = 0 \\ \Sigma_{2,\alpha\gamma}^{(2,0)} : \gamma_1 E_2 + \gamma_2 E_3 + u_1(E_2 + \gamma_3 E_3) + u_2(\alpha E_3) & & c_3 \neq 0. \end{aligned}$$

(iii) *Every two-input inhomogeneous system Σ is equivalent to exactly one of the following systems*

$$\begin{aligned} \Sigma_{1,\alpha\beta\gamma}^{(2,1)} : \alpha E_3 + u_1(E_1 + \gamma_1 E_2) + u_2(\beta E_2) & & b_3 = 0, c_3 = 0 \\ \Sigma_{2,\alpha\beta\gamma}^{(2,1)} : \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(\alpha E_3) + u_2 E_2 & & b_3 \neq 0, c_3 = 0 \\ \Sigma_{3,\alpha\beta\gamma}^{(2,1)} : \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 & & \\ & + u_1(E_2 + \gamma_3 E_3) + u_2(\alpha E_3) & c_3 \neq 0. \end{aligned}$$

(iv) Every three-input system Σ is equivalent to exactly one of the following systems

$$\begin{aligned} \Sigma_{1,\alpha\beta\gamma}^{(3,0)} &: \gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 + u_1(\alpha E_3) \\ &\quad + u_2(E_1 + \gamma_4 E_2) + u_3(\beta E_2) \quad c_3 = 0, d_3 = 0 \\ \Sigma_{2,\alpha\beta\gamma}^{(3,0)} &: \gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 + u_1(E_1 + \gamma_4 E_2 \\ &\quad + \gamma_5 E_3) + u_2(\alpha E_3) + u_3(\beta E_2) \quad c_3 \neq 0, d_3 = 0 \\ \Sigma_{3,\alpha\beta\gamma}^{(3,0)} &: \gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 + u_1(E_1 + \gamma_4 E_2 + \gamma_5 E_3) \\ &\quad + u_2(\beta E_2 + \gamma_6 E_3) + u_3(\alpha E_3) \quad d_3 \neq 0. \end{aligned}$$

Here, $\alpha > 0$, $\beta \neq 0$ and $\gamma_i \in \mathbb{R}$, with different values of these parameters yielding distinct (non-equivalent) class representatives.

4. Final remarks

The results obtained in this paper suggest that the number of parameters, involved in the classification of left-invariant control affine systems, is quite large. Accordingly, such a classification for systems on higher-dimensional Lie groups may become cumbersome. However, it seems that a classification for systems on lower-dimensional Lie groups (at least in three dimensions) is feasible.

Alternative equivalence relations may be considered, e.g., global state space equivalence and detached feedback equivalence (see Biggs and Remsing, no date). Remarkably, for $SE(2)$, the classifications under local and global state space equivalence are identical. In general, this is far from being the case. Then again, detached feedback equivalence (a weaker equivalence relation) yields far fewer equivalence classes.

We append a tabulation of the classification in matrix form.

References

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Type	Equivalence representatives ($\alpha > 0, \beta \neq 0, \gamma_i \in \mathbb{R}$)		
(1,1)	$\left[\begin{array}{c c} 0 & 0 \\ 0 & 1 \\ \alpha & 0 \end{array} \right]$	$\left[\begin{array}{c c} 0 & 0 \\ 1 & 0 \\ \gamma_1 & \alpha \end{array} \right]$	
(2,0)	$\left[\begin{array}{c cc} 0 & 0 & 0 \\ \gamma_1 & 0 & 1 \\ \gamma_2 & \alpha & 0 \end{array} \right]$	$\left[\begin{array}{c cc} 0 & 0 & 0 \\ \gamma_1 & 1 & 0 \\ \gamma_2 & \gamma_3 & \alpha \end{array} \right]$	
(2,1)	$\left[\begin{array}{c cc} 0 & 1 & 0 \\ 0 & \gamma_1 & \beta \\ \alpha & 0 & 0 \end{array} \right]$	$\left[\begin{array}{c cc} \beta & 0 & 0 \\ \gamma_1 & 0 & 1 \\ \gamma_2 & \alpha & 0 \end{array} \right]$	$\left[\begin{array}{c cc} \beta & 0 & 0 \\ \gamma_1 & 1 & 0 \\ \gamma_2 & \gamma_3 & \alpha \end{array} \right]$
(3,0)	$\left[\begin{array}{c ccc} \gamma_1 & 0 & 1 & 0 \\ \gamma_2 & 0 & \gamma_4 & \beta \\ \gamma_3 & \alpha & 0 & 0 \end{array} \right]$	$\left[\begin{array}{c ccc} \gamma_1 & 1 & 0 & 0 \\ \gamma_2 & \gamma_4 & \beta & 0 \\ \gamma_3 & \gamma_5 & \alpha & 0 \end{array} \right]$	$\left[\begin{array}{c ccc} \gamma_1 & 1 & 0 & 0 \\ \gamma_2 & \gamma_4 & \beta & 0 \\ \gamma_3 & \gamma_5 & \gamma_6 & \alpha \end{array} \right]$
$[A \mid B_1 \ \cdots \ B_\ell] \longleftrightarrow A + u_1 B_1 + \cdots + u_\ell B_\ell$			

Classification of systems on SE(2) (matrix form)