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Algebraic observability of linear differential–algebraic systems with delay*

by

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Abstract: The paper deals with the problem of algebraic observability for linear differential-algebraic systems with delay. For such systems, we present the observability matrix. By algebraic properties of the matrix we define some concepts of observability. We give necessary and sufficient conditions of these algebraic observabilities. We prove relations between these types of observabilities along with spectral observability. Practical verifiability of the conditions is demonstrated on several examples.

Keywords: observability, differential-algebraic systems, time-delay

1. Introduction

Linear differential-algebraic systems with delays (DAD) have been studied since the beginning of this century (Marchenko and Poddubnaya, 2002a,b). DAD systems involve hybrid structure i.e. some equations are differential, the other - difference, some variables are continuous (or piecewise smooth) the other - piecewise continuous. There are examples of DAD systems that can be regarded as some kinds of neutral type time-delay and discrete-continuous hybrid systems (for details see Marchenko, Poddubnaya and Zaczkiewicz 2006).

The main purpose of this paper is to propose algebraic observabilities for the analysis of linear differential-algebraic systems with delays. The concept of algebraic observability of systems with delay has been studies by Kamen (1978), Morse (1976), Sontag (1976), Olbrot and Zak (1980), Lee and Olbrot (1981), and others. These works show that some functional observabilities imply or are equivalent to some algebraic types of observabilities for systems with retarded type delay (equivalence of spectral and essential observabilities). The implications and equalities, presented here, are derived from the Hautus's result,

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Hautus (1969), for systems without delay. The new results concern: for DAD systems the Hautus's result is not true in general (Remark 1); this implies different result for $\mathbb{R}^{n_1+n_2}[d]$, $\mathbb{R}^{n_1+n_2}(d)$ observabilities and hyperobservability (Theorems 1, 2 and 6); relations between spectral observability and different types of observabilities (Theorem 3 and 4); the main result (Theorem 7).

The paper can be summarized as follows. In Section 2, we introduce DAD systems. Section 3 contains the result concerning Hautus lemma and definitions of $\mathbb{R}^{n_1+n_2}[d]$, and $\mathbb{R}^{n_1+n_2}(d)$ observabilities with necessary and sufficient conditions. Section 4 deals with spectral observability. Section 5 contains the main result, where we first present two cases for the extension of solutions to $(-\infty,\infty)$, and next we consider essential observability and hyperobservability for solutions on $(-\infty,\infty)$, then we present characterization of algebraic observabilities. Some illustrative examples are given in Sections 3 and 5, and Section 6 contains two other examples. Finally, we conclude in Section 7.

2. Preliminaries

In this paper, we concentrate on the simplest DAD system in the following form:

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t),\ t > 0, \tag{1a}$$

$$x_2(t) = A_{21}x_1(t) + A_{22}x_2(t-h),\ t \geq 0, \tag{1b}$$

with output

$$y(t) = B_1x_1(t) + B_2x_2(t), \tag{1c}$$

where $x_1(t) \in \mathbb{R}^{n_1},\ x_2(t) \in \mathbb{R}^{n_2},\ y(t) \in \mathbb{R}^r, t \geq 0;andA_{11} \in \mathbb{R}^{n_1\times n_1},\ A_{12} \in \mathbb{R}^{n_1\times n_2},\ A_{21} \in \mathbb{R}^{n_2\times n_1},\ A_{22} \in \mathbb{R}^{n_2\times n_2},\ B_1 \in \mathbb{R}^{r\times n_1},\ B_2 \in \mathbb{R}^{r\times n_2}$ are constant (real) matrices, while $0 < h$ is a constant delay. We consider an absolutely continuous n_1-vector function $x_1(\cdot)$ and a piecewise continuous n_2-vector function $x_2(\cdot)$ to be a solution of System (1) if they satisfy equation (1a) for almost all $t > 0$ and (1b) for all $t \geq 0$.

System (1) should be complemented with initial conditions:

$$x_1(+0) = x_0,\ x_2(\tau) = \psi(\tau),\ \tau \in [-h, 0), \tag{2}$$

where $x_0 \in \mathbb{R}^{n_1};\ \psi \in PC([-h,0),\mathbb{R}^{n_2})$ and $PC([-h,0),\mathbb{R}^{n_2})$ denotes the set of piecewise continuous n_2-vector-functions in $[-h,0]$. Observe that $x_2(t)$ at $t = 0$ is determined from Equation (1b).

