Control and Cybernetics

vol. **42** (2013) No. 2

On the fuzzy control stochastic differential systems*

by

Tran Thanh Tung

Faculty of Natural Science and Technology,
Tay Nguyen University, Daklak, Viet Nam
thanhtung_bmt@yahoo.com; tttung@ttn.edu.vn

Abstract: In this paper, fuzzy control stochastic differential systems are introduced. The existence and some comparison results on solutions of fuzzy control stochastic differential systems and on sheaf-solutions of sheaf fuzzy control stochastic systems are provided. The continuous dependence of solutions and sheaf-solutions on initials and controls is investigated. The results obtained are correct and meaningful for the theory control.

Keywords: fuzzy theory; differential equations; fuzzy differential equation; fuzzy stochastic differential system; control theory

1. Introduction

A large class of physical problems can be described by differential systems which combine fuzziness and randomness. Recently, these systems have gained much attention and were investigated in many directions. Feng studied the existence result on solutions, a simple comparison result on solutions of fuzzy stochastic differential systems (FSDS) based on the Hukuhara derivative, see Feng (2000B). Some properties of linear fuzzy stochastic differential systems were given in Feng (2003). Fei (2007) investigated the existence and uniqueness of solutions for systems, whose differentiability is different from the concept in Feng (2000B, 2003) and in Fei (2013), author studied fuzzy stochastic differential equations driven by a continuous local martingale. Kim researched the systems using the stochastic integrals in the Itô sense, see Kim (2005). Very recently, Michta and Malinowski (2011) studied the existence and uniqueness of solutions to the stochastic fuzzy differential equations driven by Brownian motion and the continuous dependence on initial condition and stability properties. In Michta and Malinowski (2010), the authors proposed a new approach to fuzzy stochastic integrals of Itô and Aumann type. Michta extended the notion of set-valued and fuzzy stochastic integrals to semimartingale integrators, presented their main properties and the existence of solutions of a fuzzy

*Submitted: March 2012; Accepted: March 2013

integral stochastic equation driven by a Brownian motion, see Michta (2011). In Michta and Malinowski (2010, 2011) and Michta (2011), the authors studied stochastic differential equations and fuzzy integral stochastic equation driven by a Brownian motion.

In this paper, using Hukuhara derivative provided by Feng, see Feng (2000B, 2003), fuzzy control stochastic differential systems and sheaf fuzzy control stochastic systems are introduced. These control systems combine fuzziness and randomness, and are natural extensions of ordinary and fuzzy differential equations. The existence and some comparison results on solutions of fuzzy control stochastic differential systems and on sheaf-solutions of sheaf fuzzy control stochastic systems are studied. The continuous dependence of solutions and sheaf-solutions on initials and controls is investigated.

The paper is organized as follows: Section 2 reviews some concepts of fuzzy sets, Hausdorff distance, second-order fuzzy stochastic process, the mean-square Riemann integral and mean-square differentiable. Section 3 reviews the existence result on solutions of FSDS and the comparison result on solutions of FSDS. Section 4 provides a simple existence of solutions, some comparison results on solutions and approximation solutions of fuzzy control stochastic differential systems. Similar results on sheaf-solutions of sheaf fuzzy control stochastic systems are presented. The continuous dependence of solutions and sheaf-solutions on initials and controls is studied in this section.

2. Preliminaries and notation

The following notations and concepts were presented in detail in Diamond and Kloeden (1994), Lakshmikantham (2000, 2005), Lakshmikantham, Bhaskar and Devi (2006) and Lakshmikantham and Mohapatra (2003).
Let $K_c(\mathbb{R}^n)$ denote the collection of all nonempty, compact, convex subsets of $\mathbb{R}^n$. Let A, B be two nonempty bounded subsets of $\mathbb{R}^n$. The Hausdorff distance between A and B is defined as

$$D[A,B] = \max\{\sup_{a\in A}\inf_{b\in B} \parallel a-b \parallel, \sup_{b\in B}\inf_{a\in A} \parallel a-b \parallel\}. \tag{1}$$

It is known that $K_c(\mathbb{R}^n)$, with the metric D is a complete metric space, see Puri and Ralescu (1986) and Tolstonogov (2000) and if the space $K_c(\mathbb{R}^n)$ is equipped with the natural algebraic operations of addition and nonnegative scalar multiplication, then $K_c(\mathbb{R}^n)$ becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space, see Tolstonogov (2000).

Set $\mathbb{E}^n = \{u : \mathbb{R}^n \to [0,1]$ such that u satisfies (i) to (iv) mentioned below$\}$
(i) u is normal, that is, there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$;
(ii) u is fuzzy convex, that is, for $0 \leq \lambda \leq 1$

$$u(\lambda x_1 + (1-\lambda)x_2) \geq \min\{u(x_1), u(x_2)\};$$

(iii) u is upper semicontinuous;

(iv) $[u]^0 = \mathrm{cl}\{x \in \mathbb{R}^n : u(x) > 0\}$ is compact.

The element $u \in \mathbb{E}^n$ is called a fuzzy set.

For $0 < \alpha \leq 1$, the set $[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}$ is called the α-level set. From (i)- (iv), it follows that the α-level sets are in $K_c(\mathbb{R}^n)$, for $0 \leq \alpha \leq 1$.
Let us denote

$$D_0[u, v] = \sup\left\{D\Big[[u]^\alpha, [v]^\alpha\Big] : 0 \leq \alpha \leq 1\right\},$$

the distance between u and v in $\mathbb{E}^n$, where $D\Big[[u]^\alpha, [v]^\alpha\Big]$ is Hausdorff distance between two sets $[u]^\alpha, [v]^\alpha$ of $K_c(\mathbb{R}^n)$. Then, $(\mathbb{E}^n, D_0)$ is a complete space, see Diamond and Kloeden (1994) and Puri and Ralescu (1986).
Let us denote $\theta \in \mathbb{E}^n$ the zero element of $\mathbb{E}^n$ as follows

$$\theta(z) = \begin{cases} 1 & \text{if } z = \hat{0}, \\ 0 & \text{if } z \neq \hat{0}, \end{cases}$$

where $\hat{0}$ is the zero element of $\mathbb{R}^n$.

Now, we recall some useful concepts from Feng (1999A,B, 2000A,B, 2001, 2003). The norm $\| u \|$ of a fuzzy number $u \in \mathbb{E}^n$ is defined as $\| u \| = D_0[u, \theta]$.
Let $(\Omega, \mathcal{A}, P)$ be a complete probability space. A fuzzy random variable (f.r.v. for short) is a Borel measurable function $X : (\Omega, \mathcal{A}) \to (\mathbb{E}^n, D_0)$. If $E \| X \| < +\infty$, then the expected value EX exists.
Let $L_2 = \{X | X \text{ is an f.r.v. with} E \| X \|^2 < +\infty\}$. Two f.r.v's, X and Y are called equivalent if $P(X \neq Y) = 0$. All equivalent elements in L_2 are identified.
Define $\rho[X, Y] = (ED_0^2[X, Y])^{1/2}, X, Y \in L_2$.
The norm $\| X \|_2$ of an element $X \in L_2$ is defined by

$$\| X \|_2 = \rho[X, \theta] = (E(\| X \|^2))^{1/2}.$$

The (L_2, ρ) is a complete space, see Feng (1999A) and ρ satisfies

$$\rho[X + Z, Y + Z] = \rho[X, Y], \quad \rho[\lambda X, \lambda Y] = | \lambda | \rho[X, Y] |, \tag{2}$$

$$\rho[\lambda X, kX] \leq | \lambda - k | \| X \|_2, \tag{3}$$

for any $X, Y, Z \in L_2$ and $\lambda, k \in \mathbb{R}$.
Let $u, v \in \mathbb{E}^n$. The set $w \in \mathbb{E}^n$ satisfying $u = v + w$ is known as the H-difference of the sets u and v and is denoted by the symbol $u - v$. Because $u - v : \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{E}^n$ is continuous, if X and Y are f.r.v's and the H-difference of X and Y exists a.s., i.e. $P(X - Y \text{exists}) = 1$, then $X - Y$ is an f.r.v and $X - Y \in L_2$ provided $X, Y \in L_2$.
Let $(X_n)_n \geq 1$ be a sequence in L_2. We call that X_n converges in mean square or m.s. converges to X as $n \to \infty$ if $\rho[X_n; X] \to 0$, and write $X_n \xrightarrow{\text{m.s.}} X$ or $\lim_{n \to \infty} X_n = X$.

Definition 1. [Feng (1999A)]. Let I be a finite or an infinite interval in $\mathbb{R}$. A mapping $X : I \to L_2$ is called a second-order fuzzy stochastic process (f.s.p, for short). If X is continuous at a $t \in I$ with respect to the metric ρ then we call X continuous in mean square or m.s. continuous at t. If X is m.s. continuous at every $t \in I$ then we call X m.s. continuous. An f.s.p. $\{X(t), t \in I\}$ is called stochastic continuous provided $D_0[X(s), X(t)] \xrightarrow{p} 0$, as $s \to t$, for each $t \in I$.

Definition 2. [Feng (1999A)]. Let $\{X(t), t \in I\}$ be a second-order f.s.p. defined on I=[a,b]. For each finite partition Δ_n of [a,b]: $\Delta_n : a = t_0 < t_1 < ... < t_n = b$, and for arbitrary points t'_i, $t_{i-1} \leq t'_i \leq t_i, i = 1, 2, ..., n$, let $S_n = \sum_{i=1}^n \Delta t_i X(t'_i)$ and $\mid \Delta_n \mid = max_{1 \leq i \leq n} \Delta t_i$, where $\Delta t_i = t_i - t_{i-1}$. Then the mean-square Riemann integral or m.s. integral of $X(t)$ on the interval [a,b] is defined by

$$\int_a^b X(t)dt = \lim_{|\Delta_n| \to 0} S_n$$

provided this limit exists and it is independent of the partition as well as the selected points t_i. And we say that $X(t)$ is m.s. integrable on [a,b].

