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Cone solutions of multi-order fractional difference systems*

by

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Abstract: The fractional difference system of equations with different fractional orders is considered. We obtain the existence and uniqueness results for the initial value problem. Cone solutions are presented. An example is given to illustrate the results.

Keywords: fractional difference operator, fractional difference equation, cone solutions

1. Introduction

Many papers and books devoted to fractional calculus and fractional differential equations appeared recently (see, for example, Kilbas, Srivastava and Trujillo, 2006, or Podlubny, 1999). There are also quite a lot of studies already on fractional difference calculus and equations, see, e.g., Abdeljawad (2011), Atici and Eloe (2007), Chen, Luo and Zhou (2011), Holm (2011), Miller and Ross (1988), Mozyrska and Girejko (2013), Ostalczyk (2012), and references therein. On the other hand, theory of cone solutions is not well developed yet, especially existence of cone solutions of fractional difference systems. We use the case of the Riemann-Liouville type difference operator, used, for example, in Holm (2011) and Atici and Eloe (2007). Comparing to the Grünwald-Letnikov type operator used, for example, in Kaczorek (2007) it is important to define systems with the operator which has its inverse. However, in the investigations of cone solutions we stay with the recurrence methods of solutions similar to presented in Kaczorek (2011).

We introduce a system of multi-order nonlinear fractional difference equations with a set of initial conditions and obtain the existence and uniqueness of a solution. In contrast to the case that we consider in Mozyrska, Girejko and Wyrwas (2012), here we propose the evaluation of f_i, which are the right hand sides of the examining system, at the neutral moment $k \in \mathbb{N}$. Since we consider

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different fractional orders in one system, the gather-function to be a solution to the system is proposed. Similarly as in Mozyrska, Girejko and Wyrwas (2012), where systems with continuous time are considered, we formulate conditions for the system to be viable with respect to a cone with definitions of cones inspired by Kaczorek (2006, 2009).

The paper is organized in the following way. In Section 2 we gather preliminary definitions and facts and we also prove some results needed in the sequel. Section 3 is devoted to systems of fractional difference equations. Section 4 concerns cone solutions to the system considered in Section 3 and it contains an illustrative example.

2. Preliminaries

Let

$$R^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\} \tag{1}$$

and

$$\mathbb{N}_a := \{a, a+1, \ldots\} \tag{2}$$

for a being a real number.

Let us denote, due to Atici and Eloe (2008),

$$t^{(\alpha)} := \frac{\Gamma(t+1)}{\Gamma(t-\alpha+1)}\,.$$

Fractional differences were originally defined in papers by Atici and Eloe (2008) and Miller and Ross (1988). Here we state definitions following Holm (2011).

Definition 1. *Let $\varphi : \mathbb{N}_a \to \mathbb{R}$ and $\alpha > 0$. Then the α-th order fractional sum of φ started at a is defined by*

$$\left(\Delta_a^{-\alpha}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-s-1)^{(\alpha-1)}\varphi(s), \tag{3}$$

where $\Delta_a^{-\alpha}\varphi$ is defined for $t \in \mathbb{N}_{a+\alpha}$. Moreover, we additionally define $\left(\Delta_a^{0}\varphi\right)(t)$ $:= \varphi(t)$ for $t \in \mathbb{N}_a$.

For $\alpha = 1$, formula (3) takes the form $\left(\Delta_a^{-1}\varphi\right)(t) = \sum_{s=a}^{t-1}\varphi(s) = \int_a^t \varphi(s)\Delta s$, which is the delta integral of φ on the set $[a,t]\cap\mathbb{N}_0$.

The following definition, theorems, remarks and lemma come from Holm (2011) and Atici and Eloe (2008). As we consider the case of systems with orders from the interval $(0,1]$, we decide to give definition of the fractional difference for that case.

DEFINITION 2. *Let $\alpha \in (0,1]$. Then the difference operator is defined as*

$$(\Delta_a^{\alpha}\varphi)(t) = \left(\Delta\left(\Delta_a^{-(1-\alpha)}\varphi\right)\right)(t)\,, \quad t \in \mathbb{N}_{a+1-\alpha}, \tag{4}$$

where $(\Delta\varphi)(t) = \varphi(t+1) - \varphi(t)$ and $\varphi : \mathbb{N}_a \to \mathbb{R}$.

