Control and Cybernetics

vol. **42** (2013) No. 2

Positive realizations on time scales*

by

Zbigniew Bartosiewicz†

Faculty of Computer Science, Bialystok University of Technology
Wiejska 45A, 15-351 Bialystok, Poland
z.bartosiewicz@pb.edu.pl

Abstract: The problem of realization of a linear input-output map as a positive linear system on a time scale is studied. To state the criteria of existence of realization, modified Markov parameters corresponding to the input-output map are introduced. It is necessary for the existence of a positive realization that the modified Markov parameters be nonnegative. A necessary and sufficient condition for realizability is expressed in the language of positive cones in an infinite dimensional space. The sequence of modified Markov parameters generates one of the cones that appear in the criterion of realizability.

Keywords: positive system, time scale, realization, input-output map, Markov parameters

1. Introduction

In many applications variables take only positive or nonnegative values. This concerns, in particular, chemical systems, biological systems and compartmental systems. There exist parallel theories of linear discrete-time and continuous-time positive control systems (see Farina and Rinaldi, 2000; Kaczorek, 2002). The results for the two cases are often similar but sometimes essentially different.

Calculus on time scales allows for unification of both theories. The calculus is based on the concept of delta derivative which looks like an ordinary derivative if the time scale is the set of reals and looks like the forward difference for functions defined on the time scale of integers. Since a time scale may be an arbitrary closed subset of the reals, dynamical systems on time scales can describe more complicated phenomena where time is not homogeneous.

Positive linear systems on time scales were studied in Bartosiewicz (2012 and 2013), where controllability and observability were examined. Modified Gram matrices were used to state criteria of positive reachability and positive observability for linear systems on arbitrary time scales. It was also shown that

*Submitted: September 2012; Accepted: April 2013.

†Supported by Bialystok University of Technology grant S/WI/2/2011

for specific time scales, $\mathbb{R}$ and $\mathbb{Z}$, the general statement splits into essentially different conditions (see Bru, Romero and Sanchez, 2000; Commault, 2004; Commault and Alamir, 2007; Coxson and Shapiro, 1987; Damm and Ethington, 2009; Fanti, Maione and Turchiano, 1990; Kaczorek, 2002, 2007; Ohta, Maeda and Kodama, 1984; Valcher, 1996, 2009, for results on specific time scales).

Realizations of linear systems on time scales were studied in Bartosiewicz and Pawłuszewicz (2006), where necessary and sufficient conditions for an input-output map to have a finite-dimensional realization were developed. However, the systems did not have to be positive. Positive realizations were studied in both cases of continuous-time and discrete-time systems, with different approaches and techniques in Anderson, Deistler, Farina and Benvenuti (1996), Farina (1995, 1996), Farina and Rinaldi (2000), Kaczorek (2002), Maeda and Kodama (1981), Ohta, Maeda and Kodama (1984), van den Hof (1987).

In this paper we follow the ideas of J.M. van den Hof (1987) and use the language of cones in an infinite dimensional space to state the main result - the necessary and sufficient conditions for the existence of positive realizations. The result holds for an arbitrary infinite time scale. We use the modified Markov parameters corresponding to the input-output map. Contrary to the standard Markov parameters, they depend on the particular time scale.

2. Calculus on time scales

We provide here basic information on calculus on time scales. For more information the reader can consult, e.g., Bohner and Peterson (2001).

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. It is a topological space with the topology induced from $\mathbb{R}$.

EXAMPLE 1. $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = h\mathbb{Z}$ *for* $h > 0$ *and* $\mathbb{T} = q^{\mathbb{N}} := \{q^k, k \in \mathbb{N}\}$ *for* $q > 1$.

Definition 1. *For* $t \in \mathbb{T}$ *we define:*

- *the forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *by* $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ *if* $t \neq \sup \mathbb{T}$ *and* $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$ *when* $\sup \mathbb{T}$ *is finite;*
- *the backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ *by* $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ *if* $t \neq \inf \mathbb{T}$ *and* $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$ *when* $\inf \mathbb{T}$ *is finite;*
- *the forward graininess function* $\mu : \mathbb{T} \to [0, \infty)$ *by* $\mu(t) := \sigma(t) - t$*;*
- *the backward graininess function* $\nu : \mathbb{T} \to [0, \infty)$ *by* $\nu(t) := t - \rho(t)$.