If $\{X(t), t \in [a, b]\}$ is non-random, the m.s. convergence equals the convergence in D_0, this time the m.s. integral is called R-integral. If $X(t)$ is mean-square continuous except for finitely many points of [a,b] then $X(t)$ is mean-square integrable on [a,b].

The properties of m.s. integral are the following.

Theorem 1. [Feng (1999A)]. *Let $X(t)$ and $Y(t)$ be m.s. integrable on* [a,b].

(i) *For each* $r \in [0, 1], [\int_a^b X(t)dt]^r = \int_a^b [X(t)]^r dt.$

(ii) *For each* $\alpha, \beta \in \mathbb{R}, \alpha X(t) + \beta Y(t)$ *is m.s. integrable on* [a, b] *and* $\int_a^b (\alpha X(t) + \beta Y(t))dt = \alpha \int_a^b X(t)dt + \beta \int_a^b Y(t)dt.$

(iii) $X(t)$ *is m.s. integrable on any subinterval of* [a,b], *and* $\int_a^b X(t)dt = \int_a^c X(t)dt + \int_c^b X(t)dt, a \leq c \leq b.$

(iv) $EX(t)$ *is R-integrable on* [a,b] *and* $E \int_a^b X(t)dt = \int_a^b EX(t)dt.$

(v) *If* $\rho[X(t), Y(t)]$ *is Riemann integrable on* [a,b] *then* $\rho[\int_a^b X(t)dt, \int_a^b Y(t)dt] \leq \int_a^b \rho[X(t), Y(t)]dt.$

Definition 3. [Feng (1999A)]. A second-order f.s.p. $\{X(t), t \in I = [a, b]\}$, is m.s. differentiable at $\tau \in I$ if there exists an $X'(\tau) \in L_2$ such that the m.s. limits

$$\lim_{h \to 0^+} \frac{X(\tau + h) - X(\tau)}{h} \quad \text{and} \quad \lim_{h \to 0+} \frac{X(\tau) - X(\tau - h)}{h}$$

exist and is equal to $X'(\tau)$. At the end points of I we consider only the one-sided derivatives. If $X(t)$ is m.s. differentiable at every $t \in I$ then we call $X(t)$ m.s. differentiable on I.

This definition is called Hukuhara derivative and is based on the Hukuhara difference of fuzzy sets. The restriction of this definition is that the existence of Hukuhara derivative is not easy, even for some simple fuzzy mappings, see Diamond and Kloeden (1994). Some properties of m.s. differentiation are the following.

Theorem 2. [Feng (1999A)].

(i) *If* $X(t)$ *is m.s. differentiable at* $t_0 \in I$ *then* $X(t)$ *is m.s. continuous at* $t_0 \in I$.

(ii) *If* $X(t)$ *and* $Y(t)$ *are m.s. differentiable on* I*, then for any* $\alpha, \beta \in \mathbb{R}, \alpha X(t)+\beta Y(t)$ *is m.s. differentiable on* I *and* $(\alpha X(t) + \beta Y(t))' = \alpha X'(t) + \beta Y'(t)$.

(iii) *If* $X(t), t \in [\mathrm{a}, \mathrm{b}]$*, is m.s. continuous then the m.s. integral* $Y(t) = \int_a^t X(s)ds, t \in [\mathrm{a}, \mathrm{b}]$*;, is m.s. differentiable and* $Y'(t) = X(t)$.

(iv) *If* $X(t), t \in I$*, is m.s. differentiable then* $[X(t)]^r$ *is m.s. differentiable and* $[X'(t)]^r = ([X(t)]^r)'$*, for all* $r \in [0, 1]$*. In the case of* E^1*, we take* $[X(t)]^r = [f_r(t), g_r(t)]$*, then* $f_r(t)$ *and* $g_r(t)$ *are m.s. differentiable and* $[X'(t)]^r = [f_r'(t), g_r'(t)]$.

(v) *(Newton - Leibniz formula) If* $X'(t)$ *is m.s. integrable on* [a,b] *then for each* $t \in [\mathrm{a}, \mathrm{b}], X(t) = X(a) + \int_a^t X'(s)ds$.

For further details on mean-square calculus the reader is referred to Feng (1999A,B, 2000A,B, 2001, 20003).

3. Fuzzy stochastic differential systems

The following concepts and results on fuzzy stochastic differential systems are taken from Feng (1999B, 2000A). Let $X_1, ..., X_m$ be f.r.v.'s. $\boldsymbol{X} = (X_1, ..., X_m)^T$ is called an m-dimensional fuzzy random vector, where T denotes the transpose of the vector. It is a Borel measurable function $\boldsymbol{X} : \Omega \to (\mathbb{E}^n)^m = \mathbb{E}^n \times ... \times \mathbb{E}^n$. Let $L_2^m = \{\boldsymbol{X} | \boldsymbol{X} = (X_1, ..., X_m)^T, X_i \in L_2, \, i = 1, ..., m\}$. Define

$$\rho\big[\boldsymbol{X}, \boldsymbol{Y}\big] = max\{\rho\big[X_i, Y_i\big], \, X_i, Y_i \in L_2, 1 \le i \le m\}.$$

The norm $\| \boldsymbol{X} \|_2$ of an element $\boldsymbol{X} \in L_2^m$ is defined by $\| \boldsymbol{X} \|_2 = \rho\big[\boldsymbol{X}, \theta^m\big] = max\{\| X_i \|_2, 1 \le i \le m\}$, where θ^m is the zero element of L_2^m.

By the completeness of (L_2, ρ) and (2), (3), a standard proof applies that (L_2^m, ρ) is a complete metric space and ρ satisfies some properties as in (2)- (3).

A second-order m-dimensional vector f.s.p. is characterized by a mapping of the interval I into L_2^m. For the sake of convenience, we shall adopt the notation $\boldsymbol{X} : I \to L_2^m$ in what follows. The m.s. continuity, m.s. differentiation, and m.s. integration associated with a second-order m-dimensional vector f.s.p. are defined with respect to the metric ρ in L_2^m . Hence, an m-dimensional vector f.r.p. $\{\boldsymbol{X}(t), t \in I\}$, is m.s. continuous at t, for example, if $\rho\big[\boldsymbol{X}(t+h), \boldsymbol{X}(t)\big] \to 0$, as $h \to 0$. In view of this definition, it is clear that the m-dimensional vector f.s.p. $\{\boldsymbol{X}(t), t \in I\}$, is m.s. continuous at $t \in I$ if and only if each of its component processes is m.s. continuous at t. Similar definitions and observations can be formulated with regard to m.s. differentiation and m.s. integration of the second-order m-dimensional vector f.s.p. $\{\boldsymbol{X}(t), t \in I\}$.

Feng (1999B) considered the fuzzy stochastic differential systems (FSDS) in the form

$$\boldsymbol{X}'(t) = \boldsymbol{F}(t, \boldsymbol{X}(t)), t \in I = [t_0, T] \subset \mathbb{R}_+, \tag{1}$$

with the initial value $\boldsymbol{X}(t_0) = \boldsymbol{X}_0 \in L_2^m$, where $\boldsymbol{F}$ is a mapping: $I \times L_2^m \to L_2^m$. Systems (1) are fuzzy stochastic differential systems. In the case of $m = 1$, (1) is a fuzzy stochastic differential equation. Systems(1) with out randomness, are fuzzy differential systems. Systems (1) without fuzziness, are stochastic differential systems. And, systems (1) without randomness and fuzziness, are ordinary differential equations. We now consider the solution of (1) in the mean square sense. From Theorem 2 we know that $\boldsymbol{X}(t)$ is a solution of (1) if and only if it is m.s. continuous and satisfies the integral equation

$$\boldsymbol{X}(t) = \boldsymbol{X}_0 + \int_{t_0}^{t} \boldsymbol{F}(s, \boldsymbol{X}(s))ds. \tag{2}$$

Some results on solutions of fuzzy stochastic differential systems are the following.

Theorem 3. [Feng (1999B)] *Let $\boldsymbol{F}$ be m.s. continuous with respect to t and there exists a $k > 0$ such that*

$$\rho\big[\boldsymbol{F}(t, \boldsymbol{X}), \boldsymbol{F}(t, \boldsymbol{Y})\big] \leq k\rho\big[\boldsymbol{X}, \boldsymbol{Y}\big], \tag{3}$$

for all $t \in I$ and $\boldsymbol{X}, \boldsymbol{Y} \in L_2^m$. Then (1) has a unique solution.

Theorem 4. [Feng (1999B)] *Let $\boldsymbol{F}$ be as in Theorem 3.1 Then there exist constants c_1 and c_2 such that*

(i) $\rho[\boldsymbol{X}(t, \boldsymbol{X}_0), \boldsymbol{X}(t, \boldsymbol{Y}_0] \leq c_1 \rho[\boldsymbol{X}_0, \boldsymbol{Y}_0]$, *for any $\boldsymbol{X}_0, \boldsymbol{Y}_0 \in L_2^m, t \in I$.*

(ii) $sup_{t \in I} \parallel \boldsymbol{X}(t, \boldsymbol{X}_0) \parallel_2 \leq c_1\big(c_2 + \parallel \boldsymbol{X}_0 \parallel_2\big)$.

