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# OWA operators in the weighted average and their application in decision making*

by

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**Abstract:** We introduce a new aggregation operator that unifies the weighted average (WA) and the ordered weighted averaging (OWA) operator in a single formulation. We call it the ordered weighted averaging – weighted average (OWAWA) operator. This aggregation operator provides a more complete representation of the weighted average and the OWA operator because it considers the degree of importance that each concept has in the aggregation and includes them as particular cases of a more general context. We study different properties and families of the OWAWA operator. The applicability of this method is very broad because any study that uses the weighted average or the OWA can be revised and extended with our approach. We focus on a multi-person decision-making application in the selection of financial strategies.

**Keywords:** weighted average; OWA operator; decision-making; aggregation operators

## 1. Introduction

Weighted average (WA) is one of the most common aggregation operators. It can be used in a wide range of different problems including statistics, economics and engineering. The ordered weighted averaging (OWA) operator is another interesting aggregation operator though its use has not been reported so much in the literature since it first appeared in 1988 (Yager, 1988). The OWA operator provides a parameterized family of aggregation operators that range from the maximum to the minimum values. For further reading on the OWA operator and its applications, consult Beliakov et al. (2007); Grabisch et al. (2009); Merigó and Gil-Lafuente (2009); Yager and Kacprzyk (1997); Yager et al. (2011).

Recently, some authors have tried to unify both concepts in the same formulation. Notable works include that of Torra (1997), who introduced the weighted

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OWA (WOWA) operator, and Xu and Da (2003), who reported the hybrid averaging (HA) operator. Both models unified the OWA and the WA because both concepts were included in the formulation as particular cases. However, these models seem to be a partial unification, and they fail to completely unify the concepts or weight their relevance to a specific problem. For example, in some problems, we may prefer to give more importance to the OWA operator because we believe that it is more relevant. This issue cannot be addressed with WOWA or HA. Moreover, there are other studies that have unified WAs with OWAs (see Merigó and Casanovas, 2010a; Liu, 2011; Merigó et al., 2010; Zhao et al., 2010; Zhou and Chen, 2011). An interesting approach is that of immediate probabilities (Engemann et al., 1996; Merigó, 2010; Yager et al., 1995), which is used for unifying probability with the OWA operator. Sometimes probability can be seen as a weighted average, so it is possible to extend this approach to the case with WAs. However, this case does not consider how relevant each concept is in the aggregation.

In this paper, we present a new approach that unifies the OWA operator with the WA. We call it the ordered weighted averaging – weighted average (OWAWA) operator. We could also refer to it as the WOWA operator, but there is another approach that already uses this name (Torra, 1997). The main advantage of this approach is that it unifies the OWA and the WA, taking into account the degree of importance that each concept has in the formulation.

We study different properties of the OWAWA operator. We analyze some of the common measures for characterizing the weighting vector of OWA aggregation and apply these to the case OWAWA. We develop a new degree of or-ness that considers the or-ness of OWA and WA. We also introduce a new entropy measure that unifies Yager entropy with Shannon entropy. Furthermore, we present a new divergence measure and a new balance operator for aggregation that uses OWAs and WAs.

We also develop different families of OWAWA operators. Moreover, we are also able to unify arithmetic mean (or simple average) with OWA operator when the weights of WA are equal, obtaining the arithmetic-OWA. We also find a similar result with the WA: the arithmetic-WA. We study other families such as the step-OWAWA, median-OWAWA, olympic-OWAWA, S-OWAWA, centered-OWAWA and maximal entropy OWAWA (MEOWAWA) operator.

We analyze the construction of interval numbers and related structures by using OWA and OWAWA operators in an aggregation process. We see that the OWA operator permits to organize information in the form of interval numbers where we consider the minimum, the maximum and further internal aggregations obtained with it. We extend this approach for the construction of some basic fuzzy numbers. We also consider the use of OWAWA operators obtaining a more complete representation that includes the subjective importance of the arguments. Thus, we arrive at the concept of the subjective interval number and the subjective fuzzy number. We analyze this methodology from different perspectives, focussing on the aggregation of arguments when considering a wide range of alternatives and states of nature.

We also study the applicability of the new approach. It is possible to develop an astonishingly wide range of applications because the OWAWA includes the WA and the OWA as special cases. Therefore, all studies that use either the WA or the OWA can be revised and extended with this new approach. For example, we can apply OWAWA to statistics, economics, engineering, decision theory and biology. In this paper, we focus on a multi-person decision-making problem of selection of financial strategies. The main advantage of the OWAWA in these problems is that it is possible to consider the degree of importance and the attitudinal character of the decision-maker in the same formulation. Moreover, depending on the particular type of OWAWA operator used, the results may lead to different decisions.

This paper is organized as follows. In Section 2, we briefly review the WA, the OWA operator and some previous models integrating WA and OWA in the same formulation. In Section 3, we present the new approach, and in Section 4, we study different measures for characterizing the weighting vector. Section 5 analyzes different families of OWAWA operators. In Section 6 we explain how to construct interval numbers and related structures with OWAWA operators. In Section 7, we study applications of the new approach, Section 8 analyzes the application to a multi-person decision-making problem and Section 9 presents a numerical example. Section 10 presents the conclusions of the paper.

## 2. Preliminaries

In this section, we briefly review some basic concepts used throughout the paper. We analyze weighted average (WA), OWA operator and some previous models considered the possibility of using the OWA operator in the weighted average, such as the WOWA operator and the hybrid averaging method.

### 2.1. The weighted average

Weighted average (WA) is one of the most common aggregation operators found in the literature. It has been used in an incredibly wide range of applications including statistics, economics and engineering. It can be defined as follows.

Definition 1 *A WA operator of dimension $n$ is a mapping $R^n \rightarrow R$ that has an associated weighting vector $V$, with $v_i \in$ [0, 1] and $\sum_{i=1}^{n} v_i = 1$, such that:*

$$WA(a_1, ..., a_n) = \sum_{i=1}^{n} v_i a_i, \qquad (1)$$

*where $a_i$ represents the argument variable.*

The WA operator satisfies the usual properties of aggregation operators. For further reading on different extensions and generalizations of WA, see for example Beliakov et al. (2007); Grabisch et al. (2011); Merigó (2011).

An interesting aspect of the OWAWA operator is that it is possible to reorder the arguments and the weights of the WA, although it is not necessary. This is relevant because in order to unify the WA with the OWA, we need to adapt either the WA or the OWA. If we reorder the WA to the OWA, we have to do the following:

$$WA(a_1, ..., a_n) = \sum_{j=1}^{n} v_j b_j, \tag{2}$$

where $b_j$ is the $j$th largest argument $a_i$, and $v_j$ is the weight $v_i$ reordered according to the reordering of the arguments $a_i$ in the form of $b_j$ such that $v_j \in [0, 1]$ and $\sum_{j=1}^{n} v_j = 1$.

Obviously, we get the same result in both cases. The recording is a key feature for unifying the OWA with the WA and will be explained in Section 3. Note that it is possible to apply the second method to all of the types of WA discussed in the literature, including the quasi-arithmetic WA (Quasi-WA), the weighted distance and the fuzzy WA (FWA).

### 2.2. The OWA operator

The OWA operator (Yager, 1988) is an aggregation operator that provides a parameterized family of aggregation operators between the minimum and the maximum values. It can be defined as follows.

Definition 2 *An OWA operator of dimension n is a mapping OWA: $R^n \rightarrow R$ that has an associated weighting vector $W$ of dimension n with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:*

$$OWA(a_1, ..., a_n) = \sum_{j=1}^{n} w_j b_j, \tag{3}$$

*where $b_j$ is the $j$th largest of the $a_i$.*

Note that different properties could be studied such as the distinction between descending and ascending orders, different measures for characterizing the weighting vector and different families of OWA operators. For further reading, refer, e.g., to Merigó and Casanovas (2010b; 2011a; 2011b; 2011c); Xu and Xia (2011); Yager (1993); Zeng (2011); Zhou and Chen (2011).

The reordering process of the aggregation is an interesting aspect to consider. In most of the OWA literature, we see that the arguments are reordered according to an established weighting vector. However, we may also develop the reordering process by reordering the weighting vector according to the positions of the arguments (Yager, 1998a). That is,

$$OWA(a_1, ..., a_n) = \sum_{i=1}^{n} a_i w_i, \tag{4}$$

where $w_i$ is the $i$th weight $w_j$ reordered according to the positions of the $a_i$.

Note also that (4) is especially useful in the unification between OWA and WA in the situation where we adapt the reordering of the OWA to the WA.

It is possible to extend this analysis to the whole OWA literature by considering it in different extensions, such as the uncertain OWA (UOWA) (Xu and Da, 2002), the induced quasi-OWA (Quasi-IOWA) (Merigó and Gil-Lafuente, 2009) and the OWA distance (OWAD) (Merigó and Gil-Lafuente, 2010).

### 2.3. Some previous approaches using the OWA operator in the weighted average

Some previous models already considered the possibility of using OWA operators and WAs in the same formulation. The main models are the weighted OWA (WOWA) operator (Torra, 1997; Torra and Narukawa, 2010), the hybrid averaging (HA) operator (Xu and Da, 2003) and the importance OWA operator (Yager, 1998b). These techniques can unify OWAs and WAs in the same model, but they are unable to consider the degree of importance that each case may have in the aggregation process. Moreover, in some particular cases, we also find inconsistencies. Other methods could be considered, such as the concept of immediate probability (Engemann et al., 1996; Yager et al., 1995). This method is focused on probability but it is easy to extend it to use WAs because WA may be interpreted as a subjective probability. To reiterate, these and other approaches are useful for some particular situations, but they are less complete than the OWAWA, because, although they can unify OWAs with WAs, they cannot unify them with different degrees of importance. In future research, we will also prove that these models can be seen as a special case of a general OWAWA operator (or its respective model with probabilities), which uses quasi-arithmetic means. Obviously, it is possible to develop more complex models of the WOWA, the HA, the I-OWA and the IP-OWA to account for the degree of importance of the OWAs and the WAs in the model, but these seem to be artificial and not a natural unification as it will be shown below. The WOWA operator can be defined as follows.