THEOREM 1. *Let φ be a real-valued function defined on $\mathbb{N}_a$ and let $\alpha, \beta > 0$. Then the following equalities hold:*

$$(\Delta_{a+\beta}^{-\alpha}(\Delta_a^{-\beta}\varphi))(t) = (\Delta_a^{-(\alpha+\beta)}\varphi)(t) = (\Delta_{a+\alpha}^{-\beta}(\Delta_a^{-\alpha}\varphi))(t).$$

THEOREM 2. *For any $\alpha > 0$ the following holds:*

$$\left(\Delta_a^{-\alpha}\left(\Delta\varphi\right)\right)(t) = \left(\Delta\left(\Delta_a^{-\alpha}\varphi\right)\right)(t) - \frac{(t-a)^{(\alpha-1)}}{\Gamma(\alpha)}\varphi(a), \tag{5}$$

where φ is defined on $\mathbb{N}_a$.

One of the crucial tools in our considerations is the fractional power rule formula. The proof for such a rule can be found in Atici and Eloe (2008) and Holm (2011).

LEMMA 1. *Let $a \in \mathbb{R}$ and $p > 0$. Then*

$$\Delta(t-a)^{(p)} = p(t-a)^{(p-1)} \tag{6}$$

for any t for which both sides are well-defined. Furthermore, for $\alpha > 0$,

$$\Delta_{a+p}^{-\alpha}(t-a)^{(p)} = p^{(-\alpha)}(t-a)^{(p+\alpha)}\,, \quad t \in \mathbb{N}_{a+p+\alpha} \tag{7}$$

and

$$\Delta_{a+p}^{\alpha}(t-a)^{(p)} = p^{(\alpha)}(t-a)^{(p-\alpha)}\,, \quad t \in \mathbb{N}_{a+p+1-\alpha}\,.$$

Equation (7) can be also transformed as follows: let $\varphi(s) = (s-a+p)^{(p)}$, then, for $s \in \mathbb{N}_a$ $\left(\Delta_a^{-\alpha}\varphi\right)(s+\alpha) = \frac{\Gamma(p+1)}{\Gamma(p+\alpha+1)}(k+p+\alpha)^{(p+\alpha)},\ s = a+k.$

The general version of the next theorem was established in the presented form in Holm (2011), and with a slight difference in notations, also in Atici and Eloe (2008). Here we use particular case of the result.

THEOREM 3. *Let $\alpha \in (0,1]$. Then, for $t \in \mathbb{N}_a$ and $x : \mathbb{N}_{\alpha-1} \to \mathbb{R}$ the following formula holds*

$$\left(\Delta_0^{-\alpha}\left(\Delta_{\alpha-1}^{\alpha}x\right)\right)(t) = x(t) - \frac{t^{(\alpha-1)}}{\Gamma(\alpha)}x(\alpha-1)\,, \quad t \in \mathbb{N}_{\alpha}\,. \tag{8}$$

3. System of fractional difference equations

Let us consider a set of known functions where $f_i : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots n$ and let $(\alpha) = (\alpha_1, \dots, \alpha_n)$, $(a) = (a_1, \dots, a_n)$ be finite sequences of orders $\alpha_1, \dots, \alpha_n \in (0,1]$ and $a_i = \alpha_i - 1$. Moreover, let $x_i : \mathbb{N}_{a_i} \to \mathbb{R}$, $i = 1, \dots, n$ be unknown functions on a different time set. Then, we introduce the evaluation of f_i at the neutral moment $k \in \mathbb{N}_0$ by the following function $\varphi_i : \mathbb{N} \to \mathbb{R}$ defined by

$$\varphi_i(k) := f_i(k + \alpha_i - 1, x_1(k + \alpha_1 - 1), \dots, x_n(k + \alpha_n - 1)) . \tag{9}$$

Now, we are ready to state the notion of the system of fractional difference equations with multi-order. We introduce here a system of nonlinear fractional difference equations with a set of initial conditions and obtain the existence and uniqueness of a solution. We state a multi-order initial value problem using the following notation:

$$\begin{cases} \left(\Delta_{a_i}^{\alpha_i} x_i\right)(k) = \varphi_i(k) \\ \left(\Delta_{a_i}^{\alpha_i - 1} x_i\right)(0) = x_{0i} , \end{cases} \tag{10}$$

for $i = 1, \dots, n$, where $k \in \mathbb{N}_0$, $\alpha_i \in (0,1]$ and $k + \alpha_i \in \mathbb{N}_{\alpha_i} \cap [0,T]$.

