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Stability and chaotic properties of multidimensional Lasota equation*

by

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Abstract: In this paper we study asymptotic properties of multidimensional Lasota equation. We give the conditions of its stability and chaos in the sense of Devaney in Orlicz spaces L^p for any $p > 0$. We also give criteria when the semigroup generated by the equation has not asymptotic behaviour.

Keywords: Lasota equation, chaos, stability

1. Introduction

The purpose of the present paper is to show some asymptotic properties of the dynamical systems described by some first order partial differential equations. Our main object is the multidimensional equation

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{d} c_i(x) \frac{\partial u}{\partial x_i} = \gamma u \quad \text{for} \ \ t \geqslant 0, \ x \in D, \ \gamma \in \mathbb{R}, \ d \in \mathbb{N} \tag{1.1}$$

with the initial condition

$$u(0,x) = v(x) \qquad \text{for} \ \ x \in D \tag{1.2}$$

where v belongs to some normed vector space V of functions defined on D. We assume that

$$D = \left\{ x \in \mathbb{R}_+^d \setminus \{0\} : |x| \leqslant g\left(\frac{x}{|x|}\right) \right\} \cup \{0\}$$

is a compact set. Here $|x|$ denotes the distance of the point x from the origin, $g : S^{d-1} \cap \mathbb{R}_+^d \to \mathbb{R}_+ \setminus \{0\}$ is a continuous function, S^{d-1} denoting the unit sphere in $\mathbb{R}^d$.

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Here and subsequently $c : D \to \mathbb{R}^d$, $c = (c_1, \ldots, c_d)$ is continuously differentiable function satisfying

$$c_i(0) = 0, \quad c_i(x) > 0 \quad \text{for } x \in D \setminus \{0\}, \; i = 1, \ldots, d. \tag{1.3}$$

We call (1.1) the multidimensional Lasota equation. In general, equation (1.1) may be used to describe the growth of cell populations, which constantly differentiate (change their properties) in time. In the model, $x = (x_1, \ldots, x_d)$ denotes the degrees of differentiation (maturity) for different d cell groups. The vector c denotes the velocities of the cell differentiation. The inequality in condition (1.3) points out irreversibility of the differentiation process.

The Lasota equation in its basic form is

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = f(x, u).$$

This equation is a mathematical description of a particular population, such as a population of red blood cells, see Ważewska–Czyżewska and Lasota (1976). It is the element of the so-called precursor cells model (see Lasota, Mackey and Ważewska–Czyżewska, 1981). Multidimensional version of the equation is a particular case of the Fredrickson equation describing structured, segregated model of microbial growth (see, for example, Ramkrishna, 1979). In the Fredrickson model physiological state of a cell is described by a finite dimensional vector and D is the region of the admissible states. Because of biological application, the Lasota equation as well as its multidimensional version is a matter of interest of many mathematicians: Brzeźniak and Dawidowicz (2009), Łoskot (1985, 1991), Rudnicki (1985), Lasota and Szarek (2004), Bielaczyc (2010), Leszczyński, (2008), Leszczyński and Zwierkowski (2007).

Equation (1.1) with the initial condition (1.2) generates a semigroup $(T_t)_{t \geqslant 0}$ acting on some space V. The behaviour of this dynamical system depends upon the parameter γ. We consider asymptotic properties of the multidimensional Lasota equation in L^p space for $p > 0$. These spaces are particular cases of Orlicz spaces. Since $\mu(D) < \infty$, the space $L^p(D)$ is also separable for $p < 1$ (see, for instance, Maligranda, 1989). Hence all results can be considered not only for $p \geqslant 1$, but also $p < 1$.

This work is a generalization of some results from Brzeźniak and Dawidowicz (2007) and Dawidowicz and Poskrobko (2008). These papers treat the asymptotic properties of the one-dimensional Lasota equation with $c(x) = x$ in L^p spaces. In these cases the decisive value of the coefficient γ is $-\frac{1}{p}$. It means that the solution of the one-dimensional equation displays chaotic behaviour in the sens of Devaney for $\gamma > -\frac{1}{p}$ and is strongly stable for $\gamma \leqslant -\frac{1}{p}$. There are no situations of the lack of regularity in asymptotic behaviour of the dynamical system $(T_t)_{t \geqslant 0}$ for one-dimensional equation. The novel contribution of our research is introducing multidimensionality into the equation and also latitude in the choice of the function $c(x)$. Furthermore, we apply topological properties of the set D and divergence of the vector c to the estimation of the decisive value of the coefficient γ.