Definition 3 *Let $P$ and $W$ be two weighting vectors of dimension $n$ [$P = (p_1, p_2, \ldots, p_n)$], [$W = (w_1, w_2, \ldots, w_n)$], such that $p_i \in [0, 1]$ and $\sum_{i=1}^{n} p_i = 1$, and $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. In this case, a mapping WOWA: $R^n \to R$ is a WOWA operator of dimension $n$ if:*

$$WOWA(a_1, ...., a_n) = \sum_{i=1}^{n} \omega_i a_{\sigma(i)}, \tag{5}$$

*where $\{\sigma(1), \ldots, \sigma(n)\}$is a permutation of $\{1, \ldots, n\}$such that $a_{\sigma(i-1)} \geq a_{\sigma(i)}$ for all $i = 2, \ldots, n$. (i.e. $a_{\sigma(i)}$ is the $i$th largest in the collection $a_1, \ldots, a_n$),* *and the weight* $\omega_i$ *is defined as:*

$$\omega_i = w*\left(\sum_{j\le i} p_{\sigma(j)}\right) - w*\left(\sum_{j<i} p_{\sigma(j)}\right), \tag{6}$$

*with* $w^*$ *a monotonically increasing function that interpolates the points* $(i/n, \sum_{j\le i} w_j)$ *together with the point (0, 0).* $w^*$ *is required to be a straight line when the points can be interpolated in this way.*

The hybrid averaging operator developed by Xu and Da (2003) can be defined as follows:

Definition 4 *A HA operator of dimension* $n$ *is a mapping HA:* $R^n \to R$ *that has an associated weighting vector* $W$ *of dimensionnwith* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$*, such that:*

$$HA(a_1, a_2, ..., a_n) = \sum_{j=1}^{n} w_j b_j, \tag{7}$$

*where* $b_j$*is the* $j$*th largest of the* $\hat{a}\ _i(\hat{a}_i = n\omega_i a_i, i = 1,2,\ldots,n), \omega = (\omega_1, \omega_2, \ldots, \omega_n)$ *is the weighting vector of the* $a_i$*,with* $\omega_i \in [0, 1]$ *and the sum of the weights is 1.*

As we have mentioned before, there are other methods that could be considered. We especially want to consider the concept of immediate probability. Note that in our definition we extend the immediate probability to the case where we use WAs instead of probabilities. Thus, we can refer to this model as the immediate weighted average (IWA). It can be defined as follows:

Definition 5 *An IWA operator of dimension* $n$ *is a mapping IWA:* $R^n \to R$ *that has an associated weighting vector* $W$ *of dimension* $n$ *with*$w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$*, such that:*

$$IWA(a_1, a_2, ..., a_n) = \sum_{j=1}^{n} \hat{v}_j b_j, \tag{8}$$

*where* $b_j$ *is the* $j$*th largest of the* $a_i$*, each* $a_i$ *has associated a WA* $v_i$*,*$v_j$ *is the associated WA of* $b_j$*, and* $\hat{v}_j = (w_j v_j / \sum_{j=1}^{n} w_j v_j)$*.*

## 3. The ordered weighted averaging – weighted average operator

The ordered weighted averaging – weighted average (OWAWA) operator is a new model that unifies the OWA operator and the weighted average (WA). Therefore, both concepts can be seen as particular cases of the more general OWAWA.

This approach seems to be complete, at least as an initial real unification between OWA operators and WAs. It can also be seen as a unification between decision-making problems with uncertainty (with OWA operators) and in a risk environment (with probabilities). The main advantage of the OWAWA operator is that it can unify the OWA and the WA considering the degree of importance that each concept has in the aggregation. Thus, we are able to represent situations where either the OWA or the WA is more relevant in the analysis. This aspect is fundamental because it gives high flexibility of adaptation to different situations. Therefore, the OWAWA is very useful for implementing different applications in the available models in the literature by adding the OWA or the WA because the formulation permits the addition of this concept in a flexible way depending on the relevance it may have in the application. For example, in some situations we may consider that using the OWA in a problem that already uses the WA is not very relevant but should be considered. Thus, by using this model we can introduce the OWA with a low degree of importance (1%, 2%, etc.) and consider the alterations in the results. On the other hand, we may also introduce the OWA in such a way that it is more important than the WA, giving it a high degree of importance (80%, 90%, etc.). Along the same line, we can introduce the WA to a problem formulated with the OWA. However, this is less likely, because much more research has been published using the WA than the OWA in fields such as statistics, economics, engineering and biology.

In the following, we are going to analyze the OWAWA operator, which is defined as follows.

Definition 6 *An OWAWA operator of dimension n is a mapping OWAWA:* $R^n \rightarrow R$ *that has an associated weighting vector W of dimension n such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$*, according to the following formula:*

$$OWAWA(a_1, ..., a_n) = \sum_{j=1}^{n} \hat{v}_j b_j, \tag{9}$$

*where* $b_j$ *is the* $j$*th largest of the* $a_i$*, each argument* $a_i$ *has an associated weight (WA)* $v_i$ *with* $\sum_{i=1}^{n} v_i = 1$ *and* $v_i \in [0, 1]$*,* $\hat{v}_j = \beta w_j + (1-\beta) v_j$ *with* $\beta \in [0, 1]$ *and* $v_j$ *is the weight (WA)* $v_i$ *ordered according to* $b_j$*, that is, according to the* $j$*th largest of the* $a_i$*.*

Note that it is also possible to formulate the OWAWA operator separating the part that strictly affects the OWA operator and the part that affects the WA. This representation is useful to show both models in the same formulation, but it does not seem to be a unique equation that unifies both approaches.

Theorem 1 *An OWAWA operator is a mapping OWAWA:* $R^n \rightarrow R$ *of dimension n, if it has an associated weighting vector W, with* $\sum_{j=1}^{n} w_j = 1$ *and* $w_j \in [0, 1]$ *and a weighting vector V that affects the WA, with* $\sum_{i=1}^{n} v_i = 1$ *and* $v_i \in [0, 1]$*, such that:*

$$OWAWA(a_1, ..., a_n) = \beta \sum_{j=1}^{n} w_j b_j + (1-\beta) \sum_{i=1}^{n} v_i a_i, \quad (10)$$

*where $b_j$ is the jth largest of the arguments $a_i$ and $\beta \in$ [0, 1].*

Proof. Assuming (10), first we have to adapt the reordering of either $b_j$ or $a_i$ so they can be integrated in the same equation. For example, if we reorder $a_i$, we get:

$$OWAWA(a_1, ..., a_n) = \beta \sum_{j=1}^{n} w_j b_j + (1-\beta) \sum_{i=1}^{n} v_j b_j, \quad (11)$$

where $v_j$ is the weight $v_i$ ordered according to $b_j$ and $b_j$ is the $j$th largest of the $a_i$. Since $b_j$ is used in both parts of the equation, now we can form:

$$OWAWA(a_1, ..., a_n) = \sum_{j=1}^{n} \hat{v}_j b_j, \quad (12)$$

where $\hat{v}_j = \beta w_j + (1-\beta) v_j$ and $\sum_{j=1}^{n} \hat{v}_j = 1$. Thus, we get 9.

Note that this unification process is in accordance with the general definition of aggregation operators when combining two aggregations into a single one (Beliakov et al., 2007). In the following example, we demonstrate how to aggregate with the OWAWA operator. We consider aggregation with both definitions.

**Example 1.** Assume the following arguments in an aggregation process: (30, 50, 20, 60). Assume the weighting vector $W = (0.2, 0.2, 0.3, 0.3)$ and the probabilistic weighting vector $V = (0.3, 0.2, 0.4, 0.1)$. Let the WA have a degree of importance of 70% and the weighting vector $W$ of the OWA have a degree of 30%. If we want to aggregate this information using the OWAWA operator, we will get the following. The aggregation can be done either with Eq. (9) or Eq. (10). With Eq. (9) we calculate the new weighting vector as:

$$\hat{v}_1 = 0.3 \times 0.2 + 0.7 \times 0.1 = 0.13,$$

$$\hat{v}_2 = 0.3 \times 0.2 + 0.7 \times 0.2 = 0.2,$$

$$\hat{v}_3 = 0.3 \times 0.3 + 0.7 \times 0.3 = 0.3,$$

$$\hat{v}_4 = 0.3 \times 0.3 + 0.7 \times 0.4 = 0.37,$$

and then, we perform the aggregation process as follows:

$OWAWA = 0.13 \times 60 + 0.2 \times 50 + 0.3 \times 30 + 0.37 \times 20 = 34.2.$

With (10), we aggregate as follows:

$OWAWA = 0.3 \times (0.2 \times 60 + 0.2 \times 50 + 0.3 \times 30 + 0.3 \times 20) + 0.7 \times (0.3 \times 30 + 0.2 \times 50 + 0.4 \times 20 + 0.1 \times 60) = 34.2.$

Obviously, we get the same results with both methods.

From a generalized perspective of the reordering step, it is possible to distinguish between the descending OWAWA (DOWAWA) and the ascending OWAWA (AOWAWA) operator by using $w_j \quad = w*_{n-j+1}$, where $w_j$ is the $j$th weight of the DOWAWA and $w*_{n-j+1}$ the $j$th weight of the AOWAWA operator.

Note that in 9 we have presented the OWAWA adapting the ordering of the WA to the reordering of the OWA. It is also possible to develop the unification by adapting the reordering of the OWA to the ordering of the WA. In this case, we get the following:

$$OWAWA(a_1, ..., a_n) = \sum_{i=1}^{n} \hat{v}_i a_i, \tag{13}$$

where each argument $a_i$ has an associated weight (WA) $v_i$ with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,\, 1]$, $\hat{v}_i = \beta w_i + (1 - \beta) v_i$ with $\beta \in [0,\, 1]$ and $w_i$ is the weight (OWA) $w_j$ ordered according to the $i$th position of the $a_i$, being the $j$th largest argument$a_i$.

If $B$ is a vector corresponding to the ordered arguments $b_j$, we will call this the ordered argument vector and $W^T$ is the transpose of the weighting vector; then, the OWAWA operator can be expressed as:

$$OWAWA(a_1, ..., a_n) = W^T B. \tag{14}$$

Note that if the weighting vector is not normalized, i.e., $\hat{V} = \sum_{j=1}^{n} \hat{v}_j \neq 1$, then the OWAWA operator can be expressed as:

$$OWAWA(a_1, ..., a_n) = \frac{1}{\hat{V}} \sum_{j=1}^{n} \hat{v}_j b_j. \tag{15}$$

Some other interesting generalizations can be developed following Merigó and Casanovas (2011a); Mesiar and Pap (2008); Mesiar and Spirkova (2006); Spirkova (2009); Torra and Narukawa (2010). Following the ideas of Spirkova (2009), we can develop a generating function for the arguments of the OWAWA operator that represents the internal formation of this information, such that $s$: $R^m \to R$. This generating function expresses the formation of the arguments when there exists a previous analysis, such as the use of a multi-person process where each argument is constituted by the opinion of $m$ persons. Moreover, we will also use a weighting function $f$ for the weighting vector. Note that the use of a weighting function $f_i$ in the weighting vector of the weighted average is known as the Losonczi mean. If the function is equal for all the weights $f$, then we get the simple Losonczi mean (Losonczi, 1971) or the mixture operator (Spirkova, 2009). In this case, we directly extend this approach, obtaining the mixture OWAWA (MOWAWA) operator as follows.

Definition 7 *A MOWAWA operator of dimension n is a mapping MOWAWA: $R^n \to R$ that has associated a vector of weighting functions $f$, $s$: $R^m \to R$, such that:*

$$MOWAWA(s_y(a_1), ..., s_y(a_n)) = \frac{\sum_{j=1}^{n} f_j(s_y(b_j))s_y(b_j)}{\sum_{j=1}^{n} f_j(s_y(b_j))}, \quad (16)$$

*where $s_y(b_j)$ is the jth largest of the $s_y(a_i)$,$a_i$ is the argument variable, and y indicates that each argument is formed using a different function.*

Note that a more general expression of the previous formula can be found assuming that the generating function of the weighting vector does not depend on the arguments $b_j$ and can depend on a lot of other circumstances. Thus, we get the following expression:

$$OWAWA*(s_y(a_1), ..., s_y(a_n)) = \frac{\sum_{j=1}^{n} f_j(\hat{v}_j)s_y(b_j)}{\sum_{j=1}^{n} f_j(\hat{v}_j)}, \quad (17)$$

which can be simplified to an equation that implicitly uses the normalization process:

$$OWAWA*(s_y(a_1), \ldots, s_y(a_n)) = \sum_{j=1}^{n} f_j(\hat{v}_j)s_y(b_j) \quad (18)$$

in its generating function.