By the solution to the problem (10) we mean *the gather-function* X with values defined by

$$X(k) = \begin{bmatrix} x_1(k + a_1) \\ \vdots \\ x_n(k + a_n) \end{bmatrix} , \ k \in \mathbb{N}_0, k + a_i \in \mathbb{N}_{a_i} , \tag{11}$$

where each component x_i is defined on $\mathbb{N}_{a_i}$ but a gather-function X can be treated as the mapping on $\mathbb{N}_0$. Notice that $X_i(k) = x_i(k + a_i)$.

Let us introduce the following notation

$$\left(\Delta_{(a)}^{(\alpha)} x\right)(k) := \begin{bmatrix} \left(\Delta_{a_1}^{\alpha_1} x_1\right) \\ \vdots \\ \left(\Delta_{a_n}^{\alpha_n} x_n\right) \end{bmatrix} (k)$$

and

$$F(k, X(k)) := F(k, x_1(k + a_1), \dots, x_n(k + a_n)) = \begin{bmatrix} \varphi_1(k) \\ \vdots \\ \varphi_n(k) \end{bmatrix} ,$$

then system (10) can be written shortly as

$$\left(\Delta_{(a)}^{(\alpha)} x\right)(nh) = F(k, X(k)) , \quad X(0) = \begin{bmatrix} x_{01} \\ \vdots \\ x_{0n} \end{bmatrix} . \tag{12}$$

By $k \in \mathbb{N}_0$ we mean the neutral time, while times $k+\alpha_i$ represent measurements of time in different time zones. To construct equivalent summation equations to the multi-order initial value problem (10) we apply the $\Delta_0^{-\alpha_i}$ operators separately to the equations $\left(\Delta_{a_i}^{\alpha_i} x_i\right)(k) = \varphi_i(k)$:

$$\left(\Delta_0^{-\alpha_i}\left(\Delta_{a_i}^{\alpha_i} x_i\right)\right)(k+\alpha_i) = \left(\Delta_0^{-\alpha_i}\varphi_i\right)(k+\alpha_i),\ k \in \mathbb{N}. \tag{13}$$

Now we can apply Theorems 1 and 3 and get

$$\begin{aligned}
\left(\Delta_0^{-\alpha_i}\left(\Delta_{a_i}^{\alpha_i} x_i\right)\right)(k+\alpha_i) &= \left(\Delta_0^{-\alpha_i}\left(\Delta\left(\Delta_{a_i}^{-(1-\alpha_i)} x_i\right)\right)\right)(k+\alpha_i) \\
= \left(\Delta\left(\Delta_0^{-\alpha_i}\left(\Delta_{a_i}^{-(1-\alpha_i)} x_i\right)\right)\right)(k+\alpha_i) &- \frac{(k+\alpha_i)^{(\alpha_i-1)}}{\Gamma(\alpha_i)}\left(\Delta_{a_i}^{-(1-\alpha_i)} x_i\right)(0) \\
= x_i(k+\alpha_i) - \frac{(k+\alpha_i)^{(\alpha_i-1)}}{\Gamma(\alpha_i)} x_i(a_i) &= \left(\Delta_0^{-\alpha_i}\varphi_i\right)(k+\alpha_i),
\end{aligned}$$

where

$$\begin{aligned}
\left(\Delta_0^{-\alpha_i}\varphi_i\right)(k+\alpha_i) &= \frac{1}{\Gamma(\alpha_i)}\sum_{s=0}^{k}\left(k+\alpha_i-\sigma(s)\right)^{(\alpha_i-1)}\varphi_i(s) \\
&= \sum_{s=0}^{k}\frac{\left(k+\alpha_i-\sigma(s)\right)^{(\alpha_i-1)}}{\Gamma(\alpha_i)}\varphi_i(s)
\end{aligned}$$

with $\sigma(s) = s+1$, according to the definition (9). Hence, for $k \in \mathbb{N}_0$ we can state the recursive formula for the solutions for each component separately:

$$x_i(k+\alpha_i) = \frac{(k+\alpha_i)^{(\alpha_i-1)}}{\Gamma(\alpha_i)} x_{0i} + \sum_{s=0}^{k}\frac{\left(k+\alpha_i-\sigma(s)\right)^{(\alpha_i-1)}}{\Gamma(\alpha_i)}\varphi_i(s)$$

or equivalently

$$X_i(k) = \frac{(k-1+\alpha_i)^{(\alpha_i-1)}}{\Gamma(\alpha_i)} x_{0i} + \sum_{s=0}^{k-1}\frac{(k-2+\alpha_i-s)^{(\alpha_i-1)}}{\Gamma(\alpha_i)}\varphi_i(s),\ k \in \mathbb{N}_1.$$

