Control and Cybernetics

vol. **41** (2012) No. 3

Nonconvex minimization related to quadratic double-well energy – approximation by convex problems *†

by

$\boxed{\textbf{Zdzisław Naniewicz}}$[1] **and Piotr Puchała**[2]

[1]Cardinal Stefan Wyszyński University, Faculty of Mathematics and Natural Sciences, Dewajtis 5, 01-815 Warsaw, Poland

[2] Częstochowa University of Technology, Institute of Mathematics, Dąbrowskiego 73, 42-200 Częstochowa, Poland, piotr.puchala@im.pcz.pl

Abstract: A double-well energy expressed as a minimum of two quadratic functions, called phase energies, is studied taking into account minimization of the corresponding integral functional. Such integral, as being not sequentially weakly lower semicontinuous, does not admit classical minimizers. To derive the relaxation formula for the infimum, the appropriate minimizing sequence is constructed. It consists of solutions of some approximating convex problems involving characteristic functions related to the phase energies. The weak limit of this sequence and the weak limit of the sequence of solutions of dual problems combined with the weak-star limits of the characteristic functions related to the phase energies allow to establish the final relaxation formula. It is also shown that infimum can be expressed by the Young measure associated with constructed minimizing sequence. An explicit form of Young measure in some regions of the involved domain is calculated.

Keywords: nonconvex integrand, minimum of convex functions, duality, parameterized Young measures.

1. Introduction

1.1. Generalities

One of important problems in the calculus of variations and mechanics of solids is minimization of the functional of the form

$$\mathcal{J}(u) = \int_{\Omega} f(x, u(x), Du(x))dx,$$

*Submitted: September 2011; Accepted: September 2012

†This paper was submitted to the journal by late Professor Zdzisław Naniewicz, but since his passing away the reviewing and modification process was carried out with participation of Dr Piotr Puchała only

where Ω is a nonempty Lebesgue measurable bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary, $u\colon \Omega \to \mathbb{R}^m$ is a function from a suitable Sobolev space and $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{mn} \to \mathbb{R}$ is a given function. (In physics or engineering we have $m = n$, u is called the displacement of the elastic body Ω and f is called the density of the internal energy.)

It is known that if the integrand f satisfies Carathéodory and suitable growth conditions, then its quasiconvexity (in the sense of Morrey, 1966) with respect to the third variable is equivalent to the sequential lower semicontinuity (in an appropriate topology) of $\mathcal{J}$. This, in turn, guarantees that the limit v_0 of convergent (in appropriate topology) subsequence of the minimizing for $\mathcal{J}$ sequence $\{v^k\}$ is a minimum of $\mathcal{J}$. This is the basis of the direct method in the calculus of variations. Usually the most challenging task of this method is guaranteeing sequential lower semicontinuity of $\mathcal{J}$.

Situation becomes much more complicated if the integrand is not quasiconvex. In this case the minimized functional does not generally attain its infimum. Basically, there are two ways to proceed in this case.

The first one is to 'quasiconvexify' the original functional and to gather "nonconvexities" into its quasiconvex envelope (Morrey, 1966; Ball, 1977; Ball and Murat, 1984; Acerbi and Fusco, 1984; Dacorogna, 1989; Kohn, 1991; Kohn and Strang, 1986; Tartar, 1975; Murat, 1979; Tartar, 1979; Fonseca, 1988; Fonseca and Müller, 1993; Fonseca and Rybka, 1992; Buttazzo, 1989; Dal Maso, 1993; Ambrosio, 1990; Bouchitte, Braides and Buttazzo, 1995; Allaire and Francroft, 1998; Allaire and Lods, 1999; and the references quoted there). However, computing explicit form of the quasiconvex envelope is very difficult in practice. Further, carrying out this procedure (when possible) erases some important information concerning behaviour of the minimizing sequences. Minimizers of quasiconvexifications themselves are not sufficient to characterize properly oscillatory phenomena of such problems (microstructural features describing fine mixtures of the phases in the phase transition problems, for instance). Another way is to enlarge the space of admissible functions from Sobolev spaces to the space of parameterized Young measures, Young (1937). In this approach the Young measures can be regarded as means of summarizing the spatial oscillatory properties of minimizing sequences, thus conserving some of that information. With this respect we refer the reader to Young (1969), Kinderlehrer and Pedregal (1991), Chipot and Kinderlehrer (1988), Ball and James (1987), James and Kinderlehrer (1989), Ball and Murat (1984), Murat (1979), Ericksen (1980), Pedregal (1997), Tartar (1991) and the references therein.

From the application point of view, the detailed structure of minimizing sequences including the behavioral characteristics of the phases involved appears to be as much important as the minimizers themselves. Unfortunately, it is a very difficult task to compute the parameterized Young measures associated with a minimizing sequence.

1.2. About the contents

The aim of this paper is to solve a nonconvex optimization problem in the case when the functional to be minimized has integrand expressed as a minimum of two quadratic functions. Such integrand is not, in general, quasiconvex. Namely, the nonconvex minimization problem of the form

$$\inf_{v\in H_0^1(\Omega;\mathbb{R}^n)} \int_\Omega \min\left\{\tfrac{1}{2}a\left|\varepsilon(v)+C\right|^2,\ \tfrac{1}{2}b\left|\varepsilon(v)+D\right|^2\right\} dx := \alpha \tag{P}$$

is considered, where $\varepsilon(v)$ is the symmetrized gradient of $v \in H_0^1(\Omega;\mathbb{R}^n)$, i.e. $\varepsilon(v) := \frac{1}{2}(\nabla v + \nabla^T v)$.
Here the symbol "$|\cdot|$" stands for the Euclidean norm in $\mathbb{R}^{n\times n}_{sym}$, while $H_0^1(\Omega;\mathbb{R}^n)$ denotes the Sobolev space of vector-valued functions square integrable together with their first partial distributional derivatives in a bounded region $\Omega \subset \mathbb{R}^n$.
The idea is to approximate nonconvex problem by convex ones as proposed in Naniewicz (2001). As a result we obtain the basic Theorem 1 in which we give explicit formulas for the infimum α of the nonconvex problem (P). Infimum α is a limit of a sequence whose elements are infima of the suitably formulated convex problems. The latter are treated by the well established methods of Convex Analysis.

The paper is organized as follows:

- in Section 2 Theorem 4 of Naniewicz (2001) is used to formulate convex optimization problems (P^k) approximating (P) together with corresponding dual (in the Fenchel sense) problems $(P^k)^\star$. This makes it possible to write down the representations of the elements of the sequence $\{\alpha^k\}$ of infima in (P^k).
- Section 3 contains the compensated compactness type results for the sequences $\{u^k\}$ and $\{p^k\}$. Namely, the weak limit of the sequence $\{p^k\cdot\varepsilon(u^k)\}$ is obtained, where p^k is the solution of the dual problem $(P^k)^\star$, $\{u^k\}$ is the minimizing sequence for (P) and $\varepsilon(\cdot)$ is the symmetrized gradient in the functional to be minimized.
- the results of the above two Sections are then used in Section 4 to proceed with calculations leading to the main result of the paper. This is Theorem 1, where explicit formulas are established for $\alpha = \inf(\mathcal{J})$ – the infimum in the nonconvex problem. These formulas involve
  - weak limit of the minimizing sequence $\{u^k\}$;
  - weak limit of sequence $\{p^k\}$ of solutions of the problems dual to approximating (P) convex problems;
  - weak* limits of the sequences of characteristic functions $\{\chi_a^k\}$ and $\{\chi_b^k\}$ related to the phases $\frac{1}{2}a\left|\varepsilon(v)+C\right|^2$ and $\frac{1}{2}b\left|\varepsilon(v)+D\right|^2$, respectively.
- Theorem 2 of Section 5 gives the infimum α fully expressed by the parametrized Young measure associated with the minimizing sequence. It also es-

tablishes some relations between the weak* limit of the sequence $\{\psi^k\} = \{\chi_b^k - \chi_a^k\}$ and the related parameterized measures.

