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Reducibility condition for nonlinear discrete-time systems: behavioral approach*

by

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Abstract: The paper addresses the problem of reducibility of nonlinear discrete-time systems, described by implicit higher order difference equations where no a priori distinction is made between input and output variables. The reducibility definition is based on the concept of autonomous element. We prove necessary reducibility condition, presented in terms of the left submodule, generated by the row matrix, describing the behavior of the linearized system, over the ring of left difference polynomials. Then the reducibility of the system implies the closedness of the submodule, like in the linear time-invariant case. In the special case, when the variables may be specified as inputs and outputs and the system equations are given in the explicit form, the results of this paper yield the known results.

Keywords: nonlinear discrete-time systems, reducibility, behavioral approach, left difference polynomials

1. Introduction

The standard way to look at control systems is from the input-output (i/o) point of view. Input corresponds to action (cause) and output to reaction (effect). However, laws of physics merely impose relations on the system variables but do not inherently involve signal flows. Therefore, the behavioral model treats all the system variables on an equal footing. Partitioning the variables into

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inputs and outputs is important in numerous situations, like feedback control, but in modeling, including the problem of system reduction that we study in this paper, it is unnecessary. Behavioral modeling does not eliminate input-output models but places them in a larger context. Note also that the partitioning into inputs and outputs depends heavily on the purpose for which the model is used, and is sometimes just impossible. One typical example is diode which is neither current nor voltage driven (Willems, 2007); many more may be found in Polderman and Willems (1998) and Willems (2007). Though the i/o approach is still a mainstream in control theory, the behavioral approach is gaining more and more popularity during the recent years. Van der Shaft (2011) addresses the problem of state space realization from external system description based on the linear behavioral model. Such models are popular also in the theory of linear parameter–varying systems (Toth, 2010). Note that if we discretize a nonlinear electrical (e.g. RLC) circuit, we get the difference equation with external variables being current and voltage. Then, having the electric circuit model described by the second order difference system one can study its behavior by checking its reducibility. Another possibility is to study the behavior of a discrete–time macro–economic model that relates the product output, the growth rate of money and the inflation rate.

Reducibility is an important system property since it allows for replacing the original system equation by another equation of lower order, being in certain sense equivalent to the original system description. Reducibility and system reduction have been studied earlier for discrete–time nonlinear systems in Halas et al.(2009) and Kotta and Tõnso (2012) (for the continuous-time case, see, for example, Conte et al., 2007 and the references therein). The main difference between Halas et al. (2009); Kotta and Tõnso (2012) and our paper is that we consider implicit difference equation and do not distinguish the input and output variables. This is the reason why, for instance, in Halas et al. (2009) the conditions are given in terms of the greatest common left devisor of two left difference polynomial matrices but our condition is presented in terms of the left submodule, generated by the matrix over the ring of left difference polynomials.

The goal of this paper is to extend the results on reducibility for discrete-time nonlinear systems where the system model does not distinguish between inputs and outputs. That is, we work with the system model used in the behavioral approach. Like in Halas et al.(2009) and Kotta and Tõnso (2012), our definition of reducibility is based on the concept of autonomous element (Pommaret, 2001). Again, like in Halas et al.(2009), we use module theory to present the necessary reducibility condition. However, our setup is different from that in Halas et al.(2009) and whereas Halas et al.(2009) assume only rational systems, we address analytic systems. Note that the algebraic module theory has been used earlier in the studies of structural properties (including controllability) of linear systems with time-varying coefficients, see for example Fliess (1990); Bourles (2005); Marinescu and Bourles (2009) and also in the studies of linear systems, governed by partial differential equations (Pommaret, 2001). The module-theoretic setting of linear systems, developed by Fliess and the behavioral theory, developed by Willems, are shown to be strictly dual, if the signal space W is a cogenerator (Bourles, 2005). Finally, note that already in Blomberg and Ylinen (1983) the relationship between the system solutions (behavior) and the modules, defined by linear time-invariant system, was pointed out.

As said above, the goal is to find reducibility conditions for implicit nonlinear difference equation in several system variables, not divided into inputs and outputs. We will show that one can associate with such a system a row polynomial matrix over the ring of left difference polynomials. This matrix describes the behavior of the linearized system. The set of row matrices has the mathematical structure of a module, so reducibility of difference systems from a behavioral point of view means working with modules over the ring of left difference polynomials. Similarly as in Willems (2007) by the behavior of the system we mean the set of its solutions, so the definition of autonomous element given for instance in Kotta et al. (2001) is presented here in the equivalent form. Taking the algebraic ideals we compare the set of solutions of two equations describing the system, one is the original equation and another is related to the autonomous element.

The paper is organized as follows. The next section describes the difference rings and their ideals associated with the implicit equation defined by the discrete-time system. In Section 3 we introduce the non-commutative ring of left difference polynomials and then present the polynomial description of the considered system. Section 4 is devoted to giving the necessary reducibility condition and to showing an illustrative example that describes our result. In Section 5 conclusions are drawn and possible future research direction is suggested.

2. Difference rings

Let $s \geqslant 2$ and $\mathcal{A}$ denote the ring of analytic functions in a finite number of variables from the set $\left\{w_i^{[k]},\ k \in \mathbb{Z},\ i = 1, \ldots, s\right\}$, where $w_i^{[0]} := w_i$. Denote $w := (w_1, \ldots, w_s)$ and $w^{[k]} := \left(w_1^{[k]}, \ldots, w_s^{[k]}\right)$. Then for the function F depending on $w^{[-k]}, \ldots, w^{[-1]}, w, w^{[1]}, \ldots, w^{[l]}$ the *shift operator* $\delta : \mathcal{A} \to \mathcal{A}$ is defined as follows

$$\delta(F)\left(w^{[-k+1]}, \ldots, w^{[-1]}, w, w^{[1]}, \ldots, w^{[l+1]}\right) := F\left(w^{[-k+1]}, \ldots, w^{[-1]}, w, w^{[1]}, \ldots, w^{[l+1]}\right), \quad (1)$$

and $\delta^{-1} : \mathcal{A} \to \mathcal{A}$ is given by

$$\delta^{-1}(F)\left(w^{[-k-1]}, \ldots, w^{[-1]}, w, w^{[1]}, \ldots, w^{[l-1]}\right) = F\left(w^{[-k-1]}, \ldots, w^{[-1]}, w, w^{[1]}, \ldots, w^{[l-1]}\right). \quad (2)$$

Then, $\delta w_i^{[k]} = w_i^{[k+1]}$ and $\delta^{-1} w_i^{[k]} = w_i^{[k-1]}$ for $i = 1, 2, \ldots, s$ and $k \in \mathbb{Z}$. Note that $\mathcal{A}$ is a difference ring with the shift operator δ, being an automorphism (injective and onto). We will use sometimes the alternative shorter notations $\delta(F) = F^+$ and $\delta^{-1}(F) = F^-$.

Let $\mathcal{S}$ be a multiplicative subset of the ring $\mathcal{A}$, i.e. $1 \in \mathcal{S}$ and if $a \in \mathcal{S}$ and $b \in \mathcal{S}$, then $ab \in \mathcal{S}$. Assume that $\mathcal{S}$ is invariant with respect to both δ and δ^{-1}, i.e. $\delta^m a \in \mathcal{S}$ for all $a \in \mathcal{S}$ and $m \in \mathbb{Z}$. Then one may define, as usually, the localization of the ring $\mathcal{A}$ at $\mathcal{S}$

$$\widehat{\mathcal{A}} := \mathcal{S}^{-1}\mathcal{A} = \left\{ \frac{a}{b} \mid \ a \in \mathcal{A} \text{ and } b \in \mathcal{S} \right\} . \tag{3}$$

The operator $\delta : \mathcal{A} \to \mathcal{A}$ induces the operator $\delta : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$ by

$$\delta\left(\frac{a}{b}\right) := \frac{\delta(a)}{\delta(b)} . \tag{4}$$

Observe that $\widehat{\mathcal{A}}$ is an inversive difference ring with the shift operator δ and $\mathcal{S}$ may be interpreted as a subset of $\widehat{\mathcal{A}}$, because of the natural injection $a \mapsto \frac{a}{1}$.

