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Fractional Kalman Filter algorithms for correlated system and measurement noises*

by

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Abstract: The paper presents a generalization of the Fractional Kalman Filter to a case when correlated system and measurement noises appear. The algorithm proposed is derived in detail for a linear generalized discrete fractional order state-space system for both constant and variable order cases. In order to present the efficiency of the proposed algorithm, results of numerical simulations are presented. Results of numerical experiments are compared with the effect of estimation obtained when using the traditional Fractional Kalman Filter algorithm.

Keywords: fractional order calculus, discrete fractional state-space systems, Fractional Kalman Filter

1. Introduction

The idea of fractional calculus (generalization of a traditional integer order integral and differential calculus) idea was mentioned in 1695 by Leibniz and de L'Hospital. In the end of 19th century, Liouville and Riemann introduced the first definition of fractional derivative. However, only in the 1960s this idea started to be interesting for engineers. Especially, when it was observed that the description of some systems is more accurate when the fractional derivative is used. A very good example of such plants are systems based on heat transfer (in general - processes based on diffusion). The experimental results of modeling heat transfer process in the metal beam were presented in Dzieliński and Sierociuk (2010) and Dzieliński, Sierociuk and Sarwas (2010). The results confirmed that fractional calculus is a very efficient tool for modeling the relation between heat flux at the beginning of the beam and temperature at desired point of the beam. Moreover, when the beam is heterogeneous, the diffusion process becomes an anomalous diffusion process. In Sierociuk et al. (2013) results of

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successful modeling of heat transfer process in non–solid (heterogeneous) media using fractional order transfer function based on fractional order partial differential equation were presented. Results were obtained in frequency domain and validated also in time domain. The fractional calculus was found also to be an efficient tool in signal processing. The book of Sheng, Chen and Qiu (2012) presents several areas of application fractional calculus in signal processing e.g.: fractional order filters, fractional order noises, etc. The generalization of the Kalman Filter for discrete fractional order systems with stochastic disturbances was introduced in Sierociuk and Dzieliński (2006). The more advanced algorithms, including the case when the measurements are available by the network that introduces packet losses, were presented in Sierociuk, Tejado and Vinagre (2011). This paper presents also an algorithm for joint estimation and smoothing action. In Kaczorek (2009), a generalization of the discrete model for two dimensional (2D) positive systems was introduced. The fractional Kalman filter (FKF) algorithm has been used for estimation of unknown state variables in the system with ultracapacitor (Dzieliński and Sierociuk, 2008b; Dzieliński et al., 2010) or in a chaotic secure communication scheme (Kiani et al., 2009). Similar algorithm to FKF was also used in Benmalek and Charef (2009) for R-wave detection in the electrocardiogram signal. Moreover, as it was presented in Romanovas et al. (2009) and Kirkko–Jaakkola, Collin and Takala (2012), Fractional Kalman like algorithm also has been used to improve measurement results from MEMS sensors.

2. Fractional calculus

In this paper, the following definition of fractional order difference is used (more definitions and properties can be found in Oldham and Spanier, 1974; Podlubny, 1999; Monje et al., 2010; Samko, Kilbas and Maritchev, 1993.

Definition 1. *The fractional order difference is given by the following equation*

$$\Delta^{\alpha} x_k = \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x_{k-j} \tag{1}$$

where $\alpha \in \mathbb{R}$ is the order of the fractional difference ($\mathbb{R}$ is the set of real numbers) and k is the number of the sample for which the derivative is calculated. The binomial factor $\binom{\alpha}{j}$ can be obtained from the following relation:

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for} \quad j = 0 \\ \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!} & \text{for} \quad j > 0. \end{cases} \tag{2}$$

Basing on this definition the generalized discrete fractional order state-space system is defined as follows:

Definition 2. *(Sierociuk and Dzieliński, 2006) The generalized discrete linear fractional order stochastic system in a state-space representation is given in the* *following form*

$$\Delta^{\Upsilon} x_{k+1} = Ax_k + Bu_k + \omega_k \quad (3)$$
$$y_k = Cx_k + \nu_k, \quad (4)$$

where the state vector can be obtained from the following relation

$$x_{k+1} = \Delta^{\Upsilon} x_{k+1} - \sum_{j=1}^{k+1} (-1)^j \Upsilon_j x_{k+1-j}, \quad (5)$$

$$\Upsilon_k = \mathrm{diag}\left[\ \binom{\alpha_1}{k} \ \ \ldots \ \ \binom{\alpha_N}{k}\ \right]$$

$$\Delta^{\Upsilon} x_{k+1} = \begin{bmatrix} \Delta^{\alpha_1} x_{1,k+1} \\ \vdots \\ \Delta^{\alpha_N} x_{N,k+1} \end{bmatrix},$$

and $x_k \in \mathbb{R}^N$ *is a state vector,* $y_k \in \mathbb{R}^p$ *is an output,* $u_k \in \mathbb{R}^q$ *is an input,* $(\alpha_1 ... \alpha_N)$ *are orders of system equations and* N *is the number of these equations,* ν_k *is a measurement (output) noise and* ω_k *is system noise.*

Reachability, controllability and observability conditions for commensurate discrete fractional order state-space systems can be found in Dzieliński and Sierociuk (2007). Stability conditions for such a systems are given in Dzieliński and Sierociuk (2008a). Numerical toolkit containing Simulink S-functions for simulations of discrete fractional order systems is available on Sierociuk (2005).

2.1. Fractional Kalman Filter (FKF)

The Kalman Filter is an optimal state vector estimator using the knowledge about the system model, input and output signals (Kalman, 1960). A generalization of Kalman Filter algorithm for a linear and non-linear discrete fractional order state-space systems was introduced in Sierociuk and Dzieliń ski (2006). The generalization was obtained for the assumption that correlation of the system noise ω_{k-1} and the measurement noise ν_k is equal to zero. Modification of the FKF algorithm that includes a smoothing action and less restrictive assumptions was presented in Sierociuk, Tejado and Vinagre (2011).