2. Statement of the problem and its approximation

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. Set

$$\mathcal{J}(u) = \int_\Omega \min\left\{\tfrac{1}{2}a\left|\varepsilon(u) + C\right|^2, \tfrac{1}{2}b\left|\varepsilon(u) + D\right|^2\right\}dx, \quad u \in H_0^1(\Omega;\mathbb{R}^n).$$

The problem to be considered here is

$$\inf\Big\{\mathcal{J}(u)\colon\ u \in H_0^1(\Omega;\mathbb{R}^n)\Big\} := \alpha, \tag{P}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is a competing vector-valued function from the Sobolev space $H_0^1(\Omega;\mathbb{R}^n)$, $\varepsilon(u) \in L^2(\Omega;\mathbb{R}^{n\times n}_{sym})$ is the symmetrized gradient of $u \in H_0^1(\Omega;\mathbb{R}^n)$, $C, D \in L^\infty(\Omega;\mathbb{R}^{n\times n}_{sym})$ and where $a, b \in L^\infty(\Omega)$ are such that $a(x), b(x) \geq \delta > 0$ a.e. in Ω for a positive constant δ. Theorem 4 of Naniewicz (2001) ensures the existence of sequences $\{u^k\} \subset H_0^1(\Omega;\mathbb{R}^n)$, $\chi_a^k : \Omega \to \{0,1\}$ and $\chi_b^k : \Omega \to \{0,1\}$, $\chi_a^k + \chi_b^k \equiv 1$, with the properties that

(a) $\{u_k\}$ is a minimizing sequence for (P),

(b) $u^k \to u$ weakly in $H_0^1(\Omega;\mathbb{R}^n)$ as $k \to \infty$,

(c) $\chi_a^k \to \chi_a$, $\chi_b^k \to \chi_b$ weak* in $L^\infty(\Omega)$ as $k \to \infty$, where $\chi_a\colon \Omega \to [0,1]$, $\chi_b\colon \Omega \to [0,1]$ with $\chi_a + \chi_b \equiv 1$,

(d) $\int_\Omega [\frac{1}{2}\chi_a^k a\left|\varepsilon(u^k) + C\right|^2 + \frac{1}{2}\chi_b^k b\left|\varepsilon(u^k) + D\right|^2]\,dx := \alpha^k \to \alpha$ as $k \to \infty$,

(e) $\int_\Omega [\frac{1}{2}\chi_a^k a\left|\varepsilon(u^k) + C\right|^2 + \frac{1}{2}\chi_b^k b\left|\varepsilon(u^k) + D\right|^2]\,dx \leq$

$$\leq \int_\Omega [\tfrac{1}{2}\chi_a^k a\left|\varepsilon(w) + C\right|^2 + \tfrac{1}{2}\chi_b^k b\left|\varepsilon(w) + D\right|^2]\,dx, \quad \forall\, w \in H_0^1(\Omega;\mathbb{R}^n).$$

From the same Theorem 4 it also follows that:

(i) u^k is a solution of the convex optimization problem

$$\inf\Big\{\mathcal{J}^k(v)\colon v \in H_0^1(\Omega;\mathbb{R}^n)\Big\} := \alpha^k, \tag{P^k}$$

where

$$\mathcal{J}^k(v) = \int_\Omega \Big[\frac{1}{2}\chi_a^k a\left|\varepsilon(v) + C\right|^2 + \frac{1}{2}\chi_b^k b\left|\varepsilon(v) + D\right|^2\Big]\,dx, \quad v \in H_0^1(\Omega;\mathbb{R}^n),$$

i.e.

$$\mathcal{J}^k(u^k) = \alpha^k;$$

(ii) the sequence $\{\alpha^k\}$ of infima of convex problems (P^k) is convergent to

$\alpha = \inf(\mathcal{J})$.

Now we shall give an explicit characterization of the terms of sequence $\{\alpha^k\}$. We will use Fenchel duality theorem (Fenchel, 1951; also Ekeland and Temam, 1976; Aubin,1993).

Before formulating dual minimization problems we introduce some notation. Let ψ^k be given by

$$\psi^k = \chi_b^k - \chi_a^k. \tag{2.1}$$

Note that

$$(\psi^k)^2 = 1. \tag{2.2}$$

Further, define

$$m^k := \chi_a^k a + \chi_b^k b,$$

and

$$\left.\begin{aligned} \mathcal{A}^+ &:= \frac{aC + bD}{2} \\ \mathcal{A}^- &:= \frac{bD - aC}{2} \\ \mathcal{B}^k &:= \frac{a\,|C|^2 + b\,|D|^2}{2} + \psi^k \frac{b\,|D|^2 - a\,|C|^2}{2}. \end{aligned}\right\} \tag{2.3}$$

Using the properties of the scalar product we see that

$$\chi_a^k aC + \chi_b^k bC = \frac{aC + bD}{2} + \psi^k \frac{bD - aC}{2},$$

$$\tfrac{1}{2}\chi_a^k a\,|C|^2 + \tfrac{1}{2}\chi_b^k b\,|D|^2 = \tfrac{1}{2}\left(\frac{a\,|C|^2 + b\,|D|^2}{2} + \psi^k \frac{b\,|D|^2 - a\,|C|^2}{2}\right)$$

so that $\mathcal{J}^k(\cdot)$ admits the representation

$$\mathcal{J}^k(v) = \int_\Omega \left[\frac{1}{2} m^k\,|\varepsilon(v)|^2 + (\mathcal{A}^+ + \psi^k \mathcal{A}^-)\cdot\varepsilon(v) + \frac{1}{2}\mathcal{B}^k\right] dx. \tag{R}$$

Next, define a linear continuous operator $L : H_0^1(\Omega;\mathbb{R}^n) \to L^2(\Omega;\mathbb{R}^{n\times n}_{sym})$ as

$$Lv = \varepsilon(v), \quad v \in H_0^1(\Omega;\mathbb{R}^n),$$

with transpose $L^* : L^2(\Omega;\mathbb{R}^{n\times n}_{sym}) \to H^{-1}(\Omega;\mathbb{R}^n)$

$$\left\langle L^*p, v\right\rangle_{H_0^1(\Omega;\mathbb{R}^n)} = \int_\Omega p\cdot\varepsilon(v)\,dx, \quad \forall p \in L^2(\Omega;\mathbb{R}^{n\times n}_{sym}),\ \forall v \in H_0^1(\Omega;\mathbb{R}^n).$$

and kernel

$$\operatorname{Ker} L^* = \Big\{ p \in L^2(\Omega; \mathbb{R}^{n\times n}_{sym})\colon \int\limits_\Omega p \cdot \varepsilon(v)\, dx = 0 \Big\}.$$

Finally, define

$$\mathcal{I}^k(q) := \int\limits_\Omega \Big[\frac{1}{2m^k} \left| q - (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \right|^2 - \frac{1}{2}\mathcal{B}^k \Big]\, dx, \quad q \in L^2(\Omega; \mathbb{R}^{n\times n}_{sym}). \tag{2.4}$$

Now we are in a position to formulate the dual problem $(P^k)^\star$ which can be stated as follows

$$\inf \Big\{ \int\limits_\Omega \mathcal{I}^k(q)\, dx\colon q \in \operatorname{Ker} L^* \Big\} := \beta^k. \tag{$(P^k)^\star$}$$