Now let us consider the nonlinear system described by the following difference equation

$$f(w(k), w(k+1), w(k+2), \ldots, w(k+n)) = 0\,, \ k \geqslant 0\,, \tag{5}$$

where $f \in \widehat{\mathcal{A}}$ is a function in variables $w, w^{[1]}, \ldots, w^{[n]}$ and $w^{[i]}$ is replaced by $w(k+i)$. Then, behavior of system (5) is given by

$$\mathcal{B} = \{ w : \mathbb{N}_0 \to \mathbb{R}^s \mid \ f(w(k), w(k+1), \ldots, w(k+n)) = 0 \text{ for all } k \in \mathbb{N}_0 \} \ ,$$

where $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Assume that system (5) has at least one equilibrium point $w^{\mathrm{e}} = (w_1^{\mathrm{e}}, \ldots, w_s^{\mathrm{e}})$.

Let $I := \langle f \rangle$ be the smallest ideal of $\widehat{\mathcal{A}}$ that contains all forward and backward shifts of the function f, i.e. I is generated by

$$\left\{ \delta^k f, \ k \in \mathbb{Z} \right\} . \tag{6}$$

Since I is closed with respect to all shifts of the function f, I is called the *difference ideal*. Moreover, the difference ideal I has the following property: for every element $\varphi \in \widehat{\mathcal{A}}$, $\delta\varphi \in I$ implies $\varphi \in I$, so it is called *reflexive*, see Halas et al. (2009). Assume that

A1: I is prime, i.e. if $a \cdot b \in I$ then $a \in I$ or $b \in I$,

A2: I is proper, i.e. different from the entire ring.

Note that assumptions A1 and A2 for explicit rational systems are satisfied if and only if the system equations are submersive, that is the equation contains at least one variable w_i at time instant k.

Observe that if the assumption A1 is satisfied, then function f cannot be decomposed into a product of simpler functions.

Properness of the ideal I is equivalent to the condition

$$\mathcal{S} \cap I = \emptyset\,. \tag{7}$$

In particular, the numerator of f does not belong to $\mathcal{S}$.

Note that I may be considered as a subset of $\widetilde{S}^{-1}\mathcal{A}$ for some other multiplicative set $\widetilde{\mathcal{S}}$. For that reason, when the multiplicative set is not fixed, we will write $I_{\mathcal{S}} = \langle f \rangle_{\mathcal{S}}$ if the function f generates the difference ideal of $\mathcal{S}^{-1}\mathcal{A}$ and $I_{\widetilde{\mathcal{S}}} = \langle f \rangle_{\widetilde{\mathcal{S}}}$ in the case when f generates the difference ideal of $\widetilde{\mathcal{S}}^{-1}\mathcal{A}$.

The next proposition given in Kotta et al. (2011) shows that if $I_{\mathcal{S}}$ is prime and proper then $I_{\widetilde{\mathcal{S}}}$ is also prime and proper. Later, this will be used in showing the properties of the subset of polynomials with coefficients belonging to the ideal $I_{\widetilde{\mathcal{S}}}$ (see Proposition 2 in Section 3 below).

Proposition 1. *Assume that $\mathcal{S}_1$ and $\mathcal{S}_2$ are multiplicative subsets of $\mathcal{A}$ invariant with respect to δ and δ^{-1} and $\mathcal{S}_1 \subset \mathcal{S}_2$. Let $f \in \mathcal{S}_1^{-1}\mathcal{A}$ and $I_{\mathcal{S}_1} = \langle f \rangle_{\mathcal{S}_1}$ be a prime and proper difference ideal of the ring $\mathcal{S}_1^{-1}\mathcal{A}$. Let $\mathcal{S}_2 \cap I_{\mathcal{S}_2} = \emptyset$. Then*

(i) $\mathcal{A} \subset \mathcal{S}_1^{-1}\mathcal{A} \subset \mathcal{S}_2^{-1}\mathcal{A}$

(ii) $I_{\mathcal{S}_2}$ is a prime and proper difference ideal of $\mathcal{S}_2^{-1}\mathcal{A}$.

Note that it is possible to extend $\mathcal{S}$ to some larger subset $\widetilde{\mathcal{S}} \subset \widehat{A}$ such that $\widetilde{\mathcal{S}} \cap I_{\widetilde{\mathcal{S}}} = \emptyset$. Then, by Proposition 1 we get $\mathcal{S}^{-1}\mathcal{A} \subset \widetilde{\mathcal{S}}^{-1}\mathcal{A}$. Since we allow some variables to be in the denominator, we choose $\widetilde{\mathcal{S}}$ as large as we need in our computations. In fact, all generators of $\widetilde{\mathcal{S}}$ are given by the denominators of two functions: the function f and the *autonomous element* f_{r} which is related to the reducibility problem of the considered system, see Section 4. Then the localization of the ring $\mathcal{A}$ at $\widetilde{\mathcal{S}}$ is denoted by

$$\widetilde{\mathcal{A}} := \widetilde{\mathcal{S}}^{-1}\mathcal{A} = \left\{ \frac{a}{b} \mid \; a \in \mathcal{A} \text{ and } b \in \widetilde{\mathcal{S}} \right\} \tag{8}$$

and by Proposition 1 we have $\widehat{\mathcal{A}} \subset \widetilde{\mathcal{A}}$.

Remark 1. *Note that $\widetilde{\mathcal{A}}$ is also an inversive difference ring with the shift operator $\delta : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}$ defined by (4). By Proposition 1 we get that $I_{\mathcal{S}}$ is a prime and proper difference ideal of $\widetilde{\mathcal{A}}$. Therefore, the extension of $\mathcal{S}$ to a larger subset $\widetilde{\mathcal{S}}$ does not change the properties of the difference ideal $I_{\widetilde{\mathcal{S}}}$ which is now the ideal of the bigger difference ring $\widetilde{\mathcal{A}}$.*

If the multiplicative set $\mathcal{S}$ is fixed we will write simply I instead of $I_{\mathcal{S}}$, i.e. the index $\mathcal{S}$ is omitted.

Let us now consider an example which illustrates the construction of the difference rings described above.

Example 1. *Consider the following implicit difference equation*

$$\left(w_1(k+1)\right)^2 - w_1(k) \cdot w_2(k) = 0,\; k \in \mathbb{N}_0\,. \tag{9}$$

Then the behavior of (9) *is given by*

$$\mathcal{B} = \left\{ w = (w_1, w_2) : \mathbb{N}_0 \to \mathbb{R}^2 \mid \quad (9) \text{ holds for all } k \in \mathbb{N}_0 \right\} .$$