3. $\mathbb{R}^{n_1+n_2}[d]$ and $\mathbb{R}^{n_1+n_2}(d)$ observability

Introduce notation:

$$A = A(d) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & dA_{22} \end{pmatrix},\ B = \begin{pmatrix} B_1 & B_2 \end{pmatrix}, \tag{3}$$

$$I(\lambda) = \begin{pmatrix} \lambda I_{n_1} & 0 \\ 0 & I_{n_2} \end{pmatrix}.$$

Definition 1. *We call λ an eigenvalue of the matrix A if the following condition is satisfied:*

$$\begin{aligned}&\det(A(d) - I(\lambda)) = \\ &= \det\begin{pmatrix} A_{11} - \lambda I_{n_1} & A_{12} \\ A_{21} & dA_{22} - I_{n_2}\end{pmatrix} = 0, \forall d \in \mathbb{C}.\end{aligned}$$

Similarly, we call λ a spectral eigenvalue of the matrix A, following Marchenko and Zaczkiewicz (2005).

We shall discus the following algebraically defined notions of observability of the pair $(A(d), B)$ of System (1) (see for instance Olbrot and Zak, 1980, Lee and Olbrot, 1981):

Definition 2. *System (1) is $\mathbb{R}^{n_1+n_2}[d]$-observable if and only if*

$$\mathrm{Span}_{\mathbb{R}[d]}\left[\frac{B}{A(d)}\right] = \mathbb{R}^{n_1+n_2}[d], \tag{4}$$

where

$$\left[\frac{B}{A(d)}\right] = \begin{pmatrix} B \\ BA(d) \\ \vdots \\ BA^{n_1+n_2-1}(d)\end{pmatrix} \tag{5}$$

is the formal observability matrix and $\mathbb{R}^{n_1+n_2}[d]$ is the module over $\mathbb{R}[d]$ of all $n_1 + n_2$ by 1 column vectors with elements in $\mathbb{R}[d]$.

Definition 3. *System (1) is $\mathbb{R}^{n_1+n_2}(d)$-observable if and only if*

$$\mathrm{Span}_{\mathbb{R}(d)}\left[\frac{B}{A(d)}\right] = \mathbb{R}^{n_1+n_2}(d), \tag{6}$$

where $\mathbb{R}(d)$ is the field of rational functions with real coefficients.

Now we shall prove the following characterizations of $\mathbb{R}^{n_1+n_2}[d]$-observability.

First, let us consider the following generalization of Hautus (1969) result for the DAD systems:

$$i)\ \mathrm{rank}\left[\frac{B}{A(d)}\right] = n_1 + n_2,\ \forall d \in \mathbb{C}, \tag{7a}$$

$$\Updownarrow$$

$$ii)\ \mathrm{rank}\begin{pmatrix} \lambda I_{n_1} - A_{11} & -A_{12} \\ -A_{21} & I_{n_2} - dA_{22} \\ B_1 & B_2\end{pmatrix} = n_1 + n_2,$$

$$\forall \lambda \in \mathbb{C}\ \forall\ d \in \mathbb{C}, \tag{7b}$$

We show that the Hautus result for DAD systems does not hold:

Example 1. *Let us consider the following system:*

$$
\begin{aligned}
\dot{x}_1(t) =& (1)x_1(t) + (2,3)x_2(t),\\
x_2(t) =& \begin{pmatrix} 5 \\ 11 \end{pmatrix} x_1(t) + \begin{pmatrix} 7 & \frac{21}{2} \\ 0 & 0 \end{pmatrix} x_2(t-h),\\
y(t) =& (-3)x_1(t) + (0,0)x_2(t).
\end{aligned}
$$

Then we compute the observability matrix (5)

$$
\left[\frac{B}{A(d)}\right] = \begin{pmatrix} -3 & 0 & 0 \\ -3 & -6 & -9 \\ -132 & -6-42d & -9-63d \end{pmatrix},
$$

the Smith form is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and we check (7b):*

$$
\operatorname{rank} \begin{bmatrix} 1-\lambda & 2 & 3 \\ 5 & 7d-1 & \frac{21}{2}d \\ 11 & 0 & -1 \\ -3 & 0 & 0 \end{bmatrix} = 3.
$$

Thus, the implication (7b) $\Rightarrow$ (7a) is not true.