According to the Fenchel theorem (see Theorem 3.2, p. 38, Aubin, 1993) we get

$$\begin{aligned} \mathcal{J}^k(v) \geq \mathcal{J}^k(u^k) = \alpha^k = -\beta^k = -\mathcal{I}^k(p^k) \geq -\mathcal{I}^k(q), \\ \forall\, v \in H^1_0(\Omega; \mathbb{R}^n),\ \forall\, q \in \operatorname{Ker} L^*, \end{aligned} \tag{2.5}$$

where

$$p^k = m^k \varepsilon(u^k) + \mathcal{A}^+ + \psi^k \mathcal{A}^- \in \operatorname{Ker} L^* \tag{2.6}$$

is a solution of the dual problem $(P^k)^\star$. Since $p^k \in \operatorname{Ker} L^*$,

$$\int\limits_\Omega p^k \cdot \varepsilon(v)\, dx = 0, \quad \forall\, v \in H^1_0(\Omega; \mathbb{R}^n), \tag{2.7}$$

so, in particular,

$$\int\limits_\Omega p^k \cdot \varepsilon(u^k)\, dx = \int\limits_\Omega \Big[m^k \left| \varepsilon(u^k) \right|^2 + (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot \varepsilon(u^k) \Big] dx = 0. \tag{2.8}$$

In view of $\alpha^k = -\mathcal{I}^k(p^k)$ and (2.8) we get the following representations

$$\alpha^k = \tfrac{1}{2} \int\limits_\Omega \big[-m^k \left| \varepsilon(u^k) \right|^2 + \mathcal{B}^k \big] dx = \tfrac{1}{2} \int\limits_\Omega \big[(\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot \varepsilon(u^k) + \mathcal{B}^k \big] dx. \tag{2.9}$$

Analogously, from (2.6), (2.8) and (2.9) there follows

$$\int\limits_\Omega \Big[\frac{1}{m^k} \left| p^k \right|^2 - \frac{1}{m^k} (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot p^k \Big]\, dx = 0.$$

Therefore

$$\alpha^k = -\tfrac{1}{2}\int\limits_\Omega \left[-\frac{1}{m^k}\left|p^k\right|^2 + \frac{1}{m^k}\left|\mathcal{A}^+ + \psi^k\mathcal{A}^-\right|^2 - \mathcal{B}^k\right]dx =$$

$$= \tfrac{1}{2}\int\limits_\Omega \left[\frac{1}{m^k}(\mathcal{A}^+ + \psi^k\mathcal{A}^-)\cdot p^k - \frac{1}{m^k}\left|\mathcal{A}^+ + \psi^k\mathcal{A}^-\right|^2 + \mathcal{B}^k\right]dx \qquad (2.10)$$

and we are led to the equality

$$\int\limits_\Omega \left[m^k\left|\varepsilon(u^k)\right|^2 + \frac{1}{m^k}\left|p^k\right|^2\right]dx = \int\limits_\Omega \left[\frac{1}{m^k}\left|\mathcal{A}^+ + \psi^k\mathcal{A}^-\right|^2\right]dx. \qquad (2.11)$$

Since $\chi_a^k + \chi_b^k = 1$, $(\psi^k)^2 = 1$ and $\frac{1}{m^k} = \frac{1}{2}(\frac{1}{a}+\frac{1}{b}) + \frac{1}{2}(\frac{1}{b}-\frac{1}{a})\psi^k$, letting $k \to \infty$ yields

$$\left.\begin{aligned} &\lim_{k\to\infty}\int\limits_\Omega \left[m^k\left|\varepsilon(u^k)\right|^2 + \frac{1}{m^k}\left|p^k\right|^2\right]dx \\ &= \tfrac{1}{2}\int\limits_\Omega \Big(a\left|C\right|^2 + b\left|D\right|^2\Big)dx + \tfrac{1}{2}\int\limits_\Omega \psi\Big(b\left|D\right|^2 - a\left|C\right|^2\Big)dx = \int\limits_\Omega \mathcal{B}\,dx, \end{aligned}\right\} \qquad (2.12)$$

where $\mathcal{B} = \frac{a|C|^2+b|D|^2}{2} + \frac{b|D|^2-a|C|^2}{2}\psi$. Let us divide Ω into two disjoint sets:

$$\Omega = \Omega_0 \cup (\Omega \setminus \Omega_0)$$

where

$$\Omega_0 = \{x \in \Omega\colon a(x) = b(x)\}.$$

From (2.6), (2.9) and (2.10) we get the representations of α^k:

$$\alpha^k = \tfrac{1}{2} \int\limits_{\Omega\setminus\Omega_0} \Big[\frac{ab(C-D)}{b-a} \cdot \varepsilon(u^k) + \frac{bD-aC}{b-a} \cdot p^k + \frac{ab(|C|^2 - |D|^2)}{2(b-a)}$$
$$- \psi^k \, \frac{ab\,|C-D|^2}{2(b-a)}\Big] dx$$
$$+ \tfrac{1}{2} \int\limits_{\Omega_0} \Big[-a\,|\varepsilon(u^k)|^2 + \frac{a(|C|^2 + |D|^2)}{2} + \psi^k \frac{a(|D|^2 - |C|^2)}{2}\Big] dx$$
$$+ \tfrac{1}{2} \int\limits_{\Omega_0} p^k \cdot \varepsilon(u^k) dx$$
$$= \tfrac{1}{2} \int\limits_{\Omega\setminus\Omega_0} \Big[\frac{ab(C-D)}{b-a} \cdot \varepsilon(u^k) + \frac{bD-aC}{b-a} \cdot p^k + \frac{ab(|C|^2 - |D|^2)}{2(b-a)}$$
$$- \psi^k \frac{ab\,|C-D|^2}{2(b-a)}\Big] dx$$
$$+ \tfrac{1}{2} \int\limits_{\Omega_0} \frac{1}{a}\,|p^k|^2\, dx - \tfrac{1}{2} \int\limits_{\Omega_0} p^k \cdot \varepsilon(u^k) dx. \tag{2.13}$$

3. Weak convergence in $L^1(\Omega)$

LEMMA 1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain in $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$. Then*

$$p^k \cdot \varepsilon(u^k) \to p \cdot \varepsilon(u) \quad \textit{weakly in } L^1(\Omega). \tag{3.1}$$

Proof. Extend each function $u^k \in H_0^1(\Omega;\mathbb{R}^n)$ to all of $\mathbb{R}^n$ by setting it equal to zero on $\mathbb{R}^n \setminus \Omega$. By regularity of the boundary $\partial\Omega$ all of these extensions are elements of $H^1(\mathbb{R}^n;\mathbb{R}^n)$. For an arbitrary $\varphi \in C^\infty(\mathbb{R}^n)$ we have $\varphi u^k \in H^1(\mathbb{R}^n;\mathbb{R}^n)$ and $\varphi u^k\big|_\Omega \in H_0^1(\Omega;\mathbb{R}^n)$. We claim that

$$\int\limits_{\mathbb{R}^n} \varphi\, p^k \cdot \varepsilon(u^k)\, dx \to \int\limits_{\mathbb{R}^n} \varphi\, p \cdot \varepsilon(u)\, dx, \tag{3.2}$$

for any $\varphi \in C^\infty(\mathbb{R}^n)$. Indeed, since $\varepsilon(\varphi u^k) = \varphi\varepsilon(u^k) + u^k \otimes \nabla\varphi$, there follows

$$\int\limits_{\mathbb{R}^n} \varphi\, p^k \cdot \varepsilon(u^k)\, dx = \int\limits_{\mathbb{R}^n} p^k \cdot \varepsilon(\varphi u^k)\, dx - \int\limits_{\mathbb{R}^n} p^k \cdot (u^k \otimes \nabla\varphi)\, dx =$$
$$= -\int\limits_{\mathbb{R}^n} p^k \cdot (u^k \otimes \nabla\varphi)\, dx \to -\int\limits_{\mathbb{R}^n} p \cdot (u \otimes \nabla\varphi)\, dx = \int\limits_{\mathbb{R}^n} \varphi\, p \cdot \varepsilon(u)\, dx,$$

