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A new procedure for the design of iterative learning controllers using a 2D systems formulation of processes with uncertain spatio-temporal dynamics*†

by

Błażej Cichy[1], Krzysztof Gałkowski[1], Paweł Dąbkowski[2], Harald Aschemann[3] and Andreas Rauh[3]

[1]Institute of Control and Computation Engineering,
University of Zielona Góra, ul. Ogrodowa 3b,
65-246 Zielona Góra, Poland, {B.Cichy, K.Galkowski}@issi.uz.zgora.pl
[2]Institute of Physics, Nicolaus Copernicus University,
Grudziądzka 5, 87-100 Toruń, Poland, P.Dabkowski@fizyka.umk.pl
[2]Department of Control Engineering,
Faculty of Electrical Engineering, Czech Technical University in Prague,
Technicka 2, 16627 Praha 6, Czech Republic
[3]Chair of Mechatronics, University of Rostock,
D-18059 Rostock, Germany,
Harald.Aschemann, Andreas.Rauh@uni-rostock.de

Abstract: Iterative Learning Control (ILC) is well established in control of linear and nonlinear dynamic systems, both as to underlying theory and experimental validation. This approach specifically aims at applications with the same operation repeated over finite time intervals and reset taking place between subsequent executions (the trials). The main principle behind ILC is to suitably use information from previous trials in selection of the input signal in the current trial with the objective of performance improvement from trial to trial. In this paper, new computationally efficient results are presented for an extension of the ILC approach to the uncertain 2D systems that arise from time and space discretization of partial differential equations. This type of application implies the need to use a spatio–temporal setting for the analysis of the control procedure. The resulting control laws can be computed using Linear Matrix Inequalities (LMIs). An illustrative example is provided.

Keywords: iterative learning control, spatio-temporal dynamics, Crank-Nicolson discretization

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1. Introduction

Iterative learning control (ILC) is a technique for controlling systems operating in a repetitive (or pass-to-pass) mode with the requirement that a reference trajectory $y_{ref}(t)$, defined over a finite interval $0 \leq t \leq t_N$, where t_N is the trial length (or duration), is tracked with high accuracy. Since the original work of Arimoto, Kawamura and Miyazaki (1984), the general area of ILC has been the subject of intense research effort. Examples of systems to which ILC has been applied successfully are robotic manipulators that are required to repeat a given task, chemical batch processes or, more generally, the class of tracking systems. There exist also spatio-temporal applications where ILC could be employed. For example, Moore and Chen (2006), describe an irrigation application in dry-land farming governed by a set of partial differential equations, where the independent variables are one time and three space coordinates. It is also straightforward to see the natural connections between spatio-temporal systems and multidimensional systems theory, see e.g. Rabenstein and Trautmann (2003). The starting point for the work reported in this paper is the discretization method applied to the governing partial differential equations (PDEs). Here, we consider an implicit discretization scheme, i.e., the Crank-Nicolson method, see e.g. Crank and Nicolson (1947), Cichy, Gałkowski and Rogers (2012). The main advantage of this method is its unconditional numerical stability for many PDEs including the heat equation being the benchmark here. In this paper, we develop new modified ILC results for a class of uncertain spatio-temporal systems based on the stability analysis of 2D discrete systems in the Fornasini-Marchesini model form, see Fornasini and Marchesini (1978). In this framework, it is necessary to mention the exceptionally rich contribution of T. Kaczorek to 2D systems theory, see e.g. Kaczorek (1985) and Kaczorek (1998) and many other related publications. In particular, a major novelty of the presented approach in comparison with Cichy et al. (2012) is that there is no need in the novel approach for inverting tridiagonal model matrices, which allows us to consider process uncertainties directly and to design a robust ILC scheme in a simple manner.

Throughout this paper, $M \succ 0$ (respectively $\prec 0$) denotes a real symmetric positive (respectively negative) definite matrix. The zero matrix and the identity matrix with appropriate dimensions are denoted by 0 and I, respectively. When it is not clear from the context, we use I_t which denotes a unit matrix of the dimension $t \times t$. The symbol $\otimes$ represents the Kronecker product of matrices, and $\mathrm{diag}\,(W_1, W_2)$ is a block diagonal matrix with blocks W_1 and W_2. Finally, tridiagonal matrices are defined as follows

$$\mathrm{tri}\,(\beta,\gamma,\eta) = \begin{bmatrix} \gamma & \beta & & & 0 \\ \eta & \gamma & \beta & & \\ & \ddots & \ddots & \ddots & \\ & & \eta & \gamma & \beta \\ 0 & & & \eta & \gamma \end{bmatrix} .$$

2. Approximation of a heat transfer process as a 1D discrete linear dynamic system

To illustrate the use of ILC for spatio-temporal systems, we consider the scalar heat equation

$$\frac{\partial x(t,w)}{\partial t} = \rho^2 \frac{\partial^2 x(t,w)}{\partial w^2} + \delta u(t,w), \tag{1}$$

where $x(t,w)$ represents the temperature, $u(t,w)$ is a distributed input variable, and t and w are the time and space variables, respectively. Moreover, ρ and δ are real scalars denoting process parameters.

The starting value $x(0,w)$ is given by the initial condition $x(0,w) = \theta(w)$, where the initial value $\theta(w)$ is a given spatial function. The process is defined on a finite interval on the space axis with length L $(0 < w < L\ )$. It is spatially isotropic. The behavior at the boundaries $w = 0$ and $w = L$ is given by the boundary conditions $x(t,0) = \chi(t)$ and $x(t,L) = \vartheta(t)$, where the boundary values $\chi(t)$ and $\vartheta(t)$ are given time functions.