Note that several families of OWA operators such as those that depend on the arguments (Yager, 1993) are included in the MOWAWA. But in the new formulation, we include all the available families of OWA operators and a lot of other situations. Obviously, the OWAWA* can be used in any of the OWA literature and in future extensions.

A further interesting result consists in using infinitary aggregation operators (Mesiar and Pap, 2008). Thus, we can represent an aggregation process where there are an unlimited number of arguments that appear in the aggregation process. Note that $\sum_{j=1}^{\infty} \hat{v}_j = 1$. By using the OWAWA operator we get the infinitary OWAWA ($\infty$-OWAWA) operator as follows:

$$\infty - OWAWA(a_1, ..., a_n) = \sum_{j=1}^{\infty} \hat{v}_j b_j. \quad (19)$$

The reordering process is very complex because we have an unlimited number of arguments, so we never know the first argument to be aggregated. For further reading on the usual OWA, see Mesiar and Pap (2008).

The OWAWA is monotonic, bounded and idempotent. It is not commutative because the OWAWA operator includes the weighted average.

THEOREM 2 *(Monotonicity). Assume f is the OWAWA operator, let* $(a_1,...,a_n)$ *and* $(e_1, e_2,..., e_n)$ *be two sets of arguments. If* $a_i \geq e_i$*, for all* $i \in\{1, 2, \ldots, n\}$*, then*

$$f(a_1, ..., a_n) \geq f(e_1, e_2, ...., e_n). \tag{20}$$

Proof. It is straightforward and thus omitted.

THEOREM 3 *(Idempotency). Assume f is the OWAWA operator, if* $a_i = a$*, for all* $i \in\{1, 2, \ldots, n\}$ *then:*

$$f(a_1, a_2, ..., a_n) = a. \tag{21}$$

Proof. It is straightforward and thus omitted.

THEOREM 4 *(Bounded). Assume f is the OWAWA operator, then:*

$$Min\{a_i\} \leq f(a_1, a_2, ..., a_n) \leq Max\{a_i\}. \tag{22}$$

Proof. It is straightforward and thus omitted.

Note that the boundedness property presented in Theorem 3 is the extreme case where we only use the OWA operator in the aggregation of the OWAWA. However, the usual boundary conditions that we find when using the OWAWA is a more restrictive one because we are mixing the OWA and the WA.

THEOREM 5 *(Semi-boundary conditions). Assume f is the OWAWA operator, then:*

$$\beta \times Min\{a_i\} + (1-\beta) \times \sum_{i=1}^{n} v_i a_i \leq f(a_1, a_2, ..., a_n)$$
$$\leq \beta \times Max\{a_i\} + (1-\beta) \times \sum_{i=1}^{n} v_i a_i. \tag{23}$$

Proof. Let max$\{a_i\}$= $c$, and min$\{a_i\}$= $d$, then

$$f(a_1, a_2, ..., a_n) = \beta \sum_{j=1}^{n} w_j b_j + (1-\beta) \sum_{i=1}^{n} v_i a_i$$
$$\leq \beta \sum_{j=1}^{n} w_j c + (1-\beta) \sum_{i=1}^{n} v_i a_i = \beta c \sum_{j=1}^{n} w_j + (1-\beta) \sum_{i=1}^{n} v_i a_i, \tag{24}$$

and

$$f(a_1, a_2, ..., a_n) = \beta \sum_{j=1}^{n} w_j b_j + (1-\beta) \sum_{i=1}^{n} v_i a_i$$
$$\geq \beta \sum_{j=1}^{n} w_j d + (1-\beta) \sum_{i=1}^{n} v_i a_i = \beta d \sum_{j=1}^{n} w_j + (1-\beta) \sum_{i=1}^{n} v_i a_i, \tag{25}$$

Since $\sum_{j=1}^{n} w_j = 1$, we get

$$f(a_1, a_2, ..., a_n) \leq \beta c + (1-\beta) \sum_{i=1}^{n} v_i a_i, \tag{26}$$

and

$$f(a_1, a_2, ..., a_n) \geq \beta d + (1-\beta) \sum_{i=1}^{n} v_i a_i. \tag{27}$$

Therefore,

$$\beta \times Min\{a_i\} + (1-\beta) \times \sum_{i=1}^{n} v_i a_i$$
$$\leq f(a_1, a_2, ..., a_n) \leq \beta \times Max\{a_i\} + (1-\beta) \times \sum_{i=1}^{n} v_i a_i. \tag{28}$$

As we can see, if $\beta = 1$, we get the usual boundary conditions. Note that a similar semi-boundary condition could be analyzed from the OWA perspective. That is:

$$(1-\beta) \times Min\{a_i\} + \beta \times \sum_{j=1}^{n} w_j b_j \leq f(a_1, ..., a_n)$$
$$\leq (1-\beta) \times Max\{a_i\} + \beta \times \sum_{j=1}^{n} w_j b_j. \tag{29}$$

Note that if $w_i = 1/n$ for all $i$, the semi boundaries become the or–like and the and–like S–OWA operator (Yager, 1993).

Following Grabisch et al. (2009) we can study other properties, such as continuity. Note that the OWAWA operator is not associative because the reordering process may produce different results when combining several elements. An OWAWA operator is continuous if for each $a_k$ $\lim_{a_i \to a_k} OWAWA(a_i) = OWAWA(a_k)$.

## 4. Measures for characterizing the OWAWA weighting vector

The choice of the measures to characterize the weighting vector $\hat{V}$ ($V$ and $W$) is another interesting issue. Following a similar methodology as the development of the OWA operator (Yager, 1988; 1996a; 2002), we can formulate the attitudinal character (degree of or-ness), the entropy of dispersion, the divergence of the weighting vector and the balance operator.

The degree of or-ness – and-ness of an OWA aggregation (Yager, 1988), known in decision-making as the attitudinal character or degree of optimism, is defined as follows:

$$\alpha(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right). \tag{30}$$

Before starting the analysis with the OWAWA operator, note that this measure can also be applied in the weighted average. Thus, we obtain the or-ness measure of the weighted average. This or-ness measure, based on Eq. (2), can be formulated as follows:

$$\alpha(V) = \sum_{j=1}^{n} v_j \left(\frac{n-j}{n-1}\right). \tag{31}$$

Note that $\alpha(V) \in [0, 1]$. For the maximum we get $\alpha(V) = 1$, for the minimum, $\alpha(V) = 0$ and for the arithmetic mean, $\alpha(V) = 0.5$. It is straightforward to calculate the and-ness measure of the weighted average by using the dual:

$$Andness(V) = 1 - \alpha(V) = \sum_{j=1}^{n} v_j \left(\frac{j-1}{n-1}\right). \tag{32}$$

As we can see, we reorder the weights in an artificial way because $v_j$ is the $v_i$ weight with the $j$th largest argument $a_i$. However, this measure is able to provide the degree of or-ness of a WA aggregation, so we can see if the WA aggregation is close to the maximum or to the minimum. In the WA, we cannot manipulate the aggregation as we did for the OWA because the weights are established according to degrees of importance or subjective probabilities.

If we extend the analysis of the or-ness – and-ness measure to the OWAWA operator, we get the following expressions for the degree of or-ness:

$$\alpha(\hat{V}) = \beta \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right) + (1-\beta) \sum_{j=1}^{n} v_j \left(\frac{n-j}{n-1}\right). \tag{33}$$

This measure can be expressed in different ways, such as:

$$\alpha(\hat{V}) = \sum_{j=1}^{n} \hat{v}_j \left(\frac{n-j}{n-1}\right). \tag{34}$$

Note that $\hat{v}_j = \beta w_j + (1-\beta) v_j$ is the $j$th weight of the OWAWA aggregation explained in Eq. (9). As we can see, if $\beta = 1$, we get the usual or-ness measure of Yager (1988) presented in Eq. (27) and if $\beta = 0$, we obtain the or-ness measure of the weighted average. It is straightforward to calculate the and-ness measure by using the dual. That is, $Andness(\hat{V}) = 1 - \alpha(\hat{V})$. Thus, we get the following expression:

$$Andness(\hat{V}) = \sum_{j=1}^{n} \hat{v}_j \left(\frac{j-1}{n-1}\right). \tag{35}$$

It can be shown that $\alpha \in [0,1]$.

In the following, we present some interesting results obtained with this new or-ness – and-ness measure. For the optimistic (or maximum) criteria in the OWA, we get the following:

$$\alpha(\hat{V}) = \beta + (1-\beta)\sum_{j=1}^{n} v_j \frac{n-j}{n-1}. \tag{36}$$

Note that we can refer to this situation as the maximum weighted average (Max-WA) or weighted maximum. For the pessimistic (or minimum) criteria, we obtain:

$$\alpha(\hat{V}) = (1-\beta)\sum_{j=1}^{n} v_j \frac{n-j}{n-1}. \tag{37}$$

We can refer to this situation as the minimum weighted average (Min-WA) or weighted minimum. With the arithmetic mean in the OWA case (arithmetic weighted average), we get:

$$\alpha(\hat{V}) = 0.5\beta + (1-\beta)\sum_{j=1}^{n} v_j \frac{n-j}{n-1}, \tag{38}$$

and in the WA (arithmetic OWA), we obtain:

$$\alpha(\hat{V}) = \beta\sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right) + 0.5(1-\beta). \tag{39}$$

If we use the arithmetic mean in both the OWA and the WA, then we get $\alpha(\hat{V}) = 0.5$.

If $\beta = 1$, we get the classical results obtained by Yager (1988); that is, $\alpha(\hat{V}) = 1$ for the optimistic (or maximum) criteria, $\alpha(\hat{V}) = 0$ for the pessimistic (or minimum) criteria and $\alpha(\hat{V}) = 0.5$ for the arithmetic mean. It is also possible to calculate the maximum and the minimum with the WA (Max-OWA and Min-OWA), but this result is more artificial because the WA usually uses the weighting vector as the degree of importance or subjective probability of the aggregation process.

If both the OWA and the WA use the maximum aggregation, then $\alpha(\hat{V}) = 1$, and if both use the minimum aggregation, $\alpha(\hat{V}) = 0$. It is straightforward to calculate the results obtained with the and-ness measure by using the dual ($Andness(\hat{V}) = 1 - \alpha(\hat{V})$). Thus, the maximum is $Andness(\hat{V}) = 0$, the minimum is $Andness(\hat{V}) = 1$ and the arithmetic mean is 0.5.