where we have used (2.7) and the strong convergence $u^k \to u$ in $L^2(\Omega;\mathbb{R}^n)$ (valid due to the Rellich compactness theorem).
Further, the sequence $\{p^k \cdot \varepsilon(u^k)\}$ is uniformly bounded in $L^1(\Omega)$ because $\{p^k\}$ and $\{\varepsilon(u^k)\}$ so are in $L^2(\Omega)$. By Chacon's biting lemma (Pedregal, 1997) it follows that there exist a subsequence of $\{p^k \cdot \varepsilon(u^k)\}$, not relabeled, a nonincreasing sequence of measurable sets $\Omega_n \subset \Omega$, $\Omega_{n+1} \subset \Omega_n$, $|\Omega_n| \searrow 0$ and $f \in L^1(\Omega)$ such that

$$p^k \cdot \varepsilon(u^k) \to f \quad \text{weakly in } L^1(\Omega \setminus \Omega_n) \tag{3.3}$$

for all n. It means that $\{p^k \cdot \varepsilon(u^k)\}$ converges in the biting sense to f (Pedregal, 1997).

Now we assert that the biting limit f coincides with $p \cdot \varepsilon(u)$, i.e. $f = p \cdot \varepsilon(u)$ a.e. in Ω. To show this observe that from the biting argument (3.3) and (3.2) it follows that for any $\varphi \in C^\infty(\mathbb{R}^n)$ we get

$$\int_{\Omega\setminus\Omega_n} \varphi\, p \cdot \varepsilon(u)\, dx = \int_{\Omega\setminus\Omega_n} \varphi\, f\, dx, \tag{3.4}$$

for any n. Hence $p \cdot \varepsilon(u) = f$ a.e. in $\Omega \setminus \Omega_n$ for each n. Since $|\Omega_n| \searrow 0$ as $n \to \infty$, the equality $p\cdot\varepsilon(u) = f$ must hold a.e. in Ω. Thus, the assertion follows.

Recall that $p^k \cdot \varepsilon(u^k) = m^k \left|\varepsilon(u^k)\right|^2 + \left(\mathcal{A}^+ + \psi^k \mathcal{A}^-\right) \cdot \varepsilon(u^k)$. Therefore, one can deduce the existence of a constant $C \geq 0$ such that

$$p^k \cdot \varepsilon(u^k) + C \geq 0 \quad \text{a.e in } \Omega.$$

Obviously, $p^k \cdot \varepsilon(u^k) + C$ converges in the biting sense to $p \cdot \varepsilon(u) + C$. According to Lemma 6.9 (p.109, Pedregal, 1997) to prove its weak convergence in $L^1(\Omega)$ it suffices to show that

$$\limsup_{k\to\infty} \int_\Omega \left(p^k \cdot \varepsilon(u^k) + C\right) dx \leq \int_\Omega \left(p \cdot \varepsilon(u) + C\right) dx. \tag{3.5}$$

Our task now is to establish the foregoing inequality. For this purpose notice that (3.2) can be easily extend to the convergence

$$\int_{\mathbb{R}^n} \varphi\, p^k \cdot \varepsilon(u^k)\, dx \to \int_{\mathbb{R}^n} \varphi\, p \cdot \varepsilon(u)\, dx, \tag{3.6}$$

which is valid for any $\varphi \in C_c(\mathbb{R}^n)$, where $C_c(\mathbb{R}^n)$ is the space of continuous functions on $\mathbb{R}^n$ with compact support. Thus, $\mu^k := (p^k \cdot \varepsilon(u^k) + C)dx$ and $\mu := (p \cdot \varepsilon(u) + C)dx$ can be treated as positive Radon measures on $\mathbb{R}^n$ for which there holds

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \varphi\, d\mu^k = \int_{\mathbb{R}^n} \varphi\, d\mu, \quad \forall\, \varphi \in C_c(\mathbb{R}^n).$$

But Theorem 1(p. 54, Evans and Gariepy, 1992) asserts that this condition is equivalent to the following one

$$\lim_{k\to\infty} \mu^k(B) = \mu(B) \quad \text{for each bounded Borel set } B \subset \mathbb{R}^n \text{ with } \mu(\partial B) = 0. \tag{3.7}$$

Now we are in a position to establish (3.5). For this purpose fix $\epsilon > 0$ and choose $0 < \delta < \epsilon$ with the property that for any measurable $\omega \subset \Omega$ with $|\omega| < \delta$ there holds

$$\int\limits_{\omega} p \cdot \varepsilon(u)\, dx < \epsilon.$$

In the biting convergence take n_0 large enough to fulfill $|\Omega_{n_0}| < \frac{\delta}{2}$. By the measurability of Ω_{n_0} there exists an open $\widetilde{\Omega}_{n_0} \supset \Omega_{n_0}$ with $\left|\widetilde{\Omega}_{n_0}\right| < \delta$. Vitali covering theorem ensures the representation $\widetilde{\Omega}_{n_0} = \widetilde{\Omega}'_{n_0} \cup \widetilde{\Omega}''_{n_0}$ where $\left|\widetilde{\Omega}''_{n_0}\right| = 0$ and $\widetilde{\Omega}'_{n_0}$ stands for the union of a countable collection of disjoint closed balls in $\widetilde{\Omega}_{n_0}$. Therefore $\left|\partial\widetilde{\Omega}'_{n_0}\right| = 0$ and consequently $\mu(\partial\widetilde{\Omega}'_{n_0}) = 0$. From this we have

$$\begin{aligned}
\int\limits_{\Omega} \big(p^k \cdot \varepsilon(u^k) + C\big)\, dx &= \int\limits_{\Omega\setminus\Omega_{n_0}} \big(p^k \cdot \varepsilon(u^k) + C\big)\, dx + \int\limits_{\Omega_{n_0}} \big(p^k \cdot \varepsilon(u^k) + C\big)\, dx \\
&\le \int\limits_{\Omega\setminus\Omega_{n_0}} \big(p^k \cdot \varepsilon(u^k) + C\big)\, dx + \int\limits_{\widetilde{\Omega}_{n_0}} \big(p^k \cdot \varepsilon(u^k) + C\big)\, dx \\
&= \int\limits_{\Omega\setminus\Omega_{n_0}} \big(p^k \cdot \varepsilon(u^k) + C\big)\, dx + \mu^k(\widetilde{\Omega}'_{n_0})
\end{aligned}$$

which, owing to (3.7), by passing to the limit as $k \to \infty$ yields

$$\begin{aligned}
\limsup_{k\to\infty} \int\limits_{\Omega} \big(p^k \cdot \varepsilon(u^k) + C\big)\, dx &\le \int\limits_{\Omega\setminus\Omega_{n_0}} \big(p \cdot \varepsilon(u) + C\big)\, dx + \mu(\widetilde{\Omega}'_{n_0}) \\
&\le \int\limits_{\Omega} \big(p \cdot \varepsilon(u) + C\big)\, dx + \epsilon(1 + C),
\end{aligned}$$

because $\left|\widetilde{\Omega}'_{n_0}\right| < \delta < \epsilon$. Since $\epsilon > 0$ was chosen arbitrarily, (3.5) follows. This completes the proof of Lemma 1.

4. Explicit formulas for infimum in the nonconvex problem

In this section we carry out calculations leading to the explicit expression of the infimum of (P). We finally arrive at Theorem 1 formulated at the end of the section. Theorem 1 contains the main result of the paper.