The left-hand side of (9) *corresponds to the function*

$$f(w_1, w_2, w_1^{[1]}) = \left(w_1^{[1]} \right)^2 - w_1 \cdot w_2 .$$

Note that $S = \{1\}$ *and* $f \in \widehat{\mathcal{A}} = \mathcal{A}$*, where* $\mathcal{A}$ *is the ring of analytic functions in a finite number of the variables from the set* $\{w_1^{[k_1]}, w_2^{[k_2]},\ k_1, k_2 \in \mathbb{Z}\}$, $w_i^{[0]} = w_i$, $i = 1, 2$*. Using the operators* δ *and* δ^{-1} *defined by* (1) *and* (2)*, respectively, one gets*

$$\delta w_i^{[k_i]} = w_i^{[k_i+1]} \text{ and } \delta^{-1} w_i^{[k_i]} = w_i^{[k_i-1]}$$

and $\mathcal{A}$ *is the difference ring with the shift operator* δ*. Moreover, in* $\mathcal{A}$ *we have the difference ideal* $I = \langle f \rangle$*, so* I *is generated by* $\left(w_1^{[k+1]} \right)^2 - w_1^{[k]} \cdot w_2^{[k]} \in \mathcal{A}$*,* $k \in \mathbb{Z}$*, i.e. each element* $\varphi \in I$ *has the following form:*

$$\varphi = \sum_k a_k \left(\left(w_1^{[k+1]} \right)^2 - w_1^{[k]} \cdot w_2^{[k]} \right) ,$$

where $a_k \in \mathcal{A}$*. At this point, there is no need to extend* $\mathcal{S}$ *to a larger subset but this extension can be necessary later in the computation when we would like some variables to be in denominators.*

Note that by Proposition 1 $I_{\widetilde{\mathcal{S}}}$ *is the difference ideal of each bigger difference ring* $\widetilde{\mathcal{S}}^{-1}\mathcal{A}$ *of the form* (8) *for the multiplicative set* $\widetilde{\mathcal{S}}$ *such that* $\mathcal{S} \subset \widetilde{\mathcal{S}}$ *and all properties of* $I_{\mathcal{S}}$ *are also fulfilled for* $I_{\widetilde{\mathcal{S}}}$*.*

3. Non-commutative ring of polynomials and modules over the ring

The ring $\widetilde{\mathcal{A}}$ and the shift operator δ induce the ring of polynomials in a formal variable ∂ over $\widetilde{\mathcal{A}}$. A *left difference polynomial* is an element that can be uniquely written in the form $p(\partial) = \sum_{i=0}^{n} p_i \partial^i$, $p_i \in \widetilde{\mathcal{A}}$, where $p(\partial) \neq 0$ if and only if at least one of the coefficients p_i, $i = 0, 1, \ldots, n$, is nonzero. If $p_n \not\equiv 0$, then the positive integer n is called the degree of the left difference polynomial p and is denoted by $\deg(p)$. Moreover, we set $\deg 0 = -\infty$. The addition of left difference polynomials is defined in the standard way. For $a \in \widetilde{\mathcal{A}}$, define the multiplication by $\partial \cdot a = \delta(a)\partial$ and $a \cdot \partial = a\partial$. This rule can be uniquely extended to multiplication of monomials by $(a\partial^n) \cdot (b\partial^m) = a\delta^n(b)\partial^{n+m}$ and then, to arbitrary polynomials. The set of all left difference polynomials with so defined addition and multiplication is a ring with identity. Denote this ring by $\widetilde{\mathcal{A}}[\partial] = \left\{ \sum_{i=0}^{l} a_i \partial^l,\ l \in \mathbb{N}_0, a_i \in \widetilde{\mathcal{A}} \right\}$. In general, the ring $\widetilde{\mathcal{A}}[\partial]$ is non-commutative, since for arbitrary $a \in \widetilde{\mathcal{A}}$, $a \neq$ const, we have $a \cdot \partial \neq \partial \cdot a$. Note that for arbitrary nonzero elements $a, b \in \widetilde{\mathcal{A}}$ the product $a \cdot b \neq 0$, so the coefficient ring $\widetilde{\mathcal{A}}$ is an *integral ring*, i.e. it has no zero divisors. Then for arbitrary nonzero polynomials $p, q \in \widetilde{\mathcal{A}}[\partial]$ we get $p(\partial) \cdot q(\partial) \neq 0$ and consequently the ring $\widetilde{\mathcal{A}}[\partial]$ is also an integral ring, and additionally, $\deg(p \cdot q) = \deg(p) + \deg(q)$.

Now, we define the operator $\widetilde{\delta} : \widetilde{\mathcal{A}}[\partial] \to \widetilde{\mathcal{A}}[\partial]$ as follows

$$\widetilde{\delta}\left(\sum_i a_i \partial^i\right) := \sum_i \delta(a_i)\partial^{i+1} . \tag{10}$$

Remark 2. *The definition of the operator* $\widetilde{\delta} : \widetilde{\mathcal{A}}[\partial] \to \widetilde{\mathcal{A}}[\partial]$ *given by formula* (10) *can be rewritten in terms of left difference polynomials as follows* $\widetilde{\delta}(p(\partial)) = \partial \cdot p(\partial)$, *i.e. the image of the polynomial* p *with respect to the operator* $\widetilde{\delta}$ *is equal to a product of the monomial* ∂ *and the polynomial* p.

Over the ring $\widetilde{\mathcal{A}}[\partial]$ one can define the *free module*, i.e. a module with a basis, generated by $\mathrm{d}w_i$, $i = 1, 2, \ldots, s$, i.e.

$$\mathcal{E} := \widetilde{\mathcal{A}}[\partial]^s \mathrm{d}w , \tag{11}$$

where $\mathrm{d}w = \begin{bmatrix}\mathrm{d}w_1, & \ldots, & \mathrm{d}w_s\end{bmatrix}^{\mathrm{T}}$. The differential operator $\mathrm{d} : \widetilde{\mathcal{A}} \to \mathcal{E}$ for arbitrary nonconstant function $g \in \widetilde{\mathcal{A}}$, which depends on $w_i, w_i^{[1]}, \ldots, w_i^{[m_i]}$, $i = 1, 2, \ldots, s$, is defined by

$$\mathrm{d}g := \sum_{i=1}^{s} \sum_{k=0}^{m_i} g_{ki} \partial^k \mathrm{d}w_i \in \mathcal{E} , \tag{12}$$

where $g_{ki} := \frac{\partial g}{\partial w_i^{[k]}} \in \widetilde{\mathcal{A}}$ and $\sum_{k=0}^{m_i} g_{ki}\partial^k \in \widetilde{\mathcal{A}}[\partial]$. Let $m = \max_i m_i$, then (12) can be rewritten in the matrix form as

$$\mathrm{d}g = \sum_{k=0}^{m} g_k \partial^k \mathrm{d}w \in \mathcal{E} ,$$

where $g_k := \begin{bmatrix} g_{k1} & \ldots & g_{ks}\end{bmatrix} \in \widetilde{\mathcal{A}}^s$ and $\sum_{k=0}^{m} g_k \partial^k \in \widetilde{\mathcal{A}}[\partial]^s$ is a row polynomial matrix.

Proposition 2. *If* I *is a prime and proper difference ideal of* $\widetilde{\mathcal{A}}$, *then*

$$I[\partial] := \left\{ \sum_{i=0}^{n} \varphi_i \partial^i : \ \varphi_i \in I, \ i = 1, 2, \ldots, n \right\}$$

is a prime and proper difference ideal of $\widetilde{\mathcal{A}}[\partial]$.

Proof. Note that for arbitrary monomials $\alpha\partial^k \in \widetilde{\mathcal{A}}[\partial]$ and $\beta\partial^l \in I[\partial]$ we get $\alpha\partial^k \cdot \beta\partial^l = \alpha\tilde{\beta}\partial^{k+l} \in I[\partial]$ and $\beta\partial^l \cdot \alpha\partial^k = \beta\tilde{\alpha}\partial^{k+l} \in I[\partial]$. Then one can extend this multiplication to arbitrary polynomials and $I[\partial]$ is an ideal in $\widetilde{\mathcal{A}}[\partial]$.