In this paper, the following basic definitions and notations are used:

The state vector prediction $\tilde{x}_k$ is defined as the random variable x_k conditioned on the measurement stream z^*_{k-1} (Brown and Hwang, 1997) and given as follows:

$$\tilde{x}_k = \mathrm{E}[x_k | z^*_{k-1}]. \quad (6)$$

In addition, the state estimate vector $\hat{x}_k$ is defined as the random variable x_k conditioned on the measurement stream z^*_k. The measurement stream

z_k^* contains values of the measurement output $y_0, y_1, \ldots, y_k$ and input signal $u_0, u_1, \ldots, u_k$ and is given by the following relation:

$$\hat{x}_k = \mathrm{E}[x_k | z_k^*]. \tag{7}$$

Furthermore, the prediction of estimation error covariance matrix is

$$\tilde{P}_k = \mathrm{E}\left[(\tilde{x}_k - x_k)(\tilde{x}_k - x_k)^T\right]. \tag{8}$$

Moreover, the covariance matrix of output noise ν_k in (4) is defined as

$$R_k = \mathrm{E}\left[\nu_k \nu_k^T\right],$$

whereas the covariance matrix of system noise ω_k in (3) (see Theorem 1 below) is defined as

$$Q_k = \mathrm{E}\left[\omega_k \omega_k^T\right].$$

Additionally, the estimation error covariance matrix is given as follows:

$$P_k = \mathrm{E}\left[(\hat{x}_k - x_k)(\hat{x}_k - x_k)^T\right]. \tag{9}$$

All of those matrices are assumed to be symmetric.

Theorem 1. *(Sierociuk and Dzieliń ski, 2006) For the discrete fractional order stochastic system in a state-space representation given by Definition 2, the simplified Kalman Filter (called Fractional Kalman Filter) is given by the following set of equations:*

$$\begin{aligned} \Delta^{\Upsilon} \tilde{x}_{k+1} &= A\hat{x}_k + Bu_k \\ \tilde{P}_k &= (A + \Upsilon_1) P_{k-1} (A + \Upsilon_1)^T \\ &+ Q_{k-1} + \sum_{j=2}^{k} \Upsilon_j P_{k-j} \Upsilon_j^T \\ \hat{x}_k &= \tilde{x}_k + K_k (y_k - C\tilde{x}_k) \\ P_k &= (I - K_k C)\tilde{P}_k, \end{aligned}$$

where

$$\tilde{x}_{k+1} = \Delta^{\Upsilon} \tilde{x}_{k+1} - \sum_{j=1}^{k+1} (-1)^j \Upsilon_j \hat{x}_{k+1-j}, \tag{10}$$

$$K_k = \tilde{P}_k C^T (C\tilde{P}_k C^T + R_k)^{-1},$$

with initial conditions

$$x_0 \in \mathbb{R}^N, \quad P_0 = \mathrm{E}[(\tilde{x}_0 - x_0)(\tilde{x}_0 - x_0)^T],$$

and ν_k and ω_k are assumed to be independent and have zero mean value.

3. Fractional Kalman Filter for correlated noises

In this section, the main result presented in this article, a generalization of the FKF algorithm for the case of correlated system and measurement noises, is introduced.

THEOREM 2. *For the discrete fractional order stochastic system in a state-space representation, given by Definition 2 the simplified Kalman Filter for correlated noises (called cFKF) is given by the following set of equations*

$$\begin{aligned}
\Delta^{\Upsilon}\tilde{x}_{k+1} &= A\hat{x}_k + Bu_k && (11)\\
\tilde{P}_k &= (A+\Upsilon_1)\,P_{k-1}\,(A+\Upsilon_1)^T \\
&+ Q_{k-1} + \sum_{j=2}^{k}\Upsilon_j P_{k-j}\Upsilon_j^T && (12)\\
\hat{x}_k &= \tilde{x}_k + K_k(y_k - C\tilde{x}_k) && (13)\\
P_k &= (I-K_kC)\tilde{P}_k(I-K_kC)^T && (14)\\
&+ K_k(CM^T + MC^T + R)K^T, && (15)
\end{aligned}$$

where

$$\tilde{x}_{k+1} = \Delta^{\Upsilon}\tilde{x}_{k+1} - \sum_{j=1}^{k+1}(-1)^j\Upsilon_j\hat{x}_{k+1-j} \qquad (16)$$

$$K_k = (\tilde{P}_kC^T + M_k)(C\tilde{P}_kC^T + CM + MC + R_k)^{-1}, \qquad (17)$$

with initial conditions

$$x_0 \in \mathbb{R}^N, \quad P_0 = \mathrm{E}[(\tilde{x}_0 - x_0)(\tilde{x}_0 - x_0)^T],$$

and ν_k and ω_k are assumed to be zero mean value noises. The correlation between these noises are given as matrix M defined as follows

$$M = \mathrm{E}\left[\omega_{k-1}\nu_k^T\right]. \qquad (18)$$

Proof. The proof will be divided into parts connected with respective cFKF equations.

a) The state vector prediction $\tilde{x}_{k+1}$, given by Equations (11) and (16), is obtained analogously to state prediction in Fractional Kalman Filter Sierociuk and Dzieliń ski, 2006).

$$\begin{aligned}\tilde{x}_{k+1} &= \mathrm{E}[x_{k+1}|z_k^*]\\ &= \mathrm{E}[(Ax_k + Bu_k + \omega_k\\ &- \sum_{j=1}^{k+1}(-1)^j \Upsilon_j x_{k+1-j})|z_k^*]\\ &= A\mathrm{E}[x_k|z_k^*] + Bu_k\\ &- \sum_{j=1}^{k+1}(-1)^j \Upsilon_j \mathrm{E}[x_{k+1-j}|z_k^*].\end{aligned}$$