Here, the result (i) of Theorem 2.2 in Feng (1999B) is rewritten in form of (i) in Theorem 4.
The explicit representation of solutions for the first-order linear fuzzy stochastic differential systems with general coefficient matrix was introduced in Feng (2000A).

4. Main results

Based on the fuzzy stochastic differential systems (1), we introduce the new concept of fuzzy control stochastic differential systems (FCSDS) as follows:

$$\boldsymbol{X}'(t) = \boldsymbol{F}(t, \boldsymbol{X}(t), \boldsymbol{U}(t)), t \in I = [t_0, T] \subset \mathbb{R}_+, \boldsymbol{X}(t_0) = \boldsymbol{X}_0 \in L_2^m \tag{1}$$

with the initial value $\boldsymbol{X}(t_0) = \boldsymbol{X}_0 \in L_2^m$, where $\boldsymbol{F}$ is a mapping: $\boldsymbol{F} \in C\big[I \times L_2^m \times L_2^p, L_2^m\big]$, state $\boldsymbol{X}(t) \in L_2^m$ and control $\boldsymbol{U}(t) \in L_2^p$.
If $\boldsymbol{U} : I \to L_2^p$ is m.s. integrable, then it is called admissible control. Let $\mathbf{U^{ac}}$ be a set of all admissible controls.

Without randomness and fuzziness, (1) are classical control differential systems. In practice, we are often faced with the random experiments and fuzzy data. So, FCSDS describe the motion in physical problems better than ordinary differential systems.
We consider the solution of (1) in the mean square sense. From Theorem 2 we know that $\boldsymbol{X}(t)$ is a solution of (1) if and only if it is m.s. continuous and satisfies the integral equation

$$\boldsymbol{X}(t) = \boldsymbol{X}_0 + \int_{t_0}^{t} \boldsymbol{F}(s, \boldsymbol{X}(s), \boldsymbol{U}(s))ds. \tag{2}$$

Now, set

$$\boldsymbol{F}^*(t, \boldsymbol{X}(t)) = \boldsymbol{F}(t, \boldsymbol{X}(t), \boldsymbol{U}(t)).$$

We have the following simple existence result on solutions of FCSDS (1).

Theorem 5. *Let $\boldsymbol{F}^*$ be m.s. continuous with respect to t and there exists a $k > 0$ such that*

$$\rho\big[\boldsymbol{F}^*(t, \bar{\boldsymbol{X}}), \boldsymbol{F}^*(t, \boldsymbol{X})\big] \le k\rho\big[\bar{\boldsymbol{X}}, \boldsymbol{X}\big] \tag{3}$$

for all $t \in I$, $\bar{\boldsymbol{X}}, \boldsymbol{X} \in L_2^m$. Then (1) has a unique solution.

The conclusion of Theorem 5 follows from Theorem 3.

Remark 1. Theorems 3 and 5 also hold when k is a bounded positive continuous function, that means, $k : I \to \mathbb{R}_+$ and $k(t) < k_0$ for all $t \in I$, where k_0 is a constant.

Example 1. A simple population growth model of a specie is as follows:

$$p'(t) = a(t)p(t), \; p(t_0) = p_0 \in \mathbb{R}_+, \tag{4}$$

where $t \in [t_0, T]$, $p(t) \in \mathbb{R}_+$ is population at time t and $a(t)$ is a real-valued continuous function. It is also called the logistic law of population growth of a specie. This is an ordinary differential equation and it is easy to solve. In practice, we are often faced with random experiments and fuzzy data and we do not know exactly the population of a country or of a specie at the time t. For example, if population of a country is said to be p_0 million people at the time $t = t_0$, it is understood that the population is "about" p_0. The population is not only fuzzy but also random, so $p(t)$ and p_0 are considered to be fuzzy random variables, one has the fuzzy stochastic differential equation as follows.

$$p'(t) = a(t)p(t), p(t_0) = p_0 \in L_2, \tag{5}$$

where $t \in [t_0, T]$, $p(t) \in L_2$ is population at time t and $a(t)$ is a real-valued continuous function. The equation satisfies the Theorem 3, so it has solutions. By Example 3.6, Feng (2003), the solution of the equation (5), where $a(t)$ is positive, is

$$p(t) = p_0 e^{\int_{t_0}^{t} a(s)ds}.$$

In this simple example, the structure of solutions of (5) is similar to that of the ordinary differential equation (4), where $p(t)$ and p_0 are fuzzy random variables.

Now, a general form of (5) is

$$\boldsymbol{P}'(t) = a(t)\boldsymbol{P}(t), \boldsymbol{P}(t_0) = \boldsymbol{P}_0 \in L_2^m, \tag{6}$$

where $t \in [t_0, T]$, $\boldsymbol{P}(t) \in L_2^m$ is population at time t and $a(t)$ is a real-valued continuous function. Here, $\boldsymbol{P}(t)$ is a second-order m-dimensional vector f.s.p. that means, the vector of populations of m countries.

The control systems based on FCSDS are formed, for example

$$\boldsymbol{P}'(t) = a(t)\boldsymbol{P}(t) + b(t)\boldsymbol{U}(t), \boldsymbol{P}(t_0) = \boldsymbol{P}_0 \in L_2^m, \tag{7}$$

where $t \in [t_0, T]$, $\boldsymbol{P}(t) \in L_2^m, \boldsymbol{U}(t) \in L_2^p$ and $a(t), b(t)$ are matrices of real- valued continuous functions with appropriate dimensions. The control $\boldsymbol{U}(t)$ combines fuzziness and randomness.

In Ovsanikov (1980), the concept of sheaf-solution of classical control differential systems was introduced. Instead of studying each solution, one studies sheaf-solution, that means, a set of solutions. In Phu, Quang and Tung (2008), Phu and Tung (2005, 2006A,B, 2007), some results on sheaf-solutions in sheaf fuzzy control systems and sheaf set control systems were studied. In this paper, we introduce the concept of sheaf-solutions for FCSDS (1). Let $\boldsymbol{H}_0$ be the collection of some given initials of FCSDS in L_2^m. The notation $\boldsymbol{X}_0 \in \boldsymbol{H}_0$ means that $\boldsymbol{X}_0$ is an element of $\boldsymbol{H}_0$, and $\boldsymbol{H}_0$ is considered to be a subset of L_2^m ($\boldsymbol{H}_0 \subset L_2^m$ for short). The concept sheaf-solutions of FCSDS as following.

Definition 4. The sheaf-solution (or sheaf-trajectory) of (1) which gives at the time t a set

$$\boldsymbol{H}_{t,\boldsymbol{U}} = \{\boldsymbol{X}(t) = \boldsymbol{X}(t, \boldsymbol{X}_0, \boldsymbol{U}(t)) - \text{solution of (1)} \mid \boldsymbol{X}_0 \in \boldsymbol{H}_0\},$$

where $t \in I, \boldsymbol{H}_0 \subset L_2^m, \boldsymbol{U} \in \boldsymbol{U}^{\mathbf{ac}}$.

The $\boldsymbol{H}_{t,\boldsymbol{U}}$ is called a *cross − area* at $(t, \boldsymbol{U})$ (or $(t, \boldsymbol{U}) - cut$) of the sheaf-solution. Systems (1) with their sheaf-solutions are called *sheaf fuzzy control stochastic systems* (SFCSS). For two given initial sets $\bar{\boldsymbol{H}}_0$ and $\boldsymbol{H}_0$, we have two sheaf-solutions whose cross-areas are

$$\boldsymbol{H}_{t,\boldsymbol{U}} = \big\{\boldsymbol{X}(t) = \boldsymbol{X}(t, \boldsymbol{X}_0, \boldsymbol{U}(t)) - \text{solution of (1)} \mid \boldsymbol{X}_0 \in \boldsymbol{H}_0\big\}, \tag{8}$$

$$\bar{\boldsymbol{H}}_{t,\bar{\boldsymbol{U}}} = \big\{\bar{\boldsymbol{X}}(t) = \boldsymbol{X}(t, \bar{\boldsymbol{X}}_0, \bar{\boldsymbol{U}}(t)) - \text{solution of (1)} \mid \bar{\boldsymbol{X}}_0 \in \bar{\boldsymbol{H}}_0\big\}, \tag{9}$$

where $t \in I, \boldsymbol{H}_0, \bar{\boldsymbol{H}}_0 \subset L_2^m, \boldsymbol{U} \in \mathbf{U}^{\mathbf{ac}}$.

In this paper, instead of comparison of two sheaf-solutions, we compare their cross-areas.
Suppose that $\boldsymbol{Q}, \boldsymbol{G} \subset L_2^l$ for some positive integer l. Here are some useful notations:

$$\begin{aligned} &d^*[\boldsymbol{Q}, \boldsymbol{G}] = \sup\{\rho[\boldsymbol{X}, \boldsymbol{Y}] : \boldsymbol{X} \in \boldsymbol{Q}, \boldsymbol{Y} \in \boldsymbol{G}\},\\ &\operatorname{diam}[\boldsymbol{Q}] = \sup\{\rho[\boldsymbol{X}, \boldsymbol{Y}] : \boldsymbol{X}, \boldsymbol{Y} \in \boldsymbol{Q}\},\\ &d[\boldsymbol{Q}] = d^*[\boldsymbol{Q}, \theta^l],\ \theta^l \text{ is the zero element of } L_2^l,\\ &\| \boldsymbol{Q} \|_2 = \sup\{\| \boldsymbol{X} \|_2 : \boldsymbol{X} \in \boldsymbol{Q}\}. \end{aligned} \tag{10}$$

The set $\boldsymbol{Q}$ is said to be bounded if $d[\boldsymbol{Q}] < +\infty$.
Now, one considers the assumption on $\boldsymbol{F}$ below.
The mapping $\boldsymbol{F} : \mathbb{R}_+ \times L_2^m \times L_2^p \to L_2^m$ satisfies the condition

$$\rho\big[\boldsymbol{F}(t, \bar{\boldsymbol{X}}, \bar{\boldsymbol{U}}), \boldsymbol{F}(t, \boldsymbol{X}, \boldsymbol{U})\big] \le c(t)\big[\rho[\bar{\boldsymbol{X}}, \boldsymbol{X}] + \rho[\bar{\boldsymbol{U}}, \boldsymbol{U}]\big] \tag{11}$$

for all $t \in I; \boldsymbol{X}, \bar{\boldsymbol{X}} \in L_2^m$ where $c(t)$ is a positive, Lebesgue measurable, bounded real function on I. Let $C = \int_{t_0}^T c(t)dt$. Since $c(t)$ is Lebesgue measurable and bounded on I, it is integrable and there exists a positive number K such that $c(t) \le K$ for all $t \in I$.