The entropy of dispersion (Yager, 1988) measures the amount of information being used in the aggregation:

$$H(W) = -\sum_{j=1}^{n} w_j \ln(w_j). \tag{40}$$

If we extend the entropy of dispersion to the OWAWA operator, we get the following:

$$H(\hat{V}) = -\left(\beta \sum_{j=1}^{n} w_j \ln(w_j) + (1-\beta) \sum_{i=1}^{n} v_i \ln(v_i)\right). \tag{41}$$

Note that $v_i$ is the $i$th weight of the WA aggregation. As we can see, if $\beta = 1$, we get the Yager entropy of dispersion of the OWA presented in Eq. (40), and if $\beta = 0$, we obtain the classical Shannon entropy (Shannon, 1948). Strictly speaking, the Shannon entropy is extended by using the OWAWA operator as follows:

$$H(\hat{V}) = -\left(\beta \sum_{j=1}^{n} w_j \log_2(w_j) + (1-\beta) \sum_{i=1}^{n} v_i \log_2(v_i)\right). \tag{42}$$

Thus, we can extend the entire analysis developed by Shannon (1948) and others in information theory with this new approach. We could also consider other entropy measures that could be implemented in the OWAWA operator.

Some interesting examples by using this measure are the step aggregations (Yager, 1993). If we use a step aggregation in the OWA, we get the following entropy:

$$H(\hat{V}) = -\left((1-\beta) \sum_{i=1}^{n} v_i \ln(v_i)\right). \tag{43}$$

And if we use a step aggregation in the WA, we obtain:

$$H(\hat{V}) = -\left(\beta \sum_{j=1}^{n} w_j \ln(w_j)\right). \tag{44}$$

Finally, if we use step aggregation in both OWA and WA, we get the minimum dispersion, that is, $H(\hat{V}) = 0$. The maximum dispersion is found when both $w_j = 1/n$ and $v_i = 1/n$, for all $i$ (note that $j$ is a reordering of $i$), and it is $H(\hat{V}) = \ln n$. If we only use $w_j = 1/n$, for all $j$, we get the following:

$$H(\hat{V}) = \beta \ln n - (1-\beta) \sum_{i=1}^{n} v_i \ln(v_i), \tag{45}$$

and if we only use $v_i = 1/n$, for all $i$, we obtain:

$$H(\hat{V}) = -\left(\beta \sum_{j=1}^{n} w_j \ln(w_j)\right) + (1-\beta) \ln n. \tag{46}$$

The divergence of $W$ (Yager, 2002) measures the divergence of the weights against the degree of or-ness – and-ness measure. It is useful in various situations, especially when the attitudinal character and the entropy of dispersion do not suffice to correctly analyze the weighting vector of an aggregation:

$$Div(W) = \sum_{j=1}^{n} w_j \left( \frac{n-j}{n-1} - \alpha(W) \right)^2. \tag{47}$$

Another measure for calculating the divergence can be as follows:

$$Div*(W) = \sum_{j=1}^{n} w_j \left| \frac{n-j}{n-1} - \alpha(W) \right|. \tag{48}$$

And, more generally, we could use $g^{-1}\left( \sum_{j=1}^{n} w_j g\left( \left| \frac{n-j}{n-1} - \alpha(W) \right| \right) \right)$ where $g$ is a strictly continuous monotone function.

If we extend the divergence of $W$ to the OWAWA operator, we get the following divergence of $\hat{V}$:

$$Div(\hat{V}) = \beta \left( \sum_{j=1}^{n} w_j \left( \frac{n-j}{n-1} - \alpha(W) \right)^2 \right) + (1-\beta) \left( \sum_{j=1}^{n} v_j \left( \frac{n-j}{n-1} - \alpha(V) \right)^2 \right). \tag{49}$$

If $\beta = 1$, we get the OWA divergence, and if $\beta = 0$, we get the WA divergence. Moreover, we can also suggest new measures of divergence as it has been developed above. For example, we can use:

$$Div*(\hat{V}) = \beta \left( \sum_{j=1}^{n} w_j \left| \frac{n-j}{n-1} - \alpha(W) \right| \right) + (1-\beta) \left( \sum_{j=1}^{n} v_j \left| \frac{n-j}{n-1} - \alpha(V) \right| \right). \tag{50}$$

We can also generalize the divergence by using the following expression:

$$Div*(\hat{V}) = \beta \left( g^{-1} \left( \sum_{j=1}^{n} w_j g \left( \left| \frac{n-j}{n-1} - \alpha(W) \right| \right) \right) \right) + (1-\beta) \left( g^{-1} \left( \sum_{j=1}^{n} v_j g \left( \left| \frac{n-j}{n-1} - \alpha(V) \right| \right) \right) \right). \tag{51}$$

If $\beta = 0$, we get the WA divergence and if $\beta = 1$, the OWA divergence. When using the maximum and the minimum in both the OWA and the WA, we get $Div*(\hat{V}) = 0$. More generally, for any step aggregation we get $Div*(\hat{V}) = 0$. If we use step aggregation in the WA, we get the following expression:

$$Div(\hat{V}) = \beta \left( \sum_{j=1}^{n} w_j \left( \frac{n-j}{n-1} - \alpha(W) \right)^2 \right), \tag{52}$$

and if we use step aggregation in the OWA, we obtain:

$$Div(\hat{V}) = (1-\beta) \left( \sum_{j=1}^{n} v_j \left( \frac{n-j}{n-1} - \alpha(V) \right)^2 \right). \tag{53}$$

If $\beta = 0$, this result becomes the divergence of the WA aggregation. We have presented the results of the divergence by using the or-ness measure but it is also possible to present them using the and-ness measure, that is, the dual: or-ness = 1 − and-ness.

The balance operator (Yager, 1996a) measures the balance of the weights against the or-ness or the and-ness. It is formulated as follows:

$$Bal(W) = \sum_{j=1}^{n} \left( \frac{n+1-2j}{n-1} \right) w_j. \tag{54}$$

It can be shown that $Bal(W) \in [-1, 1]$. Note that for the maximum, $Bal(W) = 1$, and for the minimum, $Bal(W) = -1$.

The balance operator can also be extended by using generalized and quasi-arithmetic means. Using quasi-arithmetic means, we get:

$$Bal*(W) = g^{-1} \left( \sum_{j=1}^{n} g \left( \frac{n+1-2j}{n-1} \right) w_j \right). \tag{55}$$

Now, if we extend the measures for obtaining the balance operator to the OWAWA operator, we get the following expression:

$$Bal(\hat{V}) = \beta g^{-1} \left( \sum_{j=1}^{n} g \left( \frac{n+1-2j}{n-1} \right) w_j \right) + (1-\beta) g^{-1} \left( \sum_{j=1}^{n} g \left( \frac{n+1-2j}{n-1} \right) v_j \right). \tag{56}$$

And if $g(b) = b$, then, we get the usual balance operator applied to the OWAWA operator as follows:

$$Bal(\hat{V}) = \sum_{j=1}^{n} \left( \frac{n+1-2j}{n-1} \right) \hat{v}_j. \tag{57}$$

If $\beta = 1$, we get the classic balance operator developed by Yager (1996a) presented in Eq. (52) and if $\beta = 0$, we obtain the balance operator of the weighted average that is formulated as follows:

$$Bal(V) = \sum_{j=1}^{n} \left( \frac{n + 1 - 2j}{n - 1} \right) v_j. \tag{58}$$

As we can see, $Bal(V) \in [-1, 1]$. For the maximum criteria, $Bal(V) = 1$, for the minimum criteria, $Bal(V) = -1$ and for the arithmetic mean, $Bal(V) = 0$.

## 5. Families of OWAWA operators

In this section, we analyze different families of OWAWA operators. This analysis will demonstrate a wide range of particular cases that can be used in the OWAWA operator, leading to different results. Thus, we are able to provide a more complete picture of the aggregation process. However, note that each family is just a particular case useful in some special situations according to the interests of the analysis.

First we will consider the two main cases of the OWAWA operator that are found by analyzing the coefficient $\beta$. Basically, if $\beta = 0$, then we get the WA, and if $\beta = 1$, we get the OWA operator. Note that when $\beta$ increases, we are giving more importance to the OWA operator, and vice versa. From this, it is possible to consider a wide range of particular cases by giving different values and interpretations to the $\beta$ value.

By choosing different manifestations of the weighting vector in the OWAWA operator, we are able to obtain different types of aggregation operators. For example, we can obtain the maximum-WA, the minimum-WA and the step-OWAWA operator.

**Remark 1.** The maximum-WA corresponds to $w_1 = 1$ and $w_j = 0$ for all $j \neq 1$. The minimum-WA is formed when $w_n = 1$ and $w_j = 0$ for all $j \neq n$. More generally, the step-OWAWA is formed when $w_k = 1$ and $w_j = 0$ for all $j \neq k$. Note that if $k = 1$, the step-OWAWA is transformed into the maximum-WA, and if $k = n$, into the minimum-WA.

**Remark 2.** The arithmetic-WA (A-WA) is obtained when $w_j = 1/n$ for all $j$, and it can be formulated as follows:

$$A - WA(a_1, ..., a_n) = \frac{\beta}{n} \sum_{i=1}^{n} a_i + (1 - \beta) \sum_{i=1}^{n} v_i a_i, \tag{59}$$

If $v_i = 1/n$, for all $i$, then, we get the unification between the arithmetic mean (or simple average) and the OWA operator, that is, the arithmetic-OWA (A-OWA). The A-OWA operator can be formulated as follows:

$$A - OWA(a_1, ..., a_n) = \beta \sum_{j=1}^{n} w_j b_j + \frac{(1 - \beta)}{n} \sum_{i=1}^{n} a_i. \tag{60}$$

Note that if $w_1 = 1$ and $w_j = 0$ for all $j \neq 1$, the A-OWA operator becomes the A-Max, also known in the literature as the or-like S-OWA operator, and if $w_n = 1$ and $w_j = 0$ for all $j \neq n$, it becomes the A-Min, known as the and-like S-OWA operator (Yager, 1993).

**Remark 3.** Another interesting family is that of the S-OWAWA operator. It can be subdivided into three classes: the "or-like," the "and-like" and the generalized S-OWAWA operator. The generalized S-OWAWA operator is obtained if:

- $\hat{v}_1 = (1/\mathrm{n})(1 - (\alpha + \gamma)) + \alpha$,
- $\hat{v}_n = (1/\mathrm{n})(1 - (\alpha + \gamma)) + \gamma$,
- $\hat{v}_j = (1/\mathrm{n})(1 - (\alpha + \gamma))$, for $j = 2$ to $n - 1$,

where $\alpha$, $\gamma \in [0, 1]$ and $\alpha + \gamma \leq 1$. If $\alpha = 0$, the generalized S-OWAWA operator becomes the "and-like" S-OWAWA operator, and if $\gamma = 0$, it becomes the "or-like" S-OWAWA operator.

**Remark 4.** Another family of aggregation operators that could be used is the centered-OWAWA operator. We can define an OWAWA operator as a centered aggregation operator if it is symmetric, strongly decaying and inclusive.

- It is symmetric if $\hat{v}_j = \hat{v}_{j+n-1}$.
- It is strongly decaying if $i < j \leq (n+1)/2$ then $\hat{v}_i < \hat{v}_j$ and if $i > j \geq (n+1)/2$ then $\hat{v}_i < \hat{v}_j$.
- It is inclusive if $\hat{v}_j > 0$.