In the last term of the equation above we may use the same simplifying assumption as in the derivation of FKF algorithm

$$\mathrm{E}[x_{k+1-j}|z_k^*] \approx \mathrm{E}[x_{k+1-j}|z_{k+1-j}^*] \quad \text{for} \quad i = 1, \ldots, (k+1).$$

This assumption causes that the past state vector will not be updated using newer data z_k. Using this assumption, the following relation is obtained (the same as in the traditional FKF algorithm)

$$\begin{aligned}\tilde{x}_{k+1} &\approx A\hat{x}_k + Bu_k\\ &- \sum_{j=1}^{k+1}(-1)^j \Upsilon_j \hat{x}_{k+1-j}.\end{aligned}$$

b) The term $(\tilde{x}_k - x_k)$, used in prediction of the covariance error matrix, given by the Equation (12), is evaluated as follows:

$$\begin{aligned}(\tilde{x}_k - x_k) &= A\hat{x}_{k-1} + Bu_{k-1} - \sum_{j=1}^{k}\left((-1)^j \Upsilon_j \hat{x}_{k-j}\right)\\ &- Ax_{k-1} - Bu_{k-1} - \omega_{k-1} + \sum_{j=1}^{k}\left((-1)^j \Upsilon_j x_{k-j}\right)\\ &= (A - \Upsilon_1)(\hat{x}_{k-1} - x_{k-1}) - \omega_{k-1}\\ &- \sum_{j=2}^{k}\left[(-1)^j \Upsilon_j (\hat{x}_{k-j} - x_{k-j})\right].\end{aligned}$$

In order to obtain this relation, similar assumption to that used in FKF derivation, is made. It is assumed that the expected values of terms $(\hat{x}_l - x_l)(\hat{x}_m - x_m)^T$ are equal to zero when $l \neq m$, which finally gives the following equation

$$\begin{aligned}\tilde{P}_k &= \mathrm{E}\left[(\tilde{x}_k - x_k)(\tilde{x}_k - x_k)^T\right] \\ &= (A - \Upsilon_1)\mathrm{E}[(\hat{x}_{k-1} - x_{k-1}) \\ &\quad (\hat{x}_{k-1} - x_{k-1})^T](A - \Upsilon_1)^T \\ &+ \mathrm{E}[\omega_{k-1}\omega_{k-1}^T] + \sum_{j=2}^{k}\Upsilon_j\mathrm{E}[(\hat{x}_{k-j} - x_{k-j}) \\ &\quad (\hat{x}_{k-j} - x_{k-j})^T]\Upsilon_j^T \\ &= (A + \Upsilon_1)\, P_{k-1}\, (A + \Upsilon_1)^T + Q_{k-1} \\ &+ \sum_{j=2}^{k}\Upsilon_j P_{k-j}\Upsilon_j^T.\end{aligned}$$

c) The update equation is assumed to have the following form:

$$\hat{x}_k = \tilde{x}_k + K_k(y_k - C\tilde{x}_k).$$

The term $(\hat{x}_k - x_k)$ used in estimation error covariance matrix, defined by Equation (9), has the following form

$$\begin{aligned}(\hat{x}_k - x_k) &= \tilde{x}_k - K_k(Cx_k + \nu_k - C\tilde{x}_k) - x_k \\ &= (\tilde{x}_k - x_k) - K_kC(\tilde{x}_k - x_k) - K_k\nu_k,\end{aligned}$$

which gives the following relation for the covariance matrix of estimation error

$$\begin{aligned}P_k &= \mathrm{E}[(\hat{x}_k - x_k)(\hat{x}_k - x_k)^T] \\ &= (I - K_kC)\mathrm{E}[(\tilde{x}_k - x_k)(\tilde{x}_k - x_k)^T](I - K_kC)^T \\ &- (I - K_kC)\mathrm{E}[(\tilde{x}_k - x_k)\nu_k^T]K_k^T \\ &- K_k\mathrm{E}[\nu_k(\tilde{x}_k - x_k)^T](I - K_kC)^T + K_k\mathrm{E}[\nu_k\nu_k^T]K_k.\end{aligned}$$

The term $\mathrm{E}[(\tilde{x}_k - x_k)\nu_k^T]$ can be evaluated as follows

$$\begin{aligned}\mathrm{E}[(\tilde{x}_k - x_k)\nu_k^T] &= \mathrm{E}\left[(A\hat{x}_{k-1} + Bu_{k-1} + \omega_{k-1}\right. \\ &+ \sum_{j=2}^{k}(-1)^j\Upsilon_j(\hat{x}_{k-j} - x_{k-j}) \\ &- Ax_{k-1} - Bu_{k-1} - \omega_{k-1} \\ &- \left.\left.\sum_{j=2}^{k}(-1)^j\Upsilon_j(x_{k-j} - x_{k-j})\right)\nu_k^T\right] \\ &= \mathrm{E}[\omega_{k-1}\nu_k^T] = M\end{aligned}$$

which finally gives the following relation for estimation error covariance matrix

$$\begin{aligned}P_k &= (I - K_kC)\tilde{P}_k(I - K_kC)^T + K_k(CM^T + MC^T + R_k)K_k^T \\ &- M^TK_k^T - K_kM.\end{aligned}$$

d) The optimal gain K_k can be obtained from the relation

$$K_k = \min_K \mathrm{trace}(P_k),$$

for such a case the optimality condition is

$$\frac{\partial \text{trace}(P_k)}{\partial K_k} = 0,$$

which gives the relation

$$-2(I - K_k C)\tilde{P}_k C^T + 2K_k(CM^T + MC^T + R_k) - M_k - M_k = 0.$$

Finally, this gives the relation for the optimal estimation gain

$$K_k = (\tilde{P}_k C^T + M^T)(C\tilde{P}_k C^T + CM^T + MC^T + R_k)^{-1},$$

which finishes the proof. ■

4. Variable order case

In this paper the following definition of generalization of fractional order difference for time-variable order is used:

Definition 3. *The fractional variable order difference is given by the following equation:*

$$\Delta^{\alpha_k} x_k = \sum_{j=0}^{k} (-1)^j \binom{\alpha_k}{j} x_{k-j} \tag{19}$$

where $\alpha_k \in \mathbb{R}$ is a time-variable order of the fractional variable order difference, $\mathbb{R}$ is the set of real numbers and k is a number of the sample for which the derivative is calculated.