The weak lower semicontinuity of convex functionals, the upper semicontinuity of concave functionals and Lemma 1 of Section 3 yield

$$\liminf_{k\to\infty} \tfrac{1}{2}\int\limits_{\Omega_0} \frac{1}{a}\left|p^k\right|^2 dx \geq \tfrac{1}{2}\int\limits_{\Omega_0} \frac{1}{a}\left|p\right|^2 dx, \tag{4.1}$$

$$\limsup_{k\to\infty} \tfrac{1}{2}\int\limits_{\Omega_0} \Big[-a\left|\varepsilon(u^k)\right|^2 + \mathcal{B}^k\Big] dx \leq \tfrac{1}{2}\int\limits_{\Omega_0} \Big[-a\left|\varepsilon(u)\right|^2 + \mathcal{B}\Big] dx, \tag{4.2}$$

$$\lim_{k\to\infty} \tfrac{1}{2}\int\limits_{\Omega_0} p^k\cdot\varepsilon(u^k)dx = \tfrac{1}{2}\int\limits_{\Omega_0} p\cdot\varepsilon(u)dx. \tag{4.3}$$

Now we show that

$$\begin{aligned}\lim_{k\to\infty} \tfrac{1}{2}\int\limits_{\Omega\setminus\Omega_0} \Big[&\frac{ab(C-D)}{b-a}\cdot\varepsilon(u^k) + \frac{bD-aC}{b-a}\cdot p^k + \frac{ab(|C|^2-|D|^2)}{2(b-a)} \\ &-\psi^k\frac{ab\left|C-D\right|^2}{2(b-a)}\Big]dx \\ = \tfrac{1}{2}\int\limits_{\Omega\setminus\Omega_0} \Big[&\frac{ab(C-D)}{b-a}\cdot\varepsilon(u) + \frac{bD-aC}{b-a}\cdot p + \frac{ab(|C|^2-|D|^2)}{2(b-a)} \\ &-\psi\frac{ab\left|C-D\right|^2}{2(b-a)}\Big]dx.\end{aligned} \tag{4.4}$$

This is not trivial because the functions $\frac{ab(C-D)}{b-a}$ and $\frac{bD-aC}{b-a}$ are not assumed to belong to $L^2(\Omega\setminus\Omega_0;\mathbb{R}^{n\times n}_{sym})$. To overcome this disadvantage let us recall (see (2.13)) that $\Omega\setminus\Omega_0$ is a set of a finite Lebesgue measure where

$$\begin{aligned}\frac{ab(C-D)}{b-a}\cdot\varepsilon(u^k) + \frac{bD-aC}{b-a}\cdot p^k + \frac{ab(|C|^2-|D|^2)}{2(b-a)} - \psi^k\,\frac{ab\left|C-D\right|^2}{2(b-a)} \\ = \mathcal{A}^+\cdot\varepsilon(u^k) + \mathcal{A}^-\cdot\psi^k\varepsilon(u^k) + \mathcal{B}^k.\end{aligned}$$

Thus, for any $\varepsilon>0$ there exist $\omega_\varepsilon\subset\Omega\setminus\Omega_0$ and $\delta>0$ such that $|\omega_\varepsilon|<\varepsilon$ and for each $x\in(\Omega\setminus\Omega_0)\setminus\omega_\varepsilon$ one has $|a(x)-b(x)|\geq\delta$. Hence

$$\frac{C-D}{b-a}\in L^\infty\big((\Omega\setminus\Omega_0)\setminus\omega_\varepsilon;\mathbb{R}^{n\times n}_{sym}\big)\subset L^2\big((\Omega\setminus\Omega_0)\setminus\omega_\varepsilon;\mathbb{R}^{n\times n}_{sym}\big) \tag{4.5}$$

and

$$\begin{aligned}\Big|\int\limits_{\omega_\varepsilon}\Big[&\frac{ab(C-D)}{b-a}\cdot\varepsilon(u^k) + \frac{bD-aC}{b-a}\cdot p^k + \frac{ab(|C|^2-|D|^2)}{2(b-a)} \\ &-\psi^k\frac{ab\left|C-D\right|^2}{2(b-a)}\Big]dx\Big| \leq \mathrm{const}\,|\omega_\varepsilon|^{\frac{1}{2}} \leq \mathrm{const}\,\varepsilon^{\frac{1}{2}}.\end{aligned}$$

This allows to conclude that

$$\lim_{k\to\infty} \int\limits_{(\Omega\setminus\Omega_0)\setminus\omega_\varepsilon} \Big[\frac{ab(C-D)}{b-a}\cdot\varepsilon(u^k) + \frac{bD-aC}{b-a}\cdot p^k + \frac{ab(|C|^2-|D|^2)}{2(b-a)} - \psi^k \frac{ab\,|C-D|^2}{2(b-a)}\Big]dx$$

$$= \int\limits_{(\Omega\setminus\Omega_0)\setminus\omega_\varepsilon} \Big[\frac{ab(C-D)}{b-a}\cdot\varepsilon(u) + \frac{bD-aC}{b-a}\cdot p + \frac{ab(|C|^2-|D|^2)}{2(b-a)} - \psi \frac{ab\,|C-D|^2}{2(b-a)}\Big]dx$$

and due to the fact that $\varepsilon > 0$ was chosen arbitrarily we easily arrive at (4.4), as desired.
Now, for $v \in H^1_0(\Omega;\mathbb{R}^n)$ and $q \in \operatorname{Ker} L^*$ let us set

$$\widetilde{\mathcal{I}}(v,q) := \int\limits_{\Omega\setminus\Omega_0} \Big[\frac{ab(C-D)}{b-a}\cdot\varepsilon(v) + \frac{bD-aC}{b-a}\cdot q + \frac{ab(|C|^2-|D|^2)}{2(b-a)} - \psi\,\frac{ab\,|C-D|^2}{2(b-a)}\Big]dx. \tag{4.6}$$

Since $p^k \in \operatorname{Ker} L^*$, from (2.13) and (4.1)-(4.4) there follows

$$\alpha \le \tfrac12 \widetilde{\mathcal{I}}(u,p) + \tfrac12 \int\limits_{\Omega_0} \Big[-a\,|\varepsilon(u)|^2\Big]dx + \tfrac12 \int\limits_{\Omega_0} \mathcal{B}\,dx + \tfrac12 \int\limits_{\Omega_0} p\cdot\varepsilon(u)dx$$

and

$$\alpha \ge \tfrac12 \widetilde{\mathcal{I}}(u,p) + \tfrac12 \int\limits_{\Omega_0} \Big[\frac1a\,|p|^2 - \frac1a\big(\big|\mathcal{A}^+\big|^2 + \big|\mathcal{A}^-\big|^2 + 2\psi\mathcal{A}^+\cdot\mathcal{A}^-\big)\Big]dx + \tfrac12 \int\limits_{\Omega_0} \mathcal{B}\,dx.$$

Combining the above inequalities leads to

$$\begin{aligned} &\tfrac12 \int\limits_{\Omega_0} \Big[-a\,|\varepsilon(u)|^2\Big]dx + \int\limits_{\Omega_0} p\cdot\varepsilon(u)dx \ge \\ &\ge \alpha + \tfrac12 \int\limits_{\Omega_0} p\cdot\varepsilon(u)dx - \widetilde{\mathcal{I}}(u,p) - \int\limits_{\Omega_0} \mathcal{B}\,dx \ge \\ &\qquad \ge \tfrac12 \int\limits_{\Omega_0} \Big[\frac1a\,|p|^2 - \frac1a\big(\big|\mathcal{A}^+\big|^2 + \big|\mathcal{A}^-\big|^2 + 2\psi\mathcal{A}^+\cdot\mathcal{A}^-\big)\Big]dx. \end{aligned} \tag{4.7}$$