Now, we apply a finite difference discretization, where signals are considered only at discrete points in time and space according to $t = lT$ and $w = ph$, with T and h being the time and space sampling periods, respectively. Due to the aforementioned advantages, we apply the Crank-Nicolson discretization method with its characteristic approximations of signals and differentials, see Crank and Nicolson (1947), Cichy et al. (2012). For this purpose, we introduce the following approximations for $x(t,w)$ and its derivatives, as well as for $u(t,w)$, at the points of time $t = lT$ and the position $w = ph$ according to

$$\begin{aligned}
x(t,w) &\approx \frac{x_{l+1}(p) + x_l(p)}{2}, \qquad \frac{\partial x(t,w)}{\partial t} \approx \frac{x_{l+1}(p) - x_l(p)}{T} \\
\frac{\partial x(t,w)}{\partial w} &\approx \frac{x_l(p+1) - x_l(p-1)}{4h} + \frac{x_{l+1}(p+1) - x_{l+1}(p-1)}{4h} \\
\frac{\partial^2 x(t,w)}{\partial w^2} &\approx \frac{x_l(p+1) - 2x_l(p) + x_l(p-1)}{2h^2} \\
&\quad + \frac{x_{l+1}(p+1) - 2x_{l+1}(p) + x_{l+1}(p-1)}{2h^2}
\end{aligned} \tag{2}$$

and $u(t,w) \approx u_l(p)$. Then, by evaluating the model (1) over lT with $l = 0,1,\ldots,N$ and over ph with $p = 1,2,\ldots,\alpha-1$ and $\alpha h = L$, the equation

$$\begin{aligned}
&A_1 x_{l+1}(p+1) + B_1 x_{l+1}(p) + C_1 x_{l+1}(p-1) \\
&= A_2 x_l(p+1) + B_2 x_l(p) + C_2 x_l(p-1) + \delta u_l(p)
\end{aligned} \tag{3}$$

is obtained, where

$$A_1 = -A_2 = C_1 = -C_2 = -\frac{\rho^2}{2h^2}, \quad B_1 = \frac{1}{T} + \frac{\rho^2}{h^2}, \quad B_2 = \frac{1}{T} - \frac{\rho^2}{h^2}$$

holds with the associated boundary conditions

$$\begin{aligned} x_0(p) &= \theta(p), & 1 \le p \le \alpha - 1 \\ x_l(0) &= \chi_l,\ x_l(\alpha) = \vartheta_l, & l > 0\ . \end{aligned} \tag{4}$$

In (4), $\theta(p)$, $(\vartheta_l,\ \chi_l)$ are known scalar functions of p (respectively l) that come from the discretization with the sampling period h (respectively T) of the boundary conditions in (1). The sequences $\{\vartheta_l\}$, $\{\chi_l\}$ are assumed to be bounded. This discrete approximation is in the form of an implicit 2D equation that cannot be directly used to construct a discrete recursive model approximation of the process dynamics. Consequently, introduce the vector of system states

$$\mathcal{X}(l) = [\, x_l(1) \;\; x_l(2) \;\; \ldots \;\; x_l(\alpha-1)\,]^T \tag{5}$$

and boundary points

$$\mathcal{X}_B(l) = [\, x_l(0) \;\; x_l(\alpha)\,]^T. \tag{6}$$

Then, (3) can be rewritten as

$$\mathcal{A}_1\mathcal{X}(l+1) = \mathcal{A}_2\mathcal{X}(l) - \mathcal{C}_1\mathcal{X}_B(l+1) + \mathcal{C}_2\mathcal{X}_B(l) + \mathcal{B}\mathcal{U}(l) \tag{7}$$

with

$$\mathcal{U}(l) = [\, u_l(1) \;\; u_l(2) \;\; \ldots \;\; u_l(\alpha-1)\,]^T. \tag{8}$$

In addition, $\mathcal{X}(0) = \begin{bmatrix}\theta(1) & \theta(2) & \ldots & \theta(\alpha-1)\end{bmatrix}^T$ and the remaining boundary conditions are given as in (4). The remaining matrices in (7) are defined as

$$\mathcal{A}_i = \operatorname{tri}(A_i, B_i, C_i)\,, \quad \mathcal{B} = \delta I, \quad \mathcal{C}_i^T = \begin{bmatrix} C_i & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & A_i \end{bmatrix}, \quad i = 1, 2.$$

This type of model can be extended in future work to the case of PDEs depending on more than one significant space coordinate.

3. ILC problem formulation

The objective is to achieve iteratively a prescribed reference signal $x_l(p) = x_l^*(p)$ in space and time over the finite rectangle

$$R = \{(l,p) : l = 0, 1, \ldots, N; p = 1, 2, \ldots, \alpha - 1\}$$

which can be represented in vector form as

$$\mathcal{X}(l)^* = [\, x_l^*(1) \;\; x_l^*(2) \;\; \ldots \;\; x_l^*(\alpha-1)\,]^T.$$

A new positive integer variable k is introduced to denote the trial-to-trial update. Hence, the state $\mathcal{X}(l)$ is replaced by its realization in the kth trial $\mathcal{X}(k,l)$; the equation (7) now becomes

$$\mathcal{A}_1\mathcal{X}(k,l+1) = \mathcal{A}_2\mathcal{X}(k,l) - \mathcal{C}_1\mathcal{X}_B(k,l+1) + \mathcal{C}_2\mathcal{X}_B(k,l) + \mathcal{B}\mathcal{U}(k,l). \tag{9}$$