It is the first type of the variable order generalizations given in Lorenzo and Hartley (2002). More properties and applications of the fractional variable order differences and derivatives are provided in Ostalczyk (2010), Ostalczyk and Rybicki (2008), Sun, Chen, Wei and Chen (2011), Sheng et al. (2012) and Sierociuk, Podlubny and Petras (2012).

With such a definition, the discrete fractional variable order system will be defined as follows:

Definition 4. *(Sierociuk, 2012) The linear Discrete Fractional Variable Order System in state-space representation is given as follows:*

$$\Delta^{\Upsilon_{k+1}} x_{k+1} = Ax_k + Bu_k \tag{20}$$

$$y_k = Cx_k, \tag{21}$$

where

$$x_{k+1} = \Delta^{\Upsilon_{k+1}} x_{k+1} - \sum_{j=1}^{k+1} (-1)^j \Upsilon_{j,k+1} x_{k-j+1} \tag{22}$$

$$\Upsilon_{j,k} = \text{diag}\left[\begin{array}{ccc} \binom{\alpha_{1,k}}{j} & \dots & \binom{\alpha_{N,k}}{j} \end{array}\right]$$

$$\Delta^{\Upsilon_{k+1}} x_{k+1} = \left[\begin{array}{c} \Delta^{\alpha_{1,k+1}} x_{1,k+1} \\ \vdots \\ \Delta^{\alpha_{N,k+1}} x_{N,k+1} \end{array}\right]$$

and $x_k \in \mathbb{R}^N$ is a state vector, $\alpha_{i,k} \in \mathbb{R}$ are time dependent (variable) orders of system equations (where i is the number of state variable and k is the time of the order) and N is the number of these equations.

The discrete variable order system and its system properties, for a commensurate case, were introduced in Sierociuk (2012).

For the so defined system, the fractional Kalman filter for correlated noises is generalized as follows:

Theorem 3. *For the discrete fractional order stochastic system in a state-space representation, introduced by Definition 2, the simplified Kalman filter for correlated noises (called cVOFKF) is given by the following set of equations*

$$\begin{aligned} \Delta^{\Upsilon_{k+1}} \tilde{x}_{k+1} &= A\hat{x}_k + Bu_k && (23) \\ \tilde{P}_k &= (A + \Upsilon_{1,k}) P_{k-1} (A + \Upsilon_{1,k})^T + Q_{k-1} \\ &+ \sum_{j=2}^{k} \Upsilon_{j,k} P_{k-j} \Upsilon_{j,k}^T && (24) \\ \hat{x}_k &= \tilde{x}_k + K_k(y_k - C\tilde{x}_k) && (25) \\ P_k &= (I - K_k C)\tilde{P}_k (I - K_k C)^T && (26) \\ &+ K_k(CM^T + MC^T + R_k)K^T, && (27) \end{aligned}$$

where

$$\tilde{x}_{k+1} = \Delta^{\Upsilon_{k+1}} \tilde{x}_{k+1} - \sum_{j=1}^{k+1} (-1)^j \Upsilon_{j,k+1} \hat{x}_{k+1-j} \quad (28)$$

$$K_k = (\tilde{P}_k C^T + M_k)(C\tilde{P}_k C^T + CM + MC + R_k)^{-1}, \quad (29)$$

with initial conditions

$$x_0 \in \mathbb{R}^N, \quad P_0 = \text{E}[(\tilde{x}_0 - x_0)(\tilde{x}_0 - x_0)^T],$$

and ν_k and ω_k are assumed to be zero mean value noises. The correlation matrix between these noises is given as matrix M defined as follows

$$M = \text{E}\left[\omega_{k-1}\nu_k^T\right]. \quad (30)$$

Proof. Proof can be conducted analogously to the proof given for Theorem 2. The main difference is in prediction equations (23) and (24).

The state vector prediction, given by Equations (23) and (28), is obtained analogously to the state prediction in fractional Kalman filter (Sierociuk and Dzieliński, 2006).

$$
\begin{aligned}
\tilde{x}_{k+1} &= \mathrm{E}[x_{k+1}|z_k^*] \\
&= \mathrm{E}[(Ax_k + Bu_k + \omega_k - \sum_{j=1}^{k+1}(-1)^j \Upsilon_{j,k+1} x_{k+1-j})|z_k^*] \\
&\approx A\hat{x}_k + Bu_k - \sum_{j=1}^{k+1}(-1)^j \Upsilon_{j,k+1} \hat{x}_{k+1-j},
\end{aligned}
$$

with the same simplifying assumption as in derivation of FKF and cFKF algorithms.