In ordinary differential equations and control systems described by ordinary differential equations, Lipschitz condition plays an important role. Condition (11), is considered a type of Lipschitz condition on $\boldsymbol{F}$ in $\boldsymbol{X}$ and $\boldsymbol{U}$, and is widely used in the paper. The Lipschitz condition expresses the fact that $\boldsymbol{F}$ can be bounded by a linear function for all $t \in I$. Under Lipschitz condition, many properties of solutions of these systems are studied. A simple example for condition (11) is the following.

Example 2. Consider

$$\boldsymbol{X}'(t) = a(t)\boldsymbol{X}(t) + b(t)\boldsymbol{U}(t), \boldsymbol{X}(t_0) = \boldsymbol{X}_0 \in L_2,$$

where $t \in [t_0, T]$, $\boldsymbol{X}(t) \in L_2, \boldsymbol{U}(t) \in L_2$. The real functions $a(t), b(t)$ are positive and increasing. It is easy to check that the mapping $\boldsymbol{F}$ satisfies the condition (11) with real function $c(t) = max\{a(t), b(t)\}$.

Next, ones suppose that $\boldsymbol{H}_0, \bar{\boldsymbol{H}}_0, \mathbf{U^{ac}}$ are bounded. In the following theorem, the solutions of (1) depend continuously on initials and controls.

THEOREM 6. *Suppose that $\boldsymbol{F}$ is m.s. continuous and satisfies (11) and $\bar{\boldsymbol{X}}(t) = \boldsymbol{X}(t, t_0, \bar{\boldsymbol{X}}_0, \bar{\boldsymbol{U}}(t))$, $\boldsymbol{X}(t) = \boldsymbol{X}(t, t_0, \boldsymbol{X}(t), \boldsymbol{U}(t))$ are two solutions of (1) such that $\bar{\boldsymbol{X}}(t_0) = \bar{\boldsymbol{X}}_0$, $\boldsymbol{X}(t_0) = \boldsymbol{X}_0$. Then, for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that*

$$\rho\big[\bar{\boldsymbol{X}}(t), \boldsymbol{X}(t)\big] \le \epsilon \quad if \quad \rho\big[\bar{\boldsymbol{U}}(t), \boldsymbol{U}(t)\big] \le \delta(\epsilon) \quad and \quad \rho\big[\bar{\boldsymbol{X}}_0, \boldsymbol{X}_0\big] \le \delta(\epsilon)$$

where $t \in I, \bar{\boldsymbol{U}}(t), \boldsymbol{U}(t) \in \mathbf{U^{ac}}$.

Proof. Solutions of (1), originating at the points $\boldsymbol{X}_0$ and $\bar{\boldsymbol{X}}_0$, are equivalent to the following integral forms

$$\boldsymbol{X}(t) = \boldsymbol{X}_0 + \int_{t_0}^{t} \boldsymbol{F}(s, \boldsymbol{X}(s), \boldsymbol{U}(s))ds,$$

$$\bar{\boldsymbol{X}}(t) = \bar{\boldsymbol{X}}_0 + \int_{t_0}^{t} \boldsymbol{F}(s, \bar{\boldsymbol{X}}(s), \bar{\boldsymbol{U}}(s))ds.$$

Estimating $\rho\big[\bar{\boldsymbol{X}}(t), \boldsymbol{X}(t)\big]$, using property of metric ρ and Theorem 1 (v), we got

$$\rho\big[\bar{\boldsymbol{X}}(t), \boldsymbol{X}(t)\big] \le \rho\big[\bar{\boldsymbol{X}}_0, \boldsymbol{X}_0\big] + \int_{t_0}^{t} \rho\big[\boldsymbol{F}(s, \bar{\boldsymbol{X}}(s), \bar{\boldsymbol{U}}(s)), \boldsymbol{F}(s, \boldsymbol{X}(s), \boldsymbol{U}(s))\big]ds.$$

By assumption (11), one has

$$\begin{aligned}\rho\big[\bar{\boldsymbol{X}}(t), \boldsymbol{X}(t)\big] &\le \rho\big[\bar{\boldsymbol{X}}_0, \boldsymbol{X}_0\big] + \int_{t_0}^{t} c(s)\big[\rho\big[\bar{\boldsymbol{X}}(s), \boldsymbol{X}(s)\big] + \rho[\bar{\boldsymbol{U}}(s), \boldsymbol{U}(s)]\big]ds \\ &\le \rho\big[\bar{\boldsymbol{X}}_0, \boldsymbol{X}_0\big] + \int_{t_0}^{t} c(s)\rho\big[\bar{\boldsymbol{X}}(s), \boldsymbol{X}(s)\big]ds + K\int_{t_0}^{t} \rho\big[\bar{\boldsymbol{U}}(s), \boldsymbol{U}(s)\big]ds,\end{aligned} \tag{12}$$

then, using $\rho\big[\bar{\boldsymbol{U}}(t), \boldsymbol{U}(t)\big] \le \delta(\epsilon)$ and $\rho\big[\bar{\boldsymbol{X}}_0, \boldsymbol{X}_0\big] \le \delta(\epsilon)$, we obtain

$$\rho\big[\bar{\boldsymbol{X}}(t), \boldsymbol{X}(t)\big] \le \delta(\epsilon) + K(T - t_0)\delta(\epsilon) + \int_{t_0}^{t} c(s)\rho\big[\bar{\boldsymbol{X}}(s), \boldsymbol{X}(s)\big]ds.$$

By the classical Gronwall inequality, we have the following estimate

$$\rho\big[\bar{\boldsymbol{X}}(t), \boldsymbol{X}(t)\big] \le \big[1 + K(T - t_0)\big]\delta(\epsilon)\exp(C).$$

With given $\epsilon > 0$, if we choose

$$0 < \delta(\epsilon) \le \frac{\epsilon}{\big[1 + K(T - t_0)\big]\exp(C)},$$

then

$$\rho\big[\bar{\boldsymbol{X}}(t), \boldsymbol{X}(t)\big] \le \epsilon.$$

The proof is completed. ■

The following result is a consequence of Theorem 6.

Corollary 1. *Under assumptions of Theorem 6, one has* $d^*\big[\bar{\boldsymbol{H}}_{t,\bar{\boldsymbol{U}}}, \boldsymbol{H}_{t,\boldsymbol{U}}\big] \le \epsilon$ *if* $d^*\big[\bar{\boldsymbol{H}}_0, \boldsymbol{H}_0\big] \le \delta(\epsilon)$ *and* $\rho\big[\bar{\boldsymbol{U}}(t), \boldsymbol{U}(t)\big] \le \delta(\epsilon)$ *where* $t \in I$ *and* $\bar{\boldsymbol{U}}(t), \boldsymbol{U}(t) \in \mathbf{U^{ac}}$.

The sheaf-solutions of (1) depend continuously on initials and controls.

The distance between two sheaf-solutions is bounded, is content of the following theorem.

Theorem 7. *Suppose that* $\boldsymbol{F}$ *is m.s. continuous and satisfies (11) and* $\bar{\boldsymbol{H}}_0, \boldsymbol{H}_0, \mathbf{U^{ac}}$ *are bounded subsets. Then*

$$d^*\big[\bar{\boldsymbol{H}}_{t,\bar{\boldsymbol{U}}}, \boldsymbol{H}_{t,\boldsymbol{U}}\big] \le \big[d^*\big[\bar{\boldsymbol{H}}_0, \boldsymbol{H}_0\big] + K.diam[\mathbf{U^{ac}}](T-t_0)\big].exp(C) \qquad (13)$$

where $\bar{\boldsymbol{H}}_{t,\bar{\boldsymbol{U}}}, \boldsymbol{H}_{t,\boldsymbol{U}}$ *are any cross-areas of sheaf-solutions of (1), for all* $t \in I$ *and* $\bar{\boldsymbol{U}}(t), \boldsymbol{U}(t) \in \mathbf{U^{ac}}$.