The time scale $\mathbb{T}$ *is* homogeneous, *if* μ *and* ν *are constant.*

If $\mathbb{T}$ *has a finite supremum* $M \in \mathbb{T}$ *and* $\rho(M) < M$, *then* $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$. *Otherwise* $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 2. *Let* $f : \mathbb{T} \to \mathbb{R}$ *and* $t \in \mathbb{T}^\kappa$. *The* delta derivative of f at t, *denoted by* $f^\Delta(t)$ *or* $\frac{\Delta f}{\Delta t}(t)$, *is the real number with the property that for any* $\varepsilon > 0$ *there is a neighborhood* $U = (t - \delta, t + \delta) \cap \mathbb{T}$ *such that*

$$|(f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

If $f^{\Delta}(t)$ exists, then we say that f is delta differentiable at t.

EXAMPLE 2. *If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = f'(t)$.*
If $\mathbb{T} = h\mathbb{Z}$, then $f^{\Delta}(t) = \frac{f(t+h)-f(t)}{h}$.
If $\mathbb{T} = q^{\mathbb{N}}$, then $f^{\Delta}(t) = \frac{f(qt)-f(t)}{(q-1)t}$.

Higher order delta derivatives are defined inductively.

Definition 3. *A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided that $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^{\kappa}$.*

A continuous function f has an antiderivative.

Definition 4. *Let $a, b \in \mathbb{T}$ and assume that f has an antiderivative on $[a, b] \cap \mathbb{T}$. Then the* delta integral *of f on the interval $[a, b)_{\mathbb{T}} : [a, b) \cap \mathbb{T}$ is defined by*

$$\int_a^b f(\tau)\Delta\tau := \int_{[a,b)_{\mathbb{T}}} f(\tau)\Delta\tau := F(b) - F(a).$$

EXAMPLE 3. *If $\mathbb{T} = \mathbb{R}$, then $\int\limits_a^b f(\tau)\Delta\tau = \int\limits_a^b f(\tau)d\tau$, where the integral on the right is the usual Riemann integral.*
If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\int\limits_a^b f(\tau)\Delta\tau = \sum\limits_{t=\frac{a}{h}}^{\frac{b}{h}-1} f(th)h$ for $a < b$.

Definition 5. *A function $f : [a, b)_{\mathbb{T}} \to \mathbb{R}$ is* piecewise continuous *if there are $t_0, t_1, \ldots, t_k \in \mathbb{T}$ such that $a = t_0 < t_1 < \ldots < t_k = b$, such that f is continuous on $[t_{i-1}, t_i)_{\mathbb{T}}$ for $i = 1, \ldots, k$, and has a finite left-hand limit at each t_i for $i = 1, \ldots, k$. Then we define*

$$\int\limits_a^b f(\tau)\Delta\tau := \sum_{i=1}^{k} \int\limits_{t_{i-1}}^{t_i} f(\tau)\Delta\tau.$$

3. Linear systems on time scales

Let us consider the system of delta differential equations on the time scale $\mathbb{T}$:

$$x^{\Delta}(t) = Ax(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and A is a constant $n \times n$ matrix.

PROPOSITION 1 (Bohner and Peterson, 2001). *Equation (1) with initial condition $x(t_0) = x_0$ has a unique forward solution defined for all $t \in [t_0, +\infty)_{\mathbb{T}} := [t_0, +\infty) \cap \mathbb{T}$.*

Definition 6. *The* matrix exponential function *(at t_0) for A is defined as the unique forward solution of the matrix differential equation $X^\Delta = AX$, with the initial condition $X(t_0) = I$. Its value at t is denoted by $e_A(t, t_0)$.*

As we need $e_A(t, t_0)$ only for $t \geq t_0$, we do not assume regressivity of A, which would allow to have $e_A(t, t_0)$ well defined also for $t < t_0$. In particular, $e_A(t, t_0)$ does not have to be invertible.

Example 4. *a) If $\mathbb{T} = \mathbb{R}$, then $e_A(t, t_0) = e^{A(t-t_0)}$.*
b) If $\mathbb{T} = \mathbb{Z}$, then $e_A(t, t_0) = (I + A)^{(t-t_0)}$.
c) If $\mathbb{T} = q^{\mathbb{N}}$, $q > 1$, then $e_A(q^k t_0, t_0) = \prod_{i=0}^{k-1}(I + (q-1)q^i t_0 A)$ for $k \geq 1$ and $t_0 \in \mathbb{T}$.