On the other hand, from $p = a\,\varepsilon(u) + \mathcal{A}^+ + \psi\mathcal{A}^-$ in Ω_0 there follows

$$\tfrac{1}{2}\int_{\Omega_0}\Big[\frac{1}{a}\,|p|^2 - \frac{1}{a}\big(\left|\mathcal{A}^+\right|^2 + \left|\mathcal{A}^-\right|^2 + 2\psi\mathcal{A}^+\cdot\mathcal{A}^-\big)\Big]dx =$$

$$= \tfrac{1}{2}\int_{\Omega_0}\Big[-a\,|\varepsilon(u)|^2\Big]dx + \int_{\Omega_0} p\cdot\varepsilon(u)dx - 2\int_{\Omega_0}\frac{1}{a}\,\chi_a\chi_b\,\left|\mathcal{A}^-\right|^2 dx.$$

Here we have used the fact that $\psi^2 - 1 = -4\chi_a\chi_b$. Thus, in view of (4.7) we get

$$0 \geq \alpha - \tfrac{3}{2}\int_{\Omega_0} p\cdot\varepsilon(u)dx + \int_{\Omega_0} a\,|\varepsilon(u)|^2\,dx - \widetilde{\mathcal{I}}(u,p) - \int_{\Omega_0}\mathcal{B}\,dx \geq -2\int_{\Omega_0}\frac{1}{a}\,\chi_a\chi_b\,\left|\mathcal{A}^-\right|^2 dx.$$

Since

$$\int_{\Omega_0} p\cdot\varepsilon(u)dx = \int_{\Omega_0}\Big[a\,|\varepsilon(u)|^2 + (\mathcal{A}^+ + \psi\,\mathcal{A}^-)\cdot\varepsilon(u)\Big]dx,$$

from the fact that $\mathcal{A}^- = a\frac{D-C}{2}$ in Ω_0, we have

$$0 \geq \alpha - \int_{\Omega_0}\big[(\mathcal{A}^+ + \psi\,\mathcal{A}^-)\cdot\varepsilon(u) + \mathcal{B}\big]dx - \widetilde{\mathcal{I}}(u,p) - \tfrac{1}{2}\int_{\Omega_0} p\cdot\varepsilon(u)dx$$

$$\geq -\tfrac{1}{2}\int_{\Omega_0}\chi_a\chi_b\,a\,|C-D|^2\,dx. \tag{4.8}$$

Consequently, we are allowed to conclude that for some $\theta \in [0,1]$,

$$\alpha = \int_{\Omega_0}\big[(\mathcal{A}^+ + \psi\,\mathcal{A}^-)\cdot\varepsilon(u) + \mathcal{B}\big]dx + \tfrac{1}{2}\int_{\Omega_0} p\cdot\varepsilon(u)dx$$
$$+ \int_{\Omega\setminus\Omega_0}\Big[\frac{ab(C-D)}{b-a}\cdot\varepsilon(u) + \frac{bD-aC}{b-a}\cdot p + \frac{ab(|C|^2 - |D|^2)}{2(b-a)}$$
$$-\,\psi\,\frac{ab\,|C-D|^2}{2(b-a)}\Big]dx - \tfrac{\theta}{2}\int_{\Omega_0}\chi_a\chi_b\,a\,|C-D|^2\,dx.$$

In view of

$$\int_{\Omega_0}(\mathcal{A}^+ + \psi\,\mathcal{A}^-)\cdot\varepsilon(u)dx = \int_{\Omega_0}\big[-a\,|\varepsilon(u)|^2\big]dx + \int_{\Omega_0} p\cdot\varepsilon(u)dx,$$

we can summarize the above considerations by formulating the main result of the paper: the explicit formulas for the infimum in the nonconvex minimization problem.

THEOREM 1 *Let $u \in H_0^1(\Omega;\mathbb{R}^n)$ be the weak limit of $\{u^k\}$ (a minimizing sequence for (P)) and let $p \in L^2(\Omega;\mathbb{R}^{n\times n}_{sym})$ the weak limit of $\{p^k\}$ (the sequence of solutions of the dual problems $(P^k)^*$). Then there exists $\theta \in [0,1]$ such that $\alpha = \inf(\mathcal{J})$ can be expressed as*

$$\begin{aligned}
\alpha = &\int\limits_{\Omega_0} \Big[\Big(\frac{a(C+D)}{2} + \psi\frac{a(D-C)}{2}\Big)\cdot\varepsilon(u) + \frac{a(|C|^2+|D|^2)}{2} \\
&+ \psi\frac{a(|D|^2-|C|^2)}{2}\Big]dx + \tfrac{1}{2}\int\limits_{\Omega_0} p\cdot\varepsilon(u)dx \\
&+ \int\limits_{\Omega\setminus\Omega_0}\Big[\frac{ab(C-D)}{b-a}\cdot\varepsilon(u) + \frac{bD-aC}{b-a}\cdot p + \frac{ab(|C|^2-|D|^2)}{2(b-a)} \\
&- \psi\,\frac{ab\,|C-D|^2}{2(b-a)}\Big]dx - \tfrac{\theta}{2}\int\limits_{\Omega_0}\chi_a\chi_b\,a\,|C-D|^2\,dx = \\
= &\int\limits_{\Omega_0}\Big[-a\,|\varepsilon(u)|^2 + \frac{a(|C|^2+|D|^2)}{2} + \psi\frac{a(|D|^2-|C|^2)}{2}\Big]dx + \tfrac{3}{2}\int\limits_{\Omega_0} p\cdot\varepsilon(u)dx \\
&+ \int\limits_{\Omega\setminus\Omega_0}\Big[\frac{ab(C-D)}{b-a}\cdot\varepsilon(u) + \frac{bD-aC}{b-a}\cdot p + \frac{ab(|C|^2-|D|^2)}{2(b-a)} \\
&- \psi\frac{ab\,|C-D|^2}{2(b-a)}\Big]dx - \tfrac{\theta}{2}\int\limits_{\Omega_0}\chi_a\chi_b\,a\,|C-D|^2\,dx. \qquad (4.9)
\end{aligned}$$

REMARK 1 *Theorem 1 shows that $\alpha = \inf(\mathcal{J})$ admits the representation involving u, p, χ_a, χ_b, θ - the quantities which are determined by applying procedures well developed in Convex Analysis and Optimization. Having obtained α it is possible, following Naniewicz (2001), to introduce the relaxation term*

$$\mathcal{R}(u,\chi_a,\chi_b) := \int\limits_{\Omega}\Big[\chi_a\tfrac{1}{2}|\varepsilon(u)+C|^2 + \chi_b\tfrac{1}{2}|\varepsilon(u)+D|^2\Big]dx - \alpha.$$

The relaxation term $\mathcal{R}$ gathers information about oscillatory properties of the minimizing sequence $\{u^k\}$. When $\chi_a = 1$, then $\chi_b = 0$ (and vice versa) so in this case we have $\mathcal{R} \equiv 0$ and we get the classical solution of a convex problem. Therefore, convex problem is a special case of a nonconvex one.

5. Interrelations with Young measures

We will now consider the relation between Young measures associated with minimizing for $\mathcal{J}$ sequence $\{u^k\}$ and $\inf\mathcal{J} = \alpha$. We will show that the structure of the sequence $\{u^k\}$ allows the infimum to be fully expressed by Young measures associated with $\{u^k\}$. We will also establish some relations between Young measures and limits of the sequences of characteristic functions related to the phases of the integrand.

Before formulating next theorem it will be convenient to introduce some notation. Denote by $\omega_0^+ \subset \Omega$ the set of weak convergence of $\{\psi^k\} = \{\chi_b^k - \chi_a^k\}$ with weak limit equal to $+1$ and by $\omega_0^- \subset \Omega$ the set of weak convergence of $\{\psi^k\}$ with weak limit equal to -1. As Theorem 4 of Naniewicz (2001) guarantees only the existence of the sequences $\{\chi_b^k\}$ and $\{\chi_a^k\}$ related to the phases of the integrand, the sets $\omega_0^\pm$ depend on the choice of sequence $\{\psi^k\}$. Let $\omega_0 :=$ $\omega_0^+ \cup \omega_0^-$.

