

A time-scale variational approach to inflation,
unemployment and social loss^{*†}

by

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Abstract: Both inflation and unemployment inflict social losses. When a tradeoff exists between the two, what would be the best combination of inflation and unemployment? A well known approach in economics to address this question is writing the social loss as a function of the rate of inflation p and the rate of unemployment u , with different weights, and then, using known relations between p , u , and the expected rate of inflation π , to rewrite the social loss function as a function of π . The answer is achieved by applying the calculus of variations in order to find an optimal path π that minimizes total social loss over a given time interval. Economists dealing with this question use a continuous or a discrete variational problem. Here we propose to use a time-scale model, unifying the results available in the literature. Moreover, the new formalism allows for obtaining new insights into the classical models when applied to real data of inflation and unemployment.

Keywords: calculus on time scales; calculus of variations; delta derivatives; dynamic model; inflation; unemployment

1. Introduction

Time-scale calculus is a recent and exciting mathematical theory that unifies two existing approaches to dynamic modelling — difference and differential equations — into a general framework called dynamic models on time scales (Bohner and Peterson, 2001; Hilger, 1997; Mozyrska and Torres, 2009). As a

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more general approach to dynamic modelling, it allows for considering more complex time domains, such as $h\mathbb{Z}$, $q^{\mathbb{N}_0}$ or complex hybrid domains (Almeida and Torres, 2009).

Both inflation and unemployment inflict social losses. When a Phillips trade-off exists between the two, what would be the best combination of inflation and unemployment? A well-known approach consists in writing the social loss function as a function of the rate of inflation p and the rate of unemployment u , with different weights; then, using relations between p , u and the expected rate of inflation π , to rewrite the social loss function as a function of π ; finally, to apply the theory of the calculus of variations in order to find an optimal path π that minimizes the total social loss over a certain time interval $[0, T]$ under study. Economists dealing with this question implement the above approach using both continuous and discrete models (Chiang, 1992; Taylor, 1989). Here we propose a new, more general, time-scale model. We claim that such model describes better the reality.

We compare solutions to three models — the continuous, the discrete, and the time-scale model with $\mathbb{T} = h\mathbb{Z}$ — using real data from the USA over a period of 11 years, from 2000 to 2010. Our results show that the solutions to the classical continuous and discrete models do not approximate well the reality. Therefore, when predicting the future, one cannot base predictions on the two classical models only. The time-scale approach proposed here shows, however, that the classical models are adequate if one uses an appropriate data sampling process. Moreover, the proper time for data collection can be computed from the theory of time scales.

The paper is organized as follows. Section 2 provides all the necessary definitions and results of the delta-calculus on time scales, which will be used throughout the text. This section makes the paper accessible to economists with no previous contact with the time-scale calculus. In Section 3 we present the economic model under our consideration, in continuous, discrete, and time-scale settings. Section 4 contains our results. Firstly, we derive in Section 4.1 the necessary (Theorem 13 and Corollary 1) and sufficient (Theorem 14) optimality conditions for the variational problem that models the economical situation. For the time scale $\mathbb{T} = h\mathbb{Z}$ with appropriate values of h , we obtain an explicit solution for the global minimizer of the total social loss problem (Theorem 15). Secondly, we apply in Section 4.2 those conditions to the model with real data of inflation (InflationData.Com, 2000-2010), and unemployment (Unemployment-Data.com, 2000-2010). We end with Section 5 of conclusions.

2. Preliminaries

In this section we introduce basic definitions and theorems that will be useful in the sequel. For more on the theory of time scales we refer to Bohner and Peterson (2001, 2003). For general results on the calculus of variations on time scales we refer the reader to Girejko et al. (2012), Malinowska and Torres (2011), Martins and Torres (2011) and references therein.

A time scale \mathbb{T} is an arbitrary nonempty closed subset of \mathbb{R} . Let $a, b \in \mathbb{T}$ with $a < b$. We define the interval $[a, b]$ in \mathbb{T} by $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T} = \{t \in \mathbb{T} : a \leq t \leq b\}$.

Definition 1 (Bohner and Peterson, 2001). *The backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ for $t \neq \inf \mathbb{T}$ and $\rho(\inf \mathbb{T}) := \inf \mathbb{T}$ if $\inf \mathbb{T} > -\infty$. The forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ for $t \neq \sup \mathbb{T}$ and $\sigma(\sup \mathbb{T}) := \sup \mathbb{T}$ if $\sup \mathbb{T} < +\infty$. The backward graininess function $\nu : \mathbb{T} \rightarrow [0, \infty)$ is defined by $\nu(t) := t - \rho(t)$, while the forward graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$.*

Example 1. *The two classical time scales are \mathbb{R} and \mathbb{Z} , representing the continuous and the purely discrete time, respectively. The other example of interest to the present study is the periodic time scale $h\mathbb{Z}$. It follows from Definition 1 that if $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $\rho(t) = t$, and $\mu(t) = 0$ for all $t \in \mathbb{T}$; if $\mathbb{T} = h\mathbb{Z}$, then $\sigma(t) = t + h$, $\rho(t) = t - h$, and $\mu(t) = h$ for all $t \in \mathbb{T}$.*

A point $t \in \mathbb{T}$ is called *right-dense*, *right-scattered*, *left-dense* or *left-scattered* if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, and $\rho(t) < t$, respectively. We say that t is *isolated* if $\rho(t) < t < \sigma(t)$, and that t is *dense* if $\rho(t) = t = \sigma(t)$.

2.1. The delta derivative and the delta integral

We collect here the necessary theorems and properties concerning differentiation and integration on a time scale. To simplify the notation, we define $f^\sigma(t) := f(\sigma(t))$. The delta derivative is defined for points in the set

$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T} \setminus \{\sup \mathbb{T}\} & \text{if } \rho(\sup \mathbb{T}) < \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{otherwise.} \end{cases}$$

Definition 2 (Section 1.1 of Bohner and Peterson, 2001). *We say that a function $f : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable at $t \in \mathbb{T}^\kappa$ if there is a number $f^\Delta(t)$ such that for all $\varepsilon > 0$ there exists a neighborhood O of t such that*

$$|f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in O.$$

We call to $f^\Delta(t)$ the Δ -derivative of f at t .

