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Paraconvex, but not strongly, Takagi functions*

by

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Abstract: There is an important open problem in the theory of approximate convexity whether every paraconvex function on a bounded interval is strongly paraconvex. Our aim is to show that this is not the case. To do this we need the following generalization of Takagi function.

For a sequence $a = (a_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+$ we consider Takagi-like function of the form

$$T[a](x) := \sum_{i=1}^{\infty} a_i \mathrm{dist}(x, \tfrac{1}{2^{i-1}}\mathbb{Z}) \quad \text{for } x \in \mathbb{R}.$$

We give convenient conditions for verification whether $T[a]$ is paraconvex or strongly paraconvex. This enables us to construct a class of paraconvex functions which are not strongly paraconvex.

Keywords: paraconvexity, strongly paraconvex function, semiconcavity, Takagi function.

1. Introduction

In the year 1903 T. Takagi (Takagi, 1903) introduced the function

$$T(x) := \sum_{n=1}^{\infty} \mathrm{dist}(x, \tfrac{1}{2^{n-1}}\mathbb{Z}) \quad \text{for } x \in \mathbb{R},$$

which is a simple example of a continuous nowhere differentiable function. Since then, the Takagi function and its generalizations of the form

$$T[a](x) := \sum_{n=1}^{\infty} a_n \mathrm{dist}(x, \tfrac{1}{2^{n-1}}\mathbb{Z}) \quad \text{for } x \in \mathbb{R}, \tag{1}$$

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where $a = (a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, have been applied in various parts of mathematics, in particular in the theory of fractals (Allaart and Kawamura, 2010; Hata and Yamaguti, 1984; Kairies, 1997; Křuppel, 2007, 2008; Kôno, 1987), approximate convexity and functional equations (Boros, 2008; Hazy, 2005; Hazy and Pales, 2004; Kairies, 1997, 1998; Makó and Pales, 2010; Pales, 2003; Tabor and Tabor, 2009a,b), or special functions theory (Kôno, 1987). For the survey of Takagi-like functions we refer the reader to Allaart and Kawamara (2011) and Kairies (1997). It is worth mentioning that by Kôno (1987, Theorem 2.2), $T[a]$ is a real-valued function if and only if $\sum\limits_{n\in\mathbb{N}} |a_n|/2^n < \infty$.

Our aim is to show that the functions of Takagi class can serve as an important source of examples and counterexamples for paraconvex and semiconvex functions. We will show that the Takagi functions have a large variety of properties related to approximate convexity. To explain our main results, we need to recall some notions of approximate convexity (Rolewicz 1997, 2000, 2005a,b; Zajíček, 2007). We put $\mathbb{R}_+ = [0, \infty)$.

Definition 1.1 *Let* $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *be a nondecreasing function such that* $\lim\limits_{r\to 0^+} \gamma(r)/r = 0$.

Let V *be a convex subset of a normed space. We say that a function* $f : V \to \mathbb{R}$ *is* γ-paraconvex *if*

$$\mathcal{C}f(x,y;t) := f(tx + (1-t)y) - tf(x) - (1-t)f(y) \le \gamma(\|x-y\|) \quad \text{for } x,y \in V, t \in [0,1]. \tag{2}$$

We call f strongly γ-paraconvex *if*

$$\mathcal{C}f(x,y;t) \le \min(t, 1-t)\gamma(\|x-y\|) \quad \text{for } x,y \in V, t \in [0,1]. \tag{3}$$

We will say that f *is* (strongly) paraconvex *if there exists a respective function* γ *such that* f *is (strongly)* γ*-paraconvex.*

An almost equivalent notion to strong paraconvexity is the notion of semiconvexity, see Cannarsa and Sinestrari (2004). In fact, on open convex sets semiconvexity is equivalent to strong paraconvexity, Zajíček (2008). Let us mention that paraconvex, strongly paraconvex and semiconvex functions play an important role in the study of real-valued functions on normed spaces (Cannarsa and Sinestrari, 2004; Hazy, 2005; Hazy and Pales, 2005; Ngai, Luc and Théra, 2000; Rolewicz, 1979, 2000, 2005a, b; Zajíček, 2007, 2008). Important problems in the study of paraconvexity and semiconvexity are:

- proving that (under some additional assumptions) paraconvex functions are strongly paraconvex (Rolewicz, 2000, 2005b).
- showing that strongly paraconvex functions are almost everywhere differentiable, see for example Rolewicz (2005 a,b); Zajíček (2007).

In this paper we deal with, to some extent, dual problems:

- does there exist a paraconvex function $f : [0,1] \to \mathbb{R}$ which is not strongly paraconvex?

- is every paraconvex function $f : [0,1] \to \mathbb{R}$ almost everywhere differentiable?

We answer the above questions negatively by giving conditions for $T[a]$ to be paraconvex or strongly paraconvex. This, jointly with the Theorem of Kôno (1987), implies that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \mathrm{dist}(x, \tfrac{1}{2^{n-1}}\mathbb{Z})$$

is an example of a paraconvex function which is differentiable only on a set of measure zero (and, consequently, is not strongly paraconvex).

2. Preliminary results

In this section we prove a list of technical lemmas. We begin with an obvious but important result (its more general version can be found in Kôno, 1987). For the convenience of the reader we present its proof.

Proposition 2.1 *Let V be a convex subset of a normed space and let $f : V \to \mathbb{R}$ be a Lipschitz function. Then*

$$|\mathcal{C}f(x,y;t)| \le 2t(1-t)\mathrm{lip}(f)\|x-y\| \quad \textit{for } x, y \in V, t \in [0,1],$$

where $\mathrm{lip}(f)$ *denotes the Lipschitz constant of* f.