Moreover, if $\alpha\partial^k \cdot \beta\partial^l = \alpha\delta^k\beta\partial^{l+k} \in I[\partial]$, then $\alpha\delta^k\beta \in I$. If I is prime, then $\alpha \in I$ or $\delta^k\beta \in I$. Since I is a difference ideal, we get $\alpha \in I$ or $\beta \in I$ and consequently, $\alpha\partial^k \in I[\partial]$ or $\beta\partial^l \in I[\partial]$. If $(p_0 + p_1\partial) \cdot (q_0 + q_1\partial) = p_0q_0 + \left(p_0q_1 + p_1q_0^+\right)\partial + p_1q_1^+\partial^2 \in I[\partial]$, then using the assumption that I is a prime difference ideal we get

$$\begin{cases} p_0 \in I \text{ or } q_0 \in I \\ (p_0 \in I \text{ or } q_1 \in I) \text{ and } (p_1 \in I \text{ or } q_0 \in I) \\ p_1 \in I \text{ or } q_1 \in I \end{cases} . \tag{13}$$

Note that (13) is equivalent to

$$(p_0 \in I \text{ and } p_1 \in I) \text{ or } (q_0 \in I \text{ and } q_1 \in I) ,$$

so $p_0 + p_1\partial \in I[\partial]$ or $q_0 + q_1\partial \in I[\partial]$. The above computations can be extended for arbitrary polynomials $p, q \in I[\partial]$ and one gets that if $p(\partial) \cdot q(\partial) \in I[\partial]$, then $p(\partial) \in I[\partial]$ or $q(\partial) \in I[\partial]$, so the ideal $I[\partial]$ is prime. Assuming that I is proper, we get that $I \neq \widetilde{\mathcal{A}}$, so $I[\partial] \neq \widetilde{\mathcal{A}}[\partial]$. Therefore $I[\partial]$ is a proper ideal of $\widetilde{\mathcal{A}}[\partial]$. ■

Let $\widetilde{\mathcal{A}}[\partial]/I[\partial]$ be the quotient ring of $\widetilde{\mathcal{A}}[\partial]$ modulo $I[\partial]$. Note that $\widetilde{\mathcal{A}}[\partial]/I[\partial]$ consists of cosets $[p]_{I[\partial]} = p + I[\partial]$ for $p \in \widetilde{\mathcal{A}}[\partial]$. In $\widetilde{\mathcal{A}}[\partial]/I[\partial]$ we define "+" and "·" by the rules $[p_1]_{I[\partial]} + [p_2]_{I[\partial]} := [p_1 + p_2]_{I[\partial]}$ and $[p_1]_{I[\partial]} \cdot [p_2]_{I[\partial]} := [p_1 \cdot p_2]_{I[\partial]}$. These definitions do not depend on the choice of representative in a coset. Note that in particular, $[f]_{I[\partial]} = 0$. By assumptions A1 and A2 I is a prime and proper difference ideal, so from Proposition 2 ideal $I[\partial]$ is also prime and proper. Consequently, $\widetilde{\mathcal{A}}[\partial]/I[\partial]$ is an integral ring, i.e. has no zero divisors.

3.1. Polynomial matrix description of the system

By applying the differential operator d to (5) we get

$$\frac{\partial f}{\partial(w, w^{[1]}, \ldots, w^{[n]})} \begin{bmatrix} \mathrm{d}w \\ \mathrm{d}w^{[1]} \\ \vdots \\ \mathrm{d}w^{[n]} \end{bmatrix} = 0 . \tag{14}$$

From (12) we get $\mathrm{d}w^{[k]} = \partial^k \mathrm{d}w$, $k \geqslant 0$. Since $w^{[k]} = \delta^k w$, we have $\mathrm{d}\delta^k w = \partial^k \mathrm{d}w$, $k \geqslant 0$, so the operators d and δ^k commute. Then (14) can be rewritten as

$$\sum_{k=0}^{n} p_k \partial^k \mathrm{d}w = 0 ,$$

where $p_k := \frac{\partial f}{\partial w^{[k]}} \in \widetilde{\mathcal{A}}^s$ and consequently, $\sum_{k=0}^{n} p_k \partial^k \in \widetilde{\mathcal{A}}[\partial]^s$. Therefore, the system (5) can be expressed in terms of left difference polynomials as follows

$$P(\partial)\mathrm{d}w = 0 , \tag{15}$$

where $P(\partial) = [p_1(\partial), \dots, p_s(\partial)] \in \widetilde{\mathcal{A}}[\partial]^s$ is a row polynomial matrix and $p_i(\partial) = \sum_{k=0}^{n} p_{ki}\partial^k \in \widetilde{\mathcal{A}}[\partial]$, $p_{ki} = \frac{\partial f}{\partial w_i^{[k]}}$ and $i = 1, 2, \dots, s$.

From now on consider the quotient ring $\widetilde{\mathcal{A}}[\partial]/I[\partial]$, and take $[P(\partial)]_{I[\partial]} \in \left(\widetilde{\mathcal{A}}[\partial]/I[\partial]\right)^s$. Note that the set of s–dimensional polynomial row vectors $\left(\widetilde{\mathcal{A}}[\partial]/I[\partial]\right)^s$ has the mathematical structure of a module. Let $M \subset \left(\widetilde{\mathcal{A}}[\partial]/I[\partial]\right)^s$ be the $\widetilde{\mathcal{A}}[\partial]/I[\partial]$–submodule generated by the elements of row polynomial matrix $[P(\partial)]_{I[\partial]}$, i.e. $M = \text{gen}\ [P(\partial)]_{I[\partial]}$. All $\widetilde{\mathcal{A}}[\partial]/I[\partial]$–submodules of $\left(\widetilde{\mathcal{A}}[\partial]/I[\partial]\right)^s$ are finitely generated and free and thus we can speak of the dimension of such a submodule.

Definition 1. *(Willems, 2007) The* closure *of an $\widetilde{\mathcal{A}}[\partial]/I[\partial]$–submodule M of $\left(\widetilde{\mathcal{A}}[\partial]/I[\partial]\right)^s$ is defined as*

$$\overline{M} := \left\{ \left[\hat{P}\right]_{I[\partial]} \in \left(\widetilde{\mathcal{A}}[\partial]/I[\partial]\right)^s \mid \exists\, [\pi]_{I[\partial]} \in \widetilde{\mathcal{A}}[\partial]/I[\partial], [\pi]_{I[\partial]} \neq 0, [P]_{I[\partial]} \in M : [P]_{I[\partial]} = [\pi]_{I[\partial]} \cdot \left[\hat{P}\right]_{I[\partial]} \right\}.$$

$\overline{M}$ is an $\widetilde{\mathcal{A}}[\partial]/I[\partial]$-submodule of $\left(\widetilde{\mathcal{A}}[\partial]/I[\partial]\right)^s$.

A submodule M of $\left(\widetilde{\mathcal{A}}[\partial]/I[\partial]\right)^s$ is *closed* if and only if $M = \overline{M}$, i.e. M is not properly contained in any $\widetilde{\mathcal{A}}[\partial]/I[\partial]$-submodule of $\left(\widetilde{\mathcal{A}}[\partial]/I[\partial]\right)^s$ of the same dimension.

Now let us show how the submodule M looks like in the following examples.

Example 2. *The polynomial matrix description of* (9) *is as follows:*

$$\left[2w_1^{[1]}\partial - w_2, \quad -w_1\right] \cdot \begin{bmatrix} \mathrm{d}w_1 \\ \mathrm{d}w_2 \end{bmatrix} = 0\,.$$

Then $I[\partial] = \left\{ \sum_{i=0}^{n} \varphi_i \partial^i :\ \varphi_i \in \left\langle \left(w_1^{[1]}\right)^2 - w_1 w_2 \right\rangle,\ i = 1, 2, \dots, n \right\} \subset \mathcal{A}[\partial]$.