Now, let us see the next example:

Example 2.

$$
\begin{aligned}
\dot{x}_1(t) =& (1)x_1(t) + (1,1)x_2(t),\\
x_2(t) =& \begin{pmatrix} -1 \\ 0 \end{pmatrix} x_1(t) + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} x_2(t-h),\\
y(t) =& (1)x_1(t) + (1,0)x_2(t).
\end{aligned}
$$

Then we compute the observability matrix (5)

$$
\left[\frac{B}{A(d)}\right] = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1+2d \\ -1 & 0 & 2d \end{pmatrix},
$$

the Smith form is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *and we check (7b):*

$$
\operatorname{rank} \begin{bmatrix} 1-\lambda & 1 & 1 \\ -1 & -1 & 2d \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} = 2.
$$

Thus the equivalence (7a) $\Leftrightarrow$ (7b) is not true.

Remark 1. *The equivalence between (7a) and (7b) holds if and only if $\lambda = 1$ is an eigenvalue of the matrix A.*

Proof. We leave the proof to the reader. See Hautus (1969) for more details. ■

We present the characterization of $\mathbb{R}^{n_1+n_2}[d]$-observability of System (1).

Theorem 1. *$\mathbb{R}^{n_1+n_2}[d]$-observability of System (1) is equivalent to each of the following conditions*

$$i)\ \text{The Smith form of } \left[\frac{B}{A(d)}\right] \text{ is } \begin{pmatrix} I_{n_1+n_2} \\ 0 \end{pmatrix}, \tag{8a}$$

ii) there exists $Q \in \mathbb{R}^{(n_1+n_2)\times r(n_1+n_2)}[d]$ such that

$$Q\left[\frac{B}{A}\right] = I_{n_1+n_2}, \tag{8b}$$

$$iii)\ \operatorname{rank}\left[\frac{B}{A(d)}\right] = n_1 + n_2, \quad \text{for all } d \in \mathbb{C}. \tag{8c}$$

Proof. By dualization of Theorem 1 by Olbrot and Zak (1980) for DAD systems. ■

Similar to Theorem 1 is the following.

Theorem 2. *$\mathbb{R}^{n_1+n_2}(d)$-observability of System (1) is equivalent to each of the following conditions*

$$i)\ \text{The Smith form of } \left[\frac{B}{A(d)}\right]$$

$$\text{has nonzero elements on the diagonal,} \tag{9a}$$

ii) there exists $Q \in \mathbb{R}^{(n_1+n_2)\times r(n_1+n_2)}(d)$ such that

$$Q\left[\frac{B}{A}\right] = I_{n_1+n_2}, \tag{9b}$$

$$iii)\ \operatorname{rank}\left[\frac{B}{A(d)}\right] = n_1 + n_2,\ \text{for some } d \in \mathbb{C}, \tag{9c}$$

$$iii')\ \operatorname{rank}\left[\frac{B}{A(d)}\right] = n_1 + n_2,$$

$$\text{for all but finitely many } d \in \mathbb{C}. \tag{9d}$$

Proof. By dualization of Theorem 2 by Olbrot and Zak (1980) for DAD systems. ■

4. Spectral observability

In this section we introduce the notion of spectral observability following Marchenko and Zaczkiewicz (2005).

Definition 4. *System (1) is infinite-time observable if for any initial function for which $y(t) = 0$ for $t \geq 0$ there is $T \in [0, \infty)$ such that $x_1(t) = 0$ and $x_2(t) = 0$ for $t \in [T, \infty)$.*

Definition 5. *System (1) is finite-time observable at T if for any initial function such that $y(t) = 0$ for $t \geq 0$ we have $x_1(t) = 0$ and $x_2(t) = 0$ for $t \in [T, \infty)$.*

Definition 6. *System (1) is spectrally observable if all its spectral eigenvalues are observable. A spectral eigenvalue λ is observable if any corresponding eigensolution of the form $x_1(t) = exp(\lambda t)x_1(0)$, $x_2(t) = exp(\lambda t)x_2(0)$, $x_1(0) \neq 0$, $x_2(0) \neq 0$ obtains $y(t) \neq 0$ for $t \in [0, \infty)$.*

Proposition 1. *System (1) is spectrally observable if and only if*

$$\operatorname{rank}\begin{pmatrix} pI_{n_1} - A_{11} & -A_{12} \\ -A_{21} & I_{n_2} - A_{22}e^{-ph} \\ B_1 & B_2 \end{pmatrix} = n_1 + n_2, \tag{10}$$

for all complex p.