The following semigroup property will be needed later:

Proposition 2. *For $s, t, r \in \mathbb{T}$ and $s < t < r$ we have*

$$e_A(r, t)e_A(t, s) = e_A(r, s).$$

Let us consider now a nonhomogeneous system

$$x^\Delta(t) = Ax(t) + f(t) \tag{2}$$

where f is piecewise continuous.

Theorem 1 (Bartosiewicz and Pawłuszewicz, 2006). *Let $t_0 \in \mathbb{T}$. System (2) for the initial condition $x(t_0) = x_0$ has a unique forward solution of the form*

$$x(t) = e_A(t, t_0)x_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau))f(\tau)\Delta\tau. \tag{3}$$

4. Positive systems

We are going to use the following notation:
$\mathbb{R}_+$ - the set of nonnegative real numbers,
$\mathbb{Z}_+$ - the set of nonnegative integers,
$\mathbb{R}_+^k$ - the set of all column vectors in $\mathbb{R}^k$ with nonnegative components,
$\mathbb{R}_+^{k\times p}$ - the set of $k \times p$ real matrices with nonnegative elements.

Definition 7. *If $A \in \mathbb{R}_+^{k\times p}$ we write $A \geq 0$ and say that A is* nonnegative. *A nonnegative matrix A will be called* positive *if at least one of its elements is greater than 0. Then we shall write $A > 0$.*

By $\mathbb{R}^{\mathbb{Z}_+}$ we denote the set of infinite columns $z = (z_0, z_1, \ldots)^T$. Such z is *nonnegative* if all $z_i \geq 0$. Then, $\mathbb{R}_+^{\mathbb{Z}_+}$ consists of nonnegative columns.

Definition 8. *A subset C of a linear space X over $\mathbb{R}$ is called a* (positive convex) cone *if for any $\alpha \in \mathbb{R}_+$ and any $x \in C$, $\alpha x \in C$, and for any $x, y \in C$, $x + y \in C$. A cone C is* polyhedral, *if there are $x_1, \ldots, x_k \in C$ such that*

$$C = \{x = \sum_{i=1}^{k} \alpha_i x_i,\ \alpha_i \geq 0\}.$$

We shall consider cones in $\mathbb{R}^n$ and $\mathbb{R}^{\mathbb{Z}_+}$.

EXAMPLE 5. $\mathbb{R}_+^{\mathbb{Z}_+}$ *is a cone, but it is not polyhedral.*

Let us consider a linear control system, denoted by Σ, and defined on the time scale $\mathbb{T}$:

$$x^{\Delta}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t) \tag{4}$$

where $t \in \mathbb{T}$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$.

We assume that an input (control) u is piecewise continuous.

We say that system Σ is *positive* if for any $t_0 \in \mathbb{T}$, any initial condition $x_0 \in \mathbb{R}_+^n$, any control $u : [t_0, t_1) \to \mathbb{R}_+^m$ and any $t \in [t_0, t_1]$, the solution x of (4) satisfies $x(t) \in \mathbb{R}_+^n$ and also $y(t) \in \mathbb{R}_+^p$.

From the variation-of-constants formula (3) we can easily get the following characterization.

PROPOSITION 3. *The system Σ is positive if and only if $e_A(t, t_0) \in \mathbb{R}_+^{n\times n}$ for every $t, t_0 \in \mathbb{T}$ such that $t \geq t_0$, $B \in \mathbb{R}_+^{n\times m}$ and $C \in \mathbb{R}_+^{p\times n}$.*

Let $\bar{\mu} = \sup\{\mu(t) : t \in \mathbb{T}\}$. The following theorem gives a characterization of nonnegativity of the exponential matrix on the time scale. It is a modification of a characterization obtained in Bartosiewicz (2013).