Proof: For $x, y \in V, t \in [0,1]$ we have

$$|\mathcal{C}f(x,y;t)| \le t|f(tx+(1-t)y) - f(x)| + (1-t)|f(tx+(1-t)y) - f(y)|$$

$$\le t(1-t)\mathrm{lip}(f)\|x-y\| + t(1-t)\mathrm{lip}(f)\|x-y\| = 2t(1-t)\mathrm{lip}(f)\|x-y\|.$$

We denote

$$d_n(x) := \mathrm{dist}(x, \frac{1}{2^{n-1}}\mathbb{Z}) \quad \text{for } n \in \mathbb{N}, x \in \mathbb{R}.$$

It is obvious that d_n is periodic with period $1/2^{n-1}$.

Lemma 2.1 *Let $n \in \mathbb{N}$. Then*

$$d_n(x) = \begin{cases} |x - \frac{k}{2^n}| \ \textit{if } k \in 2\mathbb{Z}, \\ \frac{1}{2^n} - |x - \frac{k}{2^n}| \ \textit{if } k \in 2\mathbb{Z}+1, \end{cases} \qquad \textit{for } x \in [\tfrac{k-1}{2^n}, \tfrac{k+1}{2^n}].$$

Proof: Since d_n is periodic with period $1/2^{n-1}$, it is enough to consider the case when $k = 0$ or $k = 1$. If $k = 0$, then for $x \in [-\frac{1}{2^n}, \frac{1}{2^n}]$ we have

$$d_n(x) = \mathrm{dist}(x, \tfrac{1}{2^{n-1}}\mathbb{Z}) = \mathrm{dist}(x, \{0\}) = |x|.$$

If $k = 1$, then for $x \in [0, \frac{2}{2^n}]$ we have

$$d_n(x) = \mathrm{dist}(x, \tfrac{1}{2^{n-1}}\mathbb{Z}) = \mathrm{dist}(x, \{0, \tfrac{2}{2^n}\}) = \frac{1}{2^n} - |x - \frac{1}{2^n}|.$$

For a sequence $a = (a_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+$ and $k \in \mathbb{N}$, $l \in \mathbb{N} \cup \{\infty\}$, $l \geq k$ we consider the function $T_k^l[a] : \mathbb{R} \to [0, \infty]$ defined by

$$T_k^l[a](x) := \sum_{i=k}^{l} a_i d_i(x) \quad \text{for } x \in \mathbb{R}.$$

Instead of $T_1^\infty[a]$ we write $T[a]$. Clearly, $T_k^l[a]$ is periodic with period $1/2^{k-1}$. We use the convention $\sum_{i=1}^{0} = 0$, which implies that $T_1^0[a] = 0$.

Lemma 2.2 *Let $a = (a_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+$ be given, and let $k \in \mathbb{Z}$, $n \in \mathbb{N}$ be fixed. If $n = 1$ or $k \in 2\mathbb{Z} + 1$ then $T_1^{n-1}[a]$ is affine on $[\frac{k-1}{2^n}, \frac{k+1}{2^n}]$.*

Proof: If $n = 1$, then $T_1^{n-1}[a] = T_1^0[a] = 0$, which trivially yields the assertion. Consider now the case when $n \geq 2$. Then $k = 2m+1$ for a certain $m \in \mathbb{Z}$. Since the sum of affine functions is affine, it is enough to show that d_i is affine on

$$[\frac{k-1}{2^n}, \frac{k+1}{2^n}] = [\frac{m}{2^{n-1}}, \frac{m+1}{2^{n-1}}],$$

for every $i \in \{1, \ldots, n-1\}$. For each $i \in \{1, \ldots, n-1\}$ there exists an $m_i \in \mathbb{Z}$ such that

$$[\frac{m}{2^{n-1}}, \frac{m+1}{2^{n-1}}] \subset [\frac{m_i}{2^i}, \frac{m_i+1}{2^i}].$$

If $m_i \in 2\mathbb{Z}$, then by Lemma 2.1

$$d_i(x) = |x - \frac{m_i}{2^i}| = x - \frac{m_i}{2^i} \quad \text{for } x \in [\frac{m_i}{2^i}, \frac{m_i+1}{2^i}],$$

while if $m_i \in 2\mathbb{Z} + 1$ we get

$$d_i(x) = \frac{1}{2^i} - |x - \frac{m_i}{2^i}| = \frac{m_i+1}{2^i} - x \quad \text{for } x \in [\frac{m_i}{2^i}, \frac{m_i+1}{2^i}].$$

Lemma 2.3 *Let $a = (a_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+$ be given and let $k \in \mathbb{Z}$, $l, n \in \mathbb{N}$, $l \geq n$ be fixed. Then for $x \in [\frac{k}{2^n} - \frac{1}{2^l}, \frac{k}{2^n} + \frac{1}{2^l}]$ we have*

$$T_n^l[a](x) = \begin{cases} (a_n + \ldots + a_l)|x - \frac{k}{2^n}| \text{ if } k \in 2\mathbb{Z}, \\ \frac{a_n}{2^n} + ((-a_n) + a_{n+1} + \ldots + a_l)|x - \frac{k}{2^n}| \text{ if } k \in 2\mathbb{Z} + 1. \end{cases}$$