**Remark 5.** For the median–OWAWA, if $n$ is odd we assign $\hat{v}_{(n+1)/2} = 1$ and $\hat{v}_{j*} = 0$ for all others. If $n$ is even we assign, for example, $\hat{v}_{n/2} = \hat{v}_{(n/2)+1} = 0.5$ and $\hat{v}_{j*} = 0$ for all others. For the weighted median–OWAWA, we select the argument $b_k$ that has the $k$th largest argument such that the sum of the weights from 1 to $k$ is equal or higher than 0.5 and the sum of weights from 1 to $k - 1$ is less than 0.5.

**Remark 6.** Another type of aggregation that could be used are the E-Z OWAWA weights. In this case, we should distinguish between two classes. In the first class, we assign $\hat{v}_{j*} = (1/q)$ for $j* = 1$ to $q$ and $\hat{v}_{j*} = 0$ for $j* > q$, and in the second class, we assign $\hat{v}_{j*} = 0$ for $j* = 1$ to $n - q$ and $\hat{v}_{j*} = (1/q)$ for $j* = n - q + 1$ to $n$.

**Remark 7.** The olympic-OWAWA is generated when $\hat{v}_1 = \hat{v}_n = 0$, and for all other $\hat{v}_{j*} = 1/(n - 2)$. Note that it is possible to develop a general form of the olympic-OWAWA by considering that $\hat{v}_j = 0$ for $j = 1, 2, \ldots, k, n, n - 1, \ldots, n - k + 1$, and for all others $\hat{v}_{j*} = 1/(n - 2k)$, where $k < n/2$. Note that if $k = 1$, then this general form becomes the usual olympic-OWAWA. If $k = (n - 1)/2$, then this general form becomes the median-OWAWA aggregation. Furthermore, it is also possible to develop the contrary case of the general olympic-OWAWA operator. In this case, $\hat{v}_j = (1/2k)$ for $j = 1, 2, \ldots, k, n, n - 1, \ldots, n - k + 1$, and $\hat{v}_j = 0$, for all other values, where $k < n/2$. Note that if $k = 1$, then we obtain the contrary case for the median-OWAWA.

**Remark 8.** Another interesting type can be developed using the functional method introduced by Yager (1988) for the OWA operator. This approach can be summarized as follows. Let $f$ be a function $f$: [0, 1] → [0, 1] such that $f(0) = f(1)$ and $f(x) \geq f(y)$ for $x > y$. We call this function a basic unit interval monotonic function (BUM). Using this BUM function we obtain the OWAWA weights $\hat{v}_j$ for $j = 1$ to $n$ as:

$$\hat{v}_j = f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right). \tag{61}$$

It can be easily shown that using this method, $\hat{v}_j$ satisfy the conditions that the sum of weights is 1 and $\hat{v}_j \in [0, 1]$.

**Remark 9.** A further interesting type is the non-monotonic-OWAWA operator based on Yager (1999). It is obtained when at least one of the weights is lower than 0 and $\sum_{j=1}^{n} \hat{v}_j = 1$. A key aspect of this operator is that it does not always achieve monotonicity. Therefore, strictly speaking, this particular case is not an OWAWA operator. However, we can see it as a particular family of operators that is not monotonic but nevertheless resembles an OWAWA operator.

**Remark 10.** Using the measures explained in Section 4, we can develop another group of methods for obtaining the OWAWA weights. The maximal entropy OWAWA (MEOWAWA) method is especially noteworthy. This method seeks to maximize the entropy, subject to an established degree of or-ness. It can be solved by using the following mathematical programming problem:

$$\text{maximize } H(\hat{V}) = -\left(\beta \sum_{j=1}^{n} w_j \ln(w_j) + (1-\beta) \sum_{i=1}^{n} v_i \ln(v_i)\right),$$

subject to

$$\beta \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right) + (1-\beta) \sum_{j=1}^{n} v_j \left(\frac{n-j}{n-1}\right) = \alpha(\hat{V}), 0 \leq \alpha(\hat{V}) \leq 1, \tag{62}$$

$$\sum_{j=1}^{n} \hat{v}_j = 1, 0 \leq \hat{v}_j \leq 1, \ j = 1, 2, \ldots, n.$$

Note that if $\beta = 1$, we get the usual maximal entropy OWA (MEOWA) method (Yager, 2009) and if $\beta = 0$, as is the case for the WA, we get the maximal entropy WA (MEWA) method.

**Remark 11.** We have focused on situations where the weighting vector of the OWA and the WA satisfy the same conditions of a particular family. However, it is also possible to consider situations where only the OWA or the WA satisfies these conditions. Furthermore, we may find that the OWA is a member of a different family than the WA. For example, the OWA can use a centered aggregation while the WA uses an olympic one. Thus, we are using in the OWAWA a centered-OWA and an olympic-WA (centered-OWA-olympic-WA). In Table 1, we briefly present families that use different types of weighting vectors in OWA and in WA.

**Remark 12.** Other families of OWAWA operators could be used, following the recent literature concerning different methods for obtaining the OWA

Table 1. Mixing families of OWA and WA operators

| | | OWA | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | OWA | Max | Min | AM | Step | Olympic | Centered | ME |
| WA | WA | OWAWA | Max-WA | Min-WA | A-WA | Step-WA | Oly-WA | Cent-WA | ME-WA |
| | Max | OWA-Max | Max | Min-Max | A-Max | Step-Max | Oly-Max | Cent-Max | ME-Max |
| | Min | OWA-Min | Max-Min | Min | A-Min | Step-Min | Oly-Min | Cent-Min | ME-Min |
| | AM | OWA-AM | Max-AM | Min-AM | AM | Step-AM | Oly-AM | Cent-AM | ME-AM |
| | Step | OWA-Step | Max-Step | Min-Step | A-Step | Step | Oly-Step | Cent-Step | ME-Step |
| | Oly. | OWA-Oly. | Max-Oly. | Min-Oly. | A-Oly. | Step-Oly. | Olympic | Cent-Oly. | ME-Oly. |
| | Cent. | OWA-Cent. | Max-Cent. | Min-Cent. | A-Cent. | Step-Cent. | Oly-Cent. | Centered | ME-Cent. |
| | ME | OWA-ME | Max-ME | Min-ME | A-ME | Step-ME | Oly-ME | Cent-ME | ME |
| | Etc | | | | | | | | |

weights (Merigó and Casanovas, 2009; Merigó and Gil-Lafuente, 2009; Merigó et al. 2011; Merigó and Wei, 2011; Yager, 1993; 1996a; Yager and Kacprzyk, 1997).

## 6. Construction of interval numbers and related structures with OWAWA operators

In this Section we describe how to construct interval numbers and other related structures such as fuzzy numbers by using OWAWA operators. First, we consider the use of the classical OWA operator. Next, we analyze the use of the OWAWA operator. Finally, we discuss several implications of using this methodology.

### 6.1. Construction of interval numbers with OWAWA operators

The OWA operator provides a parameterized family of aggregation operators between the minimum and the maximum. In order to represent the information in a more complete way we can construct an interval number when aggregating the information with OWA operators. Depending on our interests this construction will only consider the simplest interval considering only the minimum and the maximum, or more complex structures by using triplets, quadruplets and so on. In the following, we present the methodology to use in the construction of interval numbers with OWA operators.

Assume a set of arguments $A = (a_1, a_2, \ldots, a_n)$. For the construction of a 2-tuple interval number (Moore, 1966), we simply aggregate the information of the OWA operator in the following way: $C = [\text{Min}\{a_i\}, \text{Max}\{a_i\}]$. Thus, we are considering an interval number that considers the lowest and the highest result of the set of arguments $A$. Note that in this construction process we aggregate the information from $n$ arguments $a_i$ into an interval number. That is, $OWA$: $R^n \to \Omega$, where $\Omega$ is the set of interval numbers.

Another type of interval number that we could construct is a triplet. In this case, we can use the minimum, the maximum and the OWA aggregation that is more in accordance with the interests of the decision maker. In this case, we get: $C = [\text{Min}\{a_i\}, OWA, \text{Max}\{a_i\}]$. And so on.

By using OWAWA operators, it is also possible to construct interval and fuzzy numbers. The main advantage of using OWAWA operators is that we are considering subjective information and the degree of orness of the decision maker in the same formulation. Thus, the OWAWA operator leads to a new type of interval and fuzzy numbers: the subjective interval number (SIN) and the subjective fuzzy number (SFN). Their main advantage is that they include the degree of importance of each argument by using the weighted average in the aggregation. Note that by using the OWAWA operator we can consider the relevance of the OWA and the WA in the aggregation process.

In the construction of subjective interval numbers with the OWAWA operator, it is worth noting the construction of subjective triplets, quadruplets, quintuplets and sextuplets. The subjective triplet and the subjective quadruplet follow the same methodology as OWA. That is: $C$ = [Min$\{a_i\}$, *OWAWA*, Max$\{a_i\}$] and $C$ = [Min$\{a_i\}$, $OWAWA_*$, $OWAWA^*$, Max$\{a_i\}$]. Note that if $\beta =$ 1, the OWAWA becomes the OWA operator and thus we get the same results as in Section 6.1. If $\beta = 0$, the OWAWA becomes the WA and thus we can construct an interval number where we do not use the OWA but we consider a subjective importance of the arguments. That is: $C$ = [Min$\{a_i\}$, WA, Max$\{a_i\}$] and $C$ = [Min$\{a_i\}$, $WA_*$, $WA^*$, Max$\{a_i\}$]. Note that in Section 6.1. we assume that we do not know the subjective importance of the arguments.

As it has been explained in Theorem 4, with the OWAWA operator we obtain semi boundary conditions when we use the WA with the OWA bounds. Thus, we can reduce the bounds considering the information given by the WA. Note that this reduction is artificial according to the information obtained from the WA but the result can always move from the minimum to the maximum. However, sometimes the usual bounds are too broad and we need to reduce them in order to reduce the uncertainty and be able to make better decisions with the available information. In these situations, it becomes useful to consider subjective quintuplets and sextuplets in the analysis. Thus, we get: $C$ = [Min$\{a_i\}$, Min–WA, OWAWA, Max–WA, Max$\{a_i\}$] and $C = [Min\{a_i\}$, *Min–WA*, $OWAWA_*$, $OWAWA^*$, *Max–WA*, Max$\{a_i\}$]; where *Min–WA* is the convex combination $\beta \times Min\{a_i\} + (1-\beta) \times WA$, and *Max–WA* is $\beta \times Max\{a_i\} + (1-\beta) \times WA$.