Proof. Starting as in the proof of Theorem 6, we arrive at (12) and use first notation in (10), then

$$\rho\big[\bar{\boldsymbol{X}}(t), \boldsymbol{X}(t)\big] \le d^*\big[\bar{\boldsymbol{H}}_0, \boldsymbol{H}_0\big] + \int_{t_0}^{t} c(s)\big[\rho\big[\bar{\boldsymbol{X}}(s), \boldsymbol{X}(s)\big]ds + K diam[\mathbf{U^{ac}}](T-t_0).$$

Using the classical Gronwall inequality, then the proof is completed. ■

If $\bar{\boldsymbol{H}}_0 = \boldsymbol{H}_0$ then $d^*\big[\bar{\boldsymbol{H}}_0, \boldsymbol{H}_0\big] = diam[\boldsymbol{H}_0]$. So, the estimate (13) becomes

$$d^*\big[\bar{\boldsymbol{H}}_{t,\bar{\boldsymbol{U}}}, \boldsymbol{H}_{t,\boldsymbol{U}}\big] \le \big[diam\big[\bar{\boldsymbol{H}}_0\big] + K.diam[\mathbf{U^{ac}}](T-t_0)\big].exp(C).$$

The above results are similar the ones in Phu and Tung (2006B, 2007) for sheaf fuzzy control systems and sheaf set control systems, but for stochastic systems.

Consider the following fuzzy control differential systems

$$\bar{\boldsymbol{X}}' = \bar{\boldsymbol{F}}(t, \bar{\boldsymbol{X}}(t), \bar{\boldsymbol{U}}(t)), \bar{\boldsymbol{X}}(t_0) = \bar{\boldsymbol{X}}_0 \in L_2^m, \qquad (14)$$

where $\bar{\boldsymbol{F}} \in C[\mathbb{R}_+ \times L_2^m \times L_2^p, L_2^m]$.

An assumption on $\boldsymbol{F}$ of (1) and $\bar{\boldsymbol{F}}$ of (14) as follows.

$$\rho\big[\bar{\boldsymbol{F}}(t, \bar{\boldsymbol{X}}, \bar{\boldsymbol{U}}), \boldsymbol{F}(t, \boldsymbol{X}, \boldsymbol{U})\big] \le c(t)\big[\rho[\bar{\boldsymbol{X}}, \boldsymbol{X}] + \rho[\bar{\boldsymbol{U}}, \boldsymbol{U}]\big], \qquad (15)$$

for all $t \in I; \bar{\boldsymbol{U}}, \boldsymbol{U} \in \mathbf{U^{ac}}; \bar{\boldsymbol{X}}, \boldsymbol{X} \in L_2^m$, where $c(t)$ satisfies the condition as in (11).

Example 3. Consider two systems

$$\boldsymbol{X}'(t) = a_1(t)\boldsymbol{X}(t) + b_1(t)\boldsymbol{U}(t),$$

$$\bar{\boldsymbol{X}}'(t) = a_2(t)\bar{\boldsymbol{X}}(t) + b_2(t)\bar{\boldsymbol{U}}(t),$$

where $a_1, a_2, b_1, b_2 \in C[I, \mathbb{R}_+]$, $\boldsymbol{X}(t), \bar{\boldsymbol{X}}(t) \in L_2^1; \boldsymbol{U}(t), \bar{\boldsymbol{U}}(t) \in L_2^1$.
The condition (15) holds for $c(t) = max\{| a_1(t) - a_2(t) |, | b_1(t) - b_2(t) |\}$.

We compare solutions of (1) to the ones of (14) for studying the influence of right-hand side in (1) on solutions.

Theorem 8. *Suppose that $\boldsymbol{F}$ of (1) and $\bar{\boldsymbol{F}}$ of (14) are m.s. continuous and satisfy (15) and $\bar{\boldsymbol{X}}(t), \boldsymbol{X}(t)$ are two solutions of (1) and (14) originating at different initials $\bar{\boldsymbol{X}}_0, \boldsymbol{X}_0$ with controls $\bar{\boldsymbol{U}}(t), \boldsymbol{U}(t)$, respectively. Then, for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that*

$$\rho\big[\bar{\boldsymbol{X}}(t), \boldsymbol{X}(t)\big] \leq \epsilon$$

$$if \quad \rho\big[\bar{\boldsymbol{X}}_0, \boldsymbol{X}_0\big] \leq \delta(\epsilon) \quad and \quad \rho[\bar{\boldsymbol{U}}(t), \boldsymbol{U}(t)] \leq \delta(\epsilon), \tag{16}$$

where $t \in I; \bar{\boldsymbol{U}}(t), \boldsymbol{U}(t) \in \mathbf{U^{ac}}$.

The proof is similar the one of Theorem 6. So, we omit the details. In Theorem 8, it is concluded that the solutions of (1) depend continuously on initials, controls and right-hand side.

An immediate consequence of Theorem 8 is the following result on comparison of two sheaf-solutions of (1) and (14).

Corollary 2. *Suppose that $\boldsymbol{F}$ of (1) and $\bar{\boldsymbol{F}}$ of (14) are m.s. continuous and satisfy (15), and $\bar{\boldsymbol{H}}_{t,\bar{\boldsymbol{U}}}, \boldsymbol{H}_{t,\boldsymbol{U}}$ are two cross-areas of sheaf-solutions of (1) and (14) corresponding to $\bar{\boldsymbol{H}}_0, \boldsymbol{H}_0$ and controls $\bar{\boldsymbol{U}}(t), \boldsymbol{U}(t) \in \mathbf{U^{ac}}$. Then, for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that*

$$d^*\big[\bar{\boldsymbol{H}}_{t,\bar{\boldsymbol{U}}}, \boldsymbol{H}_{t,\boldsymbol{U}}\big] \leq \epsilon$$

$$if \quad d^*\big[\bar{\boldsymbol{H}}_0, \boldsymbol{H}_0\big] \leq \delta(\epsilon) \quad and \quad \rho[\bar{\boldsymbol{U}}(t), \boldsymbol{U}(t)] \leq \delta(\epsilon)$$

where $t \in I$; $\bar{\boldsymbol{U}}(t), \boldsymbol{U}(t) \in \mathbf{U^{ac}}$.

Similar to solutions, the sheaf-solutions of (1) depend continuously on controls, initials and right-hand side. The following theorem is similar in proving to Theorems 6-7.

Theorem 9. *Suppose that $\boldsymbol{F}$ of (1) and $\bar{\boldsymbol{F}}$ of (14) are m.s. continuous and satisfy (15) and $\bar{\boldsymbol{H}}_{t,\bar{\boldsymbol{U}}}, \boldsymbol{H}_{t,\boldsymbol{U}}$ are two cross-areas of sheaf-solutions of (1) and (14) corresponding to $\bar{\boldsymbol{H}}_0, \boldsymbol{H}_0$, and controls $\bar{\boldsymbol{U}}(t), \boldsymbol{U}(t) \in \mathbf{U^{ac}}$ and $\bar{\boldsymbol{H}}_0, \boldsymbol{H}_0, \mathbf{U^{ac}}$ are bounded subsets. Then one has*

$$d^*\big[\bar{\boldsymbol{H}}_{t,\bar{\boldsymbol{U}}}, \boldsymbol{H}_{t,\boldsymbol{U}}\big] \leq \big[d^*[\bar{\boldsymbol{H}}_0, \boldsymbol{H}_0] + K.\operatorname{diam}[\mathbf{U^{ac}}](T - t_0)\big] \exp{(C)},$$

for all $t \in I; \bar{\boldsymbol{U}}(t), \boldsymbol{U}(t) \in \mathbf{U^{ac}}$.

With Lipschitz condition, one proves that solutions and sheaf-solutions of systems (1) depend continuously on initials, controls and right-hand side. And, for upper bounds of distances between two solutions, two sheaf-solutions are provided. This is useful in practice. In classical case, many tools are used to study control systems and nonlinear control systems can be approximated by linear ones. Up to now, unfortunately, for fuzzy and stochastic fields, there are no concepts of partial derivatives, so nonlinear control systems can not be approximated by linear ones and the distance is our main tool.

Next, we use the methods presented in Lakshmikantham (2000, 2005), Lakshmikantham, Bhaskar and Devi (2006) and Lakshmikantham and Mohapatra (2003) to compare the solutions of the FCSDS. To investigate the qualitative behavior of solutions of (1), the following comparison result can be proved via the theory of ordinary differential inequalities. The the maximal solutions of scalar differential equations are used for comparison and estimation of the solutions of FCSDS.

THEOREM 10. *Assume that* $\boldsymbol{F}$ *is m.s. continuous and*

$$\rho\big[\boldsymbol{F}(t,\bar{\boldsymbol{X}},\bar{\boldsymbol{U}}),\boldsymbol{F}(t,\boldsymbol{X},\boldsymbol{U})\big] \leq g(t,\rho[\bar{\boldsymbol{X}},\boldsymbol{X}]), \tag{17}$$

for $(t,\bar{\boldsymbol{X}},\bar{\boldsymbol{U}})$, $(t,\boldsymbol{X},\boldsymbol{U}) \in I \times L_2^m \times L_2^p$ *where* $g \in C\big[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+\big]$ *and* $g(t,w)$ *is nondecreasing in* w *for each* $t \in I$. *Suppose further that the maximal solution* $r(t) = r(t,t_0,w_0)$ *of scalar differential equation*

$$w' = g(t,w), w(t_0) = w_0 \geq 0$$

exists for $t \in I$.

Then, if $\bar{\boldsymbol{X}}(t) = \bar{\boldsymbol{X}}(t,t_0,\bar{\boldsymbol{X}}_0,\bar{\boldsymbol{U}}(t))$ *and* $\boldsymbol{X}(t) = \boldsymbol{X}(t,t_0,\boldsymbol{X}_0,\boldsymbol{U}(t))$ *are any solutions of (1) such that* $\bar{\boldsymbol{X}}(t_0) = \bar{\boldsymbol{X}}_0, \boldsymbol{X}(t_0) = \boldsymbol{X}_0$; $\bar{\boldsymbol{X}}_0, \boldsymbol{X}_0 \in L_2^m$ *existing for* $t \in I$, *one has*

$$\rho\big[\bar{\boldsymbol{X}}(t),\boldsymbol{X}(t)\big] \leq r(t,t_0,w_0), \text{ for all } t \in I \text{ and } \bar{\boldsymbol{U}}(t),\boldsymbol{U}(t) \in \mathbf{U}^{\mathbf{ac}}, \tag{18}$$

provided $\rho\big[\bar{\boldsymbol{X}}_0,\boldsymbol{X}_0\big] \leq w_0$.