Theorem 1 (Theorem 1.16 of Bohner and Peterson, 2001). *Let $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. The following holds:*

1. *If f is Δ -differentiable at t , then f is continuous at t .*
2. *If f is continuous at t and t is right-scattered, then f is Δ -differentiable at t with*

$$f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\mu(t)}.$$

3. If t is right-dense, then f is Δ -differentiable at t if, and only if, the limit

$$\lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case,

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}.$$

4. If f is Δ -differentiable at t , then $f^\sigma(t) = f(t) + \mu(t)f^\Delta(t)$.

Example 2. If $\mathbb{T} = \mathbb{R}$, then item 3 of Theorem 1 yields that $f : \mathbb{R} \rightarrow \mathbb{R}$ is Δ -differentiable at $t \in \mathbb{R}$ if, and only if,

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}$$

exists, i.e., if, and only if, f is differentiable (in the ordinary sense) at t : $f^\Delta(t) = f'(t)$. If $\mathbb{T} = h\mathbb{Z}$, then point 2 of Theorem 1 yields that $f : \mathbb{Z} \rightarrow \mathbb{R}$ is Δ -differentiable at $t \in h\mathbb{Z}$ if, and only if,

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+h) - f(t)}{h}. \quad (1)$$

In the particular case $h = 1$, $f^\Delta(t) = \Delta f(t)$, where Δ is the usual forward difference operator.

Theorem 2 (Theorem 1.20 of Bohner and Peterson, 2001). Assume $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are Δ -differentiable at $t \in \mathbb{T}^\kappa$. Then,

1. The sum $f + g : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable at t with $(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t)$.
2. For any constant α , $\alpha f : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable at t with $(\alpha f)^\Delta(t) = \alpha f^\Delta(t)$.
3. The product $fg : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable at t with

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t).$$

4. If $g(t)g^\sigma(t) \neq 0$, then f/g is Δ -differentiable at t with

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g^\sigma(t)}.$$

Definition 3 (Definition 1.71 of Bohner and Peterson, 2001). A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$ provided $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$.

Definition 4 (Bohner and Peterson, 2001). A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in \mathbb{T} and its left-sided limits exists (finite) at all left-dense points in \mathbb{T} .

The set of all rd-continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$. The set of functions $f : \mathbb{T} \rightarrow \mathbb{R}$ that are Δ -differentiable and whose derivative is rd-continuous is denoted by $C_{rd}^1 = C_{rd}^1(\mathbb{T}) = C_{rd}^1(\mathbb{T}, \mathbb{R})$.

Theorem 3 (Theorem 1.74 of Bohner and Peterson, 2001). *Every rd-continuous function f has an antiderivative F . In particular, if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) := \int_{t_0}^t f(\tau)\Delta\tau, \quad t \in \mathbb{T},$$

is an antiderivative of f .

Definition 5. *Let \mathbb{T} be a time scale and $a, b \in \mathbb{T}$. If $f : \mathbb{T}^\kappa \rightarrow \mathbb{R}$ is an rd-continuous function and $F : \mathbb{T} \rightarrow \mathbb{R}$ is an antiderivative of f , then the Δ -integral is defined by*

$$\int_a^b f(t)\Delta t := F(b) - F(a).$$

Example 3. *Let $a, b \in \mathbb{T}$ and $f : \mathbb{T} \rightarrow \mathbb{R}$ be rd-continuous. If $\mathbb{T} = \mathbb{R}$, then*

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right hand side is the usual Riemann integral. If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h, & \text{if } a < b, \\ 0, & \text{if } a = b, \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} f(kh)h, & \text{if } a > b. \end{cases}$$

Theorem 4 (Theorem 1.75 of Bohner and Peterson, 2001). *If $f \in C_{rd}$ and $t \in \mathbb{T}^\kappa$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

Theorem 5 (Theorem 1.77 of Bohner and Peterson, 2001). *If $a, b \in \mathbb{T}$, $a \leq c \leq b$, $\alpha \in \mathbb{R}$, and $f, g \in C_{rd}(\mathbb{T}, \mathbb{R})$, then:*

1. $\int_a^b (f(t) + g(t))\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t,$

2. $\int_a^b \alpha f(t) \Delta t = \alpha \int_a^b f(t) \Delta t,$
3. $\int_a^b f(t) \Delta t = - \int_b^a f(t) \Delta t,$
4. $\int_a^b f(t) \Delta t = \int_a^c f(t) \Delta t + \int_c^b f(t) \Delta t,$
5. $\int_a^a f(t) \Delta t = 0,$
6. $\int_a^b f(t) g^\Delta(t) \Delta t = f(t)g(t)|_{t=a}^{t=b} - \int_a^b f^\Delta(t)g^\sigma(t) \Delta t,$
7. $\int_a^b f^\sigma(t)g^\Delta(t) \Delta t = f(t)g(t)|_{t=a}^{t=b} - \int_a^b f^\Delta(t)g(t) \Delta t,$
8. if $f(t) \geq 0$ for all $a \leq t < b$, then $\int_a^b f(t) \Delta t \geq 0.$

2.2. Delta dynamic equations

We now recall the definition and main properties of the delta exponential function. The general solution to a linear and homogenous second-order delta differential equation with constant coefficients is given.

Definition 6 (Definition 2.25 of Bohner and Peterson, 2001). *We say that a function $p : \mathbb{T} \rightarrow \mathbb{R}$ is regressive if*

$$1 + \mu(t)p(t) \neq 0$$

for all $t \in \mathbb{T}^\kappa$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$ is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Definition 7 (Definition 2.30 of Bohner and Peterson, 2001). *If $p \in \mathcal{R}$, then we define the exponential function by*

$$e_p(t, s) := \exp \left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta \tau \right), \quad s, t \in \mathbb{T},$$

where ξ_μ is the cylinder transformation (see Definition 2.21 of Bohner and Peterson, 2001).