**Example 5.** Assume the following arguments $A$ = (20, 40, 10, 70, 80). When using one weighting vector (for triplets), we assume $W$ = (0.1, 0.2, 0.2, 0.2, 0.3). When using quadruplets, we assume $W_*$ = (0.1, 0.1, 0.2, 0.3, 0.3), $W^*$ = (0.15, 0.2, 0.2, 0.2, 0.25) and the following weighting vector for WA: $V$ = (0.3, 0.3, 0.2, 0.1, 0.1). In this example we assume that OWA has degree of importance of 40% and WA of 60%. Thus, we can construct the following subjective interval numbers (SIN):

- Subjective triplet: $C$ = [Min$\{a_i\}$, *OWAWA*, Max$\{a_i\}$] = [10, 37.8, 80].
  - Note that for *OWAWA* we use: $OWAWA = 0.4 \times (0.1 \times 80 + 0.2 \times 70 + 0.2 \times 40 + 0.2 \times 30 + 0.3 \times 20) + 0.6 \times (0.3 \times 20 + 0.3 \times 40 + 0.2 \times 10 + 0.1 \times 70 + 0.1 \times 80) = 37.8$.
- Subjective quadruplet: $C$ = [Min$\{a_i\}$, $OWAWA_*$, $OWAWA^*$, Max$\{a_i\}$] = [10, 36.2, 39, 80].
  - Note that for $OWAWA_*$ we use: $OWA_* = 0.4 \times (0.1 \times 80 + 0.1 \times 70 + 0.2 \times 40 + 0.3 \times 30 + 0.3 \times 20) + 0.6 \times (0.3 \times 20 + 0.3 \times 40 + 0.2 \times 10 + 0.1 \times 70 + 0.1 \times 80) = 36.2$.
  - For $OWAWA^*$ we use: $OWA^* = 0.4 \times (0.15 \times 80 + 0.2 \times 70 + 0.2 \times 40 + 0.2 \times 30 + 0.25 \times 20) + 0.6 \times (0.3 \times 20 + 0.3 \times 40 + 0.2 \times 10 + 0.1 \times 70 + 0.1 \times 80) = 39$.
- Subjective quintuplet: $C$ = [Min$\{a_i\}$, *Min–WA*, *OWAWA*, *Max–WA*, Max$\{a_i\}$] = [10, (25, 37.8, 53), 80].

  - Note that for Min-WA we use: $0.4 \times 10 + 0.6 \times (0.3 \times 20 + 0.3 \times 40 + 0.2 \times 10 + 0.1 \times 70 + 0.1 \times 80) = 25$.
  - For Max-WA we use: $0.4 \times 80 + 0.6 \times (0.3 \times 20 + 0.3 \times 40 + 0.2 \times 10 + 0.1 \times 70 + 0.1 \times 80) = 53$.
- Subjective sextuplet: $C$ = [Min$\{a_i\}$, *Min–WA*, $OWAWA_*$, $OWAWA^*$, *Max–WA*, Max$\{a_i\}$] = [10, (25, 36.2, 39, 53), 80].

Note that with the OWAWA operator we can also construct subjective FNs (SFNs). For subjective triplets and quadruplets, we follow the same procedure as with the OWA operator. Thus, for example, we can assume that the internal values can be represented with linear functions that connect the points of the interval obtaining the $\alpha$-cut representation of the subjective TFN (STFN) and the subjective TpFN (STpFN) as follows: $C_\alpha = [Minfa_ig + (OWAWA - Minfa_ig) \times \alpha, Maxfa_ig - (Maxfa_ig - OWAWA) \times \alpha]$ and $C_\alpha = [Minfa_ig + (OWAWA_* - Minfa_ig) \times \alpha, Maxfa_ig - (Maxfa_ig - -OWAWA^*) \times \alpha]$.

By using subjective quintuplets and sextuplets, it is also possible to construct SFNs. However, dealing with them is more complex because we have to introduce several linear functions and there are many ways for doing so. For example, when using quintuplets we can construct an $\alpha$-cut representation from the minimum and the maximum to the OWAWA and from the Min-WA and the Max-WA to the OWAWA forming an interval-valued SFN (IVSFN). Thus, we get: $C_\alpha = [Minfa_ig + (OWAWA - -Minfa_ig) \times \alpha, Min - WA + (OWAWA - -Min - WA) \times \alpha, Max - WA - -(Max - WA - -OWAWA) \times \alpha, Maxfa_ig - -(Maxfa_ig - -OWAWA) \times \alpha]$. With a sextuplet, we get the following representation: $C_\alpha = [Minfa_ig + (OWAWA_* - -Minfa_ig) \times \alpha, Min - WA + (OWAWA_* - -Min - WA) \times \alpha, Max - WA - -(Max - WA - -OWAWA^*) \times \alpha, Maxfa_ig - -(Maxfa_ig - -OWAWA^*) \times \alpha]$. Note that the interval-valued SFN formed with the quintuplet can be represented graphically as shown in Fig 1.

Following Example 5, we could form the STFN: $[10 + 27.8\alpha, 80 - 42.2\alpha]$; and the STpFN: $[10 + 26.2\alpha, 80 - 41\alpha]$. By using quintuplets and sextuplets we could form the IVSFNs: $[10 + 27.8\alpha, 25 + 12.8\alpha, 53 - 15.2\alpha, 80 - 42.2\alpha]$ and $[10 + 26.2\alpha, 25 + 11.2\alpha, 53 - 14\alpha, 80 - 41\alpha]$.

Finally, note that more complex structures obtained by using a wide range of families of OWAWA operators could be used for constructing interval and fuzzy numbers. Those presented here represent an overview of the most basic ones.

### 6.2. Analyzing different perspectives in the construction of interval numbers

In previous subsections, we have seen that we can construct interval numbers by using OWA and OWAWA operators. Thus, by aggregating a set of arguments we can simply obtain a single representative number or a more complete representation that includes this representative number and several other results, including the bounds of this set of arguments.

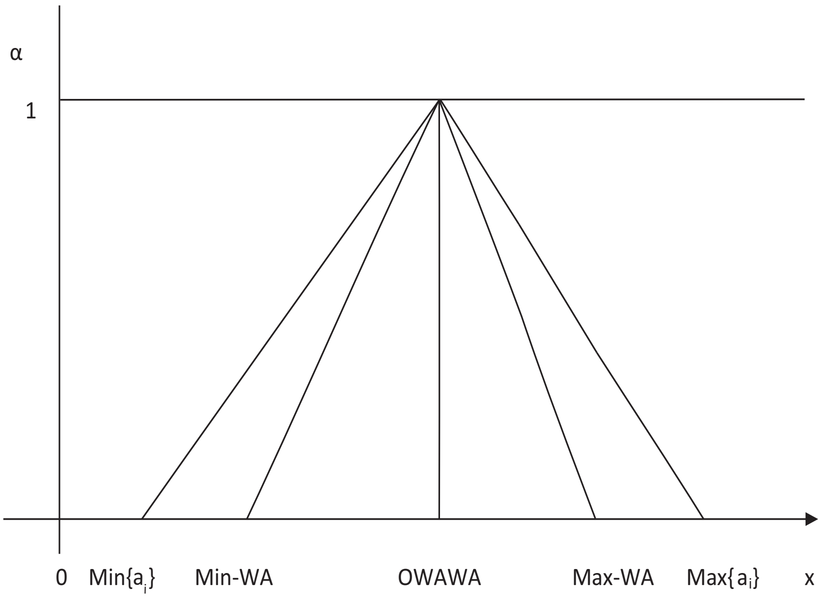


Figure 1. Interval-valued fuzzy number constructed with the OWAWA operator

Another interesting issue to consider in this analysis is the meaning of the set of arguments aggregated because this may lead to different meanings (or interpretations) of the interval numbers. Among others, we can aggregate a set of arguments that represent the results obtained according to the states of nature that may occur in the future, the different alternatives that we can select, the different criteria that we can consider, the opinion of a group of experts (or persons) and so on.

For example, in decision-making problems it is very interesting to consider a set of arguments $a$ that depend on a set of states of nature $S$ and a set of alternatives $A$. This information can be represented in the matrix shown in Table 2.

Abbreviations:
$T_1 = [Minfa_{1i}g, OWA, Maxfa_{1i}g]$;
$T_h = [Minfa_{hi}g, OWA, Maxfa_{hi}g]$;
$T_k = [Minfa_{ki}g, OWA, Maxfa_{ki}g]$;
$U_1 = [Minfa_{1i}g, \textit{Min-WA}, OWAWA, \textit{Max-WA}, Maxfa_{1i}g]$;
$U_h = [Minfa_{hi}g, \textit{Min-WA}, OWAWA, \textit{Max-WA}, Maxfa_{hi}g]$;
$U_k = [Minfa_{ki}g, \textit{Min-WA}, OWAWA, \textit{Max-WA}, Maxfa_{ki}g]$;
$X_1 = [Minfa_{h1}g, OWA, Maxfa_{h1}g]$;
$X_i = [Minfa_{h1}g, OWA, Maxfa_{h1}g]$;
$X_n = [Minfa_{h1}g, OWA, Maxfa_{h1}g]$;

Table 2. Matrix with states of nature and alternatives

| | $S_1$ | | $S_i$ | | $S_n$ | OWA | OWAWA |
|---|---|---|---|---|---|---|---|
| $A_1$ | $a_{11}$ | ... | $a_{1i}$ | ... | $a_{1n}$ | $T_1$ | $U_1$ |
| ... | ... | ... | ... | ... | ... | ... | ... |
| $A_h$ | $a_{h1}$ | ... | $a_{hi}$ | ... | $a_{hn}$ | $T_h$ | $U_h$ |
| ... | ... | ... | ... | ... | ... | ... | ... |
| $A_k$ | $a_{k1}$ | ... | $a_{ki}$ | ... | $a_{kn}$ | $T_k$ | $U_k$ |
| OWA | $X_1$ | ... | $X_i$ | ... | $X_n$ | | |
| OWAWA | $Y_1$ | ... | $Y_i$ | ... | $Y_n$ | | |

$Y_1 = [Minfa_{h1}g, \textit{Min-WA}, OWA, \textit{Max-WA}, Maxfa_{h1}g]$;
$Y_i = [Minfa_{h1}g, \textit{Min-WA}, OWA, \textit{Max-WA}, Maxfa_{h1}g]$;
$Y_n = [Minfa_{h1}g, \textit{Min-WA}, OWA, \textit{Max-WA}, Maxfa_{h1}g]$;

In order to establish a ranking of the interval numbers (or FNs) we have to establish a method for doing this. In this paper, we assume that the interval can be reduced to the OWA or OWAWA aggregation used.

As we can see, we can aggregate the arguments according to an alternative selected. In this case, we are indicating the potential results that we can get depending on the state of nature that occur in the future if we select a specific alternative. Thus, we are analyzing actions and see their potential results.

Another choice is to aggregate the arguments according to a state of nature. In this situation, we are analyzing the potential results that we can get for each state of nature according to the alternatives that we have. Thus, we are analyzing scenarios and see their potential results.

Moreover, we should note that it is possible to mix both type of aggregations in the process, considering at the same time scenarios and alternatives.

Furthermore, it is possible to include the opinion of several persons in the analysis. This case is usually considered as preceding step which results are aggregated into a collective result that represents the aggregated opinion. However, it is worth noting that in this aggregation process we can also consider the construction of interval numbers as a more complete representation of the aggregation process.

## 7. Applicability of the OWAWA operator

In this section, we study the applicability of the OWAWA operator. First, we give a general overview of potential applications. Second, we present some basic examples on how to apply the OWAWA operator to different theoretical problems.