Proof: Consider an arbitrary $x \in [\frac{k}{2^n} - \frac{1}{2^l}, \frac{k}{2^n} + \frac{1}{2^l}]$. We have

$$[\frac{k}{2^n} - \frac{1}{2^l}, \frac{k}{2^n} + \frac{1}{2^l}] \subset [\frac{2^{i-n}k}{2^i} - \frac{1}{2^i}, \frac{2^{i-n}k}{2^i} + \frac{1}{2^i}] \quad \text{for } i = n, \ldots, l. \tag{4}$$

If $k \in 2\mathbb{Z}$ then by (4) and Lemma 2.1 we obtain that

$$d_i(x) = |x - \frac{2^{i-n}k}{2^i}| = |x - \frac{k}{2^n}| \quad \text{for } i = n, \ldots, l,$$

and consequently

$$T_n^l[a](x) = \sum_{i=n}^{l} a_i d_i(x) = (\sum_{i=n}^{l} a_i)|x - \frac{k}{2^n}|.$$

Assume now that $k \in 2\mathbb{Z}+1$. Making use of (4) for $i = n$ and Lemma 2.1 we get

$$d_n(x) = \frac{1}{2^n} - |x - \frac{k}{2^n}|.$$

Since $2^{i-n}k \in 2\mathbb{Z}$ for $i = n+1, \ldots, l$, by (4) and Lemma 2.1,

$$d_i(x) = |x - \frac{k}{2^n}| \quad \text{for } i = n+1, \ldots, l.$$

Thus

$$T_n^l[a](x) = a_n d_n(x) + \sum_{i=n+1}^{l} a_i d_i(x) = \frac{a_n}{2^n} + ((-a_n) + a_{n+1} + \ldots + a_l)|x - \frac{k}{2^n}|.$$

Lemma 2.4 *For $n \in \mathbb{N}$ we have*
a) $\mathcal{C}d_n(x, y; t) \leq 2t(1-t)|x-y|$ for $x, y \in \mathbb{R}$, $t \in [0,1]$,
b) $\mathcal{C}d_n(x, y; t) \in [-\frac{1}{2^n}, \frac{1}{2^n}]$ for $x, y \in \mathbb{R}$, $t \in [0,1]$.

Proof: It is clear that d_n is Lipschitz with $\mathrm{lip}(d_n) = 1$. By Proposition 2.1 we get a). By the definition of the operator $\mathcal{C}$ we have for $x, y \in \mathbb{R}$, $t \in [0,1]$

$$\begin{array}{rcl} \mathcal{C}d_n(x,y;t) & = & d_n(tx + (1-t)y) - td_n(x) - (1-t)d_n(y) \\ & \in & [0, \frac{1}{2^n}] - [0, \frac{1}{2^n}] = [-\frac{1}{2^n}, \frac{1}{2^n}]. \end{array}$$

Lemma 2.5 *Let $a = (a_i)_{i \in \mathbb{N}} \subset \mathbb{R}_+$ be a given sequence. We assume that there exists a $q > 1/2$ such that*

$$a_{i+1} \geq q a_i \quad \textit{for } i \in \mathbb{N}.$$

Let $K_q \in \mathbb{N}$ be such that

$$q + \ldots + q^{K_q} > 1. \tag{5}$$

Let $n \in \mathbb{N}$ and $l \in \mathbb{N}$, $l \geq n + K_q$ be arbitrary.
Then $T_n^l[a]$ is convex on $[\frac{k}{2^n} - \frac{1}{2^l}, \frac{k}{2^n} + \frac{1}{2^l}]$ for every $k \in \mathbb{Z}$.

Proof: We have

$$a_{n+1} + \ldots + a_l \geq a_n(q + \ldots + q^{K_q}) \geq a_n,$$

and hence

$$(-a_n) + a_{n+1} + \ldots + a_l \geq 0.$$

Lemma 2.3 completes the proof.

LEMMA 2.6 *Let $x, y \in \mathbb{R}$, $x < y < x+1/2$. Let n be the smallest positive integer such that*

$$(x, y) \cap \tfrac{1}{2^n}\mathbb{Z} \neq \emptyset.$$

Then the following statements hold:

i) There exists a unique $k \in \mathbb{Z}$ such that $\frac{k}{2^n} \in (x, y)$. Moreover, if $n > 1$ then $k \in 2\mathbb{Z}+1$.

ii) There exists the greatest $l \in \mathbb{N}$ such that

$$[x, y] \subset [\frac{k}{2^n} - \frac{1}{2^l}, \frac{k}{2^n} + \frac{1}{2^l}]. \tag{6}$$

Moreover, then $l \geq n$ and

$$\frac{1}{4}\frac{1}{2^l} \leq y - x \leq 2\frac{1}{2^l}. \tag{7}$$

Proof: i) The existence of $k \in \mathbb{Z}$ such that $\frac{k}{2^n} \in (x, y)$ follows from the definition of n. To prove its uniqueness suppose that there exist $k_1, k_2 \in \mathbb{Z}$, $k_1 < k_2$ such that $\frac{k_1}{2^n}, \frac{k_2}{2^n} \in (x, y)$. Then $\frac{k_1}{2^n}, \frac{k_1+1}{2^n} \in (x, y)$. One of the numbers k_1, k_1+1 is even. Suppose, e.g., that $k_1 \in 2\mathbb{Z}$. Then $\frac{k_1}{2} \in \mathbb{Z}$ and $\frac{k_1/2}{2^{n-1}} = \frac{k_1}{2^n} \in \mathbb{Z}$, which contradicts the definition of n.