The term $(\tilde{x}_k - x_k)$, used in prediction of covariance error matrix is evaluated as follows:

$$
\begin{aligned}
(\tilde{x}_k - x_k) &= A\hat{x}_{k-1} + Bu_{k-1} - \sum_{j=1}^{k} \left[(-1)^j \Upsilon_{j,k} \hat{x}_{k-j}\right] \\
&= (A - \Upsilon_{1,k})(\hat{x}_{k-1} - x_{k-1}) - \omega_{k-1} - \sum_{j=2}^{k} \left[(-1)^j \Upsilon_{j,k}(\hat{x}_{k-j} - x_{k-j})\right],
\end{aligned}
$$

which under similar assumption to that taken in FKF derivation gives the following equation

$$
\begin{aligned}
\tilde{P}_k &= \mathrm{E}\left[(\tilde{x}_k - x_k)(\tilde{x}_k - x_k)^T\right] \\
&= (A - \Upsilon_1)\mathrm{E}[(\hat{x}_{k-1} - x_{k-1})(\hat{x}_{k-1} - x_{k-1})^T](A - \Upsilon_{1,k})^T \\
&+ \mathrm{E}[\omega_{k-1}\omega_{k-1}^T] + \sum_{j=2}^{k} \Upsilon_{j,k} \mathrm{E}[(\hat{x}_{k-j} - x_{k-j})(\hat{x}_{k-j} - x_{k-j})^T] \Upsilon_{j,k}^T \\
&= (A + \Upsilon_{1,k})\, P_{k-1}\, (A + \Upsilon_{1,k})^T + Q_{k-1} + \sum_{j=2}^{k} \Upsilon_{j,k} P_{k-j} \Upsilon_{j,k}^T.
\end{aligned}
$$

■

5. Numerical examples

In this section, numerical examples presenting the efficiency of the proposed algorithm are given.

Example 1. *Comparison of the proposed algorithm and traditional FKF Let us assume discrete fractional order system with the following matrices:*

$$A = \begin{bmatrix} 0 & -0.1 \\ 1 & 0.15 \end{bmatrix}, B = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 3 \end{bmatrix}, \mathcal{N} = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}^T.$$

Measured parameters of noises are

$$E[\nu_k \nu_k^T] = 0.0366, \quad E[\omega_{1,k}\omega_{1,k}^T] = 0.0234,$$

$$E[\omega_{2,k}\omega_{2,k}^T] = 0.0132, \quad E[\nu_k \omega_k^T] = \begin{bmatrix} 0.0293 & 0.022 \end{bmatrix},$$

where $\omega_{1,k}$ and $\omega_{2,k}$ are the system noises of x_1 and x_2 state variables, respectively. Fractional Kalman filter parameters used in the example are:

$$P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 0.0234 & 0 \\ 0 & 0.0132 \end{bmatrix},$$

$$R = \begin{bmatrix} 0.366 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.0293 & 0.022 \end{bmatrix}.$$

Fig.1 presents input and output signals of the estimated system. Fig.2 presents a

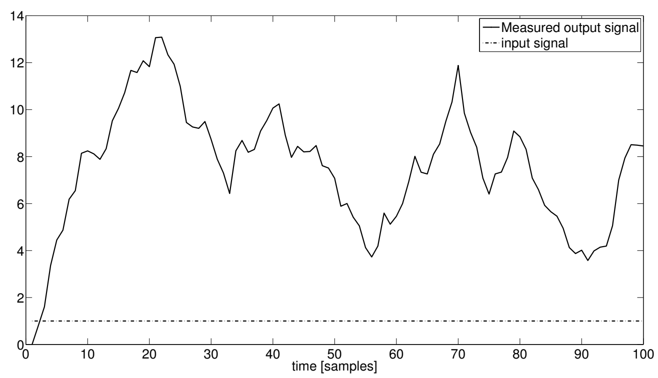

Figure 1. Input and output signals from the plant

comparison of estimates $\hat{x}_1$ obtained by the traditional FKF and proposed cFKF algorithms for the original x_1 state variable. As it can be seen, the results obtained by the proposed algorithm are much closer to the original one than those obtained by FKF. Fig.3 presents a comparison of estimates $\hat{x}_2$ obtained by the traditional FKF and proposed cFKF algorithms for the original x_2 state variable. Fig.4 shows a comparison of square error $(\hat{x}_{1,k} - x_{1,k})^2$ estimation obtained by FKF and cFKF algorithms. Fig.5 presents a comparison of square

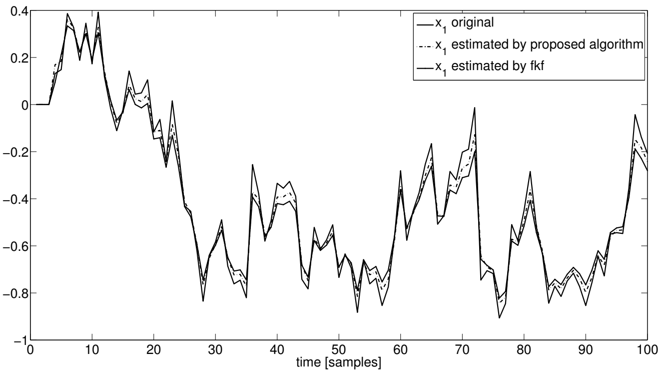

Figure 2. Comparison of estimates of x_1

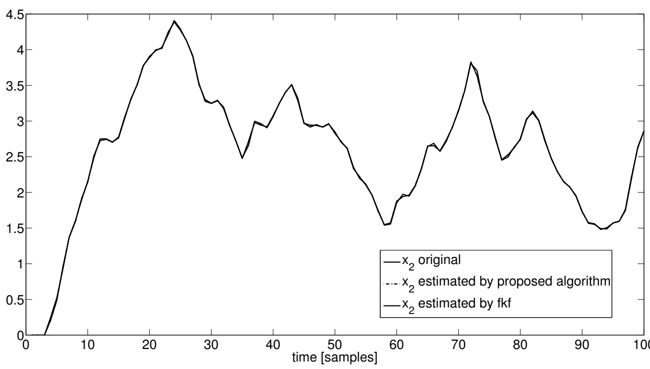

Figure 3. Comparison of estimates of x_2

error of estimation $(\hat{x}_{2,k} - x_{2,k})^2$ *obtained by FKF and cFKF algorithms.*

The sum of squares estimation error for FKF is equal 0.5937*, and for the proposed cFKF algorithm it is equal* 0.2238*. The efficiency improvement for the cFKF algorithm for this example was* 62%*. The presented example confirms higher efficiency of proposed cFKF algorithm in the case when noises are correlated.*