Let M be the $\mathcal{A}[\partial]/I[\partial]$-submodule of $(\mathcal{A}[\partial]/I[\partial])^2$ generated by the row polynomial matrix $[P(\partial)]_{I[\partial]} = \left[\left[2w_1^{[1]}\partial - w_2\right]_{I[\partial]},\quad [-w_1]_{I[\partial]}\right]$. *Since* $\deg(-w_1) = 0$, *$M = \overline{M}$ and M is closed. Note that in the considered example there is no need to extend the difference ring $\widehat{\mathcal{A}} = \mathcal{A}$.*

The next example shows that the extension of $\mathcal{S}$ to a larger subset $\widetilde{\mathcal{S}}$ is necessary.

Example 3. *Consider the following implicit difference equation*

$$w_1(k+1)\,(w_1(k+2))^2 - w_1(k)\cdot w_2(k)\cdot w_2(k+1) = 0,\;\; k\in\mathbb{N}_0\,. \tag{16}$$

The left-hand side of (16) *corresponds to the function*

$$f(w_1, w_2, w_1^{[1]}, w_2^{[1]}, w_1^{[2]}) = w_1^{[1]}\left(w_1^{[2]}\right)^2 - w_1\cdot w_2\cdot w_2^{[1]}\,.$$

Of course, at the beginning $S=\{1\}$ *and* $f\in\widehat{\mathcal{A}}=\mathcal{A}$ *where* $\mathcal{A}$ *is the ring of analytic functions in a finite number of variables from the set* $\{w_1^{[k_1]}, w_2^{[k_2]},\; k_1,k_2\in\mathbb{Z}\}$*. In the computations that will follow, we have to extend* $\mathcal{S}$ *to some larger subset* $\widetilde{\mathcal{S}}$ *that is generated by* $\left\{w_1^{[k]},\;k\in\mathbb{Z}\right\}$ *and the difference ideal*

$$I_{\widetilde{\mathcal{S}}} = \left\langle w_1^{[1]}\left(w_1^{[2]}\right)^2 - w_1\cdot w_2\cdot w_2^{[1]}\right\rangle_{\widetilde{\mathcal{S}}}$$

is the ideal of $\widetilde{\mathcal{A}}=\widetilde{\mathcal{S}}^{-1}\mathcal{A}$*, i.e. each element* $\varphi\in I_{\widetilde{\mathcal{S}}}$ *has the following form:*

$$\varphi = \sum_k a_k\left(w_1^{[1]}\left(w_1^{[2]}\right)^2 - w_1\cdot w_2\cdot w_2^{[1]}\right)\,,$$

where $a_k\in\widetilde{\mathcal{A}}$ *while* $I_{\mathcal{S}}$ *is the ideal of* $\mathcal{A}$ *and for each* $\beta\in I_{\mathcal{S}}$

$$\beta = \sum_k b_k\left(w_1^{[1]}\left(w_1^{[2]}\right)^2 - w_1\cdot w_2\cdot w_2^{[1]}\right)\,,$$

where $b_k\in\mathcal{A}$*. Therefore,* $I_{\widetilde{\mathcal{S}}}\neq I_{\mathcal{S}}$ *while the generators are the same for both ideals.*

Let us take $\widetilde{\mathcal{S}}$ *and define* $I:=I_{\widetilde{\mathcal{S}}}$*. We have the following polynomial matrix description of* (16)

$$\left[2w_1^{[2]}w_1^{[1]}\partial^2 + \left(w_1^{[2]}\right)^2\partial - w_2w_2^{[1]},\quad -w_1w_2\partial - w_1w_2^{[1]}\right]\cdot\begin{bmatrix}\mathrm{d}w_1\\ \mathrm{d}w_2\end{bmatrix} = 0\,.$$

Then $I[\partial] = \left\{\sum_{i=0}^n \varphi_i\partial^i:\;\varphi_i\in I,\; i=1,2,\dots,n\right\}\subset\widetilde{\mathcal{A}}[\partial]$*.*

Let M *be the* $\widetilde{\mathcal{A}}[\partial]/I[\partial]$*-submodule of* $\left(\widetilde{\mathcal{A}}[\partial]/I[\partial]\right)^2$ *generated by the row polynomial matrix* $[P(\partial)]_{I[\partial]} = \left[[p_1(\partial)]_{I[\partial]},\quad [p_2(\partial)]_{I[\partial]}\right]$*, where* $p_1(\partial)=2w_1^{[1]}w_1^{[2]}\partial^2+$

$\left(w_1^{[2]}\right)^2 \partial - w_2 w_2^{[1]}$ *and* $p_2(\partial) = -w_1 w_2 \partial - w_1 w_2^{[1]}$. *Since*

$$\left(-w_1 w_2 \partial - w_1 w_2^{[1]}\right) \cdot \left(-\frac{2w_2}{w_1^{[1]}}\partial + \frac{w_2}{w_1}\right) = \frac{2w_1 w_2 w_2^{[1]}}{w_1^{[2]}}\partial^2 + \frac{w_1 w_2 w_2^{[1]}}{w_1^{[1]}}\partial - w_2 w_2^{[1]}$$

$$= w_1 w_2 w_2^{[1]} \left(\frac{2}{w_1^{[2]}}\partial^2 + \frac{1}{w_1^{[1]}}\partial\right) - w_2 w_2^{[1]} =$$

$$= \left(w_1^{[1]}\left(w_1^{[2]}\right)^2 + \left(w_1 w_2 w_2^{[1]} - w_1^{[1]}\left(w_1^{[2]}\right)^2\right)\right)\left(\frac{2}{w_1^{[2]}}\partial^2 + \frac{1}{w_1^{[1]}}\partial\right) - w_2 w_2^{[1]}$$

$$\equiv w_1^{[1]}\left(w_1^{[2]}\right)^2 \left(\frac{2}{w_1^{[2]}}\partial^2 + \frac{1}{w_1^{[1]}}\partial\right) - w_2 w_2^{[1]} \ mod\ I[\partial]\,,$$

we get

$$[p_2(\partial)]_{I[\partial]} \cdot \left[-\frac{2w_2}{w_1^{[1]}}\partial + \frac{w_2}{w_1}\right]_{I[\partial]} = [p_1(\partial)]_{I[\partial]}\ .$$

Consequently, the $\widetilde{\mathcal{A}}[\partial]/I[\partial]$*-submodule* $\overline{M}$ *is generated by the row polynomial matrix* $\left[\hat{P}(\partial)\right]_{I[\partial]} = \left[\left[-\frac{2w_2}{w_1^{[1]}}\partial + \frac{w_2}{w_1}\right]_{I[\partial]},\ \ [1]_{I[\partial]}\right] \in \left(\widetilde{\mathcal{A}}[\partial]/I[\partial]\right)^2$ *and* $M \subsetneq \overline{M}$, *so* M *is not closed. Note that variables* w_1 *and* $w_1^{[1]}$ *are in the denominator, so in this case the set* $\mathcal{S}$ *must be extended to a larger subset generated by* $\left\{w_1^{[k]},\ k \in \mathbb{Z}\right\}$.

4. Reducibility of the discrete-time systems

We are interested in behavior of implicit nonlinear difference equations, so the reducibility definition introduced in Kotta et al. (2001) and later on considered in Kotta et al. (2004) is modified here to our purpose.

DEFINITION 2. *A nonconstant function* $f_{\rm r}$ *in* $\widetilde{\mathcal{A}}$ *is said to be an* autonomous element *for system* (5) *if there exist an integer* $\nu \geqslant 1$, *a constant* c, *and a non-zero meromorphic function* F *with* $F(c,\ldots,c) = 0$ *such that*

$$(f) = \left(F\left(f_{\rm r}, \delta f_{\rm r}, \ldots, \delta^\nu f_{\rm r}\right)\right) \tag{17}$$

and $F\left(f_{\rm r}, \delta f_{\rm r}, \ldots, \delta^\nu f_{\rm r}\right) \in \widetilde{\mathcal{A}}$, *where* (φ) *denotes the algebraic ideal of* $\widetilde{\mathcal{A}}$ *generated by the function* $\varphi \in \widetilde{\mathcal{A}}$.