Note that the process matrices $\{\mathcal{A}_i,\ \mathcal{B},\ \mathcal{C}_i,\ i=1,2\}$ in the state-space model (7), which are defined on the basis of (3), are frequently not precisely known, and that (according to the 1D systems case) a particular structure for the uncertainty can usually be specified. In this case, the source of uncertainty is the space discretization period h, which is motivated by the fact that the location of sensors and actuators along the space axis can often not be adjusted precisely. A further source for uncertainty can be related to the uncertain parameter ρ in the considered heat equation and to variations of the time discretization period. Here, we consider the case when the model matrices belong to a convex bounded uncertain domain $\mathcal{D}$ of polytope type, where any uncertain matrix can be written as a convex combination of its vertices

$$\begin{aligned}\mathcal{D}=\Big\{\ [\mathcal{A}_i(\xi(k,l)),\mathcal{B}(\xi(k,l)),\mathcal{C}_i(\xi(k,l))]\colon\ \mathcal{A}_i(\xi(k,l))=\sum_{v=1}^{n_v}\xi_v(k,l)\mathcal{A}_{iv},\\ \mathcal{B}(\xi(k,l))=\sum_{v=1}^{n_v}\xi_v(k,l)\mathcal{B}_v,\ \ \mathcal{C}_i(\xi(k,l))=\sum_{v=1}^{n_v}\xi_v(k,l)\mathcal{C}_{iv},\\ \sum_{v=1}^{n_v}\xi_v(k,l)=1,\ \ \xi_v(k,l)\geq 0,\ \ i=1,2,\ \ k\geq 0,\ \ 0\leq l\leq N\Big\},\end{aligned} \tag{10}$$

where $\mathcal{A}_{iv},\mathcal{B}_v,\mathcal{C}_{iv}$ are the corresponding matrix vertices and n_v denotes their number. This formulation allows us to handle also variability of the model parameters in both the time l and the trial number k but within the assumed polytope.

To construct a robust ILC scheme, the uncertain matrices of the polytope (10) are substituted for the constant, precisely known matrices in (9). Using the reference signal $\mathcal{X}(l)^*$, the tracking error

$$E(k,l)\ \widehat{=}\ \mathcal{X}(l)^*-\mathcal{X}(k,l) \tag{11}$$

can be defined over $0\leq l\leq N$ and $1\leq p\leq\alpha-1$ with the following boundary conditions and input increments

$$\begin{aligned}\Theta_B(k+1,l)&=\mathcal{X}_B(k+1,l)-\mathcal{X}_B(k,l)\\ \Delta U(k+1,l)&=\mathcal{U}(k+1,l)-\mathcal{U}(k,l).\end{aligned} \tag{12}$$

Finally, after simple algebraic manipulations, see Cichy et al. (2012), the control law

$$\Delta U(k+1,l)=\mathcal{K}_1E(k+1,l)+\mathcal{K}_2E(k,l)+\mathcal{K}_3E(k,l+1) \tag{13}$$

leads to the closed-loop system, representing an ILC scheme in a parameter-dependent form

$$\begin{aligned}\mathcal{A}_1(\xi(k,l))E(k+1,l+1)=\widehat{A}_1(\xi(k,l))E(k+1,l)+\widehat{A}_2(\xi(k,l))E(k,l)\\ +\widehat{A}_3(\xi(k,l))E(k,l+1)+\mathcal{C}_1(\xi(k,l))\Theta_B(k+1,l+1)-\mathcal{C}_2(\xi(k,l))\Theta_B(k+1,l),\end{aligned} \tag{14}$$

where the matrices

$$\begin{aligned}\widehat{A}_1(\xi(k,l)) &= \mathcal{A}_2(\xi(k,l)) - \mathcal{B}(\xi(k,l))\mathcal{K}_1\\ \widehat{A}_2(\xi(k,l)) &= -\mathcal{A}_2(\xi(k,l)) - \mathcal{B}(\xi(k,l))\mathcal{K}_2\\ \widehat{A}_3(\xi(k,l)) &= \mathcal{A}_1(\xi(k,l)) - \mathcal{B}(\xi(k,l))\mathcal{K}_3\end{aligned} \tag{15}$$

are introduced.

During the design of the controller matrices, the terms $\Theta_B(k+1,l+1)$ and $\Theta_B(k+1,l)$, arising from the boundary conditions, are omitted as they do not influence stability. Note also that (14) is in the form of a Fornasini-Marchesini (1978) 2D model.

4. ILC design

In ILC, a major objective is to achieve convergence of the trial-to-trial error. To obtain this feature, it is frequently sufficient to apply only the controller $\mathcal{K}_3$ in (13). However, in such a case we are not able to control a process directly along the trial, i.e., along the time variable l. Especially for a large time horizon N, this may result in a poor performance, see the discussion for systems governed by ODEs in Hładowski (2010). One of the approaches to avoid such problems is to employ the 2D systems approach and, particularly, the stronger stability notion for this class of systems, which ensures signal attenuation in both directions, namely along the trial and from trial to trial. Hence, we require the asymptotic stability for the associated 2D system (14) in the Fornasini-Marchesini form, which assumes that the trial time duration N tends to infinity and that the process still has to be stable in both directions. This is a very similar situation to ILC methods for systems governed by ODEs, based on the theory of repetitive process, see again Hładowski (2010).

In this paper, the problem solution is based on the use of a Lyapunov function interpretation of asymptotic stability for 2D discrete linear systems given in the Fornasini-Marchesini (1978) form according to (14). This procedure leads to a LMI design of the control law. Stability over the rectangle R of such a 2D system is achieved, when the combined energy associated with the point $(k+1,l+1)$ is less than the energy associated with all preceding points $(k+1,l)$, $(k,l+1)$, and (k,l). Hence, the following Lyapunov function candidates are defined, see Kar and Singh (2003)

$$\begin{aligned}V(k+1,l+1) &= E(k+1,l+1)^T\left(\sum_{i=1}^{3}\mathcal{P}_i\right)E(k+1,l+1)\\ V(k+1,l) &= E(k+1,l)^T\mathcal{P}_1E(k+1,l)\\ V(k,l+1) &= E(k,l+1)^T\mathcal{P}_2E(k,l+1)\\ V(k,l) &= E(k,l)^T\mathcal{P}_3E(k,l),\end{aligned} \tag{16}$$

with $\mathcal{P}_i = I \otimes P_i$, and $P_i \succ 0$, $i = 1, 2, 3$. Next, the increment