The paper is organized as follows. In Section 2 we introduce some definitions and notations appearing subsequently. In particular, we recall the notation of Orlicz spaces and introduce some of their basic properties. In Section 3 the explicit formula for the semigroup $(T_t)_{t\geqslant 0}$ is provided. Section 4 contains chaos and stability criteria for dynamical system connected with the multidimensional Lasota equation.

2. Preliminaries

In this section we list the principal definitions, notations and symbols (see Maligranda, 1989; Musielak, 1983).

Definition 2.1. *Let X be a real vector space. A functional $\rho : X \to [0,\infty]$ is called a modular, if it satisfies the conditions*
i) $\rho(x) = 0$ iff $x = 0$,
ii) $\rho(-x) = \rho(x)$,
iii) $\rho(\alpha x + \beta y) \leqslant \rho(x) + \rho(y)$ for $x, y \in X$, $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$.
If we replace iii) by the condition
iii)' $\rho(\alpha x + \beta y) \leqslant \alpha^s \rho(x) + \beta^s \rho(y)$ for $\alpha, \beta \geqslant 0$, $\alpha^s + \beta^s = 1$,
then the modular ρ is called s-convex. 1*-convex modulars are called convex.*

The modular space generated by ρ is the subspace

$$X_\rho = \left\{ x \in X : \lim_{\lambda\to 0} \rho(\lambda x) = 0 \right\}.$$

Definition 2.2. *Let X be a vector space. A functional $x \to |x|$ is called F-norm if for arbitrary $x, y \in X$ there holds*
i) $|x| = 0$ iff $x = 0$,
ii) $|x + y| \leqslant |x| + |y|$,
iii) for each scalar a there is $|ax| = |x|$ when $|a| = 1$,
iv) for each scalars a_k and a if $a_k \to a$ and $|x_k - x| \to 0$ then $|a_k x_k - ax| \to 0$.

Definition 2.3. *A φ-function is a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathbb{R}_+ = [0,\infty)$ such that*
i) φ is continuous,
ii) φ is nondecreasing,
iii) $\varphi(0) = 0$,
iv) $\varphi(u) \to \infty$ as $u \to \infty$.

Let (Ω, Σ, μ) be a measurable space, where Ω is a nonempty set, Σ is a σ-algebra of subsets of Ω and μ is a nontrivial, nonnegative, complete measure. Let X be the set of all real-valued functions on Ω, Σ-measurable and finite μ-almost everywhere functions on Ω. Then for every $x \in X$

$$\rho(x) = \int_\Omega \varphi(|x(t)|) d\mu$$

is a modular in X. Moreover, if φ is convex φ-function, then ρ is a convex modular in X.

Definition 2.4. *The modular space X_ρ will be called an Orlicz space denoted by $L^\varphi(\Omega, \Sigma, \mu)$ (or briefly L^φ):*

$$L^\varphi = \left\{ x \in X : \int_\Omega \varphi(\lambda |x(t)|) d\mu \to 0 \ \textit{as} \ \lambda \to 0^+ \right\}.$$

Moreover, the set

$$L_0^\varphi = \left\{ x \in X : \int_\Omega \varphi(|x(t)|) d\mu < \infty \right\}$$

will be called the Orlicz class. In a modular space X_ρ

$$|x|^F = \inf \left\{ s > 0 : \int_\Omega \varphi \left(\left| \frac{x(t)}{s} \right| \right) d\mu \leqslant s \right\}$$

is a F-norm. If φ is convex then the functional

$$\|x\|^L = \inf \left\{ s > 0 : \int_\Omega \varphi \left(\left| \frac{x(t)}{s} \right| \right) d\mu \leqslant 1 \right\}$$

is a norm in L^φ, called the Luxemburg norm. It is known that the space L^φ with the norm $\|x\|^L$ is a Banach space.