Proof. The solutions of (1) originating at $\boldsymbol{X}_0, \bar{\boldsymbol{X}}_0$ are equivalent to the following integral forms

$$\boldsymbol{X}(t) = \boldsymbol{X}_0 + \int_{t_0}^{t} \boldsymbol{F}(s,\boldsymbol{X}(s),\boldsymbol{U}(s))ds,$$

$$\bar{\boldsymbol{X}}(t) = \bar{\boldsymbol{X}}_0 + \int_{t_0}^{t} \boldsymbol{F}(s,\bar{\boldsymbol{X}}(s),\bar{\boldsymbol{U}}(s))ds.$$

Set $m(t) = \rho\big[\bar{\boldsymbol{X}}(t),\boldsymbol{X}(t)\big]$, so that $m(t_0) = \rho\big[\bar{\boldsymbol{X}}_0,\boldsymbol{X}_0\big] \leq w_0$. Using the properties of metric ρ and Theorem 1 (v), one has

$$\begin{aligned} m(t) &= \rho\big[\bar{\boldsymbol{X}}_0 + \int_{t_0}^{t} \boldsymbol{F}\big(s, \bar{\boldsymbol{X}}(s), \bar{\boldsymbol{U}}(s)\big)ds, \boldsymbol{X}_0 + \int_{t_0}^{t} \boldsymbol{F}\big(s, \boldsymbol{X}(s), \boldsymbol{U}(s)\big)ds\big] \\ &\le \rho\big[\bar{\boldsymbol{X}}_0 + \int_{t_0}^{t} \boldsymbol{F}\big(s, \bar{\boldsymbol{X}}(s), \bar{\boldsymbol{U}}(s)\big)ds, \bar{\boldsymbol{X}}_0 + \int_{t_0}^{t} \boldsymbol{F}(s, \boldsymbol{X}(s), \boldsymbol{U}(s)\big)ds\big] \\ &\quad + \rho\big[\bar{\boldsymbol{X}}_0 + \int_{t_0}^{t} \boldsymbol{F}\big(s, \boldsymbol{X}(s) \boldsymbol{U}(s)\big)ds, \boldsymbol{X}_0 + \int_{t_0}^{t} \boldsymbol{F}\big(s, \boldsymbol{X}(s), \boldsymbol{U}(s)\big)ds\big] \\ &= \rho\big[\int_{t_0}^{t} \boldsymbol{F}\big(s, \bar{\boldsymbol{X}}(s), \bar{\boldsymbol{U}}(s)\big)ds, \int_{t_0}^{t} \boldsymbol{F}\big(s, \boldsymbol{X}(s), \boldsymbol{U}(s)\big)ds\big] + \rho\big[\bar{\boldsymbol{X}}_0, \boldsymbol{X}_0\big] \\ &\le m(t_0) + \int_{t_0}^{t} \rho\big[\boldsymbol{F}\big(s, \bar{\boldsymbol{X}}(s), \bar{\boldsymbol{U}}(s)\big), \boldsymbol{F}\big(s, \boldsymbol{X}(s), \boldsymbol{U}(s)\big)\big]ds. \end{aligned}$$

Then, using (17), we estimate

$$\begin{aligned} m(t) &\le m(t_0) + \int_{t_0}^{t} g\big(s, \rho\big[\bar{\boldsymbol{X}}(s), \boldsymbol{X}(s)\big]\big)ds \\ &= m(t_0) + \int_{t_0}^{t} g\big(s, m(s)\big)ds, \quad t \in I. \end{aligned}$$

Applying Theorem 1.9.2 from Lakshmikantham and Leela (1969), we conclude that $m(t) \le r(t, t_0, w_0), t \in I$. The proof is completed. ■

In Example 2, if $\operatorname{diam}[\mathbf{U^{ac}}] \le L$, then condition (17) holds for the function $g(t, w) = a(t)w + b(t)L$. An immediate consequence of Theorem 10 is the following:

Corollary 3. *Under assumptions of Theorem 10, one has*

$$d^*\big[\bar{\boldsymbol{H}}_{t,\bar{\boldsymbol{U}}}, \boldsymbol{H}_{t,\boldsymbol{U}}\big] \le r(t, t_0, w_0), \ \textit{for all } t \in I \textit{ and } \bar{\boldsymbol{U}}(t), \boldsymbol{U}(t) \in \mathbf{U^{ac}}$$

provided $d^*\big[\bar{\boldsymbol{H}}_0, \boldsymbol{H}_0\big] \le w_0$.

The results in Theorem 10 and its corollary are similar to the ones for the fuzzy control differential equations from Phu and Tung (2006B) and set control differential equations Phu and Tung (2007), but in stochastic systems. Using differential inequalities, we can dispense with the monotone character of $g(t, w)$ assumed in Theorem 10.

Theorem 11. *Let the assumptions of Theorem 10 hold except for the property that $g(t, w)$ in w is nondecreasing. Then the conclusion (18) is valid.*

Proof. Set $m(t) = \rho[\bar{\boldsymbol{X}}(t), \boldsymbol{X}(t)]$, then

$$m(t + h) - m(t) = \rho[\bar{\boldsymbol{X}}(t + h), \boldsymbol{X}(t + h)] - \rho[\bar{\boldsymbol{X}}(t), \boldsymbol{X}(t)].$$

Using the property of metric ρ, we estimate

$$\begin{aligned}
\rho[\bar{\boldsymbol{X}}(t+h), \boldsymbol{X}(t+h)] \\
\leq\ & \rho[\bar{\boldsymbol{X}}(t+h), \bar{\boldsymbol{X}}(t) + h\boldsymbol{F}(t, \bar{\boldsymbol{X}}(t), \bar{\boldsymbol{U}}(t))] \\
& + \rho[\bar{\boldsymbol{X}}(t) + h\boldsymbol{F}(t, \bar{\boldsymbol{X}}(t), \bar{\boldsymbol{U}}(t)), \boldsymbol{X}(t+h)]; \\
\leq\ & \rho[\bar{\boldsymbol{X}}(t+h), \bar{\boldsymbol{X}}(t) + h\boldsymbol{F}(t, \bar{\boldsymbol{X}}(t), \bar{\boldsymbol{U}}(t))] \\
& + \rho\Big[\boldsymbol{X}(t) + h\boldsymbol{F}\Big(t, \boldsymbol{X}(t), \boldsymbol{U}(t)\Big), \boldsymbol{X}(t+h)\Big] \\
& + \rho\Big[\bar{\boldsymbol{X}}(t) + h\boldsymbol{F}\Big(t, \bar{\boldsymbol{X}}(t), \bar{\boldsymbol{U}}(t)\Big), \boldsymbol{X}(t) + h\boldsymbol{F}\Big(t, \boldsymbol{X}(t), , \boldsymbol{U}(t)\Big)\Big];
\end{aligned}$$

$$\begin{aligned}
\leq\ & \rho[\bar{\boldsymbol{X}}(t+h), \bar{\boldsymbol{X}}(t) + h\boldsymbol{F}(t, \bar{\boldsymbol{X}}(t), \bar{\boldsymbol{U}}(t))] \\
& + \rho\Big[\boldsymbol{X}(t) + h\boldsymbol{F}\Big(t, \boldsymbol{X}(t), \boldsymbol{U}(t)\Big), \boldsymbol{X}(t+h)\Big] \\
& + \rho\Big[\bar{\boldsymbol{X}}(t) + h\boldsymbol{F}\Big(t, \bar{\boldsymbol{X}}(t), \bar{\boldsymbol{U}}(t)\Big), \bar{\boldsymbol{X}}(t) + h\boldsymbol{F}\Big(t, \boldsymbol{X}(t), \boldsymbol{U}(t)\Big)\Big] \\
& + \rho\Big[\bar{\boldsymbol{X}}(t) + h\boldsymbol{F}(t, \boldsymbol{X}(t), \boldsymbol{U}(t)), \boldsymbol{X}(t) + h\boldsymbol{F}(t, \boldsymbol{X}(t), \boldsymbol{U}(t))\Big]; \\
\leq\ & \rho[\bar{\boldsymbol{X}}(t+h), \bar{\boldsymbol{X}}(t) + h\boldsymbol{F}(t, \bar{\boldsymbol{X}}(t), \bar{\boldsymbol{U}}(t))] \\
& + \rho\Big[\boldsymbol{X}(t) + h\boldsymbol{F}\Big(t, \boldsymbol{X}(t), \boldsymbol{U}(t)\Big), \boldsymbol{X}(t+h)\Big] \\
& + \rho\Big[h\boldsymbol{F}\Big(t, \bar{\boldsymbol{X}}(t), \bar{\boldsymbol{U}}(t)\Big), h\boldsymbol{F}\Big(t, \boldsymbol{X}(t), \boldsymbol{U}(t)\Big)\Big] \\
& + \rho\Big[\bar{\boldsymbol{X}}(t), \boldsymbol{X}(t)\Big].
\end{aligned}$$