Now we prove the second part of i). Suppose that $n > 1$ and that there exists a $p \in \mathbb{Z}$ such that $\frac{2p}{2^n} \in (x, y)$. Then we would get $\frac{p}{2^{n-1}} = (2p)/2^n \in (x, y)$, and since $n - 1 \in \mathbb{N}$ we again obtain a contradiction.

ii) We first prove that $l = n$ satisfies (6). We have to show that $\frac{k-1}{2^n} \leq x$ and that $\frac{k+1}{2^n} \geq y$. Suppose for an indirect proof, that either $\frac{k-1}{2^n} > x$ or $\frac{k+1}{2^n} < y$. We consider the case when $\frac{k-1}{2^n} > x$. Obviously, $\frac{k-1}{2^n} < y$. Hence $\frac{k-1}{2^n} \in (x, y)$, which contradicts i). The reasoning in the case $\frac{k+1}{2^n} < y$ is analogous.

For sufficiently large $l \in \mathbb{N}$ we have

$$(\frac{k}{2^n} - \frac{1}{2^l}, \frac{k}{2^n} + \frac{1}{2^l}) \subset (x, y).$$

It means that the set of integers l satisfying (6) is bounded above. Therefore there exists the greatest element l in this set. It remains to prove that it satisfies (7). From (6) we get

$$y - x \leq (\frac{k}{2^n} + \frac{1}{2^l}) - (\frac{k}{2^n} - \frac{1}{2^l}) = \frac{2}{2^l}.$$

Now we prove that $y - x \geq \frac{1}{4}\frac{1}{2^l}$. Suppose that it is not true, that is $y - x < \frac{1}{4 \cdot 2^l}$. Since l is the greatest integer satisfying (6), either $x < \frac{k}{2^n} - \frac{1}{2 \cdot 2^l}$ or $y > \frac{k}{2^n} + \frac{1}{2 \cdot 2^l}$. If $x < \frac{k}{2^n} - \frac{1}{2 \cdot 2^l}$ then we would get

$$y = y - x + x < \frac{1}{4 \cdot 2^l} + \frac{k}{2^n} - \frac{1}{2 \cdot 2^l} < \frac{k}{2^n},$$

a contradiction. Similarly, if $y > k/2^n + 1/(2 \cdot 2^l)$, we would get

$$x = y + (-y + x) > \frac{k}{2^n} + \frac{1}{2 \cdot 2^l} - \frac{1}{4 \cdot 2^l} > \frac{k}{2^n},$$

a contradiction.

3. Paraconvexity

In this section we investigate the problem when the function $T[a]$ is paraconvex.

THEOREM 3.1 *Let $a = (a_i)_{i \in \mathbb{N}} \subset \mathbb{R}_+$ be such that $T[a]$ is paraconvex. Then*

$$\lim_{n \to \infty} a_n = 0.$$

Proof: We have

$$a_n = \mathcal{C}T[a](0, 2^{-n-1}; \tfrac{1}{2})/(2^{-n}) \leq \frac{\gamma(2^{-(n-1)})}{2^{-n}} \to 0 \text{ as } n \to \infty.$$

It occurs that the condition $\lim\limits_{n \to \infty} a_n = 0$ does not guarantee even the local paraconvexity of the function $T[a]$.

THEOREM 3.2 *Let U be a nonempty open subinterval of $\mathbb{R}$ and let $a = (a_i)_{i \in \mathbb{N}} \subset (0, \infty)$ be such that*

$$\limsup_{i \to \infty} \frac{a_{i+1}}{a_i} \leq \frac{1}{2}.$$

Then $T[a]|_U$ is not paraconvex.

Proof: There exist $q \in (0, 1/2)$ and $n_0 \in \mathbb{N}$ satisfying

$$\frac{a_{n+1}}{a_n} \leq q \quad \text{for } n \geq n_0.$$

We can find $n \in \mathbb{N}, n \geq n_0$ and $k \in 2\mathbb{Z} + 1$ such that

$$[\frac{k-1}{2^n}, \frac{k+1}{2^n}] \subset U.$$

Fix arbitrarily an $l \in \mathbb{N}$, $l \geq n$. By Lemma 2.3 we have

$$T_n^{\infty}[a](\tfrac{k}{2^n}) = \tfrac{a_n}{2^n} \tag{8}$$

and

$$T_n^l[a](\frac{k}{2^n} + \frac{1}{2^l}) = T_n^l[a](\frac{k}{2^n} - \frac{1}{2^l}) = a_n(\frac{1}{2^n} - \frac{1}{2^l}) + \sum_{i=n+1}^{l-1} \frac{1}{2^l} a_i$$

$$\leq a_n(\frac{1}{2^n} - \frac{1}{2^l}) + \frac{1}{2^l} \sum_{i=n+1}^{l-1} a_n q^{i-n} = a_n(\frac{1}{2^n} - \frac{1}{2^l}) + \frac{1}{2^l} a_n q \frac{1 - q^{l-n-1}}{1 - q}.$$

Since $d_i(\frac{k}{2^n} \pm \frac{1}{2^l}) = 0$ for $i > l$, we obtain that

$$T_n^\infty[a](\frac{k}{2^n} \pm \frac{1}{2^l}) \leq a_n(\frac{1}{2^n} - \frac{1}{2^l}) + \frac{1}{2^l} a_n q \frac{1 - q^{l-n-1}}{1-q}. \tag{9}$$