THEOREM 2. *The exponential matrix $e_A(t, t_0)$ is nonnegative for every $t, t_0 \in \mathbb{T}$ such that $t \geq t_0$ if and only if there is $a \in \mathbb{R}_+$ such that $a \leq 1/\bar{\mu}$ and $A + aI \in \mathbb{R}_+^{n\times n}$. If $\bar{\mu} > 0$, then one can set $a = 1/\bar{\mu}$, where $1/+\infty = 0$.*

Proof. Sufficiency. Assume that $A + aI \geq 0$. If $\mu(t_0) > 0$, then $A + I/\mu(t_0) \geq 0$. This means that $e_A(\sigma(t_0), t_0) = \mu(t_0)A + I \geq 0$. If $\mu(t_0) = 0$, then for $t > t_0$ and close to t_0, $I + A(t - t_0) > 0$. The last term approximates $e_A(t, t_0)$. Since the exponential matrix is continuous (with respect to t), then also $e_A(t, t_0) > 0$ for $t > t_0$ and close to t_0. To achieve nonnegativity of $e_A(t, t_0)$ for all $t \in \mathbb{T}$, $t > t_0$, we have to use the semigroup property of the exponential matrix: $e_A(t, s)e_A(s, \tau) = e_A(t, \tau)$ for $\tau < s < t$ and $\tau, s, t \in \mathbb{T}$.
Necessity. Assume that $e_A(t, t_0)$ is nonnegative for $t, t_0 \in \mathbb{T}$ such that $t \geq t_0$. Suppose first that $\bar{\mu} > 0$ and choose $t_0 \in \mathbb{T}$ with $\mu(t_0) > 0$. Then $e_A(\sigma(t_0), t_0) = I + \mu(t_0)A \geq 0$. This means that also $A + I/\mu(t_0) \geq 0$. As it holds for all $t_0 \in \mathbb{T}$ with $\mu(t_0) > 0$, $A + I/\bar{\mu}$ is nonnegative. If $\bar{\mu} = 0$, then $\mathbb{T}$ is a standard interval. The exponential matrix is then standard $e^{A(t-t_0)}$. For t close to t_0, it may be approximated by $I + A(t - t_0)$. Nonnegativity of the exponential matrix implies that $I + A(t - t_0) > 0$ for $t > t_0$ and close to t_0. Thus, for some $a > 0$, $A + aI > 0$. ■

Corollary 1. *The system Σ is positive if and only if there is $a \in \mathbb{R}_+$ such that $a \le 1/\bar{\mu}$ and $A + aI \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$ and $C \in \mathbb{R}_+^{p\times n}$.*

Proof. This is a consequence of Proposition 3 and Theorem 2. ■

If $\mathbb{T} = \mathbb{R}$ the exponential matrix $e_A(t, t_0)$ is nonnegative for $t > t_0$ if and only if its off-diagonal elements are nonnegative (Metzler matrix).

If $\mathbb{T} = \mathbb{Z}$, then $\mu \equiv 1$ and nonnegativity of the exponential matrix is equivalent to $A + I \ge 0$. In that case the delta differential equation

$$x^{\Delta}(k) = Ax(k)$$

may be rewritten in the shift form as

$$x(k+1) = (A + I)x(k).$$

5. Realization problem

Let us fix m, p, initial time t_0 and let $x(t_0) = 0$. Let $\Phi_\Sigma(t,\tau) := Ce_A(t,\tau)B$. Then $\Phi_\Sigma(t,\tau) \in \mathbb{R}^{p\times m}$.

The *input-output map* $\mathcal{S}_\Sigma$ is given by

$$y(t) = \mathcal{S}_\Sigma(u)(t) = \int_{t_0}^{t} \Phi_\Sigma(t, \sigma(\tau))u(\tau)\Delta\tau.$$

Proposition 4. *If the system Σ is positive, then $\Phi_\Sigma(t,\tau) = Ce_A(t,\tau)B$ is a nonnegative matrix for all $t \ge \tau$.*

Now let the input-output map $\mathcal{S}$ be given by

$$y(t) = \mathcal{S}(u)(t) = \int_{t_0}^{t} \Phi(t, \sigma(\tau))u(\tau)\Delta\tau, \tag{5}$$

where $\Phi(t,\tau) \in \mathbb{R}^{p\times m}$ is a nonnegative matrix for all $t \ge \tau$. We assume that for every $t \ge t_0$ the map $\tau \mapsto \Phi(t, \sigma(\tau))$ is continuous.

The problem of positive realization:
Find conditions on $\mathcal{S}$ under which there exists a positive linear system Σ, defined by matrices A, B and C, such that $\mathcal{S}_\Sigma$ and $\mathcal{S}$ coincide.