$$\Delta V(k,l) = V(k+1,l+1) - V(k+1,l) - V(k,l) - V(k,l+1) \tag{17}$$

has to comply with the requirement

$$\Delta V(k,l) < 0 \tag{18}$$

for any choice of the positive integers N and $\alpha > 1$ and for all $k > 0$ and each system representative from the polytope (10). Fulfilling the inequality (18) is sufficient for asymptotic stability of the Fornasini-Marchesini model with N tending to infinity. It is easy to obtain an equivalent result for the situation when the space horizon is very large and the time is hidden in the process structure. Due to the fact that we only work with finite, but possibly large time and space horizons, we can limit our interest to the rectangle R. The stability of the ILC scheme will then be denoted as stability over the rectangle R, see Cichy et al. (2012). Now, as a preliminary step to the problem solution, we consider the case with no uncertainty, i.e., each model (14) is known and fixed. Hence, the dependency on $\xi(k,l)$ is omitted. First, left multiply the model (14) by $\mathcal{A}_1^{-1}$ (assuming that $\mathcal{A}_1$ is non-singular) to achieve the commonly used state-space form, which allows us to rewrite (18) in the form of

$$\widetilde{A}^T \mathcal{P} \widetilde{A} - \widetilde{P} \prec 0, \tag{19}$$

with

$$\widetilde{A} = \mathcal{A}_1^{-1}\widehat{A},\ \widehat{A} = \begin{bmatrix}\widehat{A}_1 & \widehat{A}_2 & \widehat{A}_3\end{bmatrix},\ \mathcal{P} = \sum_{i=1}^{3} \mathcal{P}_i,\ \widetilde{P} = \mathrm{diag}\left(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\right) \tag{20}$$

and matrices $\widehat{A}_1$, $\widehat{A}_2$, $\widehat{A}_3$, which are those of (15) but with vanishing uncertainty.

Now, the following theorem, which can be used for the controller design, can be stated and proved.

Theorem 1. *A discrete linear 2D system described by* (14)*, for the case with no uncertainty, is stable over a rectangle R for any choice of the positive integers N and $\alpha > 1$ if there exist scalars N_{ij}, $Q_i > 0$, $i, j = 1, 2, 3$, and some non-zero scalar G such that the following LMI is feasible*

$$\begin{bmatrix} -\widehat{Q} & Y^T \\ Y & \mathcal{Q} - \mathcal{A}_1\mathcal{G} - (\mathcal{A}_1\mathcal{G})^T \end{bmatrix} \prec 0, \tag{21}$$

where $\mathcal{G} = I_\alpha \otimes G$,

$$Y = \begin{bmatrix}\mathcal{A}_2\mathcal{G} - \mathcal{B}\mathcal{N}_1 & -\mathcal{A}_2\mathcal{G} - \mathcal{B}\mathcal{N}_2 & \mathcal{A}_1\mathcal{G} - \mathcal{B}\mathcal{N}_3\end{bmatrix}$$

and

$$\mathcal{Q}_i = I_\alpha \otimes Q_i, \quad \widehat{Q} = \mathrm{diag}\,(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3)\,, \quad \mathcal{N}_i = \mathrm{tri}\,(N_{i3}, N_{i2}, N_{i1})\,,$$
$$\mathcal{Q} = \sum_{i=1}^{3} \mathcal{Q}_i \tag{22}$$

provided that $\mathcal{A}_1$ *is non-singular. If* (21) *holds, the stabilizing control matrices in* (13) *are given by*

$$\mathcal{K}_i = \mathcal{N}_i \mathcal{G}^{-1}, \quad i = 1, 2, 3, \tag{23}$$

where the structure of the matrices $\mathcal{K}_i$ *is*

$$\mathcal{K}_i = \mathrm{tri}\,(K_{i3}, K_{i2}, K_{i1})\,. \tag{24}$$

Proof. Consider first the $(2,2)$-block of (21), that is,

$$\mathcal{Q} - \mathcal{A}_1\mathcal{G} - (\mathcal{A}_1\mathcal{G})^T \prec 0$$

which yields

$$H = -\mathcal{A}_1\mathcal{G} - (\mathcal{A}_1\mathcal{G})^T \prec 0$$

for $\mathcal{Q} \succ 0$. Assume now that $\mathcal{G}$ is singular, and hence there exists a non-zero vector z such that $\mathcal{G}z = 0$. However, then $z^T H z = 0$, which contradicts the previous inequality. Therefore, $\mathcal{G}$ is non-singular.

Next, note that (23) yields $Y = \widehat{A}\widehat{\mathcal{G}}$. Then, left multiply (21) by

$$\begin{bmatrix} \widehat{\mathcal{G}}^{-T} & \widehat{A}^T \mathcal{A}_1^{-T} \mathcal{G}^{-T} \end{bmatrix}, \quad \text{where} \quad \widehat{\mathcal{G}} = \mathrm{diag}\,(\mathcal{G}, \mathcal{G}, \mathcal{G})$$

and right multiply the result by its transpose to achieve

$$(\mathcal{A}_1^{-1}\widehat{A})^T \mathcal{G}^{-T} \mathcal{Q} \mathcal{G}^{-1} (\mathcal{A}_1^{-1}\widehat{A}) - \widehat{\mathcal{G}}^{-T} \widehat{Q} \widehat{\mathcal{G}}^{-1} \prec 0$$

which, after introduction of new variables $\widetilde{P} = \widehat{\mathcal{G}}^{-T}\widehat{Q}\widehat{\mathcal{G}}^{-1}$ and $\mathcal{P} = \mathcal{G}^{-T}\mathcal{Q}\mathcal{G}^{-1}$, gives the result (19) and completes the proof. ■

Note that the computation of the control matrices according to Theorem 1 does not require inverting the matrix $\mathcal{A}_1$. However, we require its non-singularity, which almost always is the case. This is a significant advantage of the method as a possible need of inverting the matrix $\mathcal{A}_1$ complicates drastically the extension of the approach to the uncertain case, because it is commonly very difficult to characterize the uncertainty of $\mathcal{A}_1^{-1}\mathcal{A}_2$ and $\mathcal{A}_1^{-1}\mathcal{A}_3$. Now, the major result in this paper, i.e., application of ILC to the uncertain polytopic case is presented.