Since also the value of $X(k)$ is recursively defined, it states the unique solution to the multi-order initial value problem (10).

Next, by applying formulas (3) and (4) to the left hand side of system (10) one gets (we omit for the moment all i's):

$$\begin{aligned}(\Delta^{\alpha}_{\alpha-1}x)(k)=\Delta(\Delta^{-(1-\alpha)}_{\alpha-1}x)(k)=\\ (\Delta^{-(1-\alpha)}_{\alpha-1}x)(k+1)-(\Delta^{-(1-\alpha)}_{\alpha-1}x)(k)=\\ \frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{k+1}(k+1-\alpha-s)^{(-\alpha)}x(s+\alpha-1)-\\ \frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{k}(k-\alpha-s)^{(-\alpha)}x(s+\alpha-1)=\\ \frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{k+1}\frac{\Gamma(k+2-\alpha-s)}{\Gamma(k+2-s)}x(s+\alpha-1)-\\ \frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{k}\frac{\Gamma(k+1-\alpha-s)}{\Gamma(k+1-s)}x(s+\alpha-1)=\\ \frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{k}\frac{\Gamma(k+1-\alpha-s)}{\Gamma(k+1-s)}\left(\frac{k+1-\alpha-s}{k+1-s}-1\right)x(s+\alpha-1)+\\ \frac{1}{\Gamma(1-\alpha)}\frac{\Gamma(1-\alpha)}{\Gamma(1)}x(k+\alpha)=\\ \sum_{s=0}^{k}\frac{\Gamma(k+1-\alpha-s)}{\Gamma(k+2-s)\Gamma(-\alpha)}x(s+\alpha-1)+x(k+\alpha)\,.\end{aligned}\tag{14}$$

Since $\binom{k-s-\alpha}{k-s+1}=(-1)^{k-s}\binom{\alpha}{k-s+1}$, thus, for each $i=1,...,n$, $n\in\mathbb{N}$, the gather-function (11), which is a solution, has the following form

$$x_i(k+\alpha_i)=\varphi_i(k)+\alpha_i x_i(k+a_i)+\sum_{s=0}^{k-1}(-1)^{k-s}\binom{\alpha_i}{k-s+1}x_i(s+a_i)\,,k\in\mathbb{N}_0\,,\tag{15}$$

and for $k=0$ the sum in the right hand side of formula (15) is taken as equal zero.
Taking into account the general form of the system given by formula (12), the solution to this system has the following form

$$\begin{aligned}X(k+1)=F(k,X(k))+\operatorname{diag}(\alpha_1,...,\alpha_n)X(k)+\\ \sum_{s=0}^{k-1}\operatorname{diag}\left((-1)^{k-s}\binom{\alpha_1}{k-s+1},...,(-1)^{k-s}\binom{\alpha_n}{k-s+1}\right)X(s)\,.\end{aligned}\tag{16}$$

If we define $\Lambda_{k,s}:=\operatorname{diag}\left((-1)^{k-s}\binom{\alpha_1}{k-s+1},...,(-1)^{k-s}\binom{\alpha_n}{k-s+1}\right)$, what for $s=k$ gives $\Lambda_{k,k}=\operatorname{diag}(\alpha_1,...,\alpha_n)$, then formula (16) can be rewritten shortly as

$$X(k+1) = F(k, X(k)) + \Lambda_{k,k} X(k) + \sum_{s=0}^{k-1} \Lambda_{k,s} X(s) = \\ F(k, X(k)) + \sum_{s=0}^{k} \Lambda_{k,s} X(s)\,. \tag{17}$$

LEMMA 2. *If* $0 < \alpha \le 1$*, then*

$$(-1)^s \binom{\alpha}{s+1} \ge 0 \quad \textit{for } s \in \mathbb{N}_1\,. \tag{18}$$