Example 2. *Comparison results for different noises*

In this example more results of comparison between FKF and proposed cFKF algorithms for different values of noise parameters are presented. The system matrices are the same as in Example 1. Parameters of the FKF and cFKF filters are based on measured parameters of noises. The results are summarized in Table 1. As it can be seen in this table, in all presented results, the efficiency of the proposed algorithm was higher than the efficiency of the traditional FKF

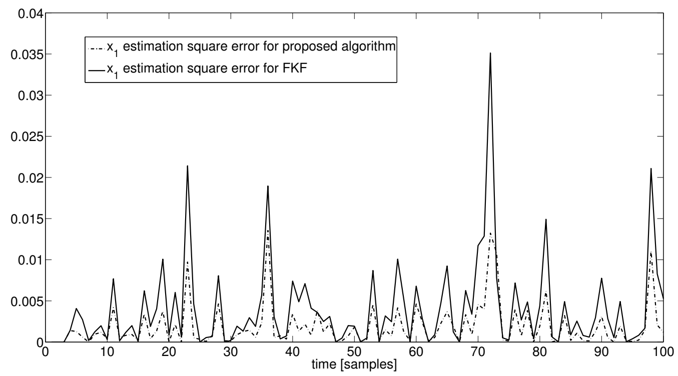

Figure 4. Comparison of estimation error for x_1

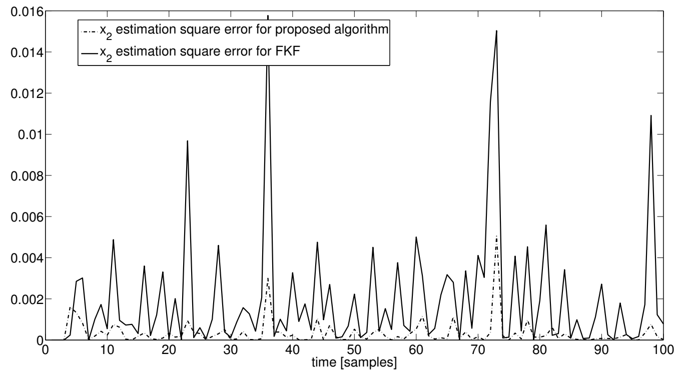

Figure 5. Comparison of estimation error for x_2

algorithm. As it could be expected, the higher correlation between system and output noises, the higher efficiency of the cFKF algorithms. The square error, presented in Table 1, is defined as follows:

$$e = \sum_{j=0}^{k} (\hat{x}_k - x_k)(\hat{x}_k - x_k)^T.$$

5.1. Variable order case example

In this section, analogous results for fractional variable order system will be presented.

Table 1. Results of numerical simulations

$E[\nu_k \omega_k^T]$	$E[\nu_k \nu_k^T]$	$\text{diag}E[\omega_k \omega_k^T]$	e_{FKF}	e_{cFKF}	improvement
[0.0117, 0.0233]	0.0388	[0.0035, 0.0140]	0.1875	0.0765	59%
[0.1511, 0.0756]	0.2519	[0.0907, 0.0227]	0.7134	0.4746	33%
[0.0885, 0.0708]	0.0885	[0.0885, 0.0567]	3.3349	1.1073	67%
[0.0762, 0.0632]	0.0777	[0.0776, 0.0540]	2.9928	1.2463	58%
[0.0049, 0.0012]	0.0100	[0.0410, 0.0359]	4.2979	3.8756	10%
[0.0023, 0.0002]	0.0090	[0.0109, 0.0101]	1.3366	1.1697	12%

Example 3. *Comparison of proposed algorithm and traditional FKF for variable order system Let us assume the discrete fractional variable order system with the following matrices:*

$$\begin{aligned} A &= \begin{bmatrix} 0 & -0.1 \\ 1 & 0.15 \end{bmatrix}, B = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 3 \end{bmatrix}, \end{aligned}$$

and the system orders

$$\mathcal{N}_k = \begin{bmatrix} \alpha_{1,k} \\ \alpha_{2,k} \end{bmatrix} = \begin{bmatrix} 0.5 + 0.1\sin(\frac{2\pi}{100}k) \\ 0.5 + 0.2\sin(\frac{2\pi}{100}k) \end{bmatrix}^T. \tag{31}$$

Measured noise parameters are the same as for the constant order case and have the following values:

$$E[\nu_k \nu_k^T] = 0.0366, \quad E[\omega_{1,k}\omega_{1,k}^T] = 0.0234, \quad E[\omega_{2,k}\omega_{2,k}^T] = 0.0132,$$

$$E[\nu_k \omega_k^T] = \begin{bmatrix} 0.0293 & 0.022 \end{bmatrix},$$

where $\omega_{1,k}$ and $\omega_{2,k}$ are the system noises of x_1 and x_2 state variables, respectively.

Fractional Kalman filter parameters used in the example are:

$$P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 0.0234 & 0 \\ 0 & 0.0132 \end{bmatrix},$$

$$R = \begin{bmatrix} 0.366 \end{bmatrix},$$

$$M = \left[\begin{array}{cc} 0.0293 & 0.022 \end{array} \right].$$

Fig.6 presents input and output signals of the estimated system, and Fig.7 presents variable orders of the system.