THEOREM 2. *An uncertain discrete linear 2D system described by* (14) *is stable over a rectangle* R *for any choice of the positive integers* N *and* $\alpha > 1$ *if there* *exist scalars* N_{ij}, $Q_i > 0$, $i, j = 1, 2, 3$, *and some non-zero scalar* G *such that the following LMI is feasible*

$$\begin{bmatrix} -\widehat{Q} & Y_v^T \\ Y_v & \mathcal{Q} - \mathcal{A}_{1v}\mathcal{G} - (\mathcal{A}_{1v}\mathcal{G})^T \end{bmatrix} \prec 0, \tag{25}$$

where $\mathcal{G} = I_\alpha \otimes G$,

$$Y_v = \begin{bmatrix} \mathcal{A}_{2v}\mathcal{G} - \mathcal{B}_v\mathcal{N}_1 & -\mathcal{A}_{2v}\mathcal{G} - \mathcal{B}_v\mathcal{N}_2 & \mathcal{A}_{1v}\mathcal{G} - \mathcal{B}_v\mathcal{N}_3 \end{bmatrix}, \tag{26}$$

with $v = 1, \ldots, n_v$ *representing the underlying vertices of the uncertain matrices. The remaining notations are as in the previous theorem, provided that* $\mathcal{A}_1(\xi(k,l))$ *is non-singular at each point of the considered polytope. If* (25) *holds, the stabilizing control matrices in* (13) *are given as in* (23) *with* (24).

Proof. We exploit the fact that a system with uncertainty modeled in the polytopic form (10) is stable if each 'vertex' system is stable. Hence, Theorem 1 holds for all matrix vertices with a vertex-independent Lyapunov function and control law, which obviously leads to some conservativeness. Finally, allowing for any convex combination of (25), we obtain that the underlying system is stable under the assumed polytopic uncertainty. ■

Finally, to apply the control law defined in (13), note that after simple algebraic manipulations we obtain

$$\begin{aligned} \mathcal{U}(k+1,l) = \mathcal{U}(k,l) &+ \mathcal{K}_1\big(\mathcal{X}(l)^* - \mathcal{X}(k+1,l)\big) + \mathcal{K}_2\big(\mathcal{X}(l)^* - \mathcal{X}(k,l)\big) \\ &+ \mathcal{K}_3\big(\mathcal{X}(l+1)^* - \mathcal{X}(k,l+1)\big). \end{aligned} \tag{27}$$

Hence, to apply this control law we need to recover a state vector $\mathcal{X}(k,l)$ for each required $\xi(k,l)$. To accomplish this task, the equation (14) has to be solved for each required $\xi(k,l)$ by applying, e.g., the Gauss method. However, when applying this approach to a real process, we may measure the state variable or — if this is impossible — an observer can be employed.

5. Numerical example

Consider again the heat equation (1) with $\rho = 0.5$ for the heated rod of the length $L = 224$ m and apply the Crank-Nicolson discretization scheme with $T = 26$ (given in s), and

$$h \in [2.4,\ 3.2] \text{ given in m}, \tag{28}$$

which is a feasible choice since the Crank-Nicolson scheme is unconditionally numerically stable. The parameters defining the rectangle R in this case are $N = 100$ and $\alpha = 81$ ($L = 80 \cdot \frac{(2.4+3.2)\,\mathrm{m}}{2} = 80 \cdot 2.8\,\mathrm{m}$). Due to the uncertainty of the space discretization period h, which is motivated by uncertainty of the placement of actuators and/or sensors, we obtain the following parameters of

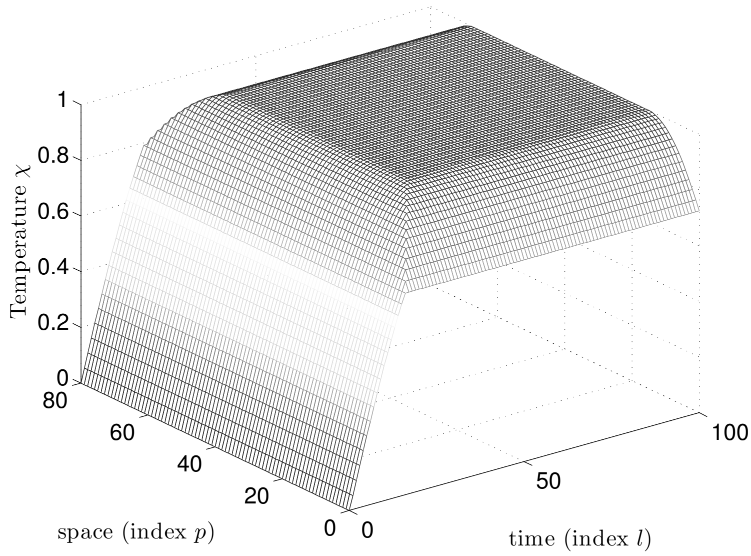


Figure 1. Reference signal $\mathcal{X}(l)^*$

the polytopic model vertex matrices ($n_v = 2$):

vertex 1:

$$A_1 = C_1 = -0.0102,\ A_2 = C_2 = 0.0102$$
$$B_1 = 0.0629,\ B_2 = 0.014,\ B = 5$$

vertex 2:

$$A_1 = C_1 = -0.0217,\ A_2 = C_2 = 0.0217$$
$$B_1 = 0.0819,\ B_2 = -4.9412 \times 10^{-3},\ B = 5.$$