Then, one has

$$\begin{aligned}
\rho[\bar{\boldsymbol{X}}(t+h), \boldsymbol{X}(t+h)] - \rho\Big[\bar{\boldsymbol{X}}(t), \boldsymbol{X}(t)\Big] \\
\leq\ & \rho[\bar{\boldsymbol{X}}(t+h), \bar{\boldsymbol{X}}(t) + h\boldsymbol{F}(t, \bar{\boldsymbol{X}}(t), \bar{\boldsymbol{U}}(t))] \\
& + \rho\Big[\boldsymbol{X}(t) + h\boldsymbol{F}\Big(t, \boldsymbol{X}(t), \boldsymbol{U}(t)\Big), \boldsymbol{X}(t+h)\Big] \\
& + \rho\Big[h\boldsymbol{F}\Big(t, \bar{\boldsymbol{X}}(t), \bar{\boldsymbol{U}}(t)\Big), h\boldsymbol{F}\Big(t, \boldsymbol{X}(t), \boldsymbol{U}(t)\Big)\Big].
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{m(t+h) - m(t)}{h} \leq\ & \frac{1}{h}\rho\Big[\bar{\boldsymbol{X}}(t+h), \bar{\boldsymbol{X}}(t) + h\boldsymbol{F}\Big(t, \bar{\boldsymbol{X}}(t), \bar{\boldsymbol{U}}(t)\Big)\Big] \\
& + \frac{1}{h}\rho\Big[\boldsymbol{X}(t) + h\boldsymbol{F}\Big(t, \boldsymbol{X}(t), \boldsymbol{U}(t)\Big), \boldsymbol{X}(t+h)\Big] \\
& + \frac{1}{h}\rho\Big[h\boldsymbol{F}\Big(t, \bar{\boldsymbol{X}}(t), \bar{\boldsymbol{U}}(t)\Big), h\boldsymbol{F}\Big(t, \boldsymbol{X}(t), \boldsymbol{U}(t)\Big)\Big].
\end{aligned}$$

Using the properties of metric ρ, we find that

$$\begin{aligned} D^{+}m(t) &= \limsup_{h\to 0^{+}} \frac{1}{h}\Big[m(t+h)-m(t)\Big]; \\ &\le \limsup_{h\to 0^{+}} \frac{1}{h}\rho\Big[\frac{\bar{\boldsymbol{X}}(t+h)-\bar{\boldsymbol{X}}(t)}{h}, \boldsymbol{F}(t,\bar{\boldsymbol{X}}(t),\bar{\boldsymbol{U}}(t))\Big] \\ &\quad + \limsup_{h\to 0^{+}} \rho\Big[\boldsymbol{F}(t,\boldsymbol{X}(t),\boldsymbol{U}(t)), \frac{\boldsymbol{X}(t+h)-\boldsymbol{X}(t)}{h}\Big] \\ &\quad + \rho\Big[\boldsymbol{F}(t,\bar{\boldsymbol{X}}(t),\bar{\boldsymbol{U}}(t)), \boldsymbol{F}(t,\boldsymbol{X}(t),\boldsymbol{U}(t))\Big], \end{aligned}$$

where $D^{+}m(t)$ is the Dini derivative of $m(t)$.
Because $\bar{\boldsymbol{X}}(t)$, $\boldsymbol{X}(t)$ are solutions of (1) and (17), $D^{+}m(t) \le g(t,m(t))$. By the results of Theorem 1.4.1 in Lakshmikantham and Leela (1969), the result (18) holds. The proof is completed. ■

It is easy to check the following corollary:

Corollary 4. *Under assumptions of Theorem 11, one has*

$$d^{*}\big[\bar{\boldsymbol{H}}_{t,\bar{\boldsymbol{U}}}\boldsymbol{H}_{t,\boldsymbol{U}}\big] \le r(t,t_0,w_0), \ \textit{for all } t\in I \textit{ and } \bar{\boldsymbol{U}}(t),\boldsymbol{U}(t)\in \mathbf{U}^{\mathbf{ac}}$$

provided $d^{*}\big[\bar{\boldsymbol{H}}_0,\boldsymbol{H}_0\big] \le w_0$.

Using the weaker assumptions than those of Theorems 10-11, we have the following theorem:

Theorem 12. *Assume that* $\boldsymbol{F}$ *is m.s. continuous and*

$$\limsup_{h\to 0^{+}} \frac{1}{h}\Big\{\rho\big[\bar{\boldsymbol{X}}+h\boldsymbol{F}(t,\bar{\boldsymbol{X}},\bar{\boldsymbol{U}}),\boldsymbol{X}+h\boldsymbol{F}(t,\boldsymbol{X},\boldsymbol{U})\big]-\rho[\bar{\boldsymbol{X}},\boldsymbol{X}]\Big\}\le g(t,\rho[\bar{\boldsymbol{X}},\boldsymbol{X}]), \tag{19}$$

where $t\in I; \bar{\boldsymbol{X}},\boldsymbol{X}\in L_2^m; \bar{\boldsymbol{U}},\boldsymbol{U}\in \mathbf{U}^{\mathbf{ac}}$ *and the maximal solution* $r(t,w_0)$ *of the scalar differential equation*

$$w' = g(t,w), w(t_0) = w_0 \ge 0$$

exists for $t\in I$, *where* $g\in \mathbb{C}[I\times\mathbb{R}_{+},\mathbb{R}_{+}]$. *Then the conclusion of Theorem 10 is valid.*

Proof. Starting as in the proof of Theorem 11, we arrive at

$$\begin{aligned} &m(t+h)-m(t) \\ &\quad = \rho\big[\bar{\boldsymbol{X}}(t+h),\ \boldsymbol{X}(t+h)\big] - \rho[\bar{\boldsymbol{X}}(t),\ \boldsymbol{X}(t)] \\ &\quad \le \rho\big[\bar{\boldsymbol{X}}(t+h),\ \bar{\boldsymbol{X}}(t)+h\boldsymbol{F}(t,\bar{\boldsymbol{X}}(t),\bar{\boldsymbol{U}}(t)\big] \\ &\qquad + \rho\left[\boldsymbol{X}(t)+h\boldsymbol{F}(t,\boldsymbol{X}(t),\boldsymbol{U}(t)),\boldsymbol{X}(t+h)\right] \\ &\qquad + \rho\big[\bar{\boldsymbol{X}}(t)+h\boldsymbol{F}(t,\bar{\boldsymbol{X}}(t),\bar{\boldsymbol{U}}(t)),\ \boldsymbol{X}(t)+h\boldsymbol{F}(t,\boldsymbol{X}(t),\boldsymbol{U}(t))\big] - \rho\big[\bar{\boldsymbol{X}}(t),\ \boldsymbol{X}(t)\big]. \end{aligned}$$

Then, we estimate

$$\begin{aligned}
D^{+}m(t) &= \limsup_{h\to 0^{+}} \frac{1}{h}\left[m(t+h)-m(t)\right] \\
&\le \limsup_{h\to 0^{+}} \frac{1}{h}\Big\{\rho\big[\bar{\boldsymbol{X}}(t)+h\boldsymbol{F}(t,\bar{\boldsymbol{X}}(t),\bar{\boldsymbol{U}}(t)) \\
&\quad -(\boldsymbol{X}(t)+h\boldsymbol{F}(t,\boldsymbol{X}(t),\boldsymbol{U}(t)))\big]-\rho[\bar{\boldsymbol{X}}(t),\boldsymbol{X}(t)]\Big\} \\
&\quad +\limsup_{h\to 0^{+}}\rho\Big[\frac{\bar{\boldsymbol{X}}(t+h)-\bar{\boldsymbol{X}}(t)}{h},\ \boldsymbol{F}(t,\bar{\boldsymbol{X}}(t),\bar{\boldsymbol{U}}(t))\Big] \\
&\quad +\limsup_{h\to 0^{+}}\rho\Big[\boldsymbol{F}(t,\boldsymbol{X}(t),\boldsymbol{U}(t)),\ \frac{\boldsymbol{X}(t+h)-\boldsymbol{X}(t)}{h}\Big] \\
&\le g(t,\rho[\bar{\boldsymbol{X}}(t),\ \boldsymbol{X}(t)]) = g(t,m(t)), t\in I.
\end{aligned}$$

Because $\bar{\boldsymbol{X}}(t)$, $\boldsymbol{X}(t)$ are solutions of (1), using (19) and using Theorem 1.4.1 in Lakshmikantham and Leela (1969), we obtain (18). The proof is completed. ■

The result in Theorem 12 of FCSDS is similar the one in Theorem 2.2.3 from Lakshmikantham, Bhaskar and Devi (2006) of the set differential equation, but in stochastic systems.

COROLLARY 5. *Under assumptions of Theorem 12, one has*

$$d^{*}\big[\bar{\boldsymbol{H}}_{t,\bar{\boldsymbol{U}}},\boldsymbol{H}_{t,\boldsymbol{U}}\big] \le r(t,t_0,w_0),\ \textit{for all } t\in I \textit{ and } \bar{\boldsymbol{U}}(t),\boldsymbol{U}(t)\in \mathbf{U}^{\mathbf{ac}}$$

provided $d^{*}\big[\bar{\boldsymbol{H}}_0,\boldsymbol{H}_0\big]\le w_0$.

We suppose that the motions of flying-objects for instance, airplanes, missiles..., can be described by FCSDS (1) and $\bar{\boldsymbol{H}}_{t,\bar{\boldsymbol{U}}},\boldsymbol{H}_{t,\boldsymbol{U}}$ are sets of positions of flyingobjects at the time t for $\bar{\boldsymbol{U}},\boldsymbol{U}$, respectively. One can estimate the sets of positions of flyingobjects at the time t and compare the set of positions of flyingobjects at time t $\bar{\boldsymbol{H}}_{t,\bar{\boldsymbol{U}}}$ to the counterpart $\boldsymbol{H}_{t,\boldsymbol{U}}$. In practice, these estimations are useful and can be applied.