By Lemma 2.2 $T_1^{n-1}[a]$ is affine on the interval $[\frac{k}{2^n} - \frac{1}{2^l}, \frac{k}{2^n} + \frac{1}{2^l}]$ and therefore $\mathcal{C}T_1^{n-1}[a]|_{[\frac{k}{2^n} - \frac{1}{2^l}, \frac{k}{2^n} + \frac{1}{2^l}]} = 0$. Whence by the above estimations and (8), (9), we get

$$\begin{aligned} \mathcal{C}T[a](\frac{k}{2^n} - \frac{1}{2^l}, \frac{k}{2^n} + \frac{1}{2^l}; \frac{1}{2}) &= \mathcal{C}T_n^\infty[a](\frac{k}{2^n} - \frac{1}{2^l}, \frac{k}{2^n} + \frac{1}{2^l}; \frac{1}{2}) \\ &\geq \frac{1}{2^l}(a_n - a_n q \frac{1 - q^{l-n-1}}{1-q}), \end{aligned}$$

and consequently

$$\mathcal{C}T[a](\frac{k}{2^n} - \frac{1}{2^l}, \frac{k}{2^n} + \frac{1}{2^l}; \frac{1}{2}) / |(\frac{k}{2^n} + \frac{1}{2^l}) - (\frac{k}{2^n} - \frac{1}{2^l})|$$

$$\geq \frac{1}{2}(a_n - a_n q \frac{1 - q^{l-n-1}}{1-q}) \to \frac{1}{2} a_n \frac{1-2q}{1-q} > 0 \text{ as } l \to \infty.$$

This proves that $T[a]|_U$ is not paraconvex.

THEOREM 3.3 *Let $a = (a_i)_{i \in \mathbb{N}} \subset (0, \infty)$ be a sequence such that $\lim_{n\to\infty} a_n = 0$. We assume that there exists a $q > 1/2$ satisfying*

$$\frac{a_{i+1}}{a_i} \geq q \quad \textit{for } i \in \mathbb{N}. \tag{10}$$

Then $T[a]$ is paraconvex.

Proof: Let K_q be the number satisfying (5). We define a function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\omega(r) := \begin{cases} 0 \text{ for } r = 0, \\ \max\{a_i \,:\, i \in \mathbb{N}, i \geq -\log_2 r - K_q - 1\} \text{ for } r > 0. \end{cases}$$

It is clear that ω is nondecreasing and that $\lim_{r\to 0^+} \omega(r) = 0$. We will show that

$$\mathcal{C}T[a](x, y; t) \leq 2^{K_q+2}|x-y|\omega(|x-y|) \quad \text{for } x, y \in \mathbb{R}, t \in [0,1]. \tag{11}$$

Consider arbitrary $x, y \in \mathbb{R}$, $x < y$, $|x - y| < 1/2$ and arbitrary $t \in [0,1]$. Let n, k, l be as in Lemma 2.6. By Lemmas 2.6 and 2.2 we obtain that $T_1^{n-1}[a]$ is affine on $[x, y]$. Therefore we have

$$\mathcal{C}T[a](x, y; t) = \mathcal{C}T_n^\infty(x, y; t) \quad \text{for } t \in [0,1].$$

Two cases may occur:

a) $l < n + K_q$,
b) $l \geq n + K_q$.

Consider first the case a). Then $n + K_q + 1 \geq l + 2$ and by (7) we get

$$\frac{1}{2^{n+K_q+1}} \leq \frac{1}{2^{l+2}} \leq |x - y|.$$

This yields that

$$n \geq -\log_2 |x - y| - K_q - 1,$$

and consequently

$$2^{-n} \leq 2^{K_q+1}|x - y|.$$

Making use of the last two inequalities, Lemma 2.4 b) and definition of ω, we obtain

$$\begin{aligned} \mathcal{C}T_n^{\infty}[a](x, y; t) &= \sum_{i=n}^{\infty} \mathcal{C}d_i(x, y; t) \\ &\leq \max\{a_i \mid i \geq n\} \sum_{i=n}^{\infty} \mathcal{C}d_i(x, y; t) \\ &\leq \max\{a_i \mid i \geq -\log_2 |x - y| - K_q - 1\} \textstyle\sum_{i=n}^{\infty} \frac{1}{2^i} \\ &= \omega(|x - y|) \tfrac{1}{2^{n-1}} \leq 2^{K_q+2}|x - y|\omega(|x - y|). \end{aligned}$$

We have proved (11).

Now we consider the case b). It follows from Lemma 2.5 that $T_n^l[a]$ is convex on the interval $[\frac{k}{2^n} - \frac{1}{2^l}, \frac{k}{2^n} + \frac{1}{2^l}]$. Hence, by (6), $T_n^l[a]$ is convex on $[x, y]$. Whence it follows that

$$\mathcal{C}T_n^{\infty}[a](x, y; t) \leq \mathcal{C}T_{l+1}^{\infty}[a](x, y; t).$$

By (7) we have

$$2^{-l-2} \leq |x - y|,$$

and consequently

$$l + 1 \geq -\log_2 |x - y| - 1.$$

From the above inequality, Lemma 2.4 b) and (7) we get

$$\begin{aligned} \mathcal{C}T_{l+1}^{\infty}[a](x, y; t) &= \sum_{i=l+1}^{\infty} a_i \mathcal{C}d_i(x, y; t) \\ &\leq \max\{a_i \mid i \geq l + 1\} \sum_{i=l+1}^{\infty} \tfrac{1}{2^i} \\ &\leq \max\{a_i \mid i \geq -\log_2 |x - y| - 1\} \tfrac{1}{2^l} \\ &\leq \omega(|x - y|) \tfrac{1}{2^l} \leq \omega(|x - y|) 4|x - y| \\ &\leq 2^{K_q+2}|x - y|\omega(|x - y|). \end{aligned}$$

We have proved (11).