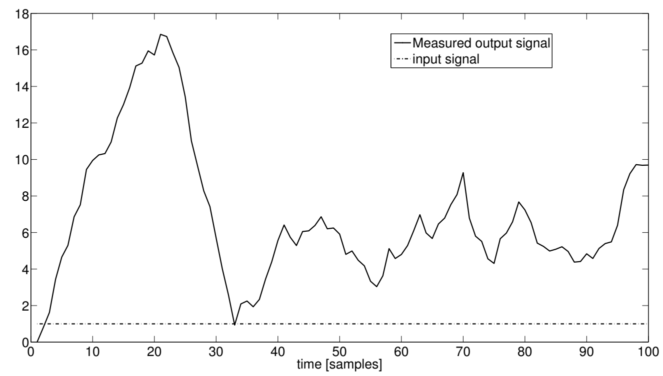

Figure 6. Input and output signals from the plant

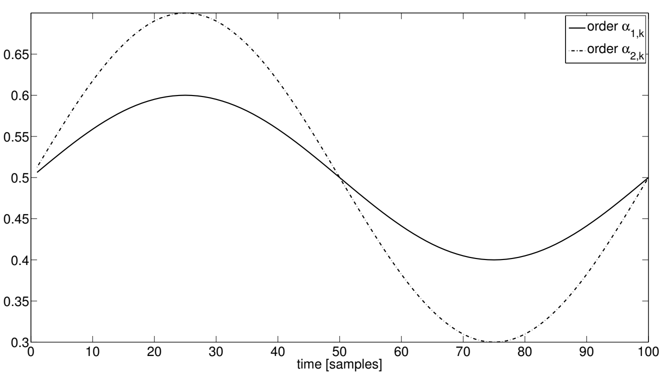

Figure 7. Variable orders of the system

Fig. 8 presents a comparison of estimates $\hat{x}_1$ *obtained by the traditional FKF and proposed cFKF algorithms for the original* x_1 *state variable. As it can be seen, the results obtained by the proposed algorithm are much closer to the original ones than obtained by FKF.*

On the other hand, Fig. 9 presents a comparison of estimates $\hat{x}_2$ *obtained by the traditional FKF and proposed cFKF algorithms for the original* x_2 *state variable.*

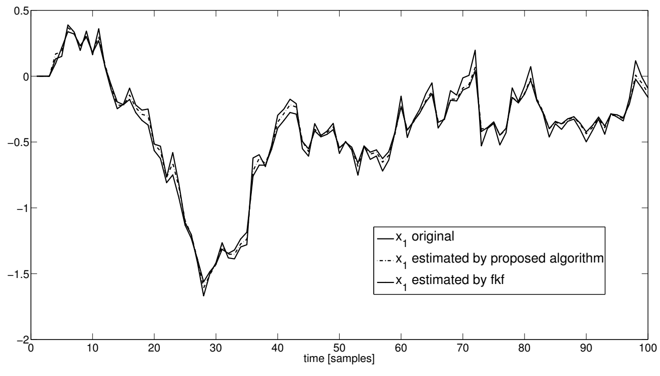

Figure 8. Comparison of estimates of x_1

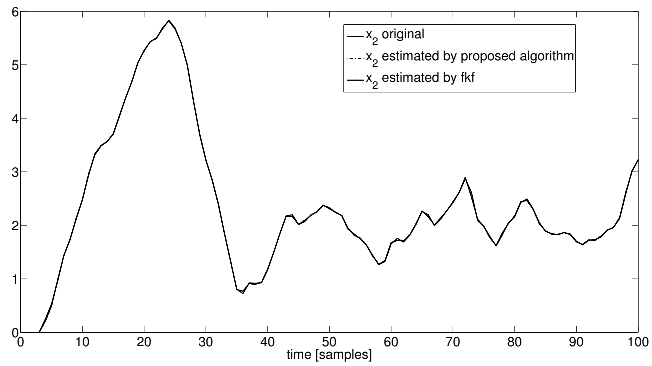

Figure 9. Comparison of estimates of x_2

Fig. 10 shows a comparison of square error estimation $(\hat{x}_{1,k} - x_{1,k})^2$ *obtained for FKF and cFKF algorithms.*

Fig. 11 presents a comparison of square error of estimation $(\hat{x}_{2,k} - x_{2,k})^2$ *obtained by FKF and cFKF algorithms.*

The sum of squares estimation error for FKF is equal 0.6057*, and for the proposed cFKF algorithm it is equal* 0.2339*. The efficiency improvement for the cFKF algorithm for this example was* 61%*. The example presented confirms higher efficiency of the proposed cFKF algorithm in the case when noises are correlated.*

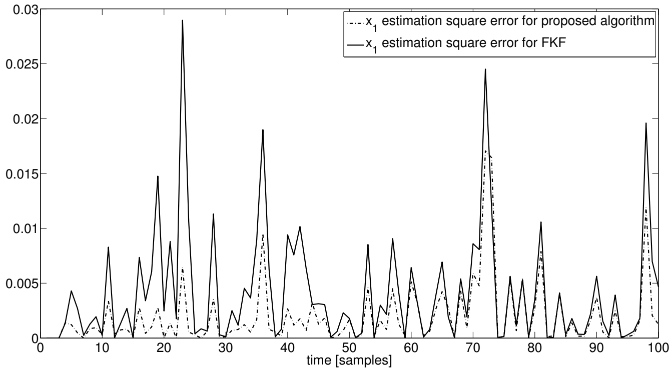

Figure 10. Comparison of estimation error for x_1

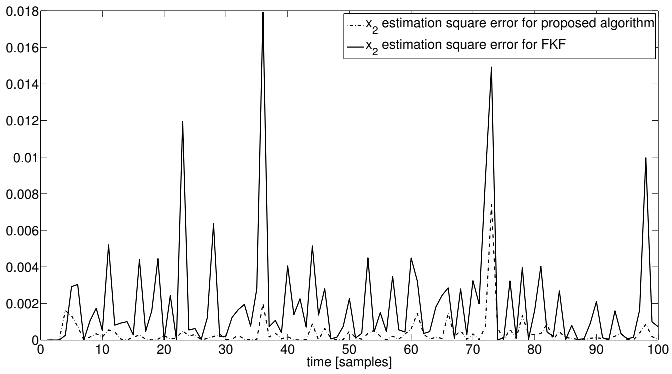

Figure 11. Comparison of estimation error for x_2

6. Conclusions

In this paper, an estimation algorithm for discrete fractional order state-space system with correlated system and measurement noises was presented. The algorithm proposed is an extension of the Fractional Kalman Filter (FKF) algorithm. Detailed derivation of the proposed algorithm was given. In order to present the efficiency of the proposed algorithm, the results of numerical simulation were shown. The results confirm that the proposed algorithm is more efficient compared to the results obtained with traditional FKF algorithm for the same situation. The higher the correlation between system and output noises, the higher the efficiency of the cFKF algorithms. The paper presents also the generalization of this algorithm for a variable order case. In such a case, numerical simulations also confirm higher efficiency of the proposed algorithm compared to the variable order FKF algorithm (also introduced in this article).