Example 1. *The L^p spaces are examples of the Orlicz spaces with the modular $\rho(x) = \int_\Omega |x(t)|^p d\mu$, which is convex for $p \geqslant 1$. The modular ρ is p-convex for $0 < p < 1$, and such Orlicz space is only the Fréchet space with the F-norm $|x|^F = \int_\Omega |x(t)|^p d\mu$. It is known that the Banach space is also the Fréchet space with F-norm $|x|^F = \|x\|^L$. In the case of L^p spaces for $p \geqslant 1$ the Luxemburg norm is given by*

$$\|x\|^L = \inf \left\{ s > 0 : \int_\Omega \left| \frac{x(t)}{s} \right|^p d\mu \leqslant 1 \right\} = \left(\int_\Omega |x(t)|^p d\mu \right)^{\frac{1}{p}}.$$

In general case L_0^φ is a convex subset of L^φ, but if $\varphi(x) = x^p$, $p > 0$, then $L_0^\varphi = L^\varphi$.

Definition 2.5. *A function $v_0 \in V$ is a periodic point of the semigroup $(T_t)_{t \geqslant 0}$ with a period $t_0 \geqslant 0$ if and only if $T_{t_0} v_0 = v_0$. A number $t_0 > 0$ is called a principal period of a periodic point v_0 if and only if the set of all periods of v_0 is equal $\mathbb{N} t_0$.*

Definition 2.6. *The semigroup $(T_t)_{t \geqslant 0}$ is strongly stable in V if and only if for every $v \in V$,*

$$\lim_{t \to \infty} T_t v = 0 \quad \textit{in } V.$$

Definition 2.7. *The semigroup* $(T_t)_{t\geqslant 0}$ *is exponentially stable when its trivial solution is exponentially stable i.e. there exists* $M < \infty$ *and* $\omega > 0$ *such that*

$$\|T_t\| \leqslant M e^{-\omega t}, \quad \textit{for } t \geqslant 0.$$

We are going to study chaotic behaviour of the semigroup $(T_t)_{t\geqslant 0}$. We use there Devaney's definition of chaos. Recall that according to Devaney a dynamical system $(F_t)_{t\geqslant 0}$ defined in a metric space (V, ϱ) is chaotic as

- $(F_t)_{t\geqslant 0}$ is transitive, that is for all nonempty open subsets $U_1, U_2 \subset V$ there exists $t > 0$ such that $F_t(U_1) \cap U_2 \neq \emptyset$;
- the set of periodic points of the system $(F_t)_{t\geqslant 0}$ is dense in V.

The original Devaney's definition (see Devaney, 1989) also contained the notion of sensitive dependence on initial conditions in the sense of Guckenhaimer but it was proved that this property appears immediately from transitivity of the system and density of the set of its periodic points. The appropriate proofs can be found in Banks et al. (1992)(for systems discrete in time) and Banasiak and Lachowicz (2002) with references therein (for the continuous case).

3. The semigroup $(T_t)_{t\geqslant 0}$

The problem (1.1)-(1.2) has a unique, nonnegative solution $u(t,x)$ (see Lasota, 1981; Lasota and Szarek, 2004). Using the method of characteristics we can write the solution in the explicit form. We quote the sketch of the proof after Lasota (1981). This gives an opportunity to introduce some notation. We denote by $\phi(t;\theta,s) = \big(\phi^1(t;\theta,s),\dots,\phi^d(t;\theta,s)\big)$, $\theta \geqslant 0$, $s \in D$, the unique solution of the system of equations

$$\frac{dx_i}{dt} = c_i(x), \qquad i = 1,\dots,d \tag{3.1}$$

with the initial conditions $x_i(\theta) = s_i$. We also write $\phi_s(t) = \phi(t;0,s)$, where $\phi_s(t) = \big(\phi_s^1(t),\dots,\phi_s^d(t)\big)$. From (1.3) it follows that $\phi(t;\theta,s)$ is nonnegative and nondecreasing function of t. Since $c_i(0) = 0$, $i = 1,\dots,d$, the solution is defined for $t \in [0,\theta]$. The function $\phi_s(t)$ is defined for $s \in D$ and $t \in [0,\tau(s)]$, where $\tau(s)$ is the first point such that $\phi_s(\tau(s)) \in \mathrm{Fr}(D)$ i.e. $\tau(s) = \inf\{\sigma : \phi_s(\sigma) \in \mathrm{Fr}(D)\}$. The solution $\phi_0(t) \equiv 0$ is defined for all $t \geqslant 0$, therefore $\tau(0) = \infty$. Let $\psi(t;s,r)$ denote the unique solution of the equation

