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On the stability of continuous-time positive switched systems with rank one difference*

by

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Abstract: Continuous-time positive systems, switching among p subsystems whose matrices differ by a rank one matrix, are introduced, and a complete characterization of the existence of a common linear copositive Lyapunov function for all the subsystems is provided. Also, for this class of systems it is proved that a well-known necessary condition for asymptotic stability, namely the fact that all convex combinations of the system matrices are Hurwitz, becomes equivalent to the generally weaker condition that the systems matrices are Hurwitz. In the special case of two-dimensional systems, this allows for drawing a complete characterization of asymptotic stability. Finally, the case when there are only two subsystems, possibly with commuting matrices, is investigated.

Keywords switched system, positive linear system, asymptotic stability, linear copositive Lyapunov function, quadratic Lyapunov function.

1. Introduction

It is a great pleasure to contribute to this special issue honoring Tadeusz Kaczorek on the occasion of his 80th birthday, as his scientific career is strictly intertwined with the evolution of the research on 2D and positive systems, which represent two of our major research areas.

Recent years have seen a growing interest in 1D and 2D systems that are subject to positivity constraint on their dynamical variables. There are several motivations for this interest, coming from different domains of science and technology. In fact, the positivity assumption is a natural one when describing physical, biological or economical processes whose variables represent quantities that are intrinsically nonnegative, such as pressures, concentrations, population levels, etc. Tadeusz's constant attention for emerging research themes led him, in the nineties, to the field of positive 2D systems. The large number of his

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papers on 2D positive systems modelling as well as many contributions on different control, estimation and stabilization strategies in a positive environment certify his constant commitment to the analysis of system theoretic problems in their different facets. Even more significant evidence of this eclectic attitude is provided by his book on 1D and 2D positive systems (Kaczorek, 2002), that constitutes a leading reference in a rather diversified scientific field.

Our contribution, concerned with the theory of p-tuples of Metzler matrices and the dynamics of the corresponding positive switching systems, falls into a circle of ideas that is closely related to Tadeusz's research interests.

By a continuous-time positive switched system (CPSS) we mean a dynamic system consisting of a family of positive state-space models (Farina and Rinaldi, 2000; Kaczorek, 2002) and a switching law, specifying when and how the switching takes place. Switching among different positive subsystems naturally arises as a way to formalize the fact that the behavior of a positive system changes under different operating conditions, and is therefore represented by different mathematical structures.

Recently, CPSSs have been the object of an intense research activity, mainly focused on stability (Fornasini and Valcher, 2010; Gurvits, Mason and Shorten, 2007; Knorn, Mason and Shorten, 2009; Mason and Shorten, 2004, 2007a, b) and stabilizability (Blanchini, Colaneri and Valcher, 2011, 2012; Zappavigna, Colaneri, Jeromel and Middleton, 2010). Special attention has been devoted to the class of CPSSs that switch among subsystems whose matrices differ by a rank one matrix (King and Nathanson, 2006; Mason and Shorten, 2007a, b; Shorten, Corless, Wulff, Klinge and Middleton, 2009; Shorten, Mason, O'Caibre and Curran, 2004; Shorten, Wirth, Mason, Wulff and King, 2007). The reason for the interest in these systems is twofold. On the one hand, they can be thought of as the possible configurations one obtains from a given SISO system, when applying different state-feedback laws that ensure the positivity of the resulting closed-loop system. For this reason, the subsystem matrices can be denoted as $A+\mathbf{b}\mathbf{c}_i^\top$, $i \in \{1, 2, \dots, p\}$. On the other hand, interesting connections have been highlighted (Shorten, Mason, O'Caibre and Curran, 2004) between the quadratic stability of CPSSs, switching between two subsystems of matrices A and $A+\mathbf{b}\mathbf{c}^\top$, and the SISO circle criterion for the transfer function $\mathbf{c}^\top(sI_n - A)^{-1}\mathbf{b}$.

In Section 2 of this note we investigate the asymptotic stability of this class CPSSs, and provide a complete characterization of the existence of a common linear copositive Lyapunov function. Also, for this class of systems we prove that a well-known necessary condition for asymptotic stability, namely the fact that all convex combinations of the system matrices are Hurwitz, becomes equivalent to the generally weaker condition that the systems matrices are Hurwitz. In the special case of two-dimensional systems, this allows for drawing a complete characterization of asymptotic stability. In Section 3, the case when there are only two subsystems is investigated, and it is shown how this additional constraint leads to a richer set of characterizations. Finally, Section 4 is devoted to the very special situation when the matrices A and $A + \mathbf{b}\mathbf{c}^\top$ commute.

Notation. $\mathbb{R}_+$ is the semiring of nonnegative real numbers and, for any pair of positive integers k, n with $k \leq n$, $[k, n]$ is the set of integers $\{k, k+1, \ldots, n\}$. The (i, j)th entry of a matrix A will be denoted by $[A]_{i,j}$, the ith entry of a vector $\mathbf{v}$ by $[\mathbf{v}]_i$. A matrix (in particular, a vector) A with entries in $\mathbb{R}_+$ is called *nonnegative*, and if so, we adopt the notation $A \geq 0$. If, in addition, A has at least one positive entry, the matrix is *positive* ($A > 0$), while if all its entries are positive, it is *strictly positive* ($A \gg 0$). We denote by $\mathbf{1}_n$ the n-dimensional vector with all unitary entries. A *Metzler matrix* is a real square matrix, whose off-diagonal entries $[A]_{i,j}$, $i \neq j$, are nonnegative.

A Metzler matrix $A \in \mathbb{R}^{n\times n}, n > 1$, is *irreducible* if no permutation matrix P can be found such that

$$P^\top AP = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

A_{11} and A_{22} being square matrices. When so, by the Perron-Frobenius theorem, A has a simple real dominant eigenvalue $\lambda_{\max}(A)$, and the corresponding (left or right) eigenvector is strictly positive.

A square symmetric matrix P is *positive definite* ($P = P^\top \succ 0$) if $\mathbf{x}^\top P\mathbf{x} > 0$ for every nonzero vector $\mathbf{x}$, and *negative definite* ($P = P^\top \prec 0$) if $-P$ is positive definite.

A set $\mathcal{K} \subset \mathbb{R}^n$ is a *cone* if $\alpha\mathcal{K} \subseteq \mathcal{K}$ for all $\alpha \geq 0$. Basic definitions and results about cones may be found, for instance, in Berman and Plemmons (1979). A cone $\mathcal{K}$ is *polyhedral* if it can be expressed as the set of nonnegative linear combinations of a finite set of vectors, called *generating vectors*; if the generating vectors are the columns of a matrix A, we adopt the notation $\mathcal{K} = \mathrm{Cone}(A)$.