### 7.1. Introduction

The OWAWA operator can be applied in an astonishingly wide range of applications. Any study that uses either the OWA or the WA can be revised and extended to use the OWAWA operator. However, we believe that in the future there will be a need to produce various degrees of underestimated and overestimated results. Thus, the use of OWAs in order to under- or over-estimate the results given by the WA will become very common. Using the model presented in this paper, we can vary the degree of importance of these different estimates depending on the problem at hand. Note that it is also possible to modify the OWA results by using the WA. In the following, we mention some of the main research application areas. Within each field, there are many potential applications.

- Statistics: The OWAWA is a key instrument to revise the majority of the statistical sciences. For example, we can extend variance, covariance, Pearson coefficient and correlation coefficient by using this new approach. We can also implement it in linear and multiple regressions as well as to probability theory and a lot of other related areas such as hypothesis testing and inference statistics.
- Fuzzy Set Theory: All aspects of fuzzy set theory that use techniques based on the WA or the OWA can be revised and extended with the OWAWA operator. For example, in methods of ranking fuzzy numbers, instead of a WA, we can use an OWAWA operator. Moreover, the whole theory of aggregation operators is strongly affected by this new approach.
- Soft Computing: A lot of new applications can be developed in neural network theory, in evolutionary computation and in chaotic computing.
- Business Administration: The OWAWA can be implement it in strategic management, financial management, accounting, marketing and human resource management. Application may especially concern business decision-making problems.
- Economics: We can use the idea for developing more complete economic theories, for example, when calculating the aggregate demand or supply in macroeconomic or microeconomic theory. On the other hand, we can also use it in a wide range of economic problems concerning decisions or analysis, e. g. when dealing with the public sector or when making economic predictions.
- Politics: For example, when making political decisions concerning national decisions. As it will be explained in the example, sometimes this area is closely related to economics.
- Decision Theory: Decision theory is critically sustained on the use of a wide range of aggregations operators. Therefore, the use of the OWAWA can imply a lot of new improvements in the current models.
- Operational Research: We can implement it in several situations regarding assignment and grouping problems.
- Biology: All the theories and techniques concerning biostatistics can be

revised with the OWAWA operator.

So, any current or future research that uses either the OWA or the WA can be revised and extended by using this new approach.

### 7.2. Some theoretical examples in statistics

Next, we present a method to apply the OWAWA operator to theories already published in the literature. In the case of statistics (McClave and Sincich, 2003), we can start revising the average and the variance of a population (discrete case) using the OWAWA operator (Eq. (9)). For the variance, we obtain the following formulation:

$$Var - OWAWA(X) = \sum_{j=1}^{n} \hat{v}_j D_j, \tag{63}$$

where $D_j$ is the $j$th largest of the $(x_i - \mu)^2$, $x_i$ is the argument variable, $\mu$ is the average (in this case, the OWAWA operator), $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, each argument $(x_i - \mu)^2$ has an associated weight (WA) $v_i$ with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1 - \beta)v_j$ with $\beta \in [0, 1]$ and $v_j$ is the weight (WA) $v_i$ ordered according to $D_j$, that is, according to the $j$th largest of the $(x_i - \mu)^2$.

Note that the use of the OWA operator in variance has been studied by Yager (1996b; 2006). Obviously, once we have variance, it is straightforward to obtain standard deviation (S.D.) with the OWAWA operator,

$$S.D. = \sqrt{\sum_{j=1}^{n} \hat{v}_j D_j}. \tag{64}$$

In a similar way, we can represent covariance by using the OWAWA operator as follows:

$$Cov - OWAWA(X, Y) = \sum_{j=1}^{n} \hat{v}_j K_j, \tag{65}$$

where $K_j$ is the $j$th largest of the $(x_i - \mu)(y_i - \nu)$, $x_i$ is the argument variable of the first set of elements $X = \{x_1, \ldots, x_n\}$ and $y_i$ the argument variable of the second set of elements $Y = \{y_1, \ldots, y_n\}$, $\mu$ and $\nu$ are the averages (in this case, the OWAWA operator) of the sets $X$ and $Y$ respectively, $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, each argument $(x_i - \mu)(y_i - \nu)$ has an associated weight (WA) $v_i$ with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1 - \beta)v_j$ with $\beta \in [0, 1]$ and $v_j$ is the weight (WA) $v_i$ ordered according to $K_j$, that is, according to the $j$th largest of the $(x_i - \mu)(y_i - \nu)$.

With this formulation, we can analyze the OWAWA covariance matrix (McClane and Sincich, 2003), or measures of correlation, such as the Pearson coefficient (P.C.). The Pearson coefficient with the OWAWA (PC – OWAWA) is formulated as follows:

$$PC - OWAWA = \frac{Cov - OWAWA(X,Y)}{\sqrt{Var - OWAWA(X) \times Var - OWAWA(Y)}}. \quad (66)$$

The PC – OWAWA is 1 in the case of an increasing linear relationship and −1 in the case of a decreasing linear relationship. If the variables $X$ and $Y$ are independent, then the PC – OWAWA is 0.

Furthermore, we can formulate linear regression using the OWAWA operator. To construct the linear regression model $y_h = \alpha + \beta x_h$, we calculate $\beta$ as follows:

$$\hat{\beta}_{OWAWA} = \frac{Cov - OWAWA(X,Y)}{Var - OWAWA(X)}. \quad (67)$$

Next, we calculate the $\hat{\alpha}_{OWAWA}$ value as follows: $\hat{\alpha}_{OWAWA} = \bar{y}_{OWAWA} - \beta\bar{x}_{OWAWA}$, where $\bar{x}_{OWAWA}$ and $\bar{y}_{OWAWA}$ are the average of the sets $X$ and $Y$ calculated by using an OWAWA operator. Once we have $\hat{\alpha}_{OWAWA}$and $\hat{\beta}_{OWAWA}$, we can construct the linear regression model with the OWAWA operator as follows:

$$y_h = \hat{\alpha}_{OWAWA} + \hat{\beta}_{OWAWA} x_h. \quad (68)$$

Note that by using the OWAWA operator, we can revise these approaches and also construct interval and fuzzy variants of these models by using the methodology from Section 6. Other existing methods can be revised and extended using the OWAWA operator in a similar fashion. We can apply the OWAWA to many other problems in descriptive statistics, inferential statistics, hypothesis testing, correlation and regression, as well as in other scientific areas.

## 8. Multi-person decision-making with the OWAWA operator

In this paper, we consider a decision-making application in the selection of financial strategies by an enterprise, using multi-person analysis. Multi-person analysis provides a more complete representation of the problem because it is based on the opinions of several people. Therefore, we can aggregate the opinion of different people to obtain a representative view of the problem. This is very useful because usually decisions are not individual, but are made by a group of people in the company forming the board of directors.

The procedure to select strategies with the OWAWA operator in multi-person decision-making is described in this section. Many other decision-making models have been discussed in the literature (Chen and Zhou, 2011; Han and Liu, 2011; Jin and Liu, 2010; Tan and Chen, 2010; Wei, 2011a; 2011b; Xu, 2010).

*Step* 1: Let $A = (a_1, a_2, \ldots, a_n)$ be a set of alternatives, $S = (s_1, s_2, \ldots, s_n)$, a set of states of nature (or attributes), forming the payoff matrix $(a_{hi})_{m\times n}$. Let $E = (e_1, e_2, \ldots, e_p)$ be a finite set of decision-makers. Let $U = (u_1, u_2, \ldots, u_p)$

be the weighting vector of the decision-makers such that $\sum_{k=1}^{p} u_k = 1$ and $u_k \in [0,1]$. Each decision-maker provides his own payoff matrix $(a_{hi}^{(k)})_{m\times n}$.

*Step* 2: Calculate the weighting vector $\hat{V} = \beta \times W + (1-\beta) \times V$ to be used in the OWAWA aggregation. Note that $W = (w_1, w_2, \ldots, w_n)$ is such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$ and $V = (v_1, v_2, \ldots, v_p)$ is such that $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$.

*Step* 3: Use the WA to aggregate the information of the decision-makers $E$ using the weighting vector $U$. The result is the collective payoff matrix $(\tilde{a}_{hi})_{m\times n}$. Thus, $a_{hi} = \sum_{k=1}^{p} u_k a_{hi}^k$. It is possible to use other types of OWAWA operators instead of the WA to aggregate this information.

*Step* 4: Calculate the aggregated results using the OWAWA operator explained in Eq. (9). Consider different families of OWAWA operators as described in Section 5.

*Step* 5: Adopt decisions according to the results found in the previous steps. Select the alternative(s) that provides the best result(s). Moreover, establish an ordering of alternatives from the most- to the least-preferred alternative, enabling consideration of more than one selection.

This aggregation process can be summarized using the following aggregation operator that we call the multi–person – OWAWA (MP–OWAWA) operator.

Definition 8.1 *An MP-OWAWA operator is a mapping MP-OWAWA* $: R^n \times R^p \to R$ *that has a weighting vector* $U$ *of dimension* $p$ *with* $\sum_{k=1}^{p} u_p = 1$ *and* $u_k \in [0,1]$ *and a weighting vector* $W$ *of dimension* $n$ *with* $\sum_{j=1}^{n} w_j = 1$ *and* $w_j \in [0,1]$*, such that:*

$$MP - OWAWA((a_1^1, ..., a_1^p), ..., (a_n^1, ..., a_n^p)) = \sum_{j=1}^{n} \hat{v}_j b_j, \tag{69}$$

*where* $b_j$ *is the* $j$*th largest of the* $a_i$*, each argument* $a_i$ *has an associated weight (WA)* $v_i$ *with* $\sum_{i=1}^{n} v_i = 1$ *and* $v_i \in [0, 1]$*,* $\hat{v}_j = \beta w_j + (1-\beta) v_j$ *with* $\beta \in [0, 1]$ *and* $v_j$ *is the weight (WA)* $v_i$ *ordered according to* $b_j$*, that is, according to the* $j$*th largest of the* $a_i$*,* $a_i = \sum_{k=1}^{p} u_k a_i^k$*,* $a_i^k$ *is the argument variable provided by each person.*

Note that the MP-OWAWA operator has similar properties to those explained in Section 3, such as the distinction between descending and ascending orders, and so on.

The MP-OWAWA operator includes a wide range of particular cases following the methodology explained in Section 5. Thus, it includes the multi-person – WA (MP-WA) operator, the multi-person – OWA (MP-OWA) operator, the multi-person – arithmetic mean (MP-AM) operator, the multi-person – arithmetic-WA (MP-AWA) operator and the multi-person – arithmetic-OWA (MP-AOWA) operator.

It is possible to consider more complex situations by using different types of aggregation operators to aggregate the expert opinions, though in Definition 10 we assume that the expert opinions were aggregated by using WA operators.

## 9. Illustrative example

In the following, we present a numerical example of the new approach in a decision-making problem regarding the selection of strategies. We analyze a business problem regarding the selection of the optimal financial strategy in an enterprise.

*Step* 1: Assume that a company that operates in Spain has to decide on the financial strategy to use next year. They consider seven alternatives:

- $A_1$ = Invest in the French market.
- $A_2$ = Invest in the Italian market.
- $A_3$ = Invest in the German market.
- $A_4$ = Invest in the British market.
- $A_5$ = Invest in the Polish market.
- $A_6$ = Invest in the Romanian market.
- $A_7$ = Do not make any investment.