$$\frac{dy}{dt} = \gamma y \quad \text{for} \quad 0 \leqslant t \leqslant \tau(s) \tag{3.2}$$

with the initial condition $y(0) = r$, where $(r,s) \in [0,\infty) \times D$.
The solutions $(\phi_s(t), \psi(t;s,r))$ are characteristics of equation (1.1). Thus we have

$$u(t,\phi_s(t)) = \psi(t;s,r),$$

$$u(t,\phi_s(t)) = re^{\gamma t}$$

where $r = u(0,s) = v(s)$. Substituting $s = \phi(0;t,x)$ we obtain $\phi_s(t) = x$ and

$$u(t,x) = \psi(t;\phi(0;t,x), v(\phi(0;t,x))) = e^{\gamma t} v(\phi(0;t,x))$$

for $(t,x) \in [0,\infty) \times D$. If the function

$$T_t v(x) = u(t,x) = e^{\gamma t} v(\phi(0;t,x)) \tag{3.3}$$

belongs to V for $v \in V$ and $t \geqslant 0$, then the family $(T_t)_{t\geqslant 0}$ is a semigroup acting on V. Our purpose is to study chaos and stability conditions of this semidynamical system.

Lemma 3.1. *Let ϕ satisfy the system of equations (3.1), $f \in L^p(D)$ and $D' = \phi(0;t,D_0)$, where $D_0 \subset D$, $\mu(D_0) > 0$. Then*

$$\int_{D'} f\left(\phi(t;0,x)\right) dx = \int_{D_0} f(z) e^{-\int_0^t \operatorname{div} c(\phi(0;s,z))ds} dz.$$

Proof: Let $\phi = \left(\phi^1, \ldots, \phi^d\right)$ be the solution of the system of equations (3.1). Fix $\alpha_{ik}(t) = \frac{\partial \phi_i(t;0,x)}{\partial x_k}$; $i,k = 1,\ldots,d$. By the differentiation with respect to t we obtain (see, for instance, Hartman, 1964)

$$\begin{cases} \alpha'_{ik}(t) = \displaystyle\sum_{j=1}^{d} \frac{\partial c_i\left(\phi(t;0,x)\right)}{\partial x_j} \cdot \alpha_{jk}(t) \\ \alpha'_{ik}(0) = \delta_{ik} \end{cases} . \tag{3.4}$$

Using the notation $X(t) = [\alpha_{ik}(t)]$ and $A(t) = \left[\frac{\partial c_i(\phi(t;0,x))}{\partial x_k}\right]$ we can express (3.4) in the form of the matrix equation $X'(t) = A(t)X(t)$. By Liouville's formula (see Teschl, 2012), $\det X(t) = e^{\int_0^t \operatorname{tr}(A(s))ds}$. In this case $\operatorname{tr}(A(t)) = \operatorname{div} c\left(\phi(t;0,x)\right)$. Integrating by substitution we get

$$\begin{aligned} \int_{D'} f\left(\phi(t;0,x)\right) dx &= \left| \begin{array}{l} z = \phi(t;0,x) \\ x = \phi(0;t,z) \\ dz = e^{\int_0^t \operatorname{div} c(\phi(s;0,x))ds} dx \\ dx = e^{-\int_0^t \operatorname{div} c(\phi(0;s,z))ds} dz \\ D_0 = \phi(t;0,D') \end{array} \right| \\ &= \int_{D_0} f(z) e^{-\int_0^t \operatorname{div} c(\phi(0;s,z))ds} dz. \end{aligned}$$

■

4. Asymptotic properties

Theorem 4.1. *If $\gamma > -\frac{1}{p} \cdot \liminf_{x\to 0} \operatorname{div} c(x)$, then for any $t_0 > 0$ there exists a periodic point $v_0 \in L^p(D)$ of the dynamical system $(T_t)_{t\geqslant 0}$.*

Proof: Let $D_0 = \{x \in D : \tau(x) \leqslant t_0\}$ and let w be an arbitrary function belonging to $L^p(D_0)$. Let $D_n = \{x \in D : nt_0 < \tau(x) \leqslant (n+1)t_0\}$. We can define a function v_0 on the set D by "squeezing" the graph of the function w into the sets D_n. We put

$$v_0(x) = \begin{cases} \mathrm{e}^{-n\gamma t_0} w\left(\phi(nt_0; 0, x)\right) & \text{for} \quad x \in D_n \\ 0 & \text{for} \quad x \in D \setminus D_n \end{cases}. \tag{4.1}$$