Recall that the functional to be minimized is of the form

$$\mathcal{J}(u) = \int\limits_\Omega \min\left\{\tfrac{1}{2}a\left|\varepsilon(u)+C\right|^2, \tfrac{1}{2}b\left|\varepsilon(u)+D\right|^2\right\}dx, \quad u \in H_0^1(\Omega;\mathbb{R}^n).$$

Set

$$h(x,\lambda) = \min\left\{\tfrac{1}{2}a(x)\left|\lambda + C(x)\right|^2, \tfrac{1}{2}b(x)\left|\lambda+D(x)\right|^2\right\}, \quad \lambda \in \mathbb{R}^{n\times n}_{sym},\ x\in\Omega.$$

Theorem 2 *Let $\{u^k\}$ be the minimizing sequence for (P) and let $\nu = \{\nu_x\}_{x\in\Omega}$ be the parametrized Young measure associated with $\{u^k\}$. Then*

$$\begin{aligned}
\alpha &= \int\limits_\Omega \int\limits_{\mathbb{R}^{n\times n}} h(x,\lambda)\,d\nu_x(\lambda)\,dx = \\
&= \int\limits_{\Omega\setminus\Omega_0} \Big[\frac{ab(C-D)}{b-a}\cdot\varepsilon(u) + \frac{bD-aC}{b-a}\cdot p + \frac{ab(|C|^2-|D|^2)}{2(b-a)} \\
&\quad - \psi\,\frac{ab\left|C-D\right|^2}{2(b-a)}\Big]dx \\
&\quad + \int\limits_{\Omega_0}\Big[-\int\limits_{\mathbb{R}^{n\times n}} a|\lambda|^2\,d\nu_x(\lambda) + \frac{a(|C|^2+|D|^2)}{2} + \psi\frac{a(|D|^2-|C|^2)}{2}\Big]dx \\
&\quad + \tfrac{3}{2}\int\limits_{\Omega_0} p\cdot\varepsilon(u)dx.
\end{aligned} \tag{5.1}$$

Moreover, we have

$$\nu_x = \delta_{\varepsilon(u(x))} \quad a.e.\ in\ \omega_0. \tag{5.2}$$

Proof. First, by making use of (2.12), (2.13), (4.4) and Lemma 1 we show that the weak limit of $\{h_1^k\}$, where

$$h_1^k = a(x)\left|\varepsilon(u^k(x))\right|^2,\ x\in\Omega_0,$$

is $\int_{\mathbb{R}^{n\times n}} a\left|\lambda\right|^2 d\nu_x(\lambda)$. Indeed, $\{p^k\cdot\varepsilon(u^k)\}$ as weakly convergent sequence in $L^1(\Omega)$ is equiintegrable according to the Dunford-Pettis criterion of weak compactness in $L^1(\Omega)$. Since $p^k \cdot \varepsilon(u^k) = m^k \left|\varepsilon(u^k)\right|^2 + \left(\mathcal{A}^+ + \psi^k \mathcal{A}^-\right) \cdot \varepsilon(u^k)$, it is easy to deduce that $\{m^k \left|\varepsilon(u^k)\right|^2\}$ is equiintegrable as well (and so is $\{h_1^k\}$). Thus, one can suppose that $\{h_1^k\}$ converges weakly in $L^1(\Omega)$ (by passing to a subsequence, if necessary). So by Theorem 6.2 p. 97 (Pedregal, 1997), its weak limit is $\int\limits_{\mathbb{R}^{n\times n}} a\,|\lambda|^2\, d\nu_x(\lambda)$, as desired.

Now, from

$$h\big(x, \varepsilon(u^k(x))\big) \le \frac{1}{2} m^k \left|\varepsilon(u^k)\right|^2 + (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot \varepsilon(u^k) + \frac{1}{2}\mathcal{B}^k$$

we conclude that the weak limit of $\{h\big(x, \varepsilon(u^k(x))\big)\}$ coincides with the right hand side of (5.1) (see Theorem 6.2, p. 97, Pedregal, 1997).
To show (5.2) it is enough to establish the strong convergence of $\{\varepsilon(u^k)\}$ in $L^2(\omega_0; \mathbb{R}^{n\times n}_{sym})$ (see Proposition 6.12, p. 111, Pedregal, 1997). The sequence $\{\psi^k\} = \{\chi_b^k - \chi_a^k\}$ takes its values in $\{-1, 1\}$. Thus the upper Kuratowski limit of the sequence of singletons $\{\psi^k(x)\}$ (i.e. the set of limit points of this sequence) is set $\{-1, 1\}$. By the Balder theorem (see Valadier, 1994) it follows that $\psi^k \to 1$ strongly in $L^1(\omega_0^+)$ and $\psi^k \to -1$ strongly in $L^1(\omega_0^-)$. Therefore, we can suppose that $\psi^k \to 1$ a.e. in ω_0^+ ($\psi^k \to -1$ a.e. in ω_0^-) (passing to a subsequence, if necessary). Further, the equiintegrability of $\{m^k \left|\varepsilon(u^k)\right|^2\}$ implies that $\{\left|\varepsilon(u^k)\right|^2\}$ is also equiintegrable. By Lemma 1 we have

$$\int\limits_{\omega_0^+} p^k \cdot \varepsilon(u^k)\, dx \to \int\limits_{\omega_0^+} p \cdot \varepsilon(u)\, dx = \int\limits_{\omega_0^+} b\,|\varepsilon(u)|^2\, dx + \int\limits_{\omega_0^+} (\mathcal{A}^+ + \mathcal{A}^-) \cdot \varepsilon(u)\, dx.$$

On the other hand,

$$\int\limits_{\omega_0^+} p^k \cdot \varepsilon(u^k)\, dx = \int\limits_{\omega_0^+} b\left|\varepsilon(u^k)\right|^2 dx + \int\limits_{\omega_{0k}^-} (a-b)\left|\varepsilon(u^k)\right|^2 dx + \int\limits_{\omega_0^+} (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot \varepsilon(u^k)\, dx,$$

where $\omega_{0k}^- = \{x \in \omega_0^+ : \psi^k(x) = -1\}$. Thus, taking into account that

$$\int\limits_{\omega_0^+} (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot \varepsilon(u^k)\, dx \to \int\limits_{\omega_0^+} (\mathcal{A}^+ + \mathcal{A}^-) \cdot \varepsilon(u)\, dx$$

and

$$\int\limits_{\omega_{0k}^-} (a-b)\left|\varepsilon(u^k)\right|^2 dx \to 0$$

(a consequence of the equiintegrability of $\{\left|\varepsilon(u^k)\right|^2\}$ and $\left|\omega_{0k}^-\right| \to 0$), we are led to

$$\int\limits_{\omega_0^+} b\left|\varepsilon(u^k)\right|^2 dx \to \int\limits_{\omega_0^+} b\,|\varepsilon(u)|^2\, dx.$$

Since, simultaneously, $\varepsilon(u^k) \rightharpoonup \varepsilon(u)$ in $L^2(\omega_0^+; \mathbb{R}^{n\times n}_{sym})$, the desired strong convergence results. Analogous reasoning holds for ω_0^-. The proof is complete.