In analogy with Conte et al. (2007) and Kotta et al. (2001) the notion of autonomous variable can be used to define reducibility of nonlinear difference system (5).

Definition 3. *The discrete-time system* (5) *is said to be* reducible *if there exists a non-zero autonomous element in* $\widetilde{\mathcal{A}}$. *Otherwise system* (5) *is called* irreducible.

If system (5) is reducible, then there exists an autonomous element $f_{\mathrm{r}} = f_{\mathrm{r}}\left(w,\ w^{[1]},\ldots,w^{[m]}\right)$ with $m < n$, and a non-zero analytic function F such that

$$f = kF\left(f_{\mathrm{r}}, \delta f_{\mathrm{r}}, \ldots, \delta^{\nu} f_{\mathrm{r}}\right), \tag{18}$$

where $\nu \geqslant 1$ and $k \neq 0$ is an invertible element of $\widetilde{\mathcal{A}}$ (in most cases $k = 1$). Since $f_{\mathrm{r}} \not\equiv \mathrm{const}$ and $\nu \geqslant 1$, $m \geqslant 1$ and $\nu + m \geqslant n$. The equation $f_{\mathrm{r}}(\cdot) = c$ is called a *reduced difference equation* of (5), where c is some constant which gives the nontrivial behavior of $f_{\mathrm{r}}(\cdot) = c$, and additionally, c may depend on an equilibrium point $(w_1^{\mathrm{e}}, \ldots, w_s^{\mathrm{e}})$ of (5) and $F(c, \ldots, c) = 0$. Then behavior of the reduced system $f_{\mathrm{r}}(\cdot) = c$ is given by

$$\mathcal{B}_{\mathrm{r}} = \left\{w : \mathbb{N}_0 \to \mathbb{R}^s \mid\ f_{\mathrm{r}}(w(k), w(k+1), \ldots, w(k+m)) = c \text{ for all } k \in \mathbb{N}_0\right\}.$$

Form Definitions 2 and 3 one gets the relation between the sets $\mathcal{B}_{\mathrm{r}}$ and $\mathcal{B}$. Note that if $f_{\mathrm{r}}(\cdot) = c$ is a reduced equation of (5), then the behavior of the reduced system is contained in the behavior of the original system, i.e. $\mathcal{B}_{\mathrm{r}} \subset \mathcal{B}$.

The following theorem gives a necessary condition for reducibility of system (5) in terms of the submodule generated by the left difference polynomials of the row matrix corresponding to system (5).

Theorem 1. *If discrete–time system* (5) *is reducible, then the* $\widetilde{\mathcal{A}}[\partial]/I[\partial]$*–submodule* $M = \mathrm{gen}\,[P(\partial)]_{I[\partial]}$ *of the module* $\left(\widetilde{\mathcal{A}}[\partial]/I[\partial]\right)^s$ *is not closed, i.e.* $M \neq \overline{M}$.

Proof. Suppose that the nonlinear system (5) is reducible. Then there exist functions $f_{\mathrm{r}} \in \widetilde{\mathcal{A}}$ and F such that (17) holds. Note that

$$\mathrm{d}f_{\mathrm{r}} = \sum_{i=0}^{m} \frac{\partial f_{\mathrm{r}}}{\partial w^{[i]}} \partial^i \mathrm{d}w\,.$$

Let $\tilde{f} := F\left(f_{\mathrm{r}}, \delta f_{\mathrm{r}}, \ldots, \delta^{\nu} f_{\mathrm{r}}\right)$. Then $\tilde{f} \in \widetilde{\mathcal{A}}$ and

$$\begin{aligned}\mathrm{d}\tilde{f} &= \sum_{k=0}^{\nu} \frac{\partial F}{\partial \delta^k f_{\mathrm{r}}}\left(f_{\mathrm{r}}, \ldots, \delta^{\nu} f_{\mathrm{r}}\right) \partial^k \mathrm{d}f_{\mathrm{r}} \\ &= \sum_{k=0}^{\nu} \frac{\partial F}{\partial \delta^k f_{\mathrm{r}}}\left(f_{\mathrm{r}}, \ldots, \delta^{\nu} f_{\mathrm{r}}\right) \partial^k \cdot \sum_{i=0}^{m} \frac{\partial f_{\mathrm{r}}}{\partial w^{[i]}} \partial^i \mathrm{d}w\,.\end{aligned}$$

Denote $\tilde{P}(\partial) = \sum_{i=0}^{m} \frac{\partial f_{\mathrm{r}}}{\partial w^{[i]}} \partial^i \in \widetilde{\mathcal{A}}[\partial]^s$ and $\pi(\partial) = \sum_{k=0}^{\nu} \frac{\partial F}{\partial \delta^k f_{\mathrm{r}}}\left(f_{\mathrm{r}}, \ldots, \delta^{\nu} f_{\mathrm{r}}\right) \partial^k \in \widetilde{\mathcal{A}}[\partial]$, where $\nu \geqslant 1$ and $m < n$. Since (18) holds, we get

$$f = k \cdot \tilde{f}\,,$$

where k is the invertible element of $\widetilde{\mathcal{A}}$. Then $\mathrm{d}f = k\cdot\mathrm{d}\tilde{f}+\tilde{f}\cdot\mathrm{d}k$ and, consequently,

$$P(\partial) = k \cdot \pi(\partial) \cdot \tilde{P}(\partial) + \tilde{f} \cdot Q(\partial)\,, \tag{19}$$

where $Q(\partial) = \sum_j \frac{\partial k}{\partial w^{[j]}}\partial^j \in \widetilde{\mathcal{A}}[\partial]$. Since $\left[\tilde{f}\right]_{I[\partial]} = 0 \in \widetilde{\mathcal{A}}[\partial]/I[\partial]$, we have

$$[P(\partial)]_{I[\partial]} = [k \cdot \pi(\partial)]_{I[\partial]} \cdot \left[\tilde{P}(\partial)\right]_{I[\partial]}\,.$$

Then $M = \mathrm{gen}\ [P(\partial)]_{I[\partial]} \subsetneq \mathrm{gen}\ \left[\tilde{P}(\partial)\right]_{I[\partial]} \subseteq \overline{M} \subseteq \left(\widetilde{\mathcal{A}}[\partial]/I[\partial]\right)^s$. Hence, M is not closed. ■

Remark 3. *We think that the necessary condition of reducibility of system* (5) *given in Theorem 1 is also a sufficient one. We suppose that in order to get the autonomous element for the considered system we need to integrate the one-form associated with the representatives of the coset which generates the submodule* $\overline{M}$. *Then having an autonomous element one can try to find an analytic function* F *such that* (17) *holds, see Example 4 below. We think that the function* F *could be related to the difference polynomial that is representative of coset* $[\pi]_{I[\partial]}$ *which describes the relation between* M *and* $\overline{M}$. *Unfortunately, in general we have found some problems in proving the existence of the autonomous element* f_{r} *and the analytic function* F*, so we leave it for our future work.*

Now let us consider an example where our result is presented.