For $\alpha = 1$ *one gets* $(-1)^s \binom{1}{s+1} = 0$ *.*

Proof. We accomplish the proof by induction. The hypothesis is true for $s = 1$, since

$$-1 \cdot \binom{\alpha}{2} > 0\,.$$

Assuming $(-1)^s \binom{\alpha}{s+1} = (-1)^s \frac{\Gamma(\alpha+1)}{\Gamma(s+2)\Gamma(\alpha-s)} > 0 \quad$ for $s \in \mathbb{N}_1$, we show that the hypothesis is valid for $s+1$. Let us write

$$(-1)^{s+1} \binom{\alpha}{s+2} = (-1)^s \cdot (-1) \frac{\Gamma(\alpha+1)}{(s+2)\Gamma(s+2)\Gamma(\alpha-s-1)} = \\ (-1)^s \cdot (-1) \frac{\Gamma(\alpha+1)\cdot(\alpha-s-1)}{(s+2)\Gamma(s+2)\Gamma(\alpha-s)} = (-1)^s \binom{\alpha}{s+1} \frac{s+1-\alpha}{s+2} > 0\,.$$

what proves the hypothesis. ■

4. Cone solutions

Based on Kaczorek (2008, 2009) we consider the following definitions:

DEFINITION 3. *Let*

$$P = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

be nonsingular and $p_i = (p_{i1}, \ldots, p_{in})$ *be its i-th row* $(i = 1, \ldots, n)$*. The set*

$$K := \{x \in \mathbb{R}^n : \forall i = 1, \ldots, n : p_i x \ge 0\}$$

is called a linear cone generated by the matrix P in $\mathbb{R}^n$*. Moreover,*

$$\mathcal{P} := \{X \in \mathcal{X}_n : \forall k \in \mathbb{N}_0 \; X(k) \in K\}$$

is called a linear cone generated by the matrix P in the space $\mathcal{X}_n$*, where* $\mathcal{X}_n := \{X : \mathbb{N}_0 \to \mathbb{R}^n\}$ *and* $x_i : \mathbb{N}_{a_i} \to \mathbb{R}$.

Remark 1. *Let us observe that $K \subset \mathcal{P}$ in the sense that K consists of constant mappings. Moreover, for $P = I$ being identity matrix we have that: $K = \mathbb{R}^n_+$.*

Definition 4. *Let a matrix $P \in \mathbb{R}^{n\times n}$ be given. The nonlinear fractional difference system (10) is called a $\mathcal{P}$ cone fractional system if $X(\cdot) \in \mathcal{P}$ for any $x_0 \in K$.*

Theorem 4. *Let K and $\mathcal{P}$ be given as in Definition 3 and $X(0) \in K$. If for every $x \in K$,*

$$F(k,x) + \Lambda_{k,k} \cdot x \in K\,, \tag{19}$$

then for every $k \in \mathbb{N}_0$ system (10) *is a $\mathcal{P}$ cone system.*

Proof. By assumption we have $X(0) \in K$. We lead the proof by induction. First, let us check if the formula holds for $k = 1$. Indeed, we obviously get $X(1) = F(0, X(0)) + \Lambda_{1,1}X(0) \in K$. Now we assume that the hypothesis is true for some k, i.e. $X(k) \in K$. Next, by assumption and Lemma 2 we see that

$$p_i \cdot X(k+1) = p_i \cdot (F(k, X(k)) + \Lambda_{k,k}X(k)) + \sum_{s=1}^{k} \Lambda_{k,s} \cdot p_i \cdot X(s) \geq 0\,, \tag{20}$$

what finishes the proof. ■

Remark 2. *Let us notice that for the system* (10) *with autonomous right hand side one can get the statement of Theorem 4 in the form "if and only if". In non-autonomous case this is obviously in general not fulfilled.*

Corollary 1. *Let P be a nonsingular $n \times n$ matrix. Then system* (10) *with the right hand side given by $F(k, X(k)) = AX(k)$ is a $\mathcal{P}$ cone system if and only if $P \cdot [A + \Lambda_{k,k}]\, P^{-1} \in \mathbb{R}^{n\times n}_+$.*