The reference signal is shown in Fig. 1. Although the boundary conditions do not influence stability of the ILC process (14), they have to be specified when considering the process dynamics. They are clearly related to those for the controlled process, i.e., for (4)–(6). Hence, they are introduced as

$$\mathcal{X}(0,l) = 0,\ \mathcal{U}(0,l) = 0, \qquad 0 \le l \le N$$
$$\mathcal{X}_B(k,l) = \mathcal{X}_B(l)^* = \begin{bmatrix} \chi \\ \vartheta \end{bmatrix}, \qquad 0 \le l \le N,\ k \ge 0$$
$$\mathcal{U}_B(k,l) = 0,\ E_B(k,l) = 0, \quad 0 \le l \le N,\ k \ge 0,$$

where

$$\mathcal{X}_B(k,l) = \begin{bmatrix} x_l(k,0) \\ x_l(k,\alpha) \end{bmatrix},\ \mathcal{X}_B(l)^* = \begin{bmatrix} x_l^*(0) \\ x_l^*(\alpha) \end{bmatrix}$$
$$\mathcal{U}_B(k,l) = \begin{bmatrix} u_l(k,0) \\ u_l(k,\alpha) \end{bmatrix},\ E_B(k,l) = \begin{bmatrix} e_l(k,0) \\ e_l(k,\alpha) \end{bmatrix} \quad .$$

Here, we have assumed that the boundary conditions for the ILC process, i.e., $\mathcal{X}_B(k,l)$ are equal to the boundary values of the reference signal $\mathcal{X}_B(l)^*$ for each $k \geq 0, \quad 0 \leq l \leq N$ and that they are the same as for the controlled process, i.e., for (4). The LMIs of Theorem 2 are feasible and their solution, determined numerically by means of SeDuMi (Sturm, 2001) in combination with YALMIP (Löfberg, 2004), yields the required matrices (23) with (24) and the following numerical values K_{ij}, $i,j = 1,2,3$, in the control law (13):

$$K_{11} = K_{13} = 3.7821 \times 10^{-3}, \ K_{12} = 3.8532 \times 10^{-4}$$
$$K_{21} = K_{23} = K_{31} = K_{33} = -3.7821 \times 10^{-3}$$
$$K_{22} = -3.8532 \times 10^{-4}, \ K_{32} = 0.015.$$

To give an averaged measure of the error signal, we use the normalized root mean square error defined as

$$E_{\mathrm{nrms}}(k) = E_{\mathrm{rms}}(k)/\mathcal{X}^*_{\mathrm{rms}} \ , \tag{29}$$

where $E_{\mathrm{rms}}(k)$ denotes the root mean square error which is given as

$$E_{\mathrm{rms}}(k) = \sqrt{\frac{1}{\alpha(N+1)} \sum_{l=0}^{N} E(k,l)^T E(k,l)} \ , \tag{30}$$

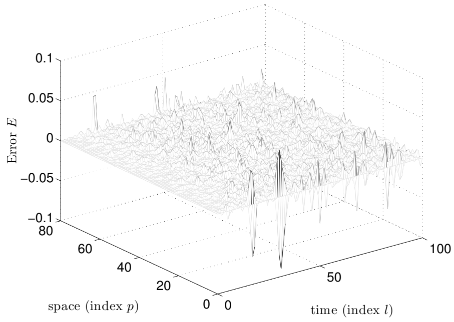
and where $\mathcal{X}^*_{\mathrm{rms}}$ is defined as

$$\mathcal{X}^*_{\mathrm{rms}} = \sqrt{\frac{1}{\alpha(N+1)} \sum_{l=0}^{N} \mathcal{X}(l)^{*T} \mathcal{X}(l)^*} \ . \tag{31}$$

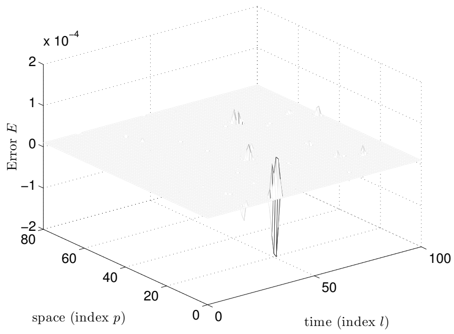
Similarly, we introduce the root mean square control signal defined as

$$\mathcal{U}_{\mathrm{rms}} = \sqrt{\frac{1}{\alpha N} \sum_{l=0}^{N-1} \mathcal{U}(k,l)^T \mathcal{U}(k,l)} \ . \tag{32}$$

In the first scenario, we consider the process described by (3), where the parameter h is chosen randomly from the range (28) for all values p and l. Here, the constraint is taken into account that the total length of the bar is constant and fixed to the overall length L. Moreover, we assume that the particular values of h, which have been generated as uniformly distributed pseudo-random numbers by using the Matlab function `rand`, remain constant over all iterations k. The control errors for the trials $k = 3$ and $k = 10$ are depicted in Fig. 2, showing strong error convergence. To further highlight the dynamics of the convergence process, the normalized root mean square error is shown in Fig. 4. Hence, it can be seen that the ILC process converges quickly and, moreover, does not require excessive control action as presented in Fig. 3.