If $\mathbb{T} = \mathbb{R}$, then (5) becomes

$$y(t) = \mathcal{S}(u)(t) = \int_{t_0}^{t} \Phi(t,\tau)u(\tau)d\tau,$$

and we are looking for a positive system $\Sigma : \dot{x} = Ax + Bu$, $y = Cx$, such that for every $t \ge t_0$

$$\mathcal{S}(u)(t) = \mathcal{S}_\Sigma(u)(t) = \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau.$$

Since the right-hand side is a convolution operator, also $\mathcal{S}$ must have such a structure. This means that $\Phi(t,\tau) = \Psi(t-\tau)$ for some nonnegative matrix-valued function Ψ. This gives a standard positive realization problem for continuous-time systems (see van den Hof, 1987). On the other hand, if $\mathbb{T} = \mathbb{Z}$, then (5) takes the form

$$y(t) = \mathcal{S}(u)(t) = \sum_{\tau=t_0}^{t-1} \Phi(t,\tau+1)u(\tau),$$

where t and t_0 are integers, and $t > t_0$. Now, Σ has the form

$$x(t+1) - x(t) = Ax(t) + Bu(t),\ y(t) = Cx(t), \tag{6}$$

and we compare $\mathcal{S}(u)(t)$ with

$$\mathcal{S}_\Sigma(u)(t) = \sum_{\tau=t_0}^{t-1} C(I+A)^{t-\tau-1}u(\tau).$$

Again, the right-hand side is a discrete convolution, which implies that $\Phi(t,\tau) = \Psi(t-\tau)$ for some nonnegative matrix-valued function Ψ defined on the set of nonnegative integers. This gives a standard positive realization problem for discrete-time systems (see van den Hof, 1987), with the only difference that the dynamics of the system is usually given in the shift form. Our system (6) is then rewritten as

$$x(t+1) = (I+A)x(t) + Bu(t),\ y(t) = Cx(t),$$

which results in powers of $I + A$ appearing in the input-output map $\mathcal{S}_\Sigma$.

As the operator $\mathcal{S}$ is uniquely defined by the function Φ, the problem may be restated as follows:
Find conditions on Φ such that $\Phi(t,\tau) = Ce_A(t,\tau)B = \Phi_\Sigma(t,\tau)$ for some matrices A, B, C defining a positive system Σ.

For every τ the function $t \mapsto \Phi_\Sigma(t,\tau)$ can be expanded in a power series originated at τ and convergent for all $t \geq \tau$ (see Bohner and Guseinov, 2007). Thus, it is necessary for the existence of a realization that Φ have the same property. This leads to the following:

Assumption 1 For every τ the function $t \mapsto \Phi(t,\tau) =: \Phi^\tau(t)$ can be expanded in a power series originated at τ and convergent for all $t \geq \tau$.

Assumption 1 means that the map Φ^τ is uniquely defined by all its delta derivatives at τ (see Bohner and Guseinov, 2007). It also implies that the time scale contains infinitely many points to the right of t_0 (otherwise we could only compute a finite number of delta derivatives).

The *Markov parameters* of the operator $\mathcal{S}$ are the matrices

$$M_k^\tau := \frac{\Delta^k}{\Delta t^k}_{|t=\tau} \Phi^\tau(t),\ \tau \geq t_0,\ k \in \mathbb{Z}_+.$$

Thus, Markov parameters uniquely define the operator $\mathcal{S}$. This means that the existence of a realization implies equality of corresponding Markov parameters for the operator $\mathcal{S}$ and its realization.

Proposition 5. *If $\Phi(t,\tau) = Ce_A(t,\tau)B$, then $M_k^\tau = CA^kB$.*

Proof. Observe that for $k = 1$

$$\frac{\Delta^k}{\Delta t^k} Ce_A(t,\tau)B = CAe_A(t,\tau)B$$

and $e_A(\tau,\tau) = I$. Thus, $M_1^\tau = CAB$. Similarly for $k > 1$. ■

Observe that the Markov parameters for $\mathcal{S}_\Sigma$ do not depend on time τ. Since the Markov parameters of the operator $\mathcal{S}$ and its realization are equal, a necessary condition for the existence of a realization is that the Markov parameters of the operator $\mathcal{S}$ do not depend on time τ. This justifies making the following

Assumption 2 The Markov parameters for $\mathcal{S}$ do not depend on time τ. We set $M_k := M_k^\tau$.

Under Assumptions 1 and 2 we have the following necessary condition for the existence of a positive realization.