Consider now the case when $|x - y| \geq 1/2$. By Lemma 2.4 b) we obtain

$$\mathcal{C}T[a](x, y; t) = \sum_{i=1}^{\infty} a_i \mathcal{C}d_i(x, y; t) \leq \sum_{i=1}^{\infty} \frac{a_i}{2^i}.$$

We are going to compare the above estimation with (11). We denote

$$\gamma(r) := 2^{K_q+2} r\omega(r) \quad \text{for } r \in \mathbb{R}_+. \tag{12}$$

We have

$$\omega(\tfrac{1}{2}) = \max\{a_i \,|\, i \geq -\log_2(\tfrac{1}{2}) - K_q - 1\} = \max\{a_i \,|\, i \in \mathbb{N}\},$$

and consequently

$$\gamma(\tfrac{1}{2}) = 2^{K_q+1} \max\{a_i \,|\, i \in \mathbb{N}\} \geq \max\{a_i \,|\, i \in \mathbb{N}\} \sum_{i=1}^{\infty} \frac{1}{2^i} \geq \sum_{i=1}^{\infty} \frac{a_i}{2^i}.$$

Hence (11) is valid also in the case when $|x - y| \geq 1/2$.

We have proved (11), which means that the function $T[a]$ is γ-paraconvex with γ defined by (12).

Example. We show that the condition (10) cannot be replaced by

$$\liminf_{i\to\infty} \frac{a_{i+1}}{a_i} > 1/2. \tag{13}$$

We define sequences $b = (b_i)_{i=1}^{\infty}$ and $c = (c_i)_{i=1}^{\infty}$ by the formulas

$$\begin{aligned} &b_1 = 3, b_i = 0 \quad \text{for } i \geq 2, \\ &c_1 = 0, c_i = (\tfrac{2}{3})^{i-1} \quad \text{for } i \geq 2. \end{aligned}$$

Observe that

$$T[b](1/2) - T[b](1/2 + h) = 3|h| \quad \text{for } h \in (-1/2, 1/2). \tag{14}$$

Clearly $a = (a_i) = (b_i + c_i)$ satisfies (13). We will show that $T[a]$ is not paraconvex on an arbitrary neighbourhood U of $1/2$.

For an indirect proof suppose that $T[a]$ is γ-paraconvex with a certain function γ. Then for arbitrary sequences $(x_n), (y_n) \subset U$ such that

$$\lim_{n\to\infty} |x_n - y_n| = 0,$$

we would get

$$\limsup_{n\to\infty} \frac{\mathcal{C}T[a](x_n, y_n; 1/2)}{|x_n - y_n|} \leq \frac{\gamma(|x_n - y_n|)}{|x_n - y_n|} = 0. \tag{15}$$

Clearly

$$\operatorname{lip}(T[c]) \leq \sum_{i=2}^{\infty} (\frac{2}{3})^{i-1} = 2. \tag{16}$$

Consider an arbitrary sequence $(h_n) \subset (0, \frac{1}{2})$ convergent to zero and such that $\frac{1}{2} \pm (h_n) \subset U$. We define sequences (x_n), (y_n) and (t_n) by the formulas

$$x_n = 1/2 + h_n,\ y_n = 1/2 - h_n,\ t_n = 1/2 \quad \text{for } n \in \mathbb{N}.$$

Then for $n \in \mathbb{N}$

$$\begin{aligned}
&\tfrac{1}{|x_n - y_n|}\mathcal{C}T[a](x_n, y_n; t_n) \\
&= \tfrac{1}{2h_n}\mathcal{C}T[b](1/2 + h_n, 1/2 - h_n; 1/2) + \tfrac{1}{2h_n}\mathcal{C}T[c](1/2 + h_n, 1/2 - h_n; 1/2) \\
&= \tfrac{1}{2h_n}\big(\tfrac{1}{2}(T[b](1/2) - T[b](1/2 + h_n)) + \tfrac{1}{2}(T[b](1/2) - T[b](1/2 - h_n))\big) \\
&\quad + \tfrac{1}{2h_n}\mathcal{C}T[c](1/2 + h_n, 1/2 - h_n; 1/2) \\
&\overset{\text{by (14)}}{=} \tfrac{3}{2} + \tfrac{1}{2h_n}\big(\tfrac{1}{2}(T[c](1/2) - T[c](1/2 + h_n)) + \tfrac{1}{2}(T[c](1/2) - T[c](1/2 - h_n))\big) \\
&\geq \tfrac{3}{2} - \tfrac{1}{2h_n}\big(\tfrac{1}{2}\mathrm{lip}(T[c])h_n + \tfrac{1}{2}\mathrm{lip}(T[c])h_n\big) \overset{\text{by (16)}}{\geq} \tfrac{1}{2},
\end{aligned}$$

which contradicts (15).

Remark. Let us observe that $q = 1/2$ is, in a sense, a boundary value. In Theorem 3.2 we have shown that if $a = (a_i) \subset (0, \infty)$ and

$$\limsup_{i \to \infty} \frac{a_{i+1}}{a_i} < \frac{1}{2} \quad \text{for } i \in \mathbb{N},$$

then $T[a]$ is not paraconvex. In the case when $a = (\frac{1}{2^i})_{i \in \mathbb{N}}$ we have (see, for example Kairies, 1997, Tabor and Tabor, 2009 b)

$$T[(2^{-i})_{i \in \mathbb{N}}](x) = x(1 - x) \quad \text{for } x \in [0, 1].$$

Whence we immediately obtain that if $a = (a_i)_{i \in \mathbb{N}} \subset (0, \infty)$ and

$$\frac{a_{i+1}}{a_i} = \frac{1}{2} \quad \text{for } i \in \mathbb{N},$$

then

$$T[a](x) = 2a_1 x(1 - x) \quad \text{for } x \in [0, 1].$$

One can easily check that this function is paraconvex with $\gamma(r) = \frac{a_1}{2}r^2$. By Theorem 3.3 if $\lim_{i \to \infty} a_i = 0$ and there exists a q such that

$$\frac{a_{i+1}}{a_i} \geq q > \frac{1}{2} \quad \text{for } i \in \mathbb{N},$$

then $T[a]$ is paraconvex.