Given a family of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ in $\mathbb{R}^n$, the *convex hull* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ is the set of vectors

$$\left\{ \sum_{i=1}^{s} \alpha_i \mathbf{v}_i : \alpha_i \geq 0, \sum_{i=1}^{s} \alpha_i = 1 \right\}.$$

2. Continuous-time positive switched systems with rank one difference

In this paper we consider continuous-time switched systems described by the following equation

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \qquad t \in \mathbb{R}_+, \tag{1}$$

where $\mathbf{x}(t)$ is the n-dimensional state variable and $\sigma(t)$ the switching sequence at time t. We assume that, at every time $t \in \mathbb{R}_+$,

$$A_\sigma(t) \in \{A + \mathbf{b}\mathbf{c}_1^\top, A + \mathbf{b}\mathbf{c}_2^\top, \ldots, A + \mathbf{b}\mathbf{c}_p^\top\}, \tag{2}$$

with $A \in \mathbb{R}^{n\times n}$, $\mathbf{b}, \mathbf{c}_i \in \mathbb{R}^n$, $i \in [1,p]$, and $A + \mathbf{b}\mathbf{c}_i^\top$ is a Metzler matrix for every index $i \in [1,p]$. This ensures, in particular, that (1) is a continuous-time positive switched system (CPSS), by this meaning that if the initial state $\mathbf{x}(0)$ is positive then the whole state trajectory remains in the positive orthant $\mathbb{R}^n_+$ for every choice of the switching sequence. For this class of systems, we want to define and characterize asymptotic stability.

Definition 1 *The CPSS (1) is* asymptotically stable *if for every initial state* $\mathbf{x}(0) > 0$ *and every switching sequence* $\sigma(t), t \in \mathbb{R}_+$, *the state trajectory* $\mathbf{x}(t), t \in \mathbb{R}_+$, *converges to zero.*

Clearly, if the CPSS (1) is asymptotically stable, then all the system matrices are Hurwitz. On the other hand, a well-known sufficient condition for asymptotic stability is the existence of a common linear copositive Lyapunov function (CLCLF), i.e. a function $V(\mathbf{x}) := \mathbf{v}^\top\mathbf{x}$ satisfying

$$\left\{\begin{array}{lcl} V(\mathbf{x}) & = & \mathbf{v}^\top\mathbf{x} > 0, \\ \dot V_i(\mathbf{x}) & := & \mathbf{v}^\top(A+\mathbf{b}\mathbf{c}_i^\top)\mathbf{x} < 0, \quad \forall\, i \in [1,p], \end{array}\right. \qquad \forall\, \mathbf{x} > 0.$$

This corresponds to the existence of a strictly positive vector $\mathbf{v}$ such that

$$\mathbf{v}^\top(A+\mathbf{b}\mathbf{c}_i^\top) \ll 0, \qquad \forall\, i \in [1,p]. \tag{3}$$

Equivalent conditions for the existence of a CLCLF for a generic CPSS (1) have been recently obtained in Fornasini and Valcher (2010), Knorn, Mason and Shorten (2009). We want to investigate what these conditions become in the special case when the subsystem matrices are described as in (2).

Proposition 1 *Given $A \in \mathbb{R}^{n\times n}$, assume that $\mathbf{b}, \mathbf{c}_i \in \mathbb{R}^n, i \in [1,p]$, are column vectors such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler for every index i. The following facts are equivalent:*

i) the matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, admit a common linear copositive Lyapunov function;

ii)
$$\nexists \begin{bmatrix} \mathbf{w} \\ \alpha \end{bmatrix} \in \mathrm{Cone}\left(\begin{bmatrix} I_n & I_n & \dots & I_n \\ \mathbf{c}_1^\top & \mathbf{c}_2^\top & \dots & \mathbf{c}_p^\top \end{bmatrix}\right), \qquad \begin{bmatrix} \mathbf{w} \\ \alpha \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
such that
$$A\mathbf{w} + \mathbf{b}\alpha \geq 0; \tag{4}$$

iii) for every choice of p diagonal matrices, $D_i \in \mathbb{R}^n_+, i \in [1,p]$, with nonnegative diagonal entries and $\sum_{i=1}^p D_i = I_n$, the Metzler matrix $\sum_{i=1}^p (A + \mathbf{b}\mathbf{c}_i^\top)D_i$ is Hurwitz.

Proof. i) $\Leftrightarrow$ ii) There exists a common linear copositive Lyapunov function if and only if there exists $\mathbf{v} \gg 0$ such that (3) holds. This happens (Fornasini and Valcher, 2010) if and only if it is not possible to find nonnegative vectors $\mathbf{z}_i, i \in [1,p]$, satisfying $\sum_{i=1}^p \mathbf{1}_n^\top \mathbf{z}_i = 1$, such that

$$[\, A + \mathbf{b}\mathbf{c}_1^\top \quad A + \mathbf{b}\mathbf{c}_2^\top \quad \dots \quad A + \mathbf{b}\mathbf{c}_p^\top \,] \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \\ \mathbf{z}_p \end{bmatrix} \geq 0. \tag{5}$$

This inequality easily proves to be equivalent to

$$A(\sum_{i=1}^p \mathbf{z}_i) + \mathbf{b}(\sum_{i=1}^p \mathbf{c}_i^\top \mathbf{z}_i) \geq 0, \tag{6}$$

and it can be rewritten as (4), provided that

$$\begin{bmatrix} \mathbf{w} \\ \alpha \end{bmatrix} \in \mathrm{Cone}\left(\begin{bmatrix} I_n & I_n & \dots & I_n \\ \mathbf{c}_1^\top & \mathbf{c}_2^\top & \dots & \mathbf{c}_p^\top \end{bmatrix} \right).$$

So, this proves that a common linear copositive Lyapunov function exists if and only if a nonzero vector $\begin{bmatrix} \mathbf{w} \\ \alpha \end{bmatrix}$, belonging to the aforementioned cone, cannot be found satisfying (4).