Example 4. *Let \mathbb{T} be a time scale, $t_0 \in \mathbb{T}$, and $\alpha \in \mathcal{R}(\mathbb{T}, \mathbb{R})$. If $\mathbb{T} = \mathbb{R}$, then $e_\alpha(t, t_0) = e^{\alpha(t-t_0)}$ for all $t \in \mathbb{T}$. If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, and $\alpha \in \mathbb{C} \setminus \{-\frac{1}{h}\}$ is a constant, then*

$$e_\alpha(t, t_0) = (1 + \alpha h)^{\frac{t-t_0}{h}} \quad \text{for all } t \in \mathbb{T}. \quad (2)$$

Theorem 6 (Theorem 2.36 of Bohner and Peterson, 2001). *Let $p, q \in \mathcal{R}$ and $\ominus p(t) := \frac{-p(t)}{1+\mu(t)p(t)}$. The following holds:*

1. $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$,
2. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$,
3. $\frac{1}{e_p(t, s)} = e_{\ominus p}(t, s)$,
4. $e_p(t, s) = \frac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$,
5. $\left(\frac{1}{e_p(t, s)}\right)^\Delta = \frac{-p(t)}{e_p^\sigma(t, s)}$.

Theorem 7 (Theorem 2.62 of Bohner and Peterson, 2001). *Suppose $y^\Delta = p(t)y$ is regressive, that is, $p \in \mathcal{R}$. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution to the initial value problem*

$$y^\Delta(t) = p(t)y(t), \quad y(t_0) = y_0,$$

is given by $y(t) = e_p(t, t_0)y_0$.

Let us consider the following linear second-order dynamic homogeneous equation with constant coefficients:

$$y^{\Delta\Delta} + \alpha y^\Delta + \beta y = 0, \quad \alpha, \beta \in \mathbb{R}. \tag{3}$$

We say that the dynamic equation (3) is regressive if $1 - \alpha\mu(t) + \beta\mu^2(t) \neq 0$ for $t \in \mathbb{T}^\kappa$, i.e., $\beta\mu - \alpha \in \mathcal{R}$.

Definition 8 (Definition 3.5 of Bohner and Peterson, 2001). *Given two delta differentiable functions y_1 and y_2 , we define the Wronskian $W(y_1, y_2)(t)$ by*

$$W(y_1, y_2)(t) := \det \begin{bmatrix} y_1(t) & y_2(t) \\ y_1^\Delta(t) & y_2^\Delta(t) \end{bmatrix}.$$

We say that two solutions y_1 and y_2 of (3) form a fundamental set of solutions (or a fundamental system) for (3), provided $W(y_1, y_2)(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$.

Theorem 8 (Theorem 3.16 of Bohner and Peterson, 2001). *If (3) is regressive and $\alpha^2 - 4\beta \neq 0$, then a fundamental system for (3) is given by $e_{\lambda_1}(\cdot, t_0)$ and $e_{\lambda_2}(\cdot, t_0)$, where $t_0 \in \mathbb{T}^\kappa$ and λ_1 and λ_2 are given by*

$$\lambda_1 := \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2}, \quad \lambda_2 := \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2}.$$

Theorem 9 (Theorem 3.32 of Bohner and Peterson, 2001). *Suppose that $\alpha^2 - 4\beta < 0$. Define $p = \frac{-\alpha}{2}$ and $q = \frac{\sqrt{4\beta - \alpha^2}}{2}$. If p and $\mu\beta - \alpha$ are regressive, then a fundamental system of (3) is given by $\cos_{\frac{q}{(1+\mu p)}}(\cdot, t_0)e_p(\cdot, t_0)$ and $\sin_{\frac{q}{(1+\mu p)}}(\cdot, t_0)e_p(\cdot, t_0)$, where $t_0 \in \mathbb{T}^\kappa$.*

Theorem 10 (Theorem 3.34 of Bohner and Peterson, 2001). *Suppose $\alpha^2 - 4\beta = 0$. Define $p = \frac{-\alpha}{2}$. If $p \in \mathcal{R}$, then a fundamental system of (3) is given by*

$$e_p(t, t_0) \quad \text{and} \quad e_p(t, t_0) \int_{t_0}^t \frac{1}{1 + p\mu(\tau)} \Delta\tau,$$

where $t_0 \in \mathbb{T}^\kappa$.

Theorem 11 (Theorem 3.7 of Bohner and Peterson, 2001). *If functions y_1 and y_2 form a fundamental system of solutions for (3), then $y(t) = \alpha y_1(t) + \beta y_2(t)$, where α, β are constants, is a general solution to (3), i.e., every function of this form is a solution to (3) and every solution of (3) is of this form.*

2.3. Calculus of variations on time scales

Consider the following problem of the calculus of variations on time scales:

$$\mathcal{L}(y) = \int_a^b L(t, y(t), y^\Delta(t)) \Delta t \longrightarrow \min \quad (4)$$

subject to the boundary conditions

$$y(a) = y_a, \quad y(b) = y_b, \quad (5)$$

where $L : [a, b]_{\mathbb{T}}^{\kappa} \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $(t, y, v) \mapsto L(t, y, v)$, is a given function, and $y_a, y_b \in \mathbb{R}$.

Definition 9. *A function $y \in C_{rd}^1([a, b]_{\mathbb{T}}, \mathbb{R})$ is said to be an admissible path to problem (4) - (5) if it satisfies the given boundary conditions (5).*

We assume that $L(t, \cdot, \cdot)$ is differentiable in (y, v) ; $L(t, \cdot, \cdot)$, $L_y(t, \cdot, \cdot)$ and $L_v(t, \cdot, \cdot)$ are continuous at (y, y^Δ) uniformly at t and rd-continuously at t for any admissible path y .

Definition 10. *We say that an admissible function \hat{y} is a local minimizer to problem (4)–(5) if there exists $\delta > 0$ such that $\mathcal{L}(\hat{y}) \leq \mathcal{L}(y)$ for all admissible functions $y \in C_{rd}^1$ satisfying the inequality $\|y - \hat{y}\| < \delta$. The following norm in C_{rd}^1 is considered:*

$$\|y\| := \sup_{t \in [a, b]_{\mathbb{T}}^{\kappa}} |y(t)| + \sup_{t \in [a, b]_{\mathbb{T}}^{\kappa}} |y^\Delta(t)|.$$

Theorem 12 (Corollary 1 of Ferreira et al., 2011). *If y is a local minimizer to problem (4)–(5), then y satisfies the Euler–Lagrange equation*

$$L_v(t, y(t), y^\Delta(t)) = \int_a^{\sigma(t)} L_y(\tau, y(\tau), y^\Delta(\tau)) \Delta \tau + c \quad (6)$$

for some constant $c \in \mathbb{R}$ and all $t \in [a, b]_{\mathbb{T}}^{\kappa}$.