Example 4. *Consider the following equation*

$$w_1^{[2]} - w_1 w_2 w_2^{[1]} = 0 \;\Leftrightarrow\; w_1^{++} - w_1 w_2 w_2^{+} = 0\,. \tag{20}$$

At the beginning, since $f(w_1, w_2, w_2^+, w_1^{++}) = w_1^{++} - w_1 w_2 w_2^+$, *we get* $\mathcal{S} = \{1\}$, *but in the computations that will follow, and taking into account the formula of the autonomous element, we have to extend* $\mathcal{S}$ *to some larger subset* $\widetilde{\mathcal{S}}$ *that is generated by* $\left\{w_i^{[k]},\ k \in \mathbb{Z},\ i = 1, 2\right\}$. *Then* $\widehat{\mathcal{A}} = \mathcal{A}$ *is the ring of analytic functions in a finite number of variables* $w_1^{[m_1]}$, $w_2^{[m_2]}$, $m_1, m_2 \in \mathbb{Z}$ *and the difference ideal* $I = \left\langle w_1^{++} - w_1 w_2 w_2^+ \right\rangle$ *is the ideal of* $\widetilde{\mathcal{A}} = \widetilde{\mathcal{S}}^{-1}\mathcal{A}$. *Compute the polynomial matrix description of* (20) *as follows*

$$\left[\partial^2 - w_2 w_2^+, \quad -w_1 w_2 \partial - w_1 w_2^+\right] \cdot \begin{bmatrix}\mathrm{d}w_1\\ \mathrm{d}w_2\end{bmatrix} = 0\,. \tag{21}$$

Then $I[\partial] = \left\{\sum_{i=0}^n \varphi_i \partial^i :\ \varphi_i \in \left\langle w_1^{++} - w_1 w_2 w_2^+ \right\rangle,\ i = 1, 2, \ldots, n\right\} \subset \widetilde{\mathcal{A}}[\partial]$.

Let M *be the* $\widetilde{\mathcal{A}}[\partial]/I[\partial]$*–submodule of* $\left(\widetilde{\mathcal{A}}[\partial]/I[\partial]\right)^2$ *generated by the row polynomial matrix* $[P(\partial)]_{I[\partial]} = \left[[p_1(\partial)]_{I[\partial]}, [p_2(\partial)]_{I[\partial]}\right]$, *where* $p_1(\partial) = \partial^2 - w_2 w_2^+$ *and* $p_2(\partial) = -w_1 w_2 \partial - w_1 w_2^+$. *Since*

$$[p_1(\partial)]_{I[\partial]} = [p_2(\partial)]_{I[\partial]} \cdot \left[-\frac{w_2}{w_1^+}\partial + \frac{w_2}{w_1}\right]_{I[\partial]}\,, \tag{22}$$

the $\widetilde{\mathcal{A}}[\partial]/I[\partial]$*–submodule* $\overline{M}$ *is generated by the row polynomial matrix* $\left[\hat{P}(\partial)\right]_{I[\partial]} = \left[\left[-\frac{w_2}{w_1^+}\partial + \frac{w_2}{w_1}\right]_{I[\partial]}, [1]_{I[\partial]}\right] \in \left(\widetilde{\mathcal{A}}[\partial]/I[\partial]\right)^2$ *and* $M \subsetneq \overline{M}$*, so* M *is not closed. Since the necessary condition of reducibility is satisfied, by Theorem 1 the system* (20) *might be reducible. We conjecture that the condition* $M \subsetneq \overline{M}$ *implies the reducibility of the considered system. Let us see how the autonomous element* f_{r} *and the function* F *can be found from the relations between submodules* M *and* $\overline{M}$*.*

In order to find the autonomous element let us take the representative of the generator of $\overline{M}$ *and multiply it by the vector* $\mathrm{d}w$*. Then we have*

$$\hat{P}(\partial)\mathrm{d}w = \left(-\frac{w_2}{w_1^+}\partial + \frac{w_2}{w_1}\right)\mathrm{d}w_1 + \mathrm{d}w_2 \in \mathcal{E}\,. \tag{23}$$

One can check that $\hat{P}(\partial)\mathrm{d}w = -\frac{w_2}{w_1^+}\mathrm{d}w_1^+ + \frac{w_2}{w_1}\mathrm{d}w_1 + \mathrm{d}w_2$ *is a closed one-form and multiplying it by the integrating factor* $-\frac{w_1^+}{w_1 w_2^2}$ *we get*

$$\frac{1}{w_1 w_2}\mathrm{d}w_1^+ - \frac{w_1^+}{w_1^2 w_2}\mathrm{d}w_1 - \frac{w_1^+}{w_1 w_2^2}\mathrm{d}w_2 = \mathrm{d}\left(\frac{w_1^+}{w_1 w_2}\right) = \mathrm{d}f_{\mathrm{r}}\,.$$

Then

$$\begin{aligned}[P(\partial)]_{I[\partial]} &= \left[\left(-w_1 w_2\partial - w_1 w_2^+\right)\left(-\frac{w_1 w_2^2}{w_1^+}\right)\cdot\left(-\frac{w_1^+}{w_1 w_2^2}\right)\hat{P}(\partial)\right]_{I[\partial]} \\ &= \left[w_1^+ w_2^+\left(\frac{w_1 w_2 w_2^+}{w_1^{++}}\partial + \left(\frac{w_1 w_2}{w_1^+}\right)^2\right)\right]_{I[\partial]}\cdot\left[\tilde{P}(\partial)\right]_{I[\partial]} \\ &= \left[w_1^+ w_2^+\left(\partial + \left(\frac{1}{f_{\mathrm{r}}}\right)^2\right)\right]_{I[\partial]}\cdot\left[\tilde{P}(\partial)\right]_{I[\partial]}\,,\end{aligned}$$

where $\tilde{P}(\partial) = -\frac{w_1^+}{w_1 w_2^2}\cdot\hat{P}(\partial)$*. Since* $\left(\partial + \left(\frac{1}{f_{\mathrm{r}}}\right)^2\right)\mathrm{d}f_{\mathrm{r}} = \mathrm{d}\left(f_{\mathrm{r}}^+ - \frac{1}{f_{\mathrm{r}}}\right)$*, we get* $k = w_1^+ w_2^+$ *and* $F(f_{\mathrm{r}}, f_{\mathrm{r}}^+) = f_{\mathrm{r}}^+ - \frac{1}{f_{\mathrm{r}}}$*. Note that* F *is not well defined at* 0*, but* $F(1,1) = F(-1,-1) = 0$*.*

Hence, there exists an autonomous element $f_{\mathrm{r}} = \frac{w_1^+}{w_1 w_2}$ *and the function* $F(f_{\mathrm{r}}, f_{\mathrm{r}}^+) = f_{\mathrm{r}}^+ - \frac{1}{f_{\mathrm{r}}}$ *such that*

$$w_1^{++} - w_1 w_2 w_2^+ = w_1^+ w_2^+\left(f_{\mathrm{r}}^+ - \frac{1}{f_{\mathrm{r}}}\right).$$

Since $\widetilde{\mathcal{S}} \ni w_1^+ w_2^+ \neq 0$*, we get*

$$(f) = (F(f_{\mathrm{r}}, f_{\mathrm{r}}^+))\,.$$

Then system (20) *can be reduced around* $(w_1^{\rm e}, w_2^{\rm e}) = (1,1)$ *to* $w_1^+ - w_1 w_2 = 0$ *(i.e.* $f_{\rm r}(w_1, w_2, w_1^+) = 1$*), but around* $(w_1^{\rm e}, w_2^{\rm e}) = (1,-1)$ *the reduced equation of* (20) *has the form* $w_1^+ + w_1 w_2 = 0$ *(i.e.* $f_{\rm r}(w_1, w_2, w_1^+) = -1$*).*