Proposition 2. *System (1) is spectrally observable if and only if System (1) is infinite-time observable.*

For more details and the proofs of Propositions 1 and 2 see Marchenko and Zaczkiewicz (2005).

Theorem 3. *System (1) is finite-time observable at n_2h if and only if System (1) is spectrally observable.*

Proof.
First we prove that spectral observability implies finite-time observability at T, where $T \leq n_2h$. Assume that spectral observability holds, then for any initial conditions such that $y(t) = 0$ for $t \geq 0$, the solution (x_1, x_2) of System (1) is an entire function of exponential type. This can be found in Marchenko and Zaczkiewicz (2005). We next claim that $x_1(t) = 0$, $x_2(t) = 0$ for $t \geq T = n_2h$. This was proved in Zaczkiewicz and Marchenko (2006). Combining this with Proposition 2 completes the proof. ■

5. Essential observability and hyperobservability

In this section we present new definitions of observability that are based on solutions defined on $(-\infty, \infty)$. We first investigate such solutions in the following lemma.

Lemma 1. *Solutions $(x_1(\cdot), x_2(\cdot))$ of System (1) can be uniquely extended to solutions of (1) on $(-\infty, \infty)$, if and only if*

$$\det A_{22} \neq 0. \tag{11}$$

Proof. We can solve (1a) for $t \in [-h, 0]$ and for any $x_2(0+t) = \psi(t)$ we have

$$A_{22}x_2(t-h) = x_2(t) - A_{21}x_1(t), \text{ for } t \in [-h, 0].$$

By (11)

$$x_2(t-h) = A_{22}^{-1}x_2(t) - A_{22}^{-1}A_{21}x_1(t), \text{ for } t \in [-h, 0].$$

Thus, $x_2(\tau)$ for $\tau \in [-2h, -h]$ is unique. We can proceed analogously to proof that $(x_1(\tau), x_2(\tau))$ for $\forall \tau \leq -h$ is unique.

Suppose that $\operatorname{rank} A_{22} < n_2$, then there exists a vector $v \in \mathbb{R}^{n_2}$ and $\forall w \in \mathbb{R}^{n_2} : A_{22}w \neq v$. Put $v = \psi(\tau)$ for $\tau \in [-h, 0]$ and for such initial conditions we will not find any extension $x_2(\tau)$ on any interval $[-h-\epsilon, -h]$, for $\epsilon > 0$, and Lemma 1 is proved. ∎

Corollary 2. *The following hold*

$$i) \text{ Any solution is unique on } (-\infty, \infty) \Leftrightarrow \det A_{22} \neq 0, \tag{12}$$

ii) There are solutions that are not unique on $(-\infty, \infty)$ if and only if the following hold

$$a) \text{ the initial data (2) satisfy } \forall \tau \in [-h, 0),\ \psi(\tau) \in range A_{22}, \tag{13}$$

$$b) \; range A_{21} \subset range A_{22} \Leftrightarrow \operatorname{rank}[A_{21}, A_{22}] = \operatorname{rank} A_{22}. \tag{14}$$

Definition 7. *We say that all solutions of System (1) can be prolonged to $-\infty$ if every solution of (1) can be extended to a solution on $(-\infty, \infty)$ i.e. if the condition (12) is satisfied.*

From here on in this section we assume that all solutions of System (1) can be prolonged to $-\infty$.

5.1. Essential observability

Now we present algebraic observabilities, the first is essential observability.

Definition 8. *Lee (1979) System (1) is essentially observable if all solutions can be prolonged to $-\infty$, and $y(t) = 0$ for $t \in (-\infty, 0]$ implies that the only solution (x_1, x_2) to (1) defined on $(-\infty, 0]$ and such that* $\left[\dfrac{B}{A(d)}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ *is $x_1 \equiv 0$ and $x_2 \equiv 0$.*

Theorem 4. *System (1) is essentially observable if and only if System (1) is spectrally observable and* $\det A_{22} \neq 0$.

Proof. We present direct proof for this theorem (see Lee and Olbrot, 1981 for example). Suppose that System (1) is not spectrally observable, then there exists eigensoluton of the form $x_1(t) = exp(\lambda t)x_1(0)$, $x_2(t) = exp(\lambda t)x_2(0)$, $x_1(0) \neq 0$, $x_2(0) \neq 0$ such that $y(t) = 0$ for $t \in (-\infty, \infty)$. Thus, System (1) is not essentially observable.