3. The economical model

The inflation rate, p , affects decisions of the society regarding consumption and saving, and therefore aggregated demand for domestic production, which, in

turn, affects the rate of unemployment, u . A relationship between the inflation rate and the rate of unemployment is described by the Phillips curve, the most commonly used term in the analysis of inflation (Samuelson and Nordhaus, 2004). Having a Phillips tradeoff between u and p , what is then the best combination of inflation and unemployment over time? To answer this question, we follow here the formulations presented in Chiang (1992) and Taylor (1989). The Phillips tradeoff between u and p is defined as

$$p := -\beta u + \pi, \quad \beta > 0, \quad (7)$$

where π is the expected rate of inflation that is captured by the equation

$$\pi' = j(p - \pi), \quad 0 < j \leq 1. \quad (8)$$

The government loss function, λ , is specified in the following quadratic form:

$$\lambda = u^2 + \alpha p^2, \quad (9)$$

where $\alpha > 0$ is the weight attached to government's distaste for inflation relative to the loss from income deviating from its equilibrium level. Combining (7) and (8), and substituting the result into (9), we obtain that

$$\lambda(\pi(t), \pi'(t)) = \left(\frac{\pi'(t)}{\beta j} \right)^2 + \alpha \left(\frac{\pi'(t)}{j} + \pi(t) \right)^2,$$

where α , β , and j are real positive parameters that describe the relations between all variables that occur in the model (Taylor, 1989). The problem is to find the optimal path π that minimizes the total social loss over the time interval $[0, T]$. The initial and the terminal values of π , π_0 and π_T , respectively, are given, with $\pi_0, \pi_T > 0$. To express the importance of the present relative to the future, all social losses are discounted to their present values via a positive discount rate δ . Two models are available in the literature: the *continuous model*

$$\Lambda_C(\pi) = \int_0^T \lambda(\pi(t), \pi'(t)) e^{-\delta t} dt \longrightarrow \min, \quad (10)$$

subject to given boundary conditions

$$\pi(0) = \pi_0, \quad \pi(T) = \pi_T, \quad (11)$$

and the *discrete model*

$$\Lambda_D(\pi) = \sum_{t=0}^{T-1} \lambda(\pi(t), \Delta\pi(t)) (1 + \delta)^{-t} \longrightarrow \min, \quad (12)$$

also subject to the boundary conditions (11). In both cases, (10) and (12),

$$\lambda(t, \pi, v) := \left(\frac{v}{\beta j}\right)^2 + \alpha \left(\frac{v}{j} + \pi\right)^2. \quad (13)$$

Here we propose the more general *time-scale model*

$$\Lambda_{\mathbb{T}}(\pi) = \int_0^T \lambda(t, \pi(t), \pi^\Delta(t)) e_{\ominus\delta}(t, 0) \Delta t \longrightarrow \min \quad (14)$$

subject to boundary conditions (11) and with λ defined by (13). Clearly, the time-scale model includes both the discrete and continuous models as special cases: our time-scale functional (14) reduces to (10) when $\mathbb{T} = \mathbb{R}$ and to (12) when $\mathbb{T} = \mathbb{Z}$.

4. Main results

Standard dynamic economic models are set up in either continuous or discrete time. Since time scale calculus can be used to model dynamic processes whose time domains are more complex than the set of integers or real numbers, the use of time scales in economy is a flexible and capable modelling technique. In this section we show the advantage of using (14) with the periodic time scale. We begin by obtaining in Section 4.1 a necessary and also a sufficient optimality condition for our economic model (14): Theorems 13 and 14, respectively. For $\mathbb{T} = h\mathbb{Z}$, $h > 0$, the explicit solution $\hat{\pi}$ to the problem (14) subject to (11) is given (Theorem 15). Afterwards, we use such results with empirical data (Section 4.2).

4.1. Theoretical results

Let us consider the problem

$$\mathcal{L}(\pi) = \int_0^T L(t, \pi(t), \pi^\Delta(t)) \Delta t \longrightarrow \min \quad (15)$$

subject to boundary conditions

$$\pi(0) = \pi_0, \quad \pi(T) = \pi_T. \quad (16)$$

As explained in Section 3, we are particularly interested in the situation where

$$L(t, \pi(t), \pi^\Delta(t)) = \left[\left(\frac{\pi^\Delta(t)}{\beta j}\right)^2 + \alpha \left(\frac{\pi^\Delta(t)}{j} + \pi(t)\right)^2 \right] e_{\ominus\delta}(t, 0). \quad (17)$$

For simplicity, in the sequel we use the notation $[\pi](t) := (t, \pi(t), \pi^\Delta(t))$.

Theorem 13. *If $\hat{\pi}$ is a local minimizer to problem (15)–(16) and the graininess function μ is a Δ -differentiable function on $[0, T]_{\mathbb{T}}^{\kappa}$, then $\hat{\pi}$ satisfies the Euler–Lagrange equation*

$$(L_v[\pi](t))^{\Delta} = (1 + \mu^{\Delta}(t)) L_y[\pi](t) + \mu^{\sigma}(t) (L_y[\pi](t))^{\Delta} \tag{18}$$

for all $t \in [0, T]_{\mathbb{T}}^{\kappa_2}$.

Proof. If $\hat{\pi}$ is a local minimizer to (15)–(16), then, by Theorem 12, $\hat{\pi}$ satisfies the following equation:

$$L_v[\pi](t) = \int_0^{\sigma(t)} L_y[\pi](\tau) \Delta\tau + c.$$

Using the properties of the Δ -integral (see Theorem 4), we can write that $\hat{\pi}$ satisfies

$$L_v[\pi](t) = \int_0^t L_y[\pi](\tau) \Delta\tau + \mu(t) L_y[\pi](t) + c. \tag{19}$$

Taking the Δ -derivative to both sides of (19), we obtain equation (18). ■

Using Theorem 13, we can immediately write the classical Euler–Lagrange equations for the continuous (10) and the discrete (12) models.