Proposition 6. *If $\mathcal{S}$ has a positive realization, then there are $n \in \mathbb{N}$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$ such for all $k \geq 0$*

$$M_{k+n} + a_{n-1}M_{k+n-1} + \ldots + a_1M_{k+1} + a_0M_k = 0.$$

Proof. Follows from Theorem 6 of Bartosiewicz and Pawłuszewicz (2006), where it was shown that the condition stated in the proposition is necessary and sufficient for the existence of some, not necessarily positive, realization. ■

Remark 1. *For $\mathbb{T} = \mathbb{Z}$, $e_A(t,\tau) = (I + A)^{t-\tau}$. Thus, we are looking for matrices A, B and C, such that $I + A$, B and C are nonnegative and $\Phi(t,\tau) = C(I + A)^{t-\tau}B$. Observe that A does not have to be nonnegative. Equivalently, we are looking for nonnegative $\tilde{A}$, B and C, such that $\Phi(t,\tau) = C\tilde{A}^{t-\tau}B$. Once we find them, we can easily obtain A from $\tilde{A}$ by setting $A = \tilde{A} - I$. As $M_k = CA^kB$ is not necessarily nonnegative, we need to modify M_k in such a way that the modified M_k is nonnegative and equal to $C\tilde{A}^kB$. Then we can apply a method similar to the one used in van den Hof (1987). For $\mathbb{T} = \mathbb{R}$, $e_A(t,\tau) = e^{A(t-\tau)}$ and the matrix A does not have to be nonnegative. But it must be a Metzler matrix, so for some positive a, $A + aI$ must be nonnegative. Thus, we again modify Markov parametres in such a way that a modified M_k is nonnegative and equal to $C(A + aI)^kB$.*

Definition 9. *Let M_k, $k \in \mathbb{Z}_+$, be the Markov parameters of $\mathcal{S}$ and let $a \geq 0$. The* modified Markov parameters *of $\mathcal{S}$ are defined by*

$$P_k^a = \sum_{i=0}^{k} \binom{k}{i} a^{k-i} M_i, \; k \in \mathbb{Z}_+.$$

Observe that for $a = 0$, $P_k^0 = M_k$. Moreover, one can uniquely recover the sequence (M_k) from the sequence (P_k^a).

PROPOSITION 7. *For any* $a \geq 0$

$$M_k = \sum_{i=0}^{k} \binom{k}{i} (-a)^{k-i} P_i^a.$$

Proof. This is an easy exercise in combinatorics. ■

PROPOSITION 8. $M_k = CA^k B$ *iff* $P_k^a = C(A + aI)^k B$.

Proof. Necessity. By expanding $(A + aI)^k$ we get

$$C(A + aI)^k B = \sum_{i=0}^{k} \binom{k}{i} C a^{k-i} A^i B = P_k^a.$$

Sufficiency. By expanding $A^k = (A + aI - aI)^k$ we get

$$CA^k B = \sum_{i=0}^{k} \binom{k}{i} C(-a)^{k-i} (A + aI)^i B = \sum_{i=0}^{k} \binom{k}{i} (-a)^{k-i} P_k^a = M_k$$

by Proposition 7. ■

Thus, the modified Markov parameters of $\mathcal{S}_\Sigma$ correspond to modification of the matrix A.

PROPOSITION 9. *Assume that* $\mathcal{S} = \mathcal{S}_\Sigma$ *for a positive system* Σ *on a time scale* $\mathbb{T}$. *Then there is* $a \in \mathbb{R}_+$ *such that* $a \leq 1/\bar{\mu}$ *and* $P_k^a \geq 0$ *for* $k \in \mathbb{Z}_+$. *If* $\mu_{\mathbb{T}} > 0$ *then* a *can be equal to* $1/\mu_{\mathbb{T}}$. *If* $\mu_{\mathbb{T}} = 0$, *then* a *depends on* Σ.

Proof. Since $\Phi_\Sigma(t, \tau) = Ce_A(t, \tau)B$, then by Propositions 5 and 8 $M_k^\tau = CA^k B$ and $P_k^a = C(A + aI)^k B$. By Theorem 2, for some $a \in \mathbb{R}_+$ such that $a \leq 1/\bar{\mu}$, $P_k^a \geq 0$ for $k \in \mathbb{Z}_+$. ■

COROLLARY 2. *If* $\mathcal{S}$ *has a positive realization, then there is* $a \in \mathbb{R}_+$ *such that* $a \leq 1/\bar{\mu}$ *and the matrices* P_k^a *are nonnegative for all* $k \in \mathbb{Z}_+$.