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References

BENMALEK M. & CHAREF A. (2009) Digital fractional order operators for r-wave detection in electrocardiogram signal. *IET Signal Processing* **3**(5), 381–391.

BROWN R.& HWANG P. (1997) *Introduction to Random Signals and Applied Kalman Filtering. With Matlab Exercises and Solutions.* John Wiley & Sons, Inc., New York.

MONJE C.A., CHEN Y. Q., VINAGRE B.M., XUE D., & FELIU-BATLLE V. (2010) *Fractional-order Systems and Controls. Fundamentals and Applications.* Springer.

DZIELIŃSKI A. & SIEROCIUK D. (2008*a*) Stability of discrete fractional order state-space systems. *JVC/Journal of Vibration and Control* **14**(9-10), 1543–1556.

DZIELIŃSKI A. & SIEROCIUK D. (2008*b*) Ultracapacitor modelling and control using discrete fractional order state-space model. *Acta Montanistica Slovaca* **13**(1), 136–145.

DZIELIŃSKI A. & SIEROCIUK D. (2010) Fractional order model of beam heating process and its experimental verification. In: D. Baleanu, Z. B. Güvenç & J. A. T. Machado, eds, *New Trends in Nanotechnology and Fractional Calculus Applications.* Springer Netherlands, 287–294.

DZIELIŃSKI A., SIEROCIUK D. & SARWAS G. (2010) Some applications of fractional order calculus. *Bulletin of the Polish Academy of Sciences - Technical Sciences* **58**(4), 583–592.

DZIELIŃSKI, A. & SIEROCIUK D. (2007) Reachability, controllability and observability of the fractional order discrete state-space system. In: *Proceedings of IEEE/IFAC International Conference on Methods and Models in Automation and Robotics, MMAR'07.* Szczecin, Poland, 129–134.

KACZOREK T. (2009) *Positive 2D fractional linear systems. COMPEL-The International Journal for Computation and Mathematics in Electrical and Electronic Engineering* **28**(2), 341–352.

KALMAN R.E. (1960) A new approach to linear filtering and prediction problems. *Transactions of the ASME–Journal of Basic Engineering* **82** (Series D), 35–45.

KIANI-B A., FALLAHI K., PARIZ N. & LEUNG H. (2009) A chaotic secure communication scheme using fractional chaotic systems based on an extended fractional Kalman filter. *Communications in Nonlinear Science and Numerical Simulation* **14**(3), 863–879.

KIRKKO–JAAKKOLA M., COLLIN J. & TAKALA J. (2012) Bias Prediction for MEMS Gyroscopes. *IEEE Sensors Journal* **12**(6).

LORENZO C.F. & HARTLEY T.T. (2002) Variable order and distributed order fractional operators. *Nonlinear Dynamics* **29**(1-4), 57–98.

OLDHAM K. B. & SPANIER J. (1974) *The Fractional Calculus.* Academic Press, New York.

OSTALCZYK P. (2010) Stability analysis of a discrete-time system with a variable-, fractional–order controller. *Bulletin of the Polish Academy of Sciences-Technical Sciences* **58**(4), 613–619.

OSTALCZYK P. & RYBICKI T. (2008) Variable-fractional-order dead-beat control of an electromagnetic servo. *Journal of Vibration and Control* **14**(9-10), 1457–1471.

PODLUBNY I. (1999) *Fractional Differential Equations.* Academic Press, San Diego.

ROMANOVAS M., KLINGBEIL L., TRAECHTLER M. & MANOLI Y. (2009) Application of fractional sensor fusion algorithms for inertial MEMS sensing. *Mathematical Modelling and Analysis* **14**(2), 199–209.

SAMKO S.G., KILBAS A.A.& MARITCHEV O.I. (1993) *Fractional Integrals and Derivatives. Theory and Applications.* Gordon & Breach Sci. Publishers, London.

SHENG H., CHEN Y. Q. & QIU T. (2012) *Fractional Processes and Fractional-Order Signal Processing: Techniques and Applications.* Springer-Verlag, London.

SIEROCIUK D. (2012) System properties of fractional variable order discrete state-space system. In: *13th International Carpathian Control Conference (ICCC), 2012*, 643–648.

SIEROCIUK D., DZIELIŃSKI A., SARWAS G., PETRAS I., PODLUBNY I. & SKOVRANEK T. (2013) Modelling heat transfer in heterogeneous media using fractional calculus. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* **371**, 1990, 20120146.

SIEROCIUK D., PODLUBNY I. & PETRAS I. (2012) Experimental evidence of variable-order behavior of ladders and nested ladders. *IEEE Trans. Control Systems Technology* **21**, 2, 459–466.

SIEROCIUK D., TEJADO I. & VINAGRE B.M. (2011) Improved fractional Kalman filter and its application to estimation over lossy networks. *Signal Processing* **91**(3), 542–552.

SIEROCIUK D. (2005) *Fractional Order Discrete State-Space System Simulink Toolkit User Guide.* http://www.ee.pw.edu.pl/~dsieroci/fsst/fsst.htm.

SIEROCIUK D. & DZIELIŃSKI A. (2006) Fractional Kalman filter algorithm for states, parameters and order of fractional system estimation. *International Journal of Applied Mathematics and Computer Science* **16**(1), 101–112.

SUN H. G., CHEN W., WEI H. & CHEN Y. Q. (2011) A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems. *European Physical Journal-Special Topics* **193**(1), 185–192.