ii) $\Leftrightarrow$ iii) By reversing the last part of the previous proof, we can claim that there exists a nonzero vector

$$\begin{bmatrix} \mathbf{w} \\ \alpha \end{bmatrix} \in \mathrm{Cone}\left(\begin{bmatrix} I_n & I_n & \dots & I_n \\ \mathbf{c}_1^\top & \mathbf{c}_2^\top & \dots & \mathbf{c}_p^\top \end{bmatrix} \right)$$

such that (4) holds if and only if there exist nonnegative vectors $\mathbf{z}_i$ such that (6) holds and $\sum_{i=1}^p \mathbf{z}_i > 0$. Clearly, $\mathbf{w} = \sum_{i=1}^p \mathbf{z}_i \geq \mathbf{z}_i$ for every $i \in [1,p]$. So, there exist diagonal matrices D_i, with diagonal entries in $[0,1]$, such that $\mathbf{z}_i = D_i \mathbf{w}$. Even more, as $(\sum_{i=1}^p D_i \mathbf{w}) = \mathbf{w}$, we can assume $\sum_{i=1}^p D_i = I_n$. Consequently, the previous condition can be rewritten by saying that there exist nonnegative diagonal matrices D_i, summing up to the identity matrix, such that

$$0 \leq A\mathbf{w} + \mathbf{b}(\sum_{i=1}^p \mathbf{c}_i^\top D_i \mathbf{w}) = \left[\sum_{i=1}^p (A + \mathbf{b}\mathbf{c}_i^\top) D_i \right] \mathbf{w},$$

for some positive vector $\mathbf{w}$. This is equivalent (Horn and Johnson, 1991) to the fact that the Metzler matrix $\sum_{i=1}^p (A + \mathbf{b}\mathbf{c}_i^\top) D_i$ is not Hurwitz, for some suitable choice of the nonnegative diagonal matrices D_i, summing up to the identity matrix. ■

Remark 1 *The equivalence of points i) and iii) in Proposition 1 holds true for arbitrary subsystem matrices $A_1, A_2, \dots, A_p$, since it relies on the fact that both these conditions are equivalent to the existence of nonnegative vectors $\mathbf{z}_i$, with $\sum_{i=1} \mathbf{1}_n^\top \mathbf{z}_i > 0$, for which (5) is satisfied, and this is independent of the assumption $A_i = A + \mathbf{b}\mathbf{c}_i^\top$.*

The existence of a CLCLF does not represent a necessary condition for the asymptotic stability of the CPSS (1), even under the restrictive assumption that the subsystem matrices take the form (2). This is shown by the following simple example.

Example 1 *Consider the two-dimensional CPSS (1), with subsystem matrices* $A + \mathbf{b}\mathbf{c}_1^\top$ *and* $A + \mathbf{b}\mathbf{c}_2^\top$, *where*

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 1/2 \\ -1 \end{bmatrix},$$

so that

$$A + \mathbf{b}\mathbf{c}_1^\top = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \quad A + \mathbf{b}\mathbf{c}_2^\top = \begin{bmatrix} -1/2 & 0 \\ 1 & -3 \end{bmatrix}.$$

It is easy to see that no strictly positive vector $\mathbf{v}$ *can be found such that* $\mathbf{v}^\top(A + \mathbf{b}\mathbf{c}_i^\top) \ll 0, i \in [1,2]$. *On the other hand, it is well-known (Gurvits, Mason and Shorten, 2007),(Shorten, Wirth, Mason, Wulff and King, 2007) that for two-dimensional positive switched systems, switching between two (subystems of) Metzler Hurwitz matrices* A_1 *and* A_2, *asymptotic stability is equivalent to the fact that* $A_1 A_2^{-1}$ *has no real negative eigenvalue, and in this case*

$$(A + \mathbf{b}\mathbf{c}_1^\top)(A + \mathbf{b}\mathbf{c}_2^\top)^{-1} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1/2 & 0 \\ 1 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} 4/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}$$

has characteristic polynomial $s^2 - 5/3s + 2/3$ *whose roots have positive real part. So, the CPSS is asymptotically stable, but a CLCLF does not exist.*

Suppose, now, that A is diagonal and Hurwitz. We want to investigate under what conditions the Metzler matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, admit a common linear copositive Lyapunov function.

Proposition 2 *Given a diagonal Hurwitz matrix* $A \in \mathbb{R}^{n\times n}$, *and vectors* $\mathbf{b} \in \mathbb{R}^n$, *and* $\mathbf{c}_i \in \mathbb{R}^n, i \in [1,p]$, *suppose that for every index* $i \in [1,p]$, *the matrix* $A + \mathbf{b}\mathbf{c}_i^\top$ *is Metzler and Hurwitz. The matrices* $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, *admit a common linear copositive Lyapunov function if and only if the matrix* $A + \mathbf{b}\mathbf{c}_*^\top$ *is Hurwitz, where*

$$[\mathbf{c}_*]_i = \begin{cases} \max_{j\in[1,p]}[\mathbf{c}_j]_i, & \text{if } [\mathbf{b}]_i > 0; \\ \min_{j\in[1,p]}[\mathbf{c}_j]_i, & \text{if } [\mathbf{b}]_i \le 0; \end{cases} \qquad i \in [1,n].$$

Proof. Assume that $A = \operatorname{diag}\{a_{11}, a_{22}, \dots, a_{nn}\}$, with $a_{ii} < 0$. Then its characteristic polynomial is

$$\det(sI_n - A) = \prod_{i=1}^{n}(s - a_{ii}) = s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_0,$$

with $\alpha_i > 0$ for every $i \in [0, n-1]$. Also, note that

$$\operatorname{adj}(sI_n - A)\mathbf{b} = \begin{bmatrix} \prod_{i\neq 1}(s-a_{ii})b_1 \\ \vdots \\ \prod_{i\neq n}(s-a_{ii})b_n \end{bmatrix} = \begin{bmatrix} b_1 \sum_{i=0}^{n-1} \beta_i^{(1)} s^i \\ \vdots \\ b_n \sum_{i=0}^{n-1} \beta_i^{(n)} s^i \end{bmatrix},$$

where all the coefficients $\beta_i^{(k)}, i \in [0, n-1], k \in [1, n]$, are positive, because they come from a Hurwitz polynomial. It is well-known (Fornasini and Valcher, 2010; Knorn, Mason and Shorten, 2009) that the matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1, p]$, admit a common linear copositive Lyapunov function if and only if for every choice of the indices $i_1, i_2, \dots, i_n \in [1, p]$, the matrix

$$A + \mathbf{b}\,[\,[\mathbf{c}_{i_1}]_1 \quad \dots \quad [\mathbf{c}_{i_n}]_n\,]$$

is Hurwitz. But since it is a Metzler matrix, this is true if and only if (Farina and Rinaldi, 2000) its characteristic polynomial