Proof. The proof is a consequence of Proposition 9. ■

This is the reason to make

Assumption 3

Let $\mathcal{S}$ be an input-output map. Assume that there is $a \in \mathbb{R}_+$ such that $a \leq 1/\bar{\mu}$ and the matrices $P_k^a \geq 0$ for $k \in \mathbb{Z}_+$.

Let $z = (z_0, z_1, \ldots)^T \in \mathbb{R}_+^{\mathbb{Z}_+}$. Let $s(z) = (z_1, z_2, \ldots)^T$ and $\varsigma = s^p = s \circ \ldots \circ s$. A cone $K \subset \mathbb{R}_+^{\mathbb{Z}_+}$ *is* ς*-invariant* if $\varsigma(K) \subset K$.

Let

$$H = \begin{pmatrix} P_0^a \\ P_1^a \\ \vdots \end{pmatrix}$$

and let K be the cone generated by H, i.e. $K = H\mathbb{R}_+^m$.

Assume that Assumptions 1, 2 and 3 hold (they are, of course, necessary for the existence of positive realizations). The following theorem is an extension of Theorem 4.4 of van den Hof (1987).

Theorem 3. *An input-output map $\mathcal{S}$ has a positive realization iff there exists a ς-invariant polyhedral cone $L \subset \mathbb{R}_+^{\mathbb{Z}_+}$ such that $K \subseteq L$.*

Proof. Necessity. Assume that $\mathcal{S}$ has a positive realization

$$x^{\Delta} = Ax + Bu, \quad y = Cx.$$

Let a be the number guaranteed by Assumption 3 and let L be the cone generated by the columns of the matrix

$$\mathcal{C} := \begin{pmatrix} C \\ C(A + aI) \\ \vdots \end{pmatrix}.$$

Then $L = \mathcal{C}\mathbb{R}_+^n$ is a polyhedral cone contained in $\mathbb{R}_+^{\mathbb{Z}_+}$. Observe now that $\varsigma(L)$ is a cone generated by the columns of the matrix $\mathcal{C}(A + aI)$, so

$$\varsigma(L) = \mathcal{C}(A + aI)\mathbb{R}_+^n = \mathcal{C}[(A + aI)\mathbb{R}_+^n] \subseteq \mathcal{C}\mathbb{R}_+^n = L$$

since $(A + aI)\mathbb{R}_+^n \subseteq \mathbb{R}_+^n$. Thus, L is ς-invariant. Moreover

$$K = H\mathbb{R}_+^m = \mathcal{C}B\mathbb{R}_+^m \subseteq \mathcal{C}\mathbb{R}_+^n = L$$

since B is a positive matrix.

Sufficiency. Let $L \subset \mathbb{R}_+^{\mathbb{Z}_+}$ be a ς-invariant polyhedral cone such that $K \subset L$. We shall first construct an abstract model of the realization. It will be a system whose state space is equal to the cone L. We shall construct maps $\alpha : L \to L$, $\beta : \mathbb{R}_+^m \to L$ and $\gamma : L \to \mathbb{R}_+^n$, such that the map $\gamma\alpha^k\beta : \mathbb{R}^m \to \mathbb{R}^p$ is given by the matrix P_k^a when the standard bases are chosen in $\mathbb{R}^m$ and $\mathbb{R}^p$. Let $\alpha := \varsigma_{|L}$. The map β is defined on the generators e_i, $i = 1, \ldots, m$, of the cone $\mathbb{R}_+^m$ by $\beta(e_i) = H_i$ - the ith column of H. The map γ cuts off the first p components from the elements of L (which are infinite sequences). Observe now that $\alpha^k(\beta(e_i))$ is the ith column of the matrix

$$\begin{pmatrix} P_k^a \\ P_{k+1}^a \\ \vdots \end{pmatrix}.$$

Thus, $\gamma(\alpha^k(\beta(e_i)))$ is the ith column of P_k^a. This means that the matrix of the map $\gamma\alpha^k\beta$ is in fact P_k^a. To find matrix representations of α, β and γ we use the generators of the cone L: $z_1, \dots, z_n$. Let $H_j = \sum_{i=1}^n b_{ij} z_i$. Then, $B := (b_{ij})$ is a positive $n \times m$ matrix. It is clear that the representation of γ is $C = P_0^a$. Finally, let $\alpha(z_j) = \sum_{i=1}^n d_{ij} z_i$. Then, $D = (d_{ij})$ is a positive $n \times n$ matrix representation of α. This gives $CD^kB = P_k^a$ for $k \in \mathbb{Z}_+$. Now it is enough to define $A := D - aI$ to achieve

$$P_k^a = C(A + aI)^k B.$$

This means that the system $x^\Delta = Ax + Bu$, $y = Cx$, is a positive realization of the input-output map $\mathcal{S}$. ■