A function $\boldsymbol{Y}_{\epsilon}(t)=\boldsymbol{Y}(t,t_0,\boldsymbol{Y}_0,\boldsymbol{U}(t),\epsilon), \epsilon>0$, is said to be an ϵ-*approximate solution* of (1) if $\boldsymbol{Y}_{\epsilon}\in C^1\big[I,L_2^m\big], \boldsymbol{Y}_{\epsilon}(t_0)=\boldsymbol{Y}_0$ and $\rho\big[\boldsymbol{Y}'_{\epsilon}(t),\boldsymbol{F}(t,\boldsymbol{Y}_{\epsilon}(t),\boldsymbol{U}(t))\big]\le \epsilon,\ t\ge t_0$.
In the case $\epsilon=0, \boldsymbol{Y}_{\epsilon=0}(t)$ is a solution of (1).

In the following Theorems 13-14, we compare solutions and ϵ- approximate solution of (1) with the same control. The proof of Theorems 13-14 are similar to the ones of Theorems 11-12. So we omit the details.

THEOREM 13. *Assume that* $\boldsymbol{F}$ *is m.s. continuous and*

$$\rho\big[\boldsymbol{F}(t,\boldsymbol{X},\boldsymbol{U}),\boldsymbol{F}(t,\boldsymbol{Y},\boldsymbol{U})\big]\le g(t,\rho[\boldsymbol{X},\boldsymbol{Y}]), \tag{20}$$

for $(t,\boldsymbol{X},\boldsymbol{U}),\ (t,\boldsymbol{Y},\boldsymbol{U})\in I\times L_2^m\times L_2^p$ *where* $g\in C\big[\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R}_+\big]$. *Suppose further that the maximal solution* $r(t)=r(t,t_0,w_0,\epsilon)$ *of the scalar differential* *equation*

$$w' = g(t, w) + \epsilon, w(t_0) = w_0 \geq 0$$

exists for $t \in I$.

Then, if $\boldsymbol{X}(t) = \boldsymbol{X}(t, t_0, \boldsymbol{X}_0, \boldsymbol{U}(t))$ *is a solution of (1) such that* $\boldsymbol{X}(t_0) = \boldsymbol{X}_0$ *and* $\boldsymbol{Y}_\epsilon(t) = \boldsymbol{Y}(t, t_0, \boldsymbol{Y}_0, \boldsymbol{U}(t), \epsilon)$ *is an* ϵ*-approximate solution of (1) such that* $\boldsymbol{Y}_\epsilon(t_0) = \boldsymbol{Y}_0$; $\boldsymbol{X}_0, \boldsymbol{Y}_0 \in L_2^m$ *existing for* $t \in I$*, one has*

$$\rho\big[\boldsymbol{X}(t), \boldsymbol{Y}_\epsilon(t)\big] \leq r(t, t_0, w_0, \epsilon), \ \textit{for all } t \in I \textit{ and } \boldsymbol{U}(t) \in \mathbf{U}^{\mathbf{ac}} \tag{21}$$

provided $\rho\big[\boldsymbol{X}_0, \boldsymbol{Y}_0\big] \leq w_0$.

Let $\boldsymbol{H}^{\boldsymbol{\epsilon}}{}_{t,\boldsymbol{U}} = \{\boldsymbol{Y}_\epsilon(t) = \boldsymbol{Y}(t, \boldsymbol{Y}_0, \boldsymbol{U}(t), \epsilon)$ is an $\epsilon-$approximate solution of (1) $|$ $\boldsymbol{Y}_0 \in \bar{\boldsymbol{H}}_0\}$ be the cross-area of the sheaf-solution of ϵ-approximate solutions $\boldsymbol{Y}_\epsilon(t)$. Then one has the following corollary.

Corollary 6. *Under assumptions of Theorem 11, one has*

$$d^*\big[\boldsymbol{H}^{\boldsymbol{\epsilon}}{}_{t,\boldsymbol{U}}, \boldsymbol{H}_{t,\boldsymbol{U}}\big] \leq r(t, t_0, w_0, \epsilon), \ \textit{for all } t \in I \textit{ and } \boldsymbol{U}(t) \in \mathbf{U}^{\mathbf{ac}}$$

provided $d^*\big[\bar{\boldsymbol{H}}_0, \boldsymbol{H}_0\big] \leq w_0$.

Using the weaker assumptions than those of Theorem 13, one has the following

Theorem 14. *Assume that* $\boldsymbol{F}$ *is m.s. continuous and*

$$\limsup_{h\to 0^+} \frac{1}{h} \Big\{ \ \rho\left[\boldsymbol{X} + h\boldsymbol{F}(t, \boldsymbol{X}, \boldsymbol{U}), \boldsymbol{Y} + h\boldsymbol{F}(t, \boldsymbol{Y}, \boldsymbol{U})\right] - \rho[\boldsymbol{X}, \boldsymbol{Y}] \ \Big\} \leq g(t, \rho[\boldsymbol{X}, \boldsymbol{Y}]), \tag{22}$$

for $(t, \boldsymbol{X}, \boldsymbol{U})$, $(t, \boldsymbol{Y}, \boldsymbol{U}) \in I \times L_2^m \times L_2^p$ *where* $g \in C\big[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+\big]$*. Suppose further that the maximal solution* $r(t) = r(t, t_0, w_0, \epsilon)$ *of the scalar differential equation*

$$w' = g(t, w) + \epsilon, w(t_0) = w_0 \geq 0$$

exists for $t \in I$.

Then, if $\boldsymbol{X}(t) = \boldsymbol{X}(t, t_0, \boldsymbol{X}_0, \boldsymbol{U}(t))$ *is a solution of (1) such that* $\boldsymbol{X}(t_0) = \boldsymbol{X}_0$ *and* $\boldsymbol{Y}_\epsilon(t) = \boldsymbol{Y}(t, t_0, \boldsymbol{Y}_0, \boldsymbol{U}(t), \epsilon)$ *is an* ϵ*-approximate solution of (1) such that* $\boldsymbol{Y}_\epsilon(t_0) = \boldsymbol{Y}_0$; $\boldsymbol{X}_0, \boldsymbol{Y}_0 \in L_2^m$ *existing for* $t \in I$*, one has*

$$\rho\big[\boldsymbol{X}(t), \boldsymbol{Y}_\epsilon(t)\big] \leq r(t, t_0, w_0, \epsilon), \ \textit{for all } t \in I \textit{ and } \boldsymbol{U}(t) \in \mathbf{U}^{\mathbf{ac}}, \tag{23}$$

provided $\rho\big[\boldsymbol{X}_0, \boldsymbol{Y}_0\big] \leq w_0$.

A simple example illustrating condition (22) of Theorem 14 is the following:

Example 4. Consider the systems in Example 2. It is easy to compute

$$\limsup_{h\to 0^+} \frac{1}{h} \left\{ \rho\left[\boldsymbol{X} + h\boldsymbol{F}(t,\boldsymbol{X},\boldsymbol{U}), \boldsymbol{Y} + h\boldsymbol{F}(t,\boldsymbol{Y},\boldsymbol{U})\right] - \rho[\boldsymbol{X},\boldsymbol{Y}] \right\} \le a(t)\rho\big[\boldsymbol{X},\boldsymbol{Y}\big].$$

Then, condition (22) of Theorem 14 holds for $g(t,w) = a(t)w$.

The direct consequence of Theorem 14 is the following:

Corollary 7. *Under assumptions of Theorem 14, one has*

$$d^*\big[\boldsymbol{H}^{\epsilon}{}_{t,\boldsymbol{U}}, \boldsymbol{H}_{t,\boldsymbol{U}}\big] \le r(t,t_0,w_0,\epsilon),\ \textit{for all } t \in I \textit{ and } \boldsymbol{U}(t) \in \mathbf{U}^{\mathbf{ac}}$$

provided $d^*\big[\bar{\boldsymbol{H}}_0, \boldsymbol{H}_0\big] \le w_0$.

5. Conclusion

(i) The paper introduces FCSDS, which are a combination of fuzziness and randomness. A simple result on existence of solutions is given. The continuous dependence of solutions on initials and controls are studied and similar properties of approximate solutions and sheaf-solutions are investigated. Some simple examples are given to illustrate the results.
(ii) It is difficult to show existence for the Hukuhara difference of two fuzzy sets, the Hukuhara derivative is restrictive. We shall discuss some properties of FCSDS with other definitions of derivatives, for instance, definition of derivative based on support-functions in future studies.

Both of fuzzy and stochastic theories have their merits and defects. Their study carries with it great promise.

6. Acknowledgements

I am grateful to the Reviewers for careful reading, valuable remarks and suggestions which improved the presentation of the paper.

This work was supported by National Foundation for Science and Technology Development (NAFOSTED), Project 101.01-2010.07, Viet Nam.

References

Diamond, P. and Kloeden, P. (1994) *Metric Spaces of Fuzzy Sets*. World Scientific, Singapore.

Fei, W. Y. (2007) Existence and uniqueness of solution for fuzzy random differential equations with non-Lipschitz coefficients. *Information Sciences* **177**, 4329-4337.

Fei, W. Y. (2013) Existence and uniqueness for solution to fuzzy stochastic differential equations driven by local martingales under the non-Lipschitzian conditions. *Nonlinear Analysis TMA* **76**, 202-214.

Feng, Y. H. (1999A) Mean square integral and differential of fuzzy stochastic processes. *Fuzzy Sets and Systems* **102**, 271