(a) Error E for trial $k = 3$

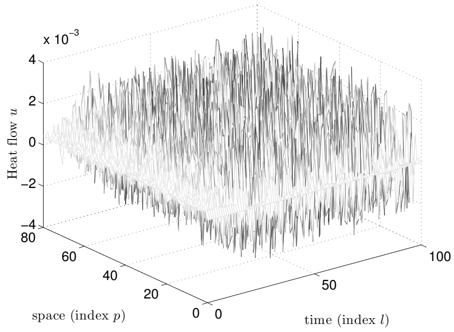


(b) Error E for trial $k = 10$

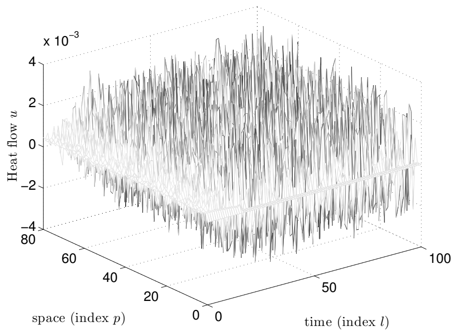
Figure 2. Scenario 1: Errors for trial $k = 3$ and $k = 10$

In the second scenario, the space discretization period h is set to piecewise constant values along the domain p. The corresponding values, furthermore, remain constant along the time l and along the iteration domain k. In the following example, we assume the minimum value $h = 2.4$ for the first 20 points after $p = 0$. Then, the parameter h is switched to the maximum value $h = 3.2$ and is kept constant for the next 40 points. Finally, at $p = 60$, it is switched back to the minimum value and remains constant until the end of rod, i.e., for $p = 80$. The error and control signals are depicted in Figs. 5 and 6. The normalized root mean square error is shown in Fig. 7. It can be seen that the controller works also very well, however the error convergence is a bit slower in comparison with Scenario 1.

Moreover, although the Crank-Nicolson discretization scheme is numerically stable, it can be claimed that it is not possible to draw conclusions from the control properties of the discretized system that has been simulated in the previous scenarios, with respect to the behavior of the continuous controlled process. To give a partial solution to this problem, we consider Scenario 3, where a discretization of the same type is employed but with much smaller discretization



(a) Input signal u for trial $k = 3$



(b) Input signal u for trial $k = 10$

Figure 3. Scenario 1: Input signal for trial $k = 3$ and $k = 10$

periods, i.e., $T' = 2$ (given in s), and

$$h' \in [0.6,\, 0.8] \text{ given in m,} \tag{33}$$

which represents a discrete approximation that is much closer to the original continuous process. Hence, we obtain the following model with polytopic uncertainty:

vertex 1:

$$A_1 = C_1 = -0.1953,\ A_2 = C_2 = 0.1953$$
$$B_1 = 0.8906,\ B_2 = 0.1094,\ B = 5.$$

vertex 2:

$$A_1 = C_1 = -0.3472,\ A_2 = C_2 = 0.3472$$
$$B_1 = 1.1944,\ B_2 = -0.1944,\ B = 5.$$

Next, for the case of a stochastically variable space discretization period, the previously calculated control law is applied to the process keeping it constant

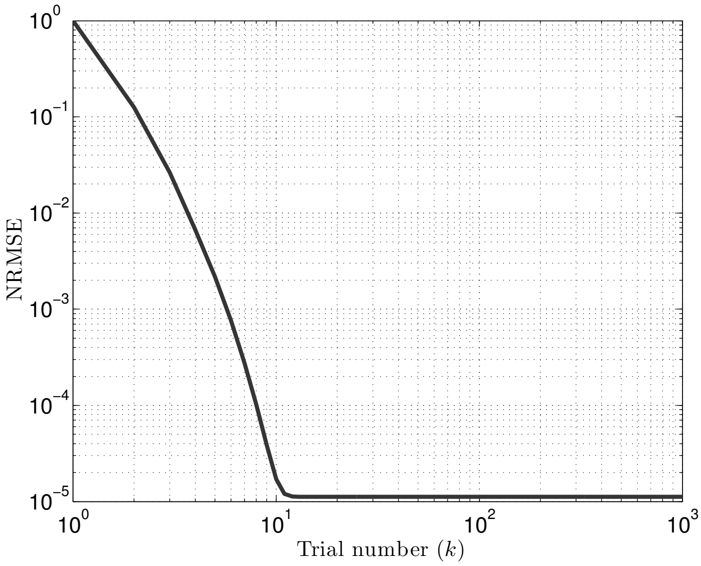


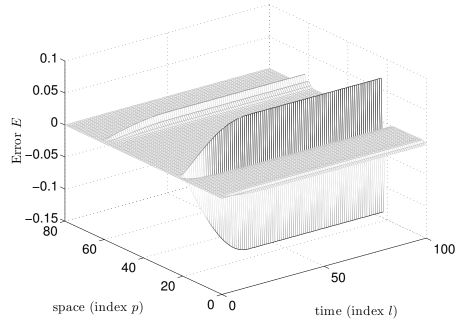
Figure 4. Scenario 1: Normalized root mean square error

within the rectangle of $13T' \times 4h'$, i.e., within the 'wide' periods: It turns out that the ILC scheme is still convergent, however more slowly, which is confirmed by Fig. 8. Moreover, Fig. 9 shows that the output during the trial 20 is quite close to the reference signal. However, there is some visible noise, which is almost absent at the trial 40.

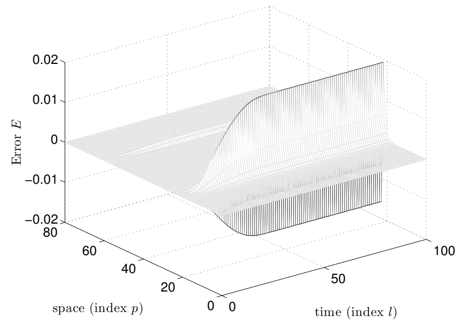
6. Conclusions

In this paper, we have presented new results on the application of ILC to uncertain spatio-temporal systems described by partial differential equations. The approach involves an implicit discretization scheme for the defining equations followed by the use of Lyapunov functions that satisfy sufficient but not necessary conditions for error convergence. Following this procedure, the control design can be carried out efficiently using LMIs.