$$\begin{aligned}
& \det\Big(sI_n - A - \mathbf{b}\,[\,[\mathbf{c}_{i_1}]_1 \quad \dots \quad [\mathbf{c}_{i_n}]_n\,]\Big) \\
= \; & \det(sI_n - A)\det\Big(I_n - (sI_n - A)^{-1}\mathbf{b}\,[\,[\mathbf{c}_{i_1}]_1 \quad \dots \quad [\mathbf{c}_{i_n}]_n\,]\Big) \\
= \; & \det(sI_n - A)\Big(1 - [\,[\mathbf{c}_{i_1}]_1 \quad \dots \quad [\mathbf{c}_{i_n}]_n\,]\,(sI_n - A)^{-1}\mathbf{b}\Big) \\
= \; & \det(sI_n - A) - [\,[\mathbf{c}_{i_1}]_1 \quad \dots \quad [\mathbf{c}_{i_n}]_n\,]\operatorname{adj}(sI_n - A)\mathbf{b} \\
= \; & [s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_0] - \sum_{k=1}^{n}[\mathbf{c}_{i_k}]_k b_k \left(\sum_{i=0}^{n-1}\beta_i^{(k)} s^i\right) \\
= \; & s^n + \Big(\alpha_{n-1} - \sum_{k=1}^{n}[\mathbf{c}_{i_k}]_k b_k \beta_{n-1}^{(k)}\Big)s^{n-1} + \ldots + \Big(\alpha_1 - \sum_{k=1}^{n}[\mathbf{c}_{i_k}]_k b_k \beta_1^{(k)}\Big)s \\
+ \; & \Big(\alpha_0 - \sum_{k=1}^{n}[\mathbf{c}_{i_k}]_k b_k \beta_0^{(k)}\Big)
\end{aligned}$$

has all positive coefficients. This is the case if and only if

$$\begin{cases} \sum_{k=1}^{n}[\mathbf{c}_{i_k}]_k b_k \beta_{n-1}^{(k)} & < & \alpha_{n-1}, \\ & \vdots & \\ \sum_{k=1}^{n}[\mathbf{c}_{i_k}]_k b_k \beta_1^{(k)} & < & \alpha_1, \\ \sum_{k=1}^{n}[\mathbf{c}_{i_k}]_k b_k \beta_0^{(k)} & < & \alpha_0. \end{cases}$$

In matrix form, this corresponds to

$$[\,[\mathbf{c}_{i_1}]_1 \quad \dots \quad [\mathbf{c}_{i_n}]_n\,] \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{bmatrix} \begin{bmatrix} \beta_0^{(1)} & \dots & \beta_{n-1}^{(1)} \\ \vdots & \ddots & \vdots \\ \beta_0^{(n)} & \dots & \beta_{n-1}^{(n)} \end{bmatrix} \ll [\,\alpha_0 \quad \dots \quad \alpha_{n-1}\,].$$

Now, as the matrix of the β coefficients is strictly positive, it clear that the above condition is satisfied for every choice of the indices $i_1, i_2, \dots, i_n$ if and only if it is satisfied in the worst case, that corresponds to choosing $[\mathbf{c}_{i_k}]_k := \max_{j\in[1,p]}[\mathbf{c}_j]_k$ when $[\mathbf{b}]_k > 0$, and $[\mathbf{c}_{i_k}]_k := \min_{j\in[1,p]}[\mathbf{c}_j]_k$ when $[\mathbf{b}]_k \le 0$. So, the worst case is the one corresponding to $\mathbf{c}_*^\top$.

But the fact that $\det\left(sI_n - A - \mathbf{b}\mathbf{c}_*^\top\right)$ has all positive coefficients is equivalent to the fact that $A + \mathbf{b}\mathbf{c}_*^\top$ is Hurwitz, and this completes the proof. ■

We now explore a well-known necessary condition for asymptotic stability, namely the Hurwitz property of all convex combinations of the subsystem matrices. In the general case of CPSSs (1), asymptotic stability implies that the convex hull generated by the subsystem matrices consists of Hurwitz matrices*. This fact, in turn, ensures that the subsystem matrices are Hurwitz. However, neither of these implications can be reversed (see, in particular, Fainshil, Margaliot and Chigansky, 2009; Gurvits, Mason and Shorten, 2007).

In the special case, when all the matrices take the form (2), the Hurwitz property of the Metzler matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, is just equivalent to the fact that every matrix belonging to their convex hull is Hurwitz. To prove this, it is convenient to introduce the following polynomials:

$$d(s) := \det(sI_n - A), \tag{7}$$

$$n_i(s) := \mathbf{c}_i^\top \operatorname{adj}(sI_n - A)\mathbf{b}, \qquad i \in [1,p]. \tag{8}$$

Proposition 3 *Given an $n \times n$ matrix A, let $\mathbf{b}, \mathbf{c}_i \in \mathbb{R}^n, i \in [1,p]$, be column vectors such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler for every $i \in [1,p]$. The following are equivalent:*

i) $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, are Hurwitz matrices;

ii) the convex combination $\sum_{i=1}^p \alpha_i(A + \mathbf{b}\mathbf{c}_i^\top)$ is Hurwitz for every choice of the parameters $\alpha_i \in [0,1]$, with $\sum_{i=1}^p \alpha_i = 1$.

Proof. i) $\Rightarrow$ ii) We first notice that

$$\begin{aligned}
\det(sI_n - \sum_{i=1}^p \alpha_i(A + \mathbf{b}\mathbf{c}_i^\top)) &= \det(sI_n - A - \mathbf{b}(\sum_{i=1}^p \alpha_i \mathbf{c}_i^\top)) \\
&= \det(sI_n - A)\det(I_n - (sI_n - A)^{-1}\mathbf{b}(\sum_{i=1}^p \alpha_i \mathbf{c}_i^\top)) \\
&= \det(sI_n - A)(1 - (\sum_{i=1}^p \alpha_i \mathbf{c}_i^\top)(sI_n - A)^{-1}\mathbf{b}) \\
&= d(s)\left(1 - \sum_{i=1}^p \alpha_i \frac{n_i(s)}{d(s)}\right) = d(s) - \sum_{i=1}^p \alpha_i n_i(s),
\end{aligned}$$

*It is obvious that the convex combination of Metzler matrices is still Metzler.

where we used the property that if $P \in \mathbb{R}^{n\times m}$ and $Q \in \mathbb{R}^{m\times n}$, then $\det(I_n - PQ) = \det(I_m - QP)$.