Example 5. *Let $\mathbb{T} = \mathbb{R}$. Then, $\mu \equiv 0$ and (18) with the Lagrangian (17) reduces to*

$$(1 + \alpha\beta^2) \pi''(t) - \delta(1 + \alpha\beta^2) \pi'(t) - \alpha j \beta^2 (\delta + j) = 0. \tag{20}$$

This is the Euler–Lagrange equation for the continuous model (10).

Example 6. *Let $\mathbb{T} = \mathbb{Z}$. Then, $\mu \equiv 1$ and (18) with the Lagrangian (17) reduces to*

$$(\alpha j \beta^2 - \alpha \beta^2 - 1) \Delta^2 \pi(t) + (\alpha j^2 \beta^2 + \delta \alpha \beta + \delta) \Delta \pi(t) + \alpha j \beta^2 (\delta + j) \pi(t) = 0. \tag{21}$$

This is the Euler–Lagrange equation for the discrete model (12).

Corollary 1. *Let $\mathbb{T} = h\mathbb{Z}$, $h > 0$, $\pi_0, \pi_T \in \mathbb{R}$, and $T = Nh$ for a certain integer $N > 2h$. If $\hat{\pi}$ is a solution to the problem*

$$\Lambda_h(\pi) = \sum_{t=0}^{T-h} L(t, \pi(t), \pi^{\Delta}(t))h \longrightarrow \min,$$

$$\pi(0) = \pi_0, \quad \pi(T) = \pi_T,$$

then $\hat{\pi}$ satisfies the Euler–Lagrange equation

$$(L_v[\pi](t))^\Delta = L_y[\pi](t) + h(L_y[\pi](t))^\Delta \quad (22)$$

for all $t \in \{0, \dots, T - 2h\}$.

Proof. Follows from Theorem 13 by choosing \mathbb{T} to be the periodic time scale $h\mathbb{Z}$. ■

Example 7. The Euler–Lagrange equation for problem (14) on $\mathbb{T} = h\mathbb{Z}$ is given by (22):

$$(1 + \alpha\beta^2 - \alpha\beta^2jh)\pi^{\Delta\Delta} + (-\delta - \alpha\beta^2\delta - \alpha\beta^2j^2h)\pi^\Delta + (-\alpha\beta^2\delta j - \alpha\beta^2j^2)\pi = 0. \quad (23)$$

Assume that $1 + \alpha\beta^2 - \alpha\beta^2jh \neq 0$. Then, equation (23) is regressive and we can use the theorems well known in the theory of dynamic equations on time scales (see Section 2.2), in order to find its general solution. Introducing the quantities

$$\Omega := 1 + \alpha\beta^2 - \alpha\beta^2jh, \quad A := -(\delta + \alpha\beta^2\delta + \alpha\beta^2j^2h), \quad B := \alpha\beta^2j(\delta + j), \quad (24)$$

we rewrite equation (23) as

$$\pi^{\Delta\Delta} + \frac{A}{\Omega}\pi^\Delta - \frac{B}{\Omega}\pi = 0. \quad (25)$$

The characteristic equation for (25) is

$$\varphi(\lambda) = \lambda^2 + \frac{A}{\Omega}\lambda - \frac{B}{\Omega} = 0$$

with determinant

$$\zeta = \frac{A^2 + 4B\Omega}{\Omega^2}. \quad (26)$$

In general, we have three different cases depending on the sign of the determinant ζ : $\zeta > 0$, $\zeta = 0$ and $\zeta < 0$. However, with our assumptions on the parameters, simple computations show that the last case cannot occur. Therefore, we consider the two possible cases:

1. If $\zeta > 0$, then we have two different characteristic roots:

$$\lambda_1 = \frac{-A + \sqrt{A^2 + 4B\Omega}}{2\Omega} > 0 \quad \text{and} \quad \lambda_2 = \frac{-A - \sqrt{A^2 + 4B\Omega}}{2\Omega} < 0,$$

and by Theorems 8 and 11 we get that

$$\pi(t) = C_1 e_{\lambda_1}(t, 0) + C_2 e_{\lambda_2}(t, 0) \quad (27)$$

is the general solution to (25), where C_1 and C_2 are constants determined using the boundary conditions (11). Using (2), we rewrite (27) as

$$\pi(t) = C_1 (1 + \lambda_1 h)^{\frac{t}{h}} + C_2 (1 + \lambda_2 h)^{\frac{t}{h}}.$$

2. If $\zeta = 0$, then by Theorems 10 and 11 we get that

$$\pi(t) = K_1 e_p(t, 0) + K_2 e_p(t, 0) \int_0^t \frac{\Delta\tau}{1 + p\mu(\tau)} \tag{28}$$

is the general solution to (25), where K_1 and K_2 are constants, determined using the boundary conditions (11), and $p = -\frac{A}{2\Omega} \in \mathcal{R}$. Using Example 3 and (2), we rewrite (28) as

$$\pi(t) = K_1 \left(1 - \frac{A}{2\Omega}h\right)^{\frac{t}{h}} + K_2 \left(1 - \frac{A}{2\Omega}h\right)^{\frac{t}{h}} \frac{2\Omega t}{2\Omega - Ah}.$$

In certain cases one can show that the Euler–Lagrange extremals are indeed minimizers. In particular, this is true for the Lagrangian (17) under study. We recall the notion of jointly convex function (see, e.g., Definition 1.6 of Malinowska and Torres, 2012).

Definition 11. Function $(t, u, v) \mapsto L(t, u, v) \in C^1([a, b]_{\mathbb{T}} \times \mathbb{R}^2; \mathbb{R})$ is jointly convex in (u, v) if

$$L(t, u + u_0, v + v_0) - L(t, u, v) \geq \partial_2 L(t, u, v)u_0 + \partial_3 L(t, u, v)v_0$$

for all $(t, u, v), (t, u + u^0, v + v^0) \in [a, b]_{\mathbb{T}} \times \mathbb{R}^2$.