The function v_0 constructed in the above manner is the periodic point of the dynamical system $(T_t)_{t\geqslant 0}$

$$T_{t_0} v_0(x) = e^{\gamma t_0} e^{-n\gamma t_0} w\left(\phi(nt_0; 0, \phi(0; t_0, x))\right) = e^{-\gamma t_0 (n-1)} w\left(\phi((n-1)t_0; 0, x)\right).$$

It is sufficient to prove that v_0 belongs to L^p space.

$$\begin{aligned} \int_D |v_0(x)|^p dx &= \sum_{n=0}^{\infty} \int_{D_n} |v_0(x)|^p dx = \sum_{n=0}^{\infty} \int_{D_n} \left| \mathrm{e}^{-n\gamma t_0} w\left(\phi(nt_0; 0, x)\right)\right|^p dx \\ &= \sum_{n=0}^{\infty} \mathrm{e}^{-n\gamma t_0 p} \int_{D_n} |w(\phi(nt_0; 0, x))|^p dx \\ &= \sum_{n=0}^{\infty} \mathrm{e}^{-n\gamma t_0 p} \int_{D_0} |w(z)|^p \mathrm{e}^{-\int_0^{nt_0} \operatorname{div} c(\phi(0; s, z)) ds} dz \\ &\leqslant \sum_{n=0}^{\infty} \mathrm{e}^{-n\gamma t_0 p} \int_{D_0} |w(z)|^p \mathrm{e}^{-nt_0 \inf_{x\in D_n} \operatorname{div} c(x)} dz \\ &= \int_{D_0} |w(z)|^p dz \sum_{n=0}^{\infty} \mathrm{e}^{-nt_0(\gamma p + \inf_{x \in D_n} \operatorname{div} c(x))}. \end{aligned}$$

For all $\varepsilon > 0$ there exists n_0 such that $\inf_{x\in D_n} \operatorname{div} c(x) > \liminf_{x\to 0} \operatorname{div} c(x) - \varepsilon$ for all $n \geqslant n_0$. For n large enough $\gamma p + \inf_{x\in D_n} \operatorname{div} c(x) > \gamma p + \liminf_{x\to 0} \operatorname{div} c(x) - \varepsilon$. This is true for all $\varepsilon > 0$, so $\gamma p + \inf_{x\in D_n} \operatorname{div} c(x) > \gamma p + \liminf_{x\to 0} \operatorname{div} c(x) > 0$. The series $\sum_{n=0}^{\infty} \mathrm{e}^{-nt_0(\gamma p + \inf_{x\in D_n} \operatorname{div} c(x))}$ is convergent. This gives the conclusion $v_0 \in L^p(D)$ because of the assumption $w \in L^p(D)$. ■

Theorem 4.2. *If $\gamma > -\frac{1}{p} \cdot \liminf_{x\to 0} \operatorname{div} c(x)$, then the set of periodic points of (1.1) is dense in the $L^p(D)$ space.*

Proof: Let w be an arbitrary function from $L^p(D)$ space and let $\varepsilon > 0$. Define v by the formula (4.1). Fix t_0 so large that $|w|^F_{D\setminus D_0} < \frac{\varepsilon}{2}$ and $|v|^F_{D\setminus D_0} < \frac{\varepsilon}{2}$. For $x \in D_0$ $v(x) = w(x)$, so finally we have

$$|v - w|^F_D = |v - w|^F_{D\setminus D_0} \leqslant |v|^F_{D\setminus D_0} + |w|^F_{D\setminus D_0} < \varepsilon.$$

This completes the proof. ■

Theorem 4.3. *If $\gamma > -\frac{1}{p} \cdot \liminf_{x\to 0} \operatorname{div} c(x)$ then the dynamical system $(T_t)_{t\geqslant 0}$ is transitive in the $L^p(D)$ space.*