In order to evaluate these strategies, the enterprise has brought together a group of experts. This group considers that the key factor is the economic situation of the world economy for the next period. They consider eight possible states of nature that could happen in the future:

- $S_1$ = Extremely bad economic situation.
- $S_2$ = Very bad economic situation.
- $S_3$ = Bad economic situation.
- $S_4$ = Regular economic situation.
- $S_5$ = Regular – Good economic situation.
- $S_6$ = Good economic situation.
- $S_7$ = Very good economic situation.
- $S_8$ = Extremely good economic situation.

The experts are classified in three groups. Each group is led by one expert and gives different opinions than the other two groups. The results of the available strategies, depending on the state of nature $S_i$ and the alternative $A_k$ that the decision-maker chooses, are shown in Tables 3, 4 and 5.

Table 3. Payoff matrix – Expert 1 (group 1)

| | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | $S_6$ | $S_7$ | $S_8$ |
|---|---|---|---|---|---|---|---|---|
| $A_1$ | 60 | 70 | 80 | 40 | 30 | 50 | 60 | 90 |
| $A_2$ | 80 | 40 | 50 | 70 | 50 | 60 | 30 | 90 |
| $A_3$ | 90 | 90 | 90 | 20 | 20 | 60 | 60 | 60 |
| $A_4$ | 40 | 40 | 40 | 60 | 60 | 60 | 90 | 90 |
| $A_5$ | 10 | 10 | 30 | 50 | 70 | 90 | 90 | 90 |
| $A_6$ | 30 | 40 | 50 | 50 | 60 | 60 | 80 | 90 |
| $A_7$ | 40 | 60 | 70 | 30 | 50 | 60 | 80 | 90 |

Table 4. Payoff matrix – Expert 2 (group 2)

| | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | $S_6$ | $S_7$ | $S_8$ |
|---|---|---|---|---|---|---|---|---|
| $A_1$ | 60 | 60 | 60 | 60 | 30 | 60 | 60 | 90 |
| $A_2$ | 70 | 30 | 50 | 70 | 40 | 60 | 50 | 90 |
| $A_3$ | 70 | 90 | 90 | 20 | 40 | 50 | 60 | 70 |
| $A_4$ | 40 | 40 | 40 | 60 | 60 | 60 | 90 | 90 |
| $A_5$ | 10 | 10 | 40 | 60 | 70 | 90 | 90 | 90 |
| $A_6$ | 40 | 50 | 50 | 50 | 60 | 70 | 80 | 80 |
| $A_7$ | 50 | 50 | 60 | 50 | 50 | 60 | 80 | 80 |

*Step* 2: In this problem, we assume the following weighting vector for the three group of experts: $U = (0.4, 0.3, 0.3)$. The experts assume the following weighting vector for the OWA: $W = (0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.2, 0.2)$. They assume the WA for each state of nature is: $V = (0.2, 0.2, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1)$. First, we aggregate the information of the three groups into one collective matrix that represents the information of all the experts of the problem. The results are shown in Table 6.

Table 5. Payoff matrix – Expert 3 (group 3)

| | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | $S_6$ | $S_7$ | $S_8$ |
|---|---|---|---|---|---|---|---|---|
| $A_1$ | 60 | 60 | 60 | 60 | 40 | 60 | 60 | 80 |
| $A_2$ | 50 | 40 | 50 | 70 | 40 | 50 | 60 | 90 |
| $A_3$ | 80 | 90 | 90 | 20 | 50 | 50 | 60 | 60 |
| $A_4$ | 10 | 10 | 40 | 50 | 50 | 80 | 90 | 90 |
| $A_5$ | 20 | 30 | 50 | 50 | 80 | 80 | 80 | 80 |
| $A_6$ | 40 | 40 | 50 | 50 | 70 | 70 | 80 | 90 |
| $A_7$ | 40 | 50 | 60 | 50 | 80 | 60 | 80 | 70 |

Table 6. Collective payoff matrix

| | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | $S_6$ | $S_7$ | $S_8$ |
|---|---|---|---|---|---|---|---|---|
| $A_1$ | 60 | 64 | 68 | 52 | 33 | 56 | 60 | 87 |
| $A_2$ | 68 | 37 | 50 | 70 | 44 | 57 | 45 | 90 |
| $A_3$ | 81 | 90 | 90 | 20 | 35 | 54 | 60 | 63 |
| $A_4$ | 31 | 31 | 40 | 57 | 57 | 66 | 90 | 90 |
| $A_5$ | 13 | 16 | 39 | 53 | 73 | 87 | 87 | 87 |
| $A_6$ | 36 | 43 | 50 | 50 | 63 | 66 | 80 | 87 |
| $A_7$ | 43 | 54 | 64 | 42 | 59 | 60 | 80 | 81 |

*Step* 3: Next, we calculate the attitudinal weights by mixing the weighting vectors $W$ and $V$. Note that the OWA operator has importance of 40% while WA has importance of 60% in this particular example. The results are shown in Table 7.

Table 7. Attitudinal weights

| | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | $S_6$ | $S_7$ | $S_8$ |
|---|---|---|---|---|---|---|---|---|
| $A_1$ | 0.1 | 0.1 | 0.16 | 0.16 | 0.1 | 0.1 | 0.14 | 0.14 |
| $A_2$ | 0.1 | 0.1 | 0.16 | 0.1 | 0.1 | 0.1 | 0.14 | 0.2 |
| $A_3$ | 0.16 | 0.1 | 0.16 | 0.1 | 0.1 | 0.1 | 0.14 | 0.14 |
| $A_4$ | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.2 | 0.2 |
| $A_5$ | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.2 | 0.2 |
| $A_6$ | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.2 | 0.2 |
| $A_7$ | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.16 | 0.2 | 0.14 |

*Step* 4: With this information, we can aggregate the expected results for each state of nature in order to make a decision. For this, we use Eq. (10) to calculate the OWAWA aggregation. In Table 8, we present the results obtained using different types of OWAWA operators.

*Step* 5: If we establish an ordering of the alternatives, as we typically would if we want to consider more than one alternative, then we get the results shown in Table 9. Note that the first alternative in each ordering is the optimal choice.

Evidently, the order preference for the financial strategies may be different, depending on the aggregation operator used. Therefore, the decisions as to which strategy to select may be also different. However, in this example it is clear that $A_3$ or $A_7$ should be the optimal choice. Note that the main advantage of the OWAWA operator is that it can provide different results regarding uncertainty according to the particular interests of the decision maker in the specific problem considered.

Table 8. Aggregated results

| | Max-WA | Min-WA | AM | WA | OWA | AM-WA | AM-OWA | OWAWA |
|---|---|---|---|---|---|---|---|---|
| $A_1$ | 71.04 | 49.44 | 60 | 60.4 | 56.5 | 60.24 | 58.6 | 58.84 |
| $A_2$ | 69.96 | 48.76 | 57.625 | 56.6 | 54.2 | 57.01 | 56.255 | 55.64 |
| $A_3$ | 75.84 | 47.84 | 61.625 | 66.4 | 54.8 | 64.49 | 58.895 | 61.76 |
| $A_4$ | 67.44 | 43.84 | 57.75 | 52.4 | 52.4 | 54.54 | 55.61 | 52.4 |
| $A_5$ | 63.84 | 34.24 | 56.875 | 48.4 | 48.4 | 51.79 | 53.485 | 48.4 |
| $A_6$ | 68.04 | 47.64 | 59.375 | 55.4 | 55.4 | 56.99 | 57.785 | 55.4 |
| $A_7$ | 67.2 | 51.6 | 60.375 | 58 | 56.8 | 58.95 | 58.945 | 57.52 |

Table 9. Ordering of the financial strategies

| | Ordering | | Ordering |
|---|---|---|---|
| Max-WA | $A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_7 \succ A_5$ | OWA | $A_7 \succ A_1 \succ A_6 \succ A_3 \succ A_2 \succ A_4 \succ A_5$ |
| Min-WA | $A_7 \succ A_1 \succ A_2 \succ ]A_3 \succ A_6 \succ A_4 \succ A_5$ | AM-WA | $A_3 \succ A_1 \succ A_7 \succ A_2 \succ A_6 \succ A_4 \succ A_5$ |
| AM | $A_3 \succ A_7 \succ A_1 \succ A_6 \succ A_4 \succ A_2 \succ A_5$ | AM-OWA | $A_7 \succ A_3 \succ A_1 \succ A_6 \succ A_2 \succ A_4 \succ A_5$ |
| WA | $A_3 \succ A_1 \succ A_7 \succ A_2 \succ A_6 \succ A_4 \succ A_5$ | OWAWA | $A_3 \succ A_1 \succ A_7 \succ A_2 \succ A_6 \succ A_4 \succ A_5$ |

## 10. Conclusions

We have developed a new aggregation operator that unifies the WA with the OWA operator. We have called it OWAWA operator. The main advantage of this unified formulation is that the WA and the OWA can be used simultaneously, with varying degrees of relative importance based on the specific application. We have studied some of the main properties of the OWAWA by analyzing its fundamental aspects such as the reordering process, the use of mixture operators, the use of infinitary aggregation operators and the analysis of new semi-boundary conditions.

We have analyzed different measures to characterize the weighting vector. We have introduced a new or-ness measure that analyzes the or-ness of an aggregation that uses both the OWA and the WA. We have presented a new entropy measure that unifies Yager entropy with Shannon entropy. We have seen that this new measure can be extended to the entirety of information theory that uses the Shannon entropy. We have also developed a generalization of the divergence measure and the balance operator applicable for OWAWA operators.

Furthermore, we have studied several families of OWAWA operators. We have seen that the OWA and the WA are particular cases of this approach and obtained a wide range of new interesting aggregation operators such as the Max-WA, the Min-WA, the arithmetic-WA and the arithmetic-OWA operator. We have also studied other families, such as the centered-OWAWA, the olympic-OWAWA, the S-OWAWA and the MEOWAWA operator.

Likewise, construction of interval numbers and FNs by using OWA and OWAWA operators was considered. We have seen that the OWA aggregation can aggregate the information creating an interval or a FN that gives a representative knowledge of the available information. By using OWAWA operators we have found the subjective interval numbers and the subjective FNs. We have seen that they permit to consider the subjective importance of the arguments in aggregation. We have also developed different contexts where this methodology could be used giving special attention to situations with a set of arguments that depends on a set of alternatives and a set of states of nature.

The applicability of the OWAWA operator is quite broad because any study that uses the WA or the OWA, can be extended using the OWAWA operator. We have focused on a multi-person decision-making problem related to selection of financial strategies. In this case, we have found a very general and interesting aggregation operator: the MP-OWAWA operator. We have seen the usefulness of using the OWAWA operator because we are able to consider WAs and OWAs at the same time, which means that we can use subjective probabilities (or degrees of importance) and the attitudinal character of the decision-maker.

In future research, we expect to develop further extensions to this approach by adding new characteristics in the problem, such as the use of order-inducing variables, uncertain information (interval numbers, fuzzy numbers, linguistic variables, etc.), generalized and quasi-arithmetic means and distance measures. We will also extend this approach to situations where we use the probability instead of the WA. We will also consider different applications, giving special attention to statistics and business decision-making problems, such as political and product management.

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