Theorem 14. Let $(t, u, v) \mapsto L(t, u, v)$ be jointly convex with respect to (u, v) for all $t \in [a, b]_{\mathbb{T}}$. If \hat{y} is a solution to the Euler–Lagrange equation (6), then \hat{y} is a global minimizer to (4)–(5).

Proof. Since L is jointly convex with respect to (u, v) for all $t \in [a, b]_{\mathbb{T}}$,

$$\begin{aligned} \mathcal{L}(y) - \mathcal{L}(\hat{y}) &= \int_a^b [L(t, y(t), y^\Delta(t)) - L(t, \hat{y}(t), \hat{y}^\Delta(t))] \Delta t \\ &\geq \int_a^b [\partial_2 L(t, \hat{y}(t), \hat{y}^\Delta(t)) \cdot (y(t) - \hat{y}(t)) + \partial_3 L(t, \hat{y}(t), \hat{y}^\Delta(t)) \cdot (y^\Delta(t) - \hat{y}^\Delta(t))] \Delta t \end{aligned}$$

for any admissible path y . Let $h(t) := y(t) - \hat{y}(t)$. Using boundary conditions

(5), we obtain that

$$\begin{aligned} \mathcal{L}(y) - \mathcal{L}(\hat{y}) &\geq \int_a^b h^\Delta(t) \left[- \int_a^{\sigma(t)} \partial_2 L(\tau, \hat{y}(\tau), \hat{y}^\Delta(\tau)) \Delta\tau + \partial_3 L(t, \hat{y}(t), \hat{y}^\Delta(t)) \right] \Delta t \\ &\quad + h(t) \int_a^b \partial_2 L(t, \hat{y}(t), \hat{y}^\Delta(t)) \Delta t \Big|_a^b \\ &= \int_a^b h^\Delta(t) \left[- \int_a^{\sigma(t)} \partial_2 L(\tau, \hat{y}(\tau), \hat{y}^\Delta(\tau)) \Delta\tau + \partial_3 L(t, \hat{y}(t), \hat{y}^\Delta(t)) \right] \Delta t. \end{aligned}$$

From (6) we get

$$\mathcal{L}(y) - \mathcal{L}(\hat{y}) \geq \int_a^b h^\Delta(t) c \Delta t = 0$$

for some $c \in \mathbb{R}$. Hence, $\mathcal{L}(y) - \mathcal{L}(\hat{y}) \geq 0$. ■

Combining Examples 4 and 7 and Theorem 14, we obtain the central result to be applied in Section 4.2.

Theorem 15 (Solution to the total social loss problem of the calculus of variations in the time scale $\mathbb{T} = h\mathbb{Z}$, $h > 0$). *Let us consider the economic problem*

$$\begin{aligned} \Lambda_h(\pi) &= \sum_{t=0}^{T-h} \left[\left(\frac{\pi^\Delta(t)}{\beta j} \right)^2 + \alpha \left(\frac{\pi^\Delta(t)}{j} + \pi(t) \right)^2 \right] \left(1 - \frac{h\delta}{1+h\delta} \right)^{\frac{t}{h}} h \longrightarrow \min, \\ \pi(0) &= \pi_0, \quad \pi(T) = \pi_T, \end{aligned} \tag{29}$$

discussed in Section 3 with $\mathbb{T} = h\mathbb{Z}$, $h > 0$, and the Δ -derivative given by (1). More precisely, let $T = Nh$ for a certain integer $N > 2h$, $\alpha, \beta, \delta, \pi_0, \pi_T \in \mathbb{R}^+$, and $0 < j \leq 1$ be such that $h > 0$ and $1 + \alpha\beta^2 - \alpha\beta^2jh \neq 0$. Let Ω , A and B be given as in (24).

1. If $A^2 + 4B\Omega > 0$, then the solution $\hat{\pi}$ to problem (29) is given by

$$\hat{\pi}(t) = C \left(1 - \frac{A - \sqrt{A^2 + 4B\Omega}}{2\Omega} h \right)^{\frac{t}{h}} + (\pi_0 - C) \left(1 - \frac{A + \sqrt{A^2 + 4B\Omega}}{2\Omega} h \right)^{\frac{t}{h}}, \tag{30}$$

$t \in \{0, \dots, T - 2h\}$, where

$$C := \frac{\pi_T - \pi_0 \left(\frac{2\Omega - hA - h\sqrt{A^2 + 4B\Omega}}{2\Omega} \right)^{\frac{T}{h}}}{\left(\frac{2\Omega - hA + h\sqrt{A^2 + 4B\Omega}}{2\Omega} \right)^{\frac{T}{h}} - \left(\frac{2\Omega - hA - h\sqrt{A^2 + 4B\Omega}}{2\Omega} \right)^{\frac{T}{h}}}.$$

2. If $A^2 + 4B\Omega = 0$, then the solution $\hat{\pi}$ to problem (29) is given by

$$\hat{\pi}(t) = \left(1 - \frac{A}{2\Omega}h\right)^{\frac{t}{h}} \pi_0 + \left(1 - \frac{A}{2\Omega}h\right)^{\frac{t}{h}} \left[\pi_T \left(\frac{2\Omega}{2\Omega - Ah}\right)^{\frac{T}{h}} - \pi_0 \right] \frac{t}{T}, \quad (31)$$

$t \in \{0, \dots, T - 2h\}$.