If there exists some choice of the parameters $\alpha_i \in [0,1]$, with $\sum_{i=1}^p \alpha_i = 1$, such that $\sum_{i=1}^p \alpha_i(A + \mathbf{b}\mathbf{c}_i^\top)$ is not Hurwitz, then, by continuity, there exists a special choice $\bar{\alpha}_i \in [0,1]$, with $\sum_{i=1}^p \bar{\alpha}_i = 1$, such that $\sum_{i=1}^p \bar{\alpha}_i(A + \mathbf{b}\mathbf{c}_i^\top)$ has dominant eigenvalue 0. This implies that

$$0 = \det(sI_n - \sum_{i=1}^p \bar{\alpha}_i(A + \mathbf{b}\mathbf{c}_i^\top))\Big|_{s=0} = d(0) - \sum_{i=1}^p \bar{\alpha}_i n_i(0). \tag{9}$$

However, if all matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, are Hurwitz, then

$$\det(sI_n - (A + \mathbf{b}\mathbf{c}_i^\top))\Big|_{s=0} = d(0) - n_i(0) > 0, \qquad \forall\, i \in [1,p].$$

But this implies that $\sum_{i=1}^p \bar{\alpha}_i n_i(0) \le \max_{i\in[1,p]} n_i(0) < d(0)$, thus contradicting (9). Therefore, condition ii) holds.

ii) $\Rightarrow$ i) is obvious. ■

When dealing with two-dimensional CPSSs (1), whose matrices are described as in (2), the Hurwitz stability of all convex combinations implies the asymptotic stability of the system, and hence we get the following corollary, previously derived in Mason and Shorten (2007b, see Theorem 4.4), by different means.

COROLLARY 1 *Given a* 2×2 *matrix* A*, let* $\mathbf{b}, \mathbf{c}_i \in \mathbb{R}^2, i \in [1,p]$*, be column vectors such that* $A + \mathbf{b}\mathbf{c}_i^\top$ *is Metzler for every* $i \in [1,p]$*. The following are equivalent:*

i) the CPSS (1) with subsystem matrices $A+\mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$*, is asymptotically stable;*

ii) $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$*, are Hurwitz matrices.*

Proof. i) $\Rightarrow$ ii) is obvious.

ii) $\Rightarrow$ i) By Proposition 3, if the subsystem matrices are all Hurwitz, then their convex hull consists of Hurwitz matrices and hence, for every pair of indices $i, j \in [1,p]$, all the convex combinations of $A + \mathbf{b}\mathbf{c}_i^\top$ and $A + \mathbf{b}\mathbf{c}_j^\top$ are Hurwitz.

By Gurvits, Mason and Shorten (2007), (Theorem 3.2), this allows to say that the two-dimensional positive switched system

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \qquad A_{\sigma(t)} \in \{A + \mathbf{b}\mathbf{c}_i^\top, A + \mathbf{b}\mathbf{c}_j^\top\},$$

is asymptotically stable for every choice of i and j. So, by Theorem 3.3 in Gurvits, Mason and Shorten (2007), condition i) holds. ■

3. The case of two subsystems

As a special case, in this section we consider CPSSs switching between two subsystems whose matrices differ by a rank one matrix, and hence described without loss of generality by the following equation

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \qquad t \in \mathbb{R}_+, \qquad A_\sigma(t) \in \{A, A + \mathbf{b}\mathbf{c}^\top\}, \tag{10}$$

where $A \in \mathbb{R}^{n\times n}$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, A and $A + \mathbf{b}\mathbf{c}^\top$ are Metzler Hurwitz matrices.

When we are dealing only with two matrices, A and $A + \mathbf{b}\mathbf{c}^\top$, the existence of a common linear copositive Lyapunov function can be characterized as a corollary to Proposition 1.

Corollary 2 *Given $A \in \mathbb{R}^{n\times n}$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, suppose that A and $A + \mathbf{b}\mathbf{c}^\top$ are Metzler and Hurwitz. These two matrices admit a common linear copositive Lyapunov function if and only if for every diagonal matrix $D = \mathrm{diag}\{d_1, d_2, \dots, d_n\}$, $d_i \in [0, 1]$, the matrix $A + \mathbf{b}\mathbf{c}^\top D$ is Metzler Hurwitz.*

Proof. By Proposition 1, there exists a CLCLF for A and $A+\mathbf{b}\mathbf{c}^\top$ if and only if for every choice of diagonal matrices D_A and D, with nonnegative diagonal entries and such that $D_A + D = I_n$, the matrix $AD_A + (A + \mathbf{b}\mathbf{c}^\top)D = A(D_A + D) + \mathbf{b}\mathbf{c}^\top D$ is Metzler Hurwitz. But since it is clear that D is an arbitrary diagonal matrix with entries in $[0, 1]$, the result is proved. ■

Similarly to what we did in the previous section, we introduce the following polynomials:

$$d(s) := \det(sI_n - A), \tag{11}$$
$$n(s) := \mathbf{c}^\top \mathrm{adj}(sI_n - A)\mathbf{b}, \tag{12}$$

and notice that for every $\alpha \in [0, 1]$ the characteristic polynomial of $A(\alpha) := (1 - \alpha)A + \alpha(A + \mathbf{b}\mathbf{c}^\top) = A + \alpha\mathbf{b}\mathbf{c}^\top$ is

$$\det(sI_n - A(\alpha)) = d(s) - \alpha n(s). \tag{13}$$

Proposition 4 *Let $A \in \mathbb{R}^{n\times n}$ be a Metzler matrix, and assume that $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ are column vectors such that $A + \mathbf{b}\mathbf{c}^\top$ is Metzler, in turn. The following are equivalent:*

i) A and $A + \mathbf{b}\mathbf{c}^\top$ are both Hurwitz matrices;
ii) the convex combination $A(\alpha)$ is Hurwitz for every $\alpha \in [0, 1]$;
iii) $1 + \mathbf{c}^\top A^{-1}\mathbf{b} > 0$ or, equivalently, $\frac{n(0)}{d(0)} < 1$.

Proof. i) ⇔ ii) Follows from Proposition 3.

ii) ⇔ iii) We preliminarily notice that since $A(1) = A+\mathbf{b}\mathbf{c}^\top$ is Metzler Hurwitz, then $\det(sI_n - A(1))|_{s=0} = d(0) - n(0) \neq 0$, and hence, $\frac{n(0)}{d(0)} \neq 1$. So, we only need to prove that ii) holds iff $\frac{n(0)}{d(0)} \leq 1$.