For comparison with existing methods, it has to be noted that a classical controller could deal with the uncertainty considered in the paper. It would be possible to parametrize such a controller in a robust way for worst-case bounds of the uncertain parameters (e.g. by using a similar LMI approach like we have used). Yet, this would result in some kind of "along the pass control". If there is a need for further improvement between repeated trials, the only thing that could be done is a parameter identification at the end of the experiment and a subsequent control re-parametrization. However, this is troublesome and becomes obsolete when using the learning-type scheme. Moreover, in the learning-



(a) Error E for trial $k = 3$



(b) Error E for trial $k = 10$

Figure 5. Scenario 2: Errors for trial $k = 3$ and $k = 10$

type controller, the error convergence from trial to trial is automatically proven, which is often not the case for a combination of parameter identification with control re-parametrization. Hence, this is the main advantage of using the ILC approach for the problem under consideration.

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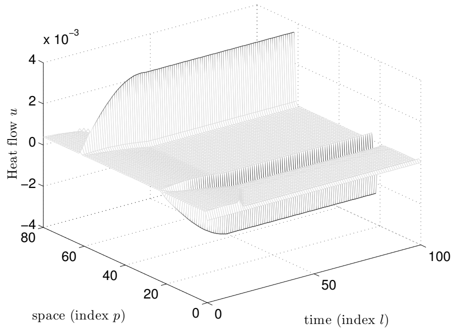
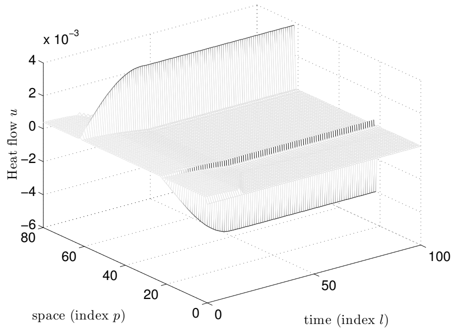


(a) Input signal u for trial $k = 3$



(b) Input signal u for trial $k = 10$

Figure 6. Scenario 2: Input signal for trial $k = 3$ and $k = 10$

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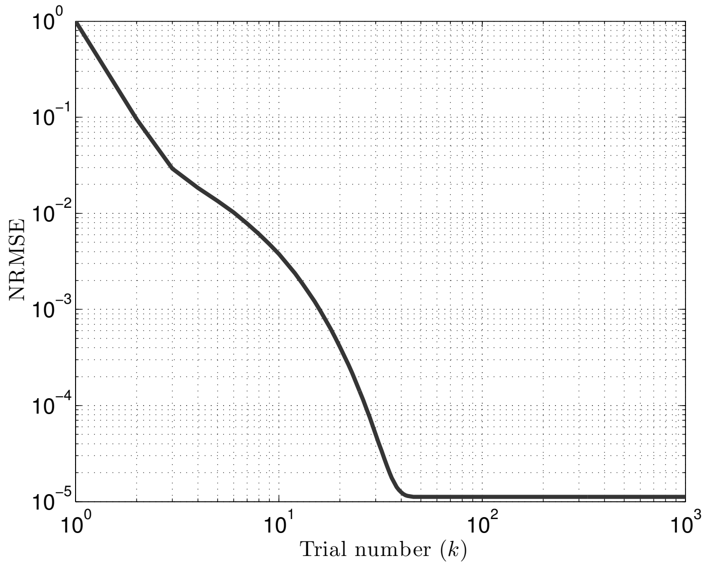

Figure 7. Scenario 2: Normalized root mean square error

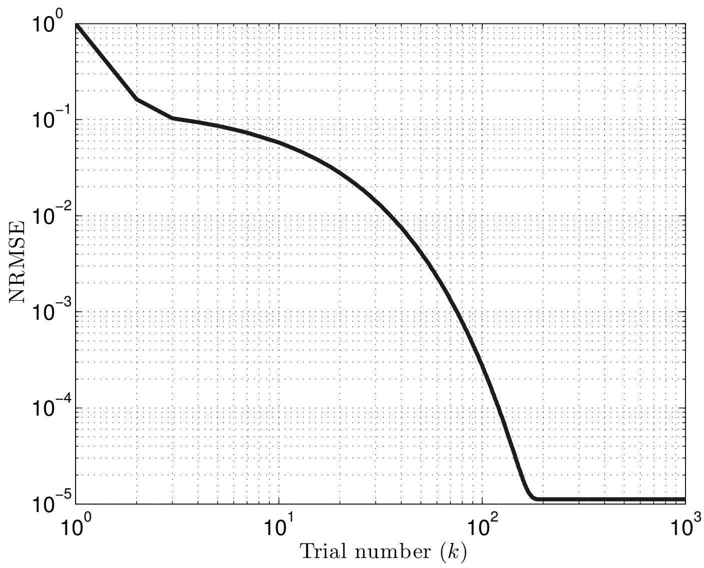

Figure 8. Scenario 3: Normalized root mean square error

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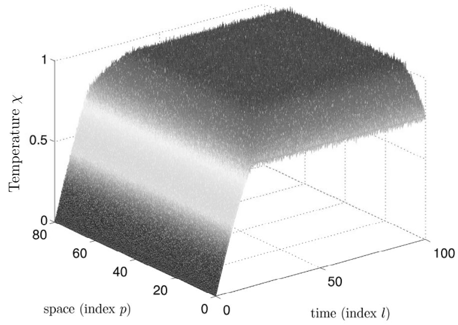
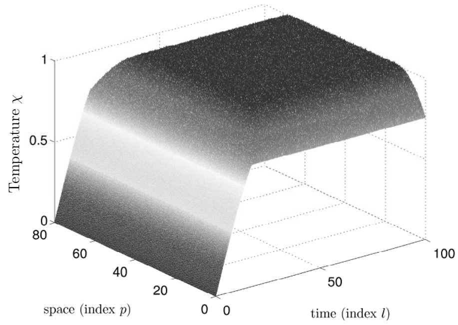


(a) Output for trial $k = 20$



(b) Output for trial $k = 40$

Figure 9. Scenario 3: Outputs at trials $k = 20$ and $k = 40$

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