Proof. From Example 7, $\hat{\pi}$ satisfies the Euler–Lagrange equation for problem (29). Moreover, the Lagrangian of functional Λ_h of (29) is a convex function because it is the sum of convex functions. Hence, by Theorem 14, $\hat{\pi}$ is a global minimizer. ■

4.2. Empirical results

We have three forms for the total social loss: continuous (10), discrete (12), and on a time scale \mathbb{T} (14). Our idea is to compare the implications of one model with those of another using empirical data: the rate of inflation p from InflationData.Com (2000-2010) and the rate of unemployment u from UnemploymentData.com (2000-2010), which were collected each month in the USA over 11 years, from 2000 to 2010. We consider the coefficients

$$\beta := 3, \quad j := \frac{3}{4}, \quad \alpha := \frac{1}{2}, \quad \delta := \frac{1}{4},$$

borrowed from Chiang (1992). Therefore, the time-scale total social loss functional for one year is

$$\Lambda_{\mathbb{T}}(\pi) = \int_0^{11} \left[\frac{16}{9} (\pi^{\Delta}(t))^2 + \frac{1}{2} \left(\frac{4}{3} \pi^{\Delta}(t) + \pi(t) \right)^2 \right] e_{\ominus\frac{1}{4}}(t, 0) \Delta t. \quad (32)$$

Empirical values π_E of the expected rate of inflation, π , for all months in each year, are calculated using (7) and appropriate values of p and u (InflationData.Com, 2000-2010; UnemploymentData.com, 2000-2010). In the sequel, the boundary conditions $\pi(0)$ and $\pi(11)$ will be selected from empirical data in January and December, respectively. We shall compare the minimum values of the total social loss functional (32) obtained from continuous and discrete models and the value for empirical data, i.e., the value of the discrete functional $\Lambda_D(\pi_E) =: \Lambda_E$ computed with empirical data π_E .

In the continuous case we use the Euler–Lagrange equation (20) with appropriate boundary conditions in order to find the optimal path that minimizes Λ_C over each year. Then, we calculate the optimal values of Λ_C for each year (see the second column of Table 1). In the third column of Table 1 we collect empirical values of total social loss Λ_E for each year, which are obtained by (12) from empirical data. We find the optimal path that minimizes Λ_D over each year using the Euler–Lagrange equation (21) with appropriate boundary conditions. The optimal values of Λ_D for each year are given in the fifth column of Table 1.

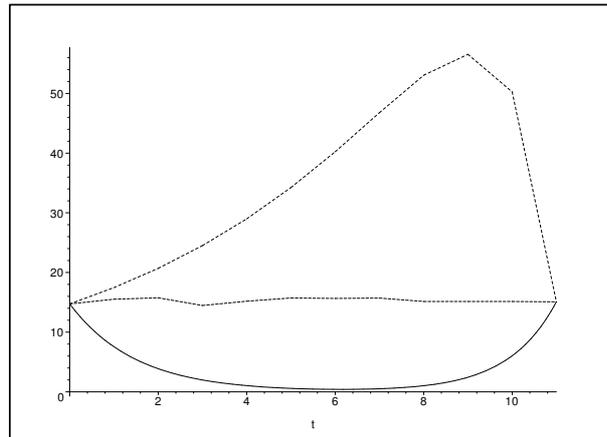


Figure 1. The expected rate of inflation $\hat{\pi}(t)$ during the year 2000 in USA, obtained from the classical discrete model (12) (upper function) and the classical continuous model (10) (lower function), with boundary conditions (11) from January ($t = 0$) and December ($t = 11$), together with the empirical rate of inflation with real data from 2000 (InflationData.Com, 2000-2010; UnemploymentData.com, 2000-2010) (function in the middle).

The paths obtained from the three approaches, using empirical data from 2000, are presented in Fig. 1. The implications obtained from the three methods in a fixed year are very different, independently of the year we chose. Table 2 shows the relative errors between Λ_C and Λ_E (the third column), Λ_D and Λ_E (the fourth column). Our research was motivated by these discrepancies. Why are the results so different? Is it caused by poor design of the model or maybe by something else?

We focus on the data collection time sampling and consider it as a cause of those differences in the results. There may exist other reasons, but we examine here the data gathering. Let us consider our time-scale model in which we consider functional (32) over a periodic time scale $\mathbb{T} = h\mathbb{Z}$. In each year we change the time scale by changing h , in such a way that the sum in the functional makes sense, and we are seeking such value of h for which the absolute error between the minimal values of the functional (32) and Λ_E is minimal. In Table 1, the sixth column presents the values of the most appropriate h and the fourth column the minimal values of the total social loss that correspond to them. Fig. 2 presents the optimal paths for the continuous, discrete and time-scale models together with the empirical path, obtained using real data from 2000 (InflationData.Com, 2000-2010; UnemploymentData.com, 2000-2010). In the second column of Table 2 we collect the relative errors between the minimal values of functional Λ_E and Λ_h .

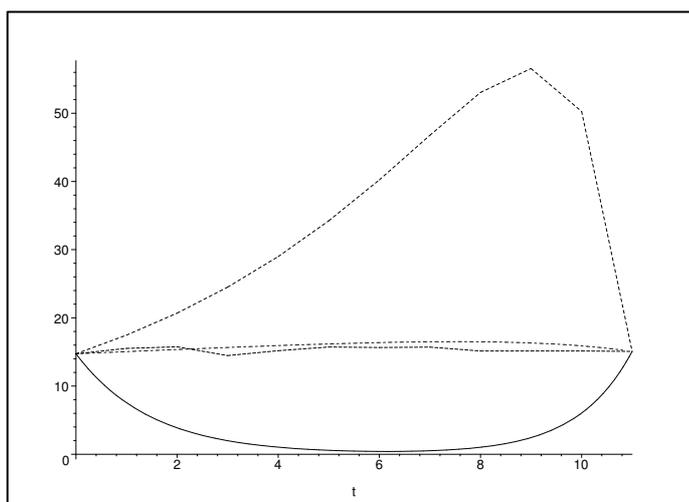


Figure 2. The three functions of Fig. 1 together with the one obtained from our time-scale model (14) and Theorem 15, illustrating the fact that the expected rate of inflation given by (30) with $h = 0.22$ approximates well the empirical rate of inflation.