To show an important consequence of Theorem 3.3 we need the result of Kôno.

Theorem of Kôno (1987, Theorem 2). *Let $a = (a_i)_{i \in \mathbb{N}} \subset \mathbb{R}$ be a sequence such that $\sum\limits_{i=1}^{\infty} \frac{|a_i|}{2^i} < \infty$. Then* *(i)* $T[a]$ *is absolutely continuous if and only if* $\sum_{i=1}^{\infty} a_i^2 < \infty$,

(ii) $T[a]$ *is differentiable on a set of continuum cardinality and the range of the derivative is a whole line but there exists no derivative almost surely if and only if* $\lim_{i\to\infty} a_i = 0$ *but* $\sum_{i=1}^{\infty} a_i^2 = \infty$,

(iii) $T[a]$ *has nowhere finite derivative if and only if* $\liminf_{i\to\infty} |a_i| > 0$.

Directly from Theorem 3.3 and Theorem of Kôno we get

COROLLARY 3.1 *Let* $a = (a_i)_{i\in\mathbb{N}} \subset (0,\infty)$ *be a sequence such that*

- $\lim_{i\to\infty} a_i = 0$,
- $\sum_{i=1}^{\infty} a_i^2 = \infty$,
- *there exists a* $q > 1/2$ *such that* $a_{i+1} \geq q a_i$ *for* $i \in \mathbb{N}$.

Then $T[a]$ *is paraconvex function which is differentiable only on a set of measure zero.*

Clearly, $a_n = \frac{1}{\sqrt{n}}$ is a sequence satisfying the assumptions of the above corollary, which implies that the function

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \operatorname{dist}(x, \tfrac{1}{2^{n-1}}\mathbb{Z})$$

is an example of a paraconvex function which is differentiable only on a set of measure zero (and consequently is not strongly paraconvex).

4. Strong paraconvexity

As we know, functions of the Takagi class are usually very irregular. In this section we will investigate the question when the elements of Takagi class are strongly paraconvex.

THEOREM 4.1 *Let* $a = (a_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+$ *be a sequence such that* $T[a]$ *is strongly paraconvex. Then*

$$\sum_{i=1}^{\infty} a_i < \infty.$$

Proof: We have

$$\frac{1}{2^n}\sum_{i=1}^{n} a_i = T[a](\frac{1}{2^n}) = \mathcal{C}T[a](1,0;\frac{1}{2^n}) \leq \min(\frac{1}{2^n}, 1-\frac{1}{2^n})\gamma(1) \leq \frac{1}{2^n}\gamma(1).$$

Whence we immediately obtain the assertion.

Now we prove a sufficient condition. The idea of the proof is similar to that of Theorem 3.3.

THEOREM 4.2 *Let* $a = (a_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+$ *be a sequence such that there exists a* $q > 1/2$ *satisfying*

$$\frac{a_{i+1}}{a_i} \ge q \quad \textit{for } i \in \mathbb{N}.$$

If

$$\sum_{i=1}^{\infty} a_i < \infty, \tag{17}$$

then the function $T[a]$ *is strongly paraconvex.*

Proof: Let K_q be the constant satisfying (5). We define the function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\omega(r) := \begin{cases} 0 & \text{for } r = 0, \\ \sum\{a_i : i \in \mathbb{N}, i \ge -\log_2 r - K_q - 1\} & \text{for } r > 0. \end{cases}$$

It is clear that ω is nondecreasing. It follows from (17) that $\lim\limits_{r\to 0} \omega(r) = 0$. We will show that

$$\mathcal{C}T[a](x,y;t) \le 2t(1-t)|x-y|\omega(|x-y|) \quad \text{for } x,y \in \mathbb{R}, t \in [0,1]. \tag{18}$$

Let $x, y \in \mathbb{R}$, $x < y$, $t \in [0,1]$. We consider first the case when $|x-y| \le 1/2$. Let n, k, l be as in Lemma 2.6. By the same argumentation as in the proof of Theorem 3.3 we obtain that

$$\mathcal{C}T[a](x,y;t) = \mathcal{C}T_n^{\infty}[a](x,y;t).$$

Again proceeding as in that proof we consider two cases:
a) $l \le n + K_q$,
b) $l \ge n + K_q$.

In the first case we get that

$$n \ge -\log_2 |x-y| - K_q - 1,$$

which means that

$$\sum_{i=n}^{\infty} a_i \le \omega(|x-y|).$$

Making use of Lemma 2.4 a) we obtain

$$\begin{aligned} &\mathcal{C}T_n^{\infty}[a](x,y;t) = \sum_{i=n}^{\infty} a_i \mathcal{C}d_i(x,y;t) \\ &\le \sum_{i=n}^{\infty} a_i \cdot \max\{\mathcal{C}d_i(x,y;t) : i \in \mathbb{N}, i \ge n\} \\ &\le \omega(|x-y|)2t(1-t)|x-y|. \end{aligned}$$

We have proved (18) in the case a).