Suppose that System (1) is spectrally observable and $\det A_{22} \neq 0$ hold. Assume that $y(t) = 0$ for $t \in (-\infty, 0]$ for $(x_1(t), x_2(t))$ solution of System (1) defined at $t \in (-\infty, 0]$. Let us set the initial time at $t_0 = -2n_2h$, then, by finite-time observability, after time n_2h such solution $x_1(t) = 0$, $x_2(t) = 0$ for $t \in (-n_2h, 0)$. By Zaczkiewicz and Marchenko (2005) $x_1(t) = 0$, $x_2(t) = 0$ for $t \in (-n_2h, 0)$, and $\det A_{22} \neq 0$ implies $x_1(t) = 0$, $x_2(t) = 0$ for $t \in (-\infty, \infty)$. This proves the theorem. ∎

5.2. Hyperobservability

Hyperobservability was studied in Lee and Olbrot (1981) for the retarded systems, here we present result concerning DAD systems:

Definition 9. *System (1) is hyperobservable when all solutions can be prolonged to* $-\infty$ *and the kernel of* $\left[\frac{B}{A(d)}\right]$ *treated as an operator on* $PC((-\infty,\infty), \mathbb{R}^{n_1+n_2})$ *is a zero subspace.*

Now we establish the relation between hyberobservability and spectral observability.

Theorem 5. *If System (1) is hyperobsevable then System (1) is essentially observable.*

Proof. The proof is by Definitions 8 i 9. ∎

The converse implication does not hold–consider the following example:

Example 3. *To see that the converse of the implication is not true, consider the following system:*

$$
\begin{aligned}
\dot{x_1}(t) =& (1)x_1(t) + (1,0)x_2(t),\\
x_2(t) =& \begin{pmatrix} 0 \\ 2 \end{pmatrix} x_1(t) + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x_2(t-h),\\
y(t) =& (-2)x_1(t) + (0,1)x_2(t).
\end{aligned}
$$

Then we compute the observability matrix (5)

$$
\left[\frac{B}{A(d)}\right] = \begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 2d \\ 4d & -2d & 4d^2 \end{pmatrix},
$$

the Smith form is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d^2+d \end{pmatrix}$ *and we check (10):*

$$\operatorname{rank}\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & e^{-\lambda h}-1 & 0 \\ 2 & 0 & 2e^{-\lambda h}-1 \\ -2 & 0 & 1 \end{bmatrix} = 3.$$

Thus, the converse of Theorem 5 is not true in general.

Now we present a theorem similar to Theorems 1 and 2 for hyperobservability.

Theorem 6. *Hyperobservability of System (1) is equivalent to each of the following conditions*

$$i)\,\textit{The Smith form of}\begin{bmatrix} B \\ \hline A(d) \end{bmatrix} \text{is diag}\left(d^{k_1},\ldots,d^{k_{n_1+n_2}}\right), 0 \le k_1 \le \ldots \le k_{n_1+n_2}. \tag{15a}$$

$$iii)\operatorname{rank}\begin{bmatrix} B \\ \hline A(d) \end{bmatrix} = n_1+n_2,\ \textit{for all complex } d \neq 0. \tag{15b}$$

Proof. By repeating the proof of Theorem 10 by Lee and Olbrot (1981) for $\psi(t) = z^{-\frac{t}{h}}$.

■

The following example shows relations between hyperobservability and $\mathbb{R}^{n_1+n_2}[d]$-observability.

Example 4. *Consider the following system*

$$\begin{aligned} \dot{x}_1(t) =& (1)x_1(t) + (0)x_2(t), \\ x_2(t) =& (-1)x_1(t) + (1)x_2(t-h), \\ y(t) =& (-2)x_1(t) + (1)x_2(t). \end{aligned} \tag{16}$$

The observability matrix (5) is $\begin{bmatrix} B \\ \hline A(d) \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & d \end{pmatrix}$, *the Smith form is* $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$. *Thus, System (16) is hyperobservable but not* $\mathbb{R}^{n_1+n_2}[d]$*-observable.*

We can now formulate our main result.