5. Conclusions

We introduced a time-scale model for the total social loss over a certain time interval under study. During examination of the proposed time-scale model for $T = h\mathbb{Z}$, $h > 0$, we changed the graininess parameter. Our goal was to obtain the most similar value of the total social loss functional Λ_h to its real value, i.e., the value from empirical data. We analyzed 11 years with real data from (InflationData.Com, 2000-2010; UnemploymentData.com, 2000-2010). With a well-chosen time scale, we found a small relative error between the real value of the total social loss and the value obtained from our time-scale model (see the second column of Table 2). We conclude that the lack of accurate results from the classical models arise due to an inappropriate frequency of data collection. Indeed, if one measures the level of inflation and unemployment about once a week, which is suggested by the values of h obtained from the time-scale model, e.g., $h = 0.11$ or $h = 0.2$ (here $h = 1$ corresponds to one month), the credibility of the results obtained from the classical methods gets much higher. In other words, similar results to the ones obtained by our time-scale model could be obtained with the classical models, if a higher frequency of data collection were used. In practical terms, however, to collect the levels of inflation and unemployment on a weekly basis is not realizable, and the calculus of variations on time scales (Bartosiewicz and Torres, 2008; Girejko et al., 2010) assumes an important role.

year	The value of the functional in different approaches				the best h
	continuous	empirical	time scales	discrete	
	Λ_C	Λ_E	Λ_h	Λ_D	
2000	37.08888039	457.1493181	487.1508715	2470	0.22
2001	52.78839446	522.8060796	536.0298868	3040	0.11
2002	63.88123645	673.399954	663.2573844	3820	0.11
2003	62.01139398	811.1909476	853.5383036	4520	0.2
2004	61.72908568	703.7663513	699.714732	4130	0.11
2005	56.01553586	672.0977499	665.8735854	4060	0.1
2006	45.73885179	592.0374216	594.1793342	3700	0.1
2007	53.65457721	505.8743517	511.5351347	2910	0.1
2008	73.4472459	785.9852316	746.8126214	4260	0.11
2009	144.2965207	1352.738181	1357.167459	6330	0.22
2010	153.4630805	1819.572063	1865.77131	11400	0.1
11 years	12.89356177	480.5729081	446.1625854	2E+91	0.11

Table 1. Comparison of the values of total social loss functionals in different approaches.

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References

- ALMEIDA, R. and TORRES, D. F. M. (2009) Isoperimetric problems on time scales with nabla derivatives. *J. Vib. Control* **15** (6), 951–958.
- BARTOSIEWICZ, Z. and TORRES, D. F. M. (2008) Noether’s theorem on time scales. *J. Math. Anal. Appl.* **342** (2), 1220–1226.
- BOHNER, M. and PETERSON, A. (2001) *Dynamic equations on time scales*. Birkhäuser Boston, Boston, MA.
- BOHNER, M. and PETERSON, A. (2003) *Advances in Dynamic Equations on*

year	Relative error between the empirical value Λ_E and the result in		
	$\mathbb{T} = h\mathbb{Z}$ with the best h	$\mathbb{T} = \mathbb{R}$	$\mathbb{T} = \mathbb{Z}$
	$\frac{\Lambda_h - \Lambda_E}{\Lambda_E}$	$\frac{\Lambda_C - \Lambda_E}{\Lambda_E}$	$\frac{\Lambda_D - \Lambda_E}{\Lambda_E}$
2000	6.562747053	91.88692208	440.3048637
2001	2.529390479	89.90287288	481.4775533
2002	1.506173195	90.51362625	467.270606
2003	5.220393068	92.35551208	457.2054291
2004	0.575705174	91.2287529	486.8424929
2005	0.926080247	91.66556712	504.0787967
2006	0.361786602	92.27433096	524.9604949
2007	1.119009687	89.39369489	475.2416564
2008	4.98388629	90.6553911	441.9949165
2009	0.327430545	89.33300451	367.939775
2010	2.539017164	91.56597952	526.5209404
11 years	7.160271027	97.31704356	4.1617E+90

Table 2. Relative errors.

Time Scales. Birkhäuser Boston, Boston, MA.

- CHIANG, A. C. (1992) *Elements of Dynamic Optimization.* McGraw-Hill, Inc., Singapore.
- FERREIRA, R. A. C., MALINOWSKA, A. B. and TORRES, D. F. M. (2011) Optimality conditions for the calculus of variations with higher-order delta derivatives. *Appl. Math. Lett.* **24** (1), 87–92.
- GIREJKO, E., MALINOWSKA, A. B. and TORRES, D. F. M. (2010) A unified approach to the calculus of variations on time scales. *Proceedings of 2010 CCDC, Xuzhou, China, May 26-28, 2010.* In: IEEE Catalog Number CFP1051D-CDR, 2010, 595–600. DOI:10.1109/CCDC.2010.5498972
- GIREJKO, E., MALINOWSKA, A. B. and TORRES, D. F. M. (2012) The contingent epiderivative and the calculus of variations on time scales. *Optimization* **61** (3), 251–264.
- HILGER, S. (1997) Differential and difference calculus-unified! *Nonlinear Anal.* **30** (5), 2683–2694.
- INFLATIONDATA.COM, Current inflation, Bureau of Labor Statistics, http://inflationdata.com/Inflation/Inflation_Rate/CurrentInflation.asp, date of retrieval: 31-Oct-2012.
- MALINOWSKA, A. B. and TORRES, D. F. M. (2011) Euler-Lagrange equations for composition functionals in calculus of variations on time scales. *Discrete Contin. Dyn. Syst.* **29** (2), 577–593.
- MALINOWSKA, A. B. and TORRES, D. F. M. (2012) *Introduction to the Fractional Calculus of Variations.* Imp. Coll. Press, London.
- MARTINS, N. and TORRES, D. F. M. (2011) Generalizing the variational theory on time scales to include the delta indefinite integral. *Comput. Math.*

- Appl.* **61** (9), 2424–2435.
- MOZYRSKA, D. and TORRES, D. F. M. (2009) A study of diamond-alpha dynamic equations on regular time scales. *Afr. Diaspora J. Math.* (N.S.) **8** (1), 35–47.
- SAMUELSON, P. A. and NORDHAUS, W. D. (2004) *Economics*, 18th ed. McGraw-Hill/Irwin, Boston, MA.
- TAYLOR, D. (1989) Stopping inflation in the Dornbusch model: Optimal monetary policies with alternate price-adjustment equations. *Journal of Macroeconomics* **11** (2), 199–216.
- UNEMPLOYMENTDATA.COM, Unemployment rate (Seasonally Adjusted U-3 Unemployment Rate), Bureau of Labor Statistics, <http://unemploymentdata.com/unemployment-rate>, date of retrieval: 31-Oct-2012.