Proof. Let us prove necessity. We assume that the system is a $\mathcal{P}$ cone system, which means that $P\cdot X(k+1) \in \mathbb{R}^{n\times n}_+$, i.e. $P(A+\Lambda)X(k)+\sum_{s=1}^{k} \Lambda_{k,s}\cdot P\cdot X(k-s) \in \mathbb{R}^{n\times n}_+$ with $\Lambda = \Lambda_{k,k}$. Therefore, $P(A + \Lambda)X(0) \in \mathbb{R}^{n\times n}_+$, since $X(0) \in K$, which means that $z = P \cdot X(0) \in \mathbb{R}^n_+$, and by Definition 4 it is arbitrary. Then, since $X(0) = P^{-1}z$, we can write $P(A + \Lambda)P^{-1}z \in \mathbb{R}^{n\times n}_+$, where z is arbitrary and we get the thesis.
Sufficiency follows directly from the theorem. ■

Example 1. *Let $a = \alpha - 1$, $b = \beta - 1$ and let us consider the following two dimensional linear initial value problem:*

$$\begin{cases} \left(\Delta_a^{\alpha} x\right)(k) = -\alpha x(k+a) + \beta y(k+b), & \\ \left(\Delta_b^{\beta} y\right)(k) = x(k+a) + (2-\beta) y(k+b), & k = 0,1,2,\ldots \\ \left(\Delta_a^{-(1-\alpha)} x\right)(0) = x_0, \quad \left(\Delta_b^{-(1-\beta)} y\right)(0) = y_0\,, & \end{cases}. \tag{21}$$

The vector of values of solutions is stated as the solution of the summation equations

$$X(k) = \left[\begin{array}{c} \frac{(k+\alpha-1)^{(\alpha-1)}x_0}{\Gamma(\alpha)} + \sum\limits_{s=0}^{k-1} \frac{(k-1+\alpha-\sigma(s))^{(\alpha-1)}}{\Gamma(\alpha)}(-\alpha x(s+a) + \beta y(s+b)) \\ \frac{(k+\beta-1)^{(\beta-1)}y_0}{\Gamma(\beta)} + \sum\limits_{s=0}^{k-1} \frac{(k-1+\beta-\sigma(s))^{(\beta-1)}}{\Gamma(\beta)}(x(s+a) + (2-\beta)y(s+b)) \end{array} \right], \tag{22}$$

where $k \in \mathbb{N}_1$. Let us take $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$. It is obvious that $P \cdot [A + \Lambda_{k,k}]\, P^{-1} = 0.5 \begin{bmatrix} 1+\beta & 3+\beta \\ 1-\beta & 3-\beta \end{bmatrix} \in \mathbb{R}_+^{n\times n}$. So, system (21) is a $(\mathcal{P})$ cone fractional system.

Let us change the cone by using the matrix $Q = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$. It means that now in the set K we have points (x, y) such that: $y \geq x$ and $y \geq 0$. It is easy to check again that $Q \cdot [A + \Lambda_{k,k}]\, Q^{-1} = \begin{bmatrix} -1 & 3-\beta \\ -1 & 3 \end{bmatrix} \notin \mathbb{R}_+^{n\times n}$. So, system (21) is not a (Q) cone fractional system, see Fig. 1. We see this, for example, with $x_0 = -1, y_0 = 0$, where we have $X(1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \notin K$.

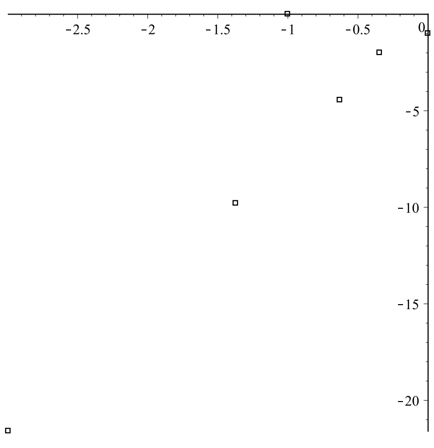

Figure 1. The trajectory of system (21) for $n = 6$ steps and being not a (Q) cone fractional system

5. Conclusions

We considered the fractional difference systems of equations with different fractional orders - multi-order systems. We obtained the recursive formula of solution to such a system, which is a unique solution to this initial value problem. We formulated conditions that guarantee for cone systems the existence of cone solutions. An example is also given to illustrate the results.

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