Note that the behavior of the reduced system $w_1^+ - w_1 w_2 = 0$ *is given by*

$$\mathcal{B}_{\rm r1} = \left\{ w : \mathbb{N}_0 \to \mathbb{R}^2 \mid \; w(n) = \left(\prod_{k=0}^{n-1} w_2(k) w_1(0), w_2(n) \right), n \in \mathbb{N}_0 \right\} ,$$

where $w_i(n) \neq 0$*,* $i = 1,2$*, and each element of* $\mathcal{B}_{\rm r1}$ *satisfies equation* (20) *because*

$$w_1(n+2) = \prod_{k=0}^{n+1} w_2(k) w_1(0) = w_2(n+1) w_2(n) w_1(n) .$$

Therefore, $\mathcal{B}_{\rm r1} \subset \mathcal{B}$*. Moreover, for* $w_1^+ + w_1 w_2 = 0$ *we get*

$$\mathcal{B}_{\rm r2} = \left\{ w : \mathbb{N}_0 \to \mathbb{R}^2 \mid \; w(n) = \left((-1)^n \prod_{k=0}^{n-1} w_2(k) w_1(0), w_2(n) \right), n \in \mathbb{N}_0 \right\} ,$$

where $w_i(n) \neq 0$*,* $i = 1,2$*, and for all elements of* $\mathcal{B}_{\rm r1}$ *equation* (20) *is satisfied, since*

$$w_1(n+2) = (-1)^{n+2} \prod_{k=0}^{n+1} w_2(k) w_1(0) = w_2(n+1) w_2(n) w_1(n) .$$

Hence, $\mathcal{B}_{\rm r2} \subset \mathcal{B}$*. Note that* $\left((-1)^n \prod_{k=0}^{n-1} w_2(k) w_1(0), w_2(n)\right) \in \mathcal{B}$ *and* $\left((-1)^n \prod_{k=0}^{n-1} w_2(k) w_1(0), w_2(n)\right) \notin \mathcal{B}_{\rm r1}$ *, so* $\mathcal{B} \not\subset \mathcal{B}_{\rm r1}$*, and similarly,* $\mathcal{B} \not\subset \mathcal{B}_{\rm r2}$ *because* $\left(\prod_{k=0}^{n-1} w_2(k) w_1(0), w_2(n)\right) \in \mathcal{B}$ *and* $\left(\prod_{k=0}^{n-1} w_2(k) w_1(0), w_2(n)\right) \notin \mathcal{B}_{\rm r2}$*.*

Let us now describe the situation when input and output variables are specified. For simplicity we restrict our explanation to single input and single output variables.

Remark 4. *Consider the nonlinear discrete-time system described by the following higher order input-output (i-o) difference equation*

$$\begin{aligned} &\phi\left(y(k), y(k), \ldots, y(k+n), u(k), u(k+1), \ldots, u(k+l)\right) = 0, \; k \geqslant 0 \\ &\qquad \Leftrightarrow \phi\left(y, y^{[1]}, \ldots, y^{[n]}, u, u^{[1]}, \ldots, u^{[l]}\right) = 0 , \end{aligned} \tag{24}$$

where u *is the input,* y *is the output of the system,* $n, l \in \mathbb{N}$*,* $l < n$ *and the function* ϕ *that relates the input, the output and a finite number of their forward time shifts, is a real analytic function belonging to the ring* $\widehat{\mathcal{A}}$*. Then the system* (24) *can be expressed in terms of left difference polynomials as follows*

$$\begin{bmatrix} P_y(\partial) & P_u(\partial) \end{bmatrix} \cdot \begin{bmatrix} {\rm d}y \\ {\rm d}u \end{bmatrix} = 0 , \tag{25}$$

where
$P_y(\partial) \in \widetilde{\mathcal{A}}[\partial]$, $P_u(\partial) \in \widetilde{\mathcal{A}}[\partial]$ *and* $P_y(\partial) = \sum_{k=0}^{n} \frac{\partial \phi}{\partial \delta^k y} \partial^k$, $P_u(\partial) = \sum_{\ell=0}^{s} \frac{\partial \phi}{\partial \delta^\ell u} \partial^\ell$. *Hence* $s = 2$, $w = [y, u]^{\mathrm{T}}$ *and* $P = [P_y, P_u]$. *Moreover,* $I = \langle \phi \rangle$ *and the* $\widetilde{\mathcal{A}}[\partial]/I[\partial]$ *-submodule* $M = \operatorname{gen}\left[[P_y(\partial)]_{I[\partial]}, [P_u(\partial)]_{I[\partial]}\right]$ *of the module* $\left(\widetilde{\mathcal{A}}[\partial]/I[\partial]\right)^2$. *Since the closedness of* M *is equivalent to the fact that polynomials* P_y *and* P_u *are left coprime, from Theorem 1 we get that the reducibility of* (24) *implies polynomials* P_y *and* P_u *have a common left devisor which is a polynomial of degree greater than 0. Note that in Kotta and Tõnso (2012) it was proved that the system* (24) *is reducible if and only if the polynomial matrices* P_y *and* P_u *are not left coprime. It is the reason we suppose the closedness of* M *is equivalent to the irreducibility of the considered system.*

Note that in the case of the linear difference equations one gets the polynomials with coefficients being real numbers, i.e. $p_{ki} = \frac{\partial f}{\partial w_i^{[k]}} \in \mathbb{R}$ and $i = 1, 2, \ldots, s$. Let us now describe this situation.

Remark 5. *Consider the linear discrete-time system described by the following higher order difference equation*

$$r_0 \cdot w(k) + r_1 \cdot w(k+1) + \ldots + r_n \cdot w(k+n) = 0 \\ \Leftrightarrow \quad r_0 \cdot w + r_1 \cdot w^{[1]} + \ldots + r_n \cdot w^{[n]} = 0\,, \quad (26)$$

where $r_i \in \mathbb{R}^{1\times s}$ *and* $w^{[i]} = w(k+i) \in \mathbb{R}^{s\times 1}$ *for* $i = 0, 1, \ldots, n$. *Then the system* (26) *can be expressed in terms of left difference polynomials as follows*

$$(r_0 + r_1\partial + \ldots + r_n\partial^n)\,\mathrm{d}w = 0\,. \quad (27)$$

Then $\widetilde{\mathcal{A}} = \mathbb{R}$ *and one gets the following row polynomial matrix describing the system* (26)*:*

$$P(\partial) = r_0 + r_1\partial + \ldots + r_n\partial^n \in \mathbb{R}[\partial]^s\,.$$

Note that the ring $\mathbb{R}[\partial]$ *is commutative while in general* $\widetilde{\mathcal{A}}[\partial]$ *is noncommutative. Now we have* $I = \left\langle r_0 \cdot w + r_1 \cdot w^{[1]} + \ldots + r_n \cdot w^{[n]} \right\rangle$ *and* $M = \operatorname{gen}\,[P(\partial)]_{I[\partial]}$ *is a submodule of the module* $(\mathbb{R}[\partial]/I[\partial])^s$. *The closedness of* M *in this case can be checked using various criteria given in Willems (2007).*

The symbolic software, implementing the results of this paper, is under development. *Mathematica* functions, allowing for checking the reducibility condition, and when possible, for finding the reduced system equation, are developed as a part of nonlinear control package NLControl. The respective functions will be made available on NLControl website (2013) using *webMathematica* software, developed by Wolfram Research. The *webMathematica* technology offers access to specific *Mathematica* applications through a web browser, so that the user does not need to install *Mathematica* into a local computer. To use the functions related to behavioral models on NLControl website, one has to choose *Behavioral approach* from the left-hand menu and then select either *Irreducibility* or *Reduction* from the submenu.

5. Conclusions

The paper presents the necessary condition for reducibility of the nonlinear discrete-time system, described by an implicit higher order difference equations where no distinction is made between input and output variables. This condition is an extension for nonlinear case of the result from Willems (2007). The condition is given in terms of the submodule, generated by the row matrix of left difference polynomials, describing the behavior of the linearized system. Our future work will be devoted to finding the sufficient condition of reducibility and to extending the results for discrete-time systems described by the set of implicit difference equations. We conjecture that the presented condition is also sufficient.

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