Now we consider the case b). As in the proof of Theorem 3.3 we obtain

$$\mathcal{C}T_n^{\infty}[a](x,y;t) \leq \mathcal{C}T_{l+1}^{\infty}[a](x,y;t)$$

and

$$l+1 \geq -\log_2 |x-y| - 1.$$

Applying the above inequalities, definition of ω and Lemma 2.4 i) we get

$$\mathcal{C}T[a]_n^{\infty}(x,y;t) \leq \mathcal{C}T_{l+1}^{\infty}[a](x,y;t) \leq \sum_{i=l+1}^{\infty} a_i \cdot \max\{\mathcal{C}d_i(x,y;t) : i \in \mathbb{N}, i \geq l+1\}$$
$$\leq \omega(|x-y|)2t(1-t)|x-y|.$$

We have proved (18). It remains to consider the case when $|x-y| \geq 1/2$. By Lemma 2.4 a) we have

$$\mathcal{C}T[a](x,y;t) = \sum_{i=1}^{\infty} a_i \mathcal{C}d_i(x,y;t) \leq (\sum_{i=1}^{\infty} a_i) 2t(1-t)|x-y|.$$

On the other hand, for $r \geq 1/2$ we have

$$\omega(r) = \sum \{a_i : i \in \mathbb{N}, i \geq -\log_2 r - K_q - 1\} = \sum_{i=1}^{\infty} a_i.$$

Thus, in the considered case we have

$$\mathcal{C}T[a](x,y;t) \leq 2t(1-t)|x-y|\omega(|x-y|),$$

which means that (18) is valid, an consequently $T[a]$ is γ-strongly paraconvex, with $\gamma(r) := 2r\omega(r)$.

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References

Allaart P. C. and Kawamura K. (2010) The improper infinite derivatives of Takagi's nowhere-differentiable function. *J. Math. Anal. Appl.* **372** (2) 656–665.

Allaart P. C. and Kawamura K. (2011) The Takagi function: a survey. *Real Anal. Exchange* **37** (1) 1–54.

Boros Z. (2008) An inequality for the Takagi function. *Math. Inequal. Appl.* **11**, 751–765.

Cannarsa P., Sinestrari C. (2004) Semiconcave Functions, Hamilton-Jacobi Equation and Optimal Control. *Progress in Nonlinear Differential Equations and their Applications*, **58**, Birkhäuser, Boston.

Falconer K. (1990) *Fractal Geometry. Mathematical Foundations and Applications.* Wiley and Sons, Chichester.

Hata M., Yamaguti M. (1984) The Takagi's function and its generalization. *Japan J. Appl. Math.* **1**, 183–199.

Házy A. (2005) On approximate t-convexity. *Mathematical Inequal. Appl.* **8** (3) 389–402.

Házy A., Páles Zs. (2004) On approximately midconvex functions. *Bulletin London Math. Soc.* **36** (3) 339–350.

Kairies H.-H. (1997) Functional equations for peculiar functions. *Aequationes Math.* **53**, 207–241.

Kairies H.-H. (1998) Takagis function and its functional equations. *Rocznik Nauk.-Dydakt. Prace Mat.* **15**, 73–83.

Kôno N. (1987) On generalized Takagi functions. *Acta Math. Hung.* **49**, 315–324.

Krüppel M. (2007) On the extrema and the improper derivatives of Takagis continuous nowhere differentiable function. *Rostocker Math. Kolloq.* **62**, 41–59.

Krüppel M. (2008) Takagis continuous nowhere differentiable function and binary digital sums. *Rostocker Math. Kolloq.* **63**, 37–54.

Lagarias J. C. (2011) The Takagi's function and its properties. *arXiv.org* no. 1112.4205.

Makó J. and Páles Zs. (2010) Approximate convexity of Takagi type functions. *J. Math. Anal. Appl.* **369**, 545–554.

Ngai H., Luc D. T., Théra M. (2000) Approximately convex functions. *Convex Anal.* **1**, 155–176.

Páles Zs. (2003) On approximately convex functions. *Proc. Amer. Math. Soc.* **131** (1) 243–252.

Rolewicz S. (1979) On γ-paraconvex multifunctions. *Math. Japonica* **24** (3) 293–300.

Rolewicz S. (2000) On $\alpha(\cdot)$-paraconvex and strongly $\alpha(\cdot)$-paraconvex functions. *Control and Cybernetics* **29**, 367–377.

Rolewicz S. (2005) On differentiability of strongly $\alpha()$-paraconvex functions in non-separable Asplund spaces. *Studia Math.* **167**, 235–244.

Rolewicz S. (2005) Paraconvex analysis. *Control and Cybernetics* **34**, 951–965.

Tabor Jacek and Tabor Józef (2009) Generalized approximate midconvexity. *Control and Cybernetics* **38** (3) 1–15.

Tabor Jacek and Tabor Józef (2009) Takagi functions and approximate midconvexity. *J. Math. Anal. Appl.* **356** (2) 729–737.

Takagi T. (1903) A simple example of continuous function without derivative. *Proc. Phy.- Math. Soc. Japan* **1**, 176–177.

Yamaguti M., Hata M., Kigami J. (1997) Mathematics of Fractals. *AMS Translations of Mathematical Monographs* **167**.

Zajíček L. (2007) A C^1 function which is nowhere strongly paraconvex and nowhere semiconcave. *Control and Cybernetics* **36**, 803–810.

Zajíček L. (2008) Differentiability of approximately convex, semiconcave and strongly paraconvex functions. *J. Convex Anal.* **15** (1) 1–15.