Example 6. *Let $m = p = 1$ and the Markov parameters of the input-output operator S be given by $M_k = (-1)^k$. They are negative for odd k. The modified Markov parameters are*

$$P_k^a = \sum_{i=0}^{k} \binom{k}{i} a^{k-i}(-1)^i = (a-1)^k, \ k = 0, 1, 2, \dots.$$

If $a \geq 1$, then $P_k^a \geq 0$ for all $k \geq 0$. Since a must be less or equal $1/\bar{\mu}$, then $\bar{\mu} \leq 1$. Now the cone K is generated by the column

$$H = \begin{pmatrix} 1 \\ a-1 \\ (a-1)^2 \\ \vdots \end{pmatrix}.$$

Since $\varsigma(H) = s(H) = (a-1)H$, then K is ς-invariant. Thus, taking $L := K$ we see that the criteria of positive realizability, stated in Theorem 3, are satisfied. One can easily compute the data of the realization: $n = 1$, $C = 1$, $B = 1$ and $A + aI = a - 1$, so $A = -1$. The realization is good for all time scales, but it is positive only for the time scales with $\bar{\mu} \leq 1$. For $\mathbb{T} = \mathbb{Z}$ the modified Markov parameters for $a = 1$ take the form $P_k^1 = 0$ for all $k > 0$ and $P_0^1 = 1$. This allows us to recover the input-output map:

$$\mathcal{S}(u)(k) = u(k-1)$$

with $\Phi(k,l) = 0$ for $k > l$ and $\Phi(k,k) = 1$. We may set the initial time $t_0 = 0$. Taking subsequent delta derivatives of the map $k \mapsto \Phi(k,0)$ we obtain the Markov parameters $M_k = (-1)^k$. Thus, the modified Markov parameters form now the impulse response (discrete) function of the input-output map $\mathcal{S}$. In van den Hof (1987) they are used to construct a positive realization, which looks like the one obtained above for $a = 1$. On the other hand, for $\mathbb{T} = \mathbb{R}$ the parameter a may be arbitrary, but to achieve positivity of the modified Markov parameters it must *be greater or equal* 1. *Thus, let us take* $a = 1$ *as before. Then* P_k^1 *are as before, but the map* $\mathcal{S}$ *looks now as follows:*

$$\mathcal{S}(u)(t) = \int_0^t e^{-(t-\tau)} u(\tau) d\tau$$

with $\Phi(t, \tau) = e^{-(t-\tau)}$. *The (standard) derivatives of the map* $t \mapsto \Phi(t, 0)$ *at* $t = 0$ *give the original Markov parameters as before. Observe that now the modified parameters are not directly related to the input-output map as this was in the case of* $\mathbb{T} = \mathbb{Z}$. *But they are used to produce matrices* A, B *and* C *exactly in the same way as before. The procedure for obtaining a positive continuous-time realization described in van den Hof (1987) is slightly different, but it results in the same modified Markov parameters. Namely, they are computed as derivatives at* $t = 0$ *of the function* $t \mapsto e^{at}\Psi(t)$, *where* Ψ *is the impulse response function. In our case* $\Psi(t) = e^{-t}$, *so* $e^{at}\Psi(t) = 1$. *Finally, observe that for* $\mathbb{T} = q^{\mathbb{N}}$ *a positive realization does not exist as* $\bar{\mu} = +\infty$.

6. Conclusion

The paper contains a general framework for studying positive linear realizations on the basis of an input-output map and the corresponding Markov parameters. The language of time scales allows for studying continuous- and discrete-time realizations simultaneously. Moreover, discrete time with variable graininess is also admitted. The main result states a necessary and sufficient condition for the existence of positive linear realizations. Future research in this area should concern the existence of minimal positive realizations and the problem of uniqueness of such realizations. Another interesting problem is the existence of nonlinear positive realizations and possible constructions of such realizations.

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