Corollary 1 *The following formula is true*

$$\theta = \begin{cases} 2\dfrac{\int\limits_{\Omega_0\setminus\omega_0}\int\limits_{\mathbb{R}^n} a\,|\lambda|^2\,d\nu_x(\lambda)\,dx - \int\limits_{\Omega_0\setminus\omega_0} a\,|\varepsilon(u)|^2\,dx}{\int\limits_{\Omega_0\setminus\omega_0} \chi_a\chi_b\,a\,|C-D|^2\,dx} & \text{if} \\ \int\limits_{\Omega_0\setminus\omega_0} \chi_a\chi_b\,a\,|C-D|^2\,dx > 0 & \\ 0 & \textit{otherwise.} \end{cases}$$

From (4.9) and (5.1) it follows that

$$\lim_{k\to\infty}\int\limits_{\Omega_0} a\left|\varepsilon(u^k)\right|^2 dx = \int\limits_{\Omega_0}\int\limits_{\mathbb{R}^n} a\,|\lambda|^2\,d\nu_x(\lambda)\,dx = \int\limits_{\Omega_0} a\,|\varepsilon(u)|^2\,dx + \tfrac{\theta}{2}\int\limits_{\Omega_0}\chi_a\chi_b\,a\,|C-D|^2\,dx. \tag{5.3}$$

This, combined with (5.2) and the fact that $\chi_a(x)\chi_b(x) = 0$, $x \in \omega_0$, implies the assertion.

Remark 2 *Thanks to Theorem 2* $\alpha = \inf(\mathcal{J})$ *can be represented by the formulas in which the associated parametrized Young measures* $\{\nu_x\}_{x\in\Omega}$ *play the fundamental role. Moreover, it is possible to compute an explicit form of the (nonhomogeneous) Young measure at least in some special subset of Ω, that is, in* ω_0. *Further, its Corollary shows that in some regions of* Ω *it is possible to determine the formula for the value of* $\theta \in [0,1]$ *(see Theorem 1, equation 4.9) in terms of the Young measures* $\{\nu_x\}_{x\in(\Omega_0\setminus\omega_0)}$ *and* $\{\chi_a(x), \chi_b(x)\}_{x\in(\Omega_0\setminus\omega_0)}$. *Unfortunately, the technique based on parametrized Young measures seems to be more difficult to be implemented in practice. Therefore, for the problem under consideration, the presented approach can be regarded as the efficient alternative for the parameterized Young measure technique.*

References

ACERBI E. and FUSCO, N. (1984) Semicontinuity problems in the calculus of variations. *Arch. Rational Mech. Anal.* **86** (2), 125–145.

ALLAIRE G. and FRANCFORT, G. (1998) Existence of minimizers for non-quasiconvex functionals arising in optimal design. *Annales de l'Institut Henri Poincaré, (C) Analyse Non-Linéaire* **15** (3), 301–339.

ALLAIRE G. and LODS, V. (1999) Minimizers for double-well problem with affine boundary conditions. *Proc. Roy. Soc. Edinburgh* **129A** (3), 439–466.

AMBROSIO L. (1990) Existence of minimal energy configurations of nematic liquid crystals with variable degree of orientation. *Manuscripta Math.* **68** (1), 215–228.

AUBIN J.–P. (1993) *Optima and Equilibria.* Springer-Verlag, Berlin, Heidelberg, New York.

BALL J.M. (1977) Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.* **63** (4), 337–403.

BALL J. M. and JAMES R. D. (1987) Fine phase mixtures as minimizers of energy. *Arch. Rational Mech. Anal.* **100** (1), 13–52.

BALL J.M. and MURAT F.(1984) $W^{1,p}$ quasiconvexity and variational problems for multiple integrals. *J. Funct. Anal.* **58** (3), 225–253.

BOUCHITTÉ G. and BRAIDES A. and BUTTAZZO G. (1995) Relaxation results for some free discontinuity problems. *J. Reine Angew Math.* **458**, 1–18.

BUTTAZZO G. (1989) *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations.* Pitman Research Notes in Mathematics Series, **207**, Longman, Harlow.

CHIPOT M. and KINDERLEHRER, D. (1988) Equilibrium configurations of crystals. *Arch. Rational Mech. Anal.* **103** (3), 237–277.

DACOROGNA B. (1989) *Direct Methods in the Calculus of Variations.* Springer Verlag, Berlin.

DAL MASO G. (1993) *An Introduction to* Γ*-Convergence.* Birkhäuser, Boston.

EVANS L. C. and GARIEPY D. F. (1992) *Measure Theory and Fine Properties of Functions.* CRC Press, Inc., Boca Raton.

ERICKSEN J. L. (1980) Some phase transitions in crystals. *Arch. Rational Mech. Anal.* **73** (2), 99–124.

EKELAND I. and TEMAM, R. (1976) *Convex Analysis and Variational Problems.* North-Holland, Amsterdam.

FENCHEL W. (1951) Convex cones, sets and functions. *Lecture notes, typescript.* Princeton University, Princeton.

FONSECA I. and MÜLLER S. (1993) Relaxation of quasiconvex functionals in $BV(\Omega, \mathbb{R}^p)$ for integrands $f(x, u, \nabla u)$. *Arch. Rational Mech. Anal.* **123** (1), 1–49.

FONSECA I. (1988) The lower quasiconvex envelope of the stored energy function for an elastic crystal. *J. Math. Pures Appl.* **67** (9), 175–195.

FONSECA I. and RYBKA P. (1992) Relaxation of multiple integrals in the space $BV(\Omega; \mathbb{R}^p)$. *Proc. Royal Soc. Edin.* **121A** (3-4), 321–348.

JAMES R. D. and KINDERLEHRER D. (1989) Theory of diffusionless phase transitions. In: D. Serre, M. Rascale, and M. Slemrod, eds., *PDE's and continuum models of phase transitions. Lecture notes in Physics*, **344**, Springer Verlag, Berlin, 51–84.

KINDERLEHRER D. and PEDREGAL, P. (1991) Characterization of Young measures generated by gradients. *Arch. Rational Mech. Anal.* **115** (4), 329–365.

KOHN R. (1991) The relaxation of a double-well energy. *Cont. Mech. Thermodyn.* **3** (3), 193–236.

KOHN R. and STRANG G. (1986) Optimal design and relaxation of variational problems I, II, III. *Comm. Pure Appl. Math.* **39** (1), (2) and (3), 113–137, 139–182 and 353–377.

MORREY B. (1966) *Multiple Integrals in the Calculus of Variations*. Springer, 1966.

MURAT F. (1979) Compacité par compensation II. In: E. De Giorgi, E. Magénes and U. Mosco, eds., *Recent Methods in Nonlinear Analysis, Proceedings*. Pitagora, 245–256.

NANIEWICZ Z. (2001) Minimization with integrands composed of minimum of convex functions. *Nonlinear Anal.* **45** (5), 629–650.

PEDREGAL P. (1997) Parametrized Measures and Variational Principles. Birk–häuser, Basel.

TARTAR L. (1975) *Topics in Nonlinear Analysis*. Preprint, University of Wisconsin, Madison.

TARTAR L. (1979) Compensated compactness and applications to partial differential equations. In: R. Knops, ed., *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium*, IV, 136–212.

TARTAR L. (1991) On mathematical tools for studying partial differential equations of continuum physics: H-measures and Young measures. In: G. Buttazzo et al., eds. *Developments in Partial Differential Equations and Applications to Mathematical Physics*. Plenum, New York, 201–217.

VALADIER M. (1994) Young masures, weak and strong convergence and the Visintin-Balder theorem. *Set-Valued Analysis* **2** (1-2), 357-367.

YOUNG L. C. (1937) Generalized curves and the existence of an attained absolute minimum in the calculus of variations. *Comptes Rendus de la Société des Sciences et des Lettres de Varsovie, classe III* **30**, 212–234.

YOUNG L. C. (1969) *Lectures on the Calculus of Variations and Optimal Control Theory*. W.B. Saunders, Philadelphia.