Theorem 7. *The following implications hold for observability properties of Sys-*

tem (1)

$$\begin{array}{c} \mathbb{R}^{n_1+n_2}[d]\text{-}\mathit{observability} \\ \Downarrow \\ \mathit{hyperobservability} \\ \Downarrow \\ \mathit{essential\ observability} \Leftrightarrow \mathit{spectral\ observability\ and}\ \det A_{22} \neq 0 \quad \Rightarrow \mathit{spectral\ observability} \\ \Downarrow \\ \mathbb{R}^{n_1+n_2}(d)\text{-}\mathit{observability} \end{array}$$

Proof. By Theorems 1 and 6 with Example 4 we have the first implication. Implication hyperobservability $\rightarrow$ essential observability follows from Theorem 5 and Example 3. The next equivalence we have by Theorem 4. Now we prove the last implication, assume System (1) is essentially observable (spectrally observable and $\det A_{22} \neq 0$). Let us define η polynomial of d by $\det \left[\frac{B}{A}\right] = \eta(d)$. By Definition 8 we have that $\eta(d) \not\equiv 0$ and $\det A_{22} \neq 0$ implies that $\eta(d)$ depends on d. Applying Theorems 5 and 6 we obtain (9a). This proves the last implication. To see that the converse is not true, consider Example 6. ■

6. Examples

The obtained results are illustrated by the following examples:

Example 5. *Consider the following system*

$$\begin{aligned} \dot{x_1}(t) &= \begin{pmatrix} 0 & 0 \\ 1 & 6 \end{pmatrix} x_1(t) + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x_2(t), \\ x_2(t) &= \begin{pmatrix} 1 & 7 \\ 1 & 0 \end{pmatrix} x_1(t) + \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} x_2(t-h), \\ y(t) &= \begin{pmatrix} 2 & 0 \end{pmatrix} x_1(t) + \begin{pmatrix} 1 & 0 \end{pmatrix} x_2(t). \end{aligned}$$

For the $\mathbb{R}^{n_1+n_2}[d]$*-observability condition (8a) we have*

$$\left[\frac{B}{A(d)}\right] =$$

$$\begin{bmatrix} 2 & 1 & 9+d & 71+9d+d^2 \\ 0 & 7 & 56+7d & 392+56d+7d^2 \\ 1 & d+2 & 8+2d+d^2 & 65+9d+2d^2+d^3 \\ 0 & 0 & 7 & 56+7d \end{bmatrix}^T,$$

where symbol $()^T$ *means transposition.*

$$\textit{The Smith form is } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The condition (10) for spectral observability is as follows:

$$\operatorname{rank}\begin{bmatrix} 0-\lambda & 0 & 1 & 0 \\ 1 & 6-\lambda & 1 & 1 \\ 1 & 7 & e^{-\lambda h}-1 & 0 \\ 1 & 0 & -e^{-\lambda h} & -1 \\ 2 & 0 & 1 & 0 \end{bmatrix} = 4.$$

Thus, the system is $\mathbb{R}^{n_1+n_2}[d]$*-observable, and so it satisfies all conditions for algebraic observabilities and spectral observability with* $\det A_{22} \neq 0$ *according to Theorem 1 and 7.*

Example 6. *This example describes the relation between spectral observability and* $\mathbb{R}^{n_1+n_2}(d)$*-observability. Consider the following system*

$$\begin{aligned} \dot{x_1}(t) =& (1)x_1(t) + (2)x_2(t), \\ x_2(t) =& (-1)x_1(t) + (-1)x_2(t-h), \\ y(t) =& (1)x_1(t) + (2)x_2(t). \end{aligned}$$

It has the unobservable eigenvalue $\lambda = 0$ *as we see below*

$$\operatorname{rank}\begin{bmatrix} 1-\lambda & 2 \\ -1 & -1-e^{-\lambda h} \\ 1 & 2 \end{bmatrix} = 1.$$

Computing the Smith form of the observability matrix we have

$$\left[\frac{B}{A(d)}\right] = \begin{bmatrix} 1 & 2 \\ -1 & 2-d \end{bmatrix},$$

$$\text{the Smith form is } \begin{bmatrix} 1 & 0 \\ 0 & d-2 \end{bmatrix}.$$

Thus, the system is $\mathbb{R}^{n_1+n_2}(d)$*-observable and it is not spectrally observable with* $\det A_{22} \neq 0$.

7. Conclusions

In this paper, we have investigated four kinds of algebraic observability for linear stationary differential-algebraic systems with retarded argument. Necessary and sufficient conditions for these algebraic observabilities have been given and we have proved relations among these types of observabilities and spectral observability. Practical verifiability of the conditions has been demonstrated on several examples.

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