Proof: Let

$$B(v_1, \varepsilon_1) = \{\sigma \in L^p(D) : |v_1 - \sigma|_D^F < \varepsilon_1\}$$

and

$$B(v_2, \varepsilon_2) = \{\sigma \in L^p(D) : |v_2 - \sigma|_D^F < \varepsilon_2\}$$

be two open balls with centers in $v_1, v_2 \in L^p(D)$. Let us define the following function

$$w(x) = \begin{cases} v_1(x) & \text{for} \quad x \in D_t \\ \mathrm{e}^{-\gamma t} v_2\left(\phi(t;0,x)\right) & \text{for} \quad x \notin D_t \end{cases}$$

at the suitable choice of t, where $D_t = \{x \in D : \tau(x) \leqslant t\}$. We should show that the above function w belongs to the space $L^p(D)$:

$$\begin{aligned} \int_D |w(x)|^p dx &= \int_{D_t} |w(x)|^p dx + \int_{D\setminus D_t} |w(x)|^p dx \\ &= \int_{D_t} |v_1(x)|^p dx + \int_{D\setminus D_t} |\mathrm{e}^{-\gamma t} v_2\left(\phi(t;0,x)\right)|^p dx \\ &= \int_{D_t} |v_1(x)|^p dx + \mathrm{e}^{-\gamma tp} \int_D |v_2(x)|^p e^{-\int_0^t \operatorname{div} c(\phi(0;s,x))ds} dx \\ &\leqslant \int_{D_t} |v_1(x)|^p dx + \mathrm{e}^{-t(\gamma p + \inf_{x\in D} \operatorname{div} c(x))} \int_D |v_2(x)|^p dx. \end{aligned}$$

The exponent $t\left(\gamma p + \inf_{x\in D} \operatorname{div} c(x)\right)$ is positive and finite at the suitable choice of t. Therefore, $w \in L^p(D)$. It results from the fact that $v_1, v_2 \in L^p(D)$. Then

$$|v_1 - w|_D^F \quad = \quad |v_1 - w|_{D\setminus D_t}^F \leqslant |v_1|_{D\setminus D_t}^F + |w|_{D\setminus D_t}^F.$$

By the estimation it turns out that for t large enough we obtain $|v_1 - w|_D^F < \varepsilon_1$, hence $w \in B(v_1, \varepsilon_1)$. Therefore, $T_t w \in T_t\left(B(v_1, \varepsilon_1)\right)$ and $v_2 = T_t w \in B(v_2, \varepsilon_2)$. We learn from the above that the intersection two sets $B(v_2, \varepsilon_2)$ and $T_t\left(B(v_1, \varepsilon_1)\right)$ is not empty. So we get the conclusion about transitivity of the dynamical system $(T_t)_{t\geqslant 0}$ in the space $L^p(D)$. ■

As proved by Banks et al. (1992) the sensitive dependence of the dynamical system on initial conditions in the sense of Guckenhaimer appears immediately from its transitivity and density of the set of its periodic points. That is expressed by the following corollary.

Corollary 1. *If $\gamma > -\frac{1}{p} \cdot \liminf_{x\to 0} \operatorname{div} c(x)$ then the dynamical system $(T_t)_{t\geqslant 0}$ is chaotic in the sense of Devaney in the $L^p(D)$ space.*

Theorem 4.4. *If $\gamma \leqslant -\frac{1}{p} \cdot \limsup_{x\to 0} \operatorname{div} c(x)$ then the semigroup $(T_t)_{t\geqslant 0}$ is strongly stable in the $L^p(D)$ space.*

Proof: Let $v \in L^p(D)$ be an arbitrary function.

$$\begin{aligned}\int_D |T_t v(x)|^p\,dx &= \int_D \left|e^{\gamma t} v\left(\phi(0;t,x)\right)\right|^p dx \\ &= e^{\gamma p t}\int_{D_t} |v(x)|^p e^{\int_0^t \operatorname{div} c(\phi(s;0,x))ds} dx \\ &\leqslant e^{tp\left(\gamma+\frac{1}{p}\sup_{x\in D_t}\operatorname{div} c(x)\right)}\int_{D_t}|v(x)|^p dx.\end{aligned}$$