It is known (see Horn and Johnson, 1991, page 127 [†]) that, given two Metzler Hurwitz matrices A and B, their convex combination $(1-\alpha)A+\alpha B$ is Metzler Hurwitz for every $\alpha \in [0,1]$ if and only if $A^{-1}B$ has no real negative eigenvalue. So, in the special case when $B = A + \mathbf{b}\mathbf{c}^\top$, the convex combination $A(\alpha)$ is Metzler Hurwitz for every $\alpha \in [0,1]$ if and only if $A^{-1}(A+\mathbf{b}\mathbf{c}^\top) = I_n + A^{-1}\mathbf{b}\mathbf{c}^\top$ has no real negative eigenvalue. It is easy to check that

$$\det(sI_n - I_n - A^{-1}\mathbf{b}\mathbf{c}^\top) \neq 0, \qquad \forall\, s \in \mathbb{R},\ s<0,$$

if and only if

$$(s-1)^n[1-(s-1)^{-1}\mathbf{c}^\top A^{-1}\mathbf{b}] \neq 0, \qquad \forall\, s \in \mathbb{R},\ s<0,$$

and this, in turn, happens if and only if

$$(s-1-\mathbf{c}^\top A^{-1}\mathbf{b}) \neq 0, \qquad \forall\, s \in\ \mathbb{R},\ s<0,$$

which means that $1+\mathbf{c}^\top A^{-1}\mathbf{b} \geq 0$. ■

As previously recalled, a two-dimensional CPSS (10), switching between two subystems, is asymptotically stable if and only if the convex hull of these two matrices consists of Hurwitz matrices, and this happens if and only if (Gurvits, Mason and Shorten, 2007) these two matrices admit a common quadratic Lyapunov function (CQLF). So, when the matrices differ by a rank one matrix, we have the following complete picture.

Corollary 3 *(Akar, Paul, Safonov and Mitra, 2006; Gurvits, Mason and Shorten, 2007; Mason and Shorten, 2007b) Let $A \in \mathbb{R}^{2\times 2}$ be a Metzler matrix, and assume that $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ are column vectors such that $A+\mathbf{b}\mathbf{c}^\top$ is Metzler, in turn. The following are equivalent:*

i) A and $A+\mathbf{b}\mathbf{c}^\top$ are both Hurwitz matrices;
ii) $A(\alpha) = A+\alpha\mathbf{b}\mathbf{c}^\top$ is Hurwitz for every $\alpha \in [0,1]$;
iii) $1+\mathbf{c}^\top A^{-1}\mathbf{b} > 0$;
iv) A and $A+\mathbf{b}\mathbf{c}^\top$ admit a CQLF, i.e. a function $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$, $P=P^\top \succ 0$, such that

$$\begin{cases} A^\top P + PA & \prec & 0, \\ (A+\mathbf{b}\mathbf{c}^\top)^\top P + P(A+\mathbf{b}\mathbf{c}^\top) & \prec & 0; \end{cases}$$

v) the CPSS (1) is asymptotically stable.

Note that the existence of a CLCLF of the matrices A and $A+\mathbf{b}\mathbf{c}^\top$ remains a stronger condition with respect to asymptotic stability, as shown in Example 1.

To conclude the section, we provide the following result that shows how the controllability and observability properties of the pairs $(A,\mathbf{b})$ and $(A,\mathbf{c}^\top)$, respectively, can be related to asymptotic stability.

[†]As a matter of fact, the result is given for M-matrices, but it is well-known that A is Metzler Hurwitz if and only if $-A$ is an M-matrix.

Proposition 5 *Let $A \in \mathbb{R}^{n\times n}$ be an irreducible Metzler matrix, and assume that $\mathbf{b},\mathbf{c} \in \mathbb{R}^n$ are column vectors such that $A + \mathbf{b}\mathbf{c}^\top$ is Metzler, in turn.*

i) If A is Hurwitz and $\lambda_{\max}(A)$ is a not controllable eigenvalue for the pair $(A, \mathbf{b})$, then the switched system (10) is asymptotically stable.

ii) If A is Hurwitz and $\lambda_{\max}(A)$ is a not observable eigenvalue for the pair $(A, \mathbf{c}^\top)$, then the switched system (10) is asymptotically stable.

Proof. i) As A is irreducible, the eigenspace corresponding to $\lambda_{\max}(A)$ has unitary dimension and it is generated by a strictly positive vector $\mathbf{v}$. So, there must be

$$\mathbf{v}^\top [\,\lambda_{\max}(A)I_n - A \quad | \quad \mathbf{b}\,] = [\,0^\top \quad 0\,].$$

This implies that

$$\mathbf{v}^\top A \ll 0 \qquad \text{and} \qquad \mathbf{v}^\top(A + \mathbf{b}\mathbf{c}^\top) \ll 0,$$

namely the matrices A and $A+\mathbf{b}\mathbf{c}^\top$ have a common linear copositive Lyapunov function $V(\mathbf{x}) := \mathbf{v}^\top\mathbf{x}$. This ensures the asymptotic stability of the system.

ii) By proceeding as in part i), we can show that there exists $\mathbf{w} \gg 0$ such that

$$A\mathbf{w} = \lambda_{\max}(A)\mathbf{w} \qquad \text{and} \qquad (A + \mathbf{b}\mathbf{c}^\top)\mathbf{w} = \lambda_{\max}(A)\mathbf{w}.$$

Clearly, for every choice of the switching sequence σ, the state trajectory starting from $\mathbf{x}(0) = \mathbf{w}$ satisfies

$$\mathbf{x}(t) = e^{\lambda_{\max}(A)t}\mathbf{w},$$

and hence converges to zero. On the other hand, for every positive state $\mathbf{x}(0)$, a positive real number $M > 0$ can be found such that $M\mathbf{w} \geq \mathbf{x}(0)$. Consequently, for every choice of the switching sequence σ, the state trajectory starting from $\mathbf{x}(0)$ satisfies

$$\mathbf{x}(t) \leq Me^{\lambda_{\max}(A)t}\mathbf{w},$$

and hence converges to zero, too. ■

4. The case of two subsystems with commuting matrices

In this section we want to investigate the asymptotic stability of continuous-time switched systems described as in (10), with $A_{\sigma(t)} \in \{A, A+\mathbf{b}\mathbf{c}^\top\}$ for every $t \in \mathbb{R}_+$, under the assumption that the matrices A and $A+\mathbf{b}\mathbf{c}^\top$ commute. We have this preliminary lemma.

Lemma 1 *Given a matrix $A \in \mathbb{R}^{n\times n}$ and two vectors $\mathbf{b},\mathbf{c} \in \mathbb{R}^n$, the matrices A and $A+\mathbf{b}\mathbf{c}^\top$ commute, by this meaning that*

$$A(A + \mathbf{b}\mathbf{c}^\top) = (A + \mathbf{b}\mathbf{c}^\top)A, \tag{14}$$

if and only if there exists $\mu \in \mathbb{R}$ such that

$$A\mathbf{b} = \mu\mathbf{b} \qquad \textit{and} \qquad \mathbf{c}^\top A = \mu\mathbf{c}^\top. \tag{15}$$

Proof. Clearly, (15) ensures that A and $A + \mathbf{b}\mathbf{c}^\top$ commute. Conversely, condition (14) is equivalent to $(A\mathbf{b})\mathbf{c}^\top = \mathbf{b}(\mathbf{c}^\top A)$. These two expressions can be regarded as two factorizations of the same real matrix of rank 1. Consequently, there must be some real number μ such that $\mathbf{c}^\top A = \mu\mathbf{c}^\top$. But this immediately leads also to the identity $A\mathbf{b} = \mu\mathbf{b}$. ■

It was proved in Barabanov (1988) that if the system matrices A and $A+\mathbf{b}\mathbf{c}^\top$ commute, the asymptotic stability of the corresponding switched system (10) is equivalent to the fact that both matrices are Hurwitz. By making use of this fact, we can provide equivalent conditions for asymptotic stability.

PROPOSITION 6 *Given a matrix $A \in \mathbb{R}^{n\times n}$ and two vectors $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, suppose that condition (15) holds for some real number μ.*

i) If $\mu < 0$, then the switched system (10) is asymptotically stable if and only if $A + \mathbf{b}\mathbf{c}^\top$ is Hurwitz;

ii) if $\mathbf{c}^\top\mathbf{b} \leq 0$, then the switched system (10) is asymptotically stable if and only if A is Hurwitz.

Proof. The "only if" parts are obvious, and hence we prove only the two sufficiencies.
i) In this case we just need to prove that if $A + \mathbf{b}\mathbf{c}^\top$ is Hurwitz, then also A is. Suppose, by contradiction, that A is not Hurwitz and hence there exists $\lambda \in \mathbb{C}, \mathrm{Re}(\lambda) \geq 0$, and $\mathbf{w} \neq 0$ such that $\mathbf{w}^\top A = \lambda\mathbf{w}^\top$. Then

$$\mathbf{w}^\top[A + \mathbf{b}\mathbf{c}^\top] = \mathbf{w}^\top A + \mathbf{w}^\top\mathbf{b}\mathbf{c}^\top = \lambda\mathbf{w}^\top + \mathbf{w}^\top\mathbf{b}\mathbf{c}^\top.$$

If we postmultiply the previous expressions by the vector $\mathbf{b}$, we get

$$\mathbf{w}^\top[A + \mathbf{b}\mathbf{c}^\top]\mathbf{b} = \lambda(\mathbf{w}^\top\mathbf{b}) + (\mathbf{w}^\top\mathbf{b})(\mathbf{c}^\top\mathbf{b}).$$

On the other hand, the left hand-side of the last identity can also be rewritten as

$$\mathbf{w}^\top[A\mathbf{b} + \mathbf{b}\mathbf{c}^\top\mathbf{b}] = \mathbf{w}^\top[\mu\mathbf{b} + \mathbf{b}\mathbf{c}^\top\mathbf{b}] = \mu(\mathbf{w}^\top\mathbf{b}) + (\mathbf{w}^\top\mathbf{b})(\mathbf{c}^\top\mathbf{b}).$$

So, by comparing the two expressions, we obtain

$$\lambda(\mathbf{w}^\top\mathbf{b}) = \mu(\mathbf{w}^\top\mathbf{b}),$$

and since $\lambda \neq \mu$, there must be $\mathbf{w}^\top\mathbf{b} = 0$. Consequently,

$$\mathbf{w}^\top[A + \mathbf{b}\mathbf{c}^\top] = \lambda\mathbf{w}^\top.$$

But this proves that λ is an eigenvalue of $A + \mathbf{b}\mathbf{c}^\top$ and hence $\mathrm{Re}(\lambda) < 0$, a contradiction.

So, as we have proved that both A and $A + \mathbf{b}\mathbf{c}^\top$ are Hurwitz and they commute, by assumption, the switched system is asymptotically stable.

ii) Notice, first, that if A is Hurwitz, then $\mu < 0$. Again, we need to prove that if A is Hurwitz, then also $A + \mathbf{b}\mathbf{c}^\top$ is. By proceeding as in part i), we assume, by contradiction, that $A + \mathbf{b}\mathbf{c}^\top$ is not Hurwitz and hence there exists $\lambda \in \mathbb{C}, \mathrm{Re}(\lambda) \geq 0$, and $\mathbf{w} \neq 0$ such that $\mathbf{w}^\top[A + \mathbf{b}\mathbf{c}^\top] = \lambda \mathbf{w}^\top$. Then

$$\mathbf{w}^\top[A + \mathbf{b}\mathbf{c}^\top]\mathbf{b} = \lambda \mathbf{w}^\top \mathbf{b}.$$

On the other hand, the left hand-side of the previous identity can also be rewritten as

$$\mathbf{w}^\top[A\mathbf{b} + \mathbf{b}\mathbf{c}^\top\mathbf{b}] = \mathbf{w}^\top[\mu\mathbf{b} + \mathbf{b}\mathbf{c}^\top\mathbf{b}] = \mu(\mathbf{w}^\top\mathbf{b}) + (\mathbf{w}^\top\mathbf{b})(\mathbf{c}^\top\mathbf{b}).$$

So, by comparing the two expressions, we obtain

$$\lambda(\mathbf{w}^\top\mathbf{b}) = \mu(\mathbf{w}^\top\mathbf{b}) + (\mathbf{w}^\top\mathbf{b})(\mathbf{c}^\top\mathbf{b}).$$

If $\mathbf{w}^\top\mathbf{b}$ were zero, then $\mathbf{w}^\top A = \lambda \mathbf{w}^\top$, thus contradicting the assumption that A is Hurwitz. So, $\mathbf{w}^\top\mathbf{b} \neq 0$ and $\lambda = \mu + \mathbf{c}^\top\mathbf{b}$. As $\mu < 0$ and $\mathbf{c}^\top\mathbf{b} \leq 0$, λ cannot be greater than or equal to 0. ■

We now consider asymptotic stability in case of Metzler commuting pairs.

Proposition 7 *Given a matrix $A \in \mathbb{R}^{n\times n}$ and two vectors $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, suppose that condition (15) holds for some real number μ, and A and $A + \mathbf{b}\mathbf{c}^\top$ are both Metzler matrices, with A irreducible. The following are equivalent:*
i) A and $A + \mathbf{b}\mathbf{c}^\top$ are both Hurwitz matrices;
ii) A and $A + \mathbf{b}\mathbf{c}^\top$ have a common linear copositive Lyapunov function;
iii) the CPSS (10) is asymptotically stable.

Proof. ii) $\Rightarrow$ iii) $\Rightarrow$ i) are obvious, and hence we need to prove only i) $\Rightarrow$ ii). As A is irreducible, there exists a strictly positive left eigenvector $\mathbf{z}^\top$ of A corresponding to the dominant eigenvalue $\lambda_{\max}(A) < 0$. From $\mathbf{z}^\top A = \lambda_{\max}(A)\mathbf{z}^\top$, there follows

$$\mathbf{z}^\top A\mathbf{b} = \lambda_{\max}(A)\mathbf{z}^\top\mathbf{b}.$$

On the other hand, it is also true that

$$\mathbf{z}^\top A\mathbf{b} = \mathbf{z}^\top[A\mathbf{b}] = \mu\mathbf{z}^\top\mathbf{b}.$$

This implies $\lambda_{\max}(A)\mathbf{z}^\top\mathbf{b} = \mu\mathbf{z}^\top\mathbf{b}$. If $\mathbf{z}^\top\mathbf{b} = 0$, then $\mathbf{z}$ defines a CLCLF for A and $A + \mathbf{b}\mathbf{c}^\top$. If $\mathbf{z}^\top\mathbf{b} \neq 0$, then $\lambda_{\max}(A) = \mu$, and hence $\mathbf{b}$ and $\mathbf{c}$ are (right and left, respectively) eigenvectors of A corresponding to the dominant eigenvalue. This implies that each of them is either strictly positive or strictly negative. Consequently, either $A \ll A + \mathbf{b}\mathbf{c}^\top$ or $A + \mathbf{b}\mathbf{c}^\top \ll A$. So, in the first case, any strictly positive vector $\mathbf{v}$ such that $\mathbf{v}^\top[A + \mathbf{b}\mathbf{c}^\top] \ll 0$ (and such a vector exists because $A + \mathbf{b}\mathbf{c}^\top$ is Metzler Hurwitz) satisfies $\mathbf{v}^\top A \ll 0$. In the second case, $\mathbf{z}^\top[A + \mathbf{b}\mathbf{c}^\top] \ll \mathbf{z}^\top A \ll 0$. So, we always have a CLCLF. ■

References

Akar, M., Paul, A., Safonov, M.G. and Mitra, U. (2006) Conditions on the stability of a class of second order switched systems. *IEEE Trans. Aut. Contr.*, **51**, 338–340.

Aliprantis, C.D. and Tourky, R. (2007) *Cones and Duality.* American Mathematical Society.

Barabanov, N.E. (1988) Absolute characteristic exponent of a class of linear nonstationary systems of differential equations. *Siberian Math. J.*, **29**, 521–530.

Berman, A. and Plemmons, R.J. (1979) *Nonnegative Matrices in the Mathematical Sciences.* Academic Press, New York (NY).

Blanchini, F., Colaneri, P. and Valcher, M.E. (2011) Is stabilization of switched positive linear systems equivalent to the existence of a Hurwitz convex combination of the system matrices? In: *Proceedings of the 50th IEEE CDC.* IEEE Orlando, FL, 1834 – 1839.

Blanchini, F., Colaneri, P. and Valcher, M.E. (2012) Co-positive Lyapunov functions for the stabilization of positive switched systems. *IEEE Trans.Aut. Contr.*, DOI 10.1109/TAC.2012.2199169.

Fainshil, L., Margaliot, M. and Chigansky, P. (2009) On the stability of positive linear switched systems under arbitrary switching laws. *IEEE Trans. Automat. Control*, **54**, 807–899.

Farina, L. and Rinaldi, S. (2000) *Positive linear Systems: Theory and Applications.* Wiley-Interscience, Series on Pure and Applied Mathematics, New York.

Fornasini, E. and Valcher, M.E. (2010) Linear copositive Lyapunov functions for continuous-time positive switched systems. *IEEE Trans. Aut. Contr.*, **55** (8), 1933–1937.

Gurvits, L., Shorten, R. and Mason, O. (2007) On the stability of switched positive linear systems. *IEEE Trans. on Aut. Contr.*, **52** (6),1099–1103.

Horn, R.A. and Johnson, C.R. (1991) *Topics in Matrix Analysis.* Cambridge University Press, Cambridge.

Kaczorek, T. (2002) *Positive 1D and 2D systems.* Springer Verlag, London.

King, C. and Nathanson, M. (2006) On the existence of common quadratic Lyapunov functions for a rank one difference. *Linear Algebra and its Appl.*, **419**, 400–416.

Knorn, F., Mason, O. and Shorten, R.N. (2009) On linear co-positive Lyapunov functions for sets of linear positive systems. *Automatica*, **45** (8), 1943–1947.

Mason, O. and Shorten, R.N. (2004) Some results on the stability of positive switched linear systems. In *Proceedings of the 43rd Conference on Decision and Control (CDC 2004)*, Paradise Island, Bahamas, 4601–4606.

Mason, O. and Shorten, R.N. (2007a) On linear copositive Lyapunov functions and the stability of switched positive linear systems. *IEEE Trans. on Aut. Contr.*, **52** (7), 1346 – 1349.

Mason, O. and Shorten, R.N. (2007b) Quadratic and copositive Lyapunov functions and the stability of positive switched linear systems. In: *Proceedings of the American Control Conference (ACC 2007)*, New York. IEEE, 657–662.

Shorten, R., Corless., M., Wulff, K., Klinge, S. and Middleton, R. (2009) Quadratic stability and singular SISO switching systems. *IEEE Trans. Aut. Contr.*, **54** (11), 2714–2718.

Shorten, R., Mason, O., O'cairbre, F. and Curran, P. (2004) A unifying framework for the SISO circle criterion and other quadratic stability criteria. *Int. J. Contr.*, **77** (1), 1–8.

Shorten, R., Wirth, F., Mason, O., Wulff, K., and King, C. (2007) Stability criteria for switched and hybrid systems. *SIAM Review*, **49**, 545–592.

Zappavigna , A., Colaneri, P., Jeromel, J.C., and Middleton, R. (2010) Stabilization of continuous-time switched linear positive systems. In *Proceedings of the 2010 American Control Conference*, Baltimore, MD. IEEE, 3275–3280.