Let $\varepsilon > 0$. We have $\sup_{x\in D_t}\operatorname{div} c(x) < \limsup_{x\to 0}\operatorname{div} c(x) + \varepsilon$ for large enough t. Thus, $\gamma + \frac{1}{p}\sup_{x\in D_t}\operatorname{div} c(x) < \gamma + \frac{1}{p}\left(\limsup_{x\to 0}\operatorname{div} c(x) + \varepsilon\right)$. This is true for all $\varepsilon > 0$, so $\gamma + \frac{1}{p}\sup_{x\in D_t}\operatorname{div} c(x) \leqslant 0$. Hence $|T_t v|_D^F \to 0$ as $t \to \infty$ in the space $L^p(D)$ for $p > 0$. This proves the strong stability of the system $(T_t)_{t\geqslant 0}$. ■

Corollary 2. *If $\gamma < -\frac{1}{p}\cdot\limsup_{x\to 0}\operatorname{div} c(x)$ then the dynamical system $(T_t)_{t\geqslant 0}$ is exponentially stabile in $L^p(D)$ for $p \geqslant 1$ with $M = 1$ and $\omega = \gamma + \frac{1}{p}\sup_{x\in D}\operatorname{div} c(x)$.*

Example 2. *Let us consider the Lasota equation (1.1) with the initial condition (1.2) in the space $L^p(D)$, $p > 0$ where $D = D_1 \cup D_2$, $D_1 \cap D_2 = \emptyset$ and D_1, D_2 are invariant with respect to multiplication. Here*

$$c(x) = \begin{cases} (x_1, x_2, \ldots, x_d) & \textit{for} \quad x \in D_1 \\ (2x_1, 2x_2, \ldots, 2x_d) & \textit{for} \quad x \in D_2 \end{cases}$$

and $-\frac{2d}{p} < \gamma \leqslant -\frac{d}{p}$. The dynamical system $(T_t)_{t\geqslant 0}$ is in the form

$$(T_t v)(x) = u(t,x) = e^{\gamma t} v(xe^{-t})\big|_{D_1} + e^{\gamma t} v(xe^{-2t})\big|_{D_2}.$$

Let us define two new dynamical systems on $L^p(D)$

$$(T_t^1 v)(x) = e^{\gamma t} v(xe^{-t}) \quad \textit{and} \quad (T_t^2 v)(x) = e^{\gamma t} v(xe^{-2t}).$$

It is clear that if supp $v \subset D_i$ then $T_t v = T_t^i v$, $i = 1, 2$. The system $(T_t^1)_{t\geqslant 0}$ is strongly stable and $(T_t^2)_{t\geqslant 0}$ is chaotic due to Theorem 4.4 and Corollary 1, respectively. We claim that the system $(T_t)_{t\geqslant 0}$ is neither chaotic nor stable. Assume that the system $(T_t)_{t\geqslant 0}$ is chaotic. Then there exists non-trivial periodic trajectory i.e. there exist $v \in L^p$ and $t_0 > 0$ such that $T_{nt_0} v = v$ for all $n \in \mathbb{N}$. Furthermore, $v = v|_{D_1} + v|_{D_2}$ where $v|_{D_i} \in L^p(D)$, $i = 1, 2$. It follows that

$$v|_{D_1} = T_{nt_0} v|_{D_1} = \left(T_{nt_0}^1 v\big|_{D_1} + T_{nt_0}^2 v\big|_{D_2}\right)\Big|_{D_1} = T_{nt_0}^1 v\big|_{D_1}.$$

We know that $T_{nt_0}^1 v\big|_{D_1} \to 0$, in contradiction with the above equality. It proves the lack of chaotic behaviour for $(T_t)_{t\geqslant 0}$.
Since $(T_t^2)_{t\geqslant 0}$ is chaotic, there exists $v \in L^p(D)$ and $t_0 > 0$ such that $T_{nt_0}^2 v = v$

for all $n \in \mathbb{N}$. By the invariance of D_2, $T_{nt_0}v|_{D_2} = T^2_{nt_0}v\big|_{D_2} = v|_{D_2}$. Hence, $T_t v|_{D_2} \nrightarrow 0$. Therefore, the system $(T_t)_{t\geqslant 0}$ is not stable.
This example shows that the dynamical system (3.3) is neither chaotic nor stable for $\gamma \in \left(-\frac{1}{p}\cdot \limsup_{x\to 0} \operatorname{div} c(x); -\frac{1}{p}\cdot \liminf_{x\to 0} \operatorname{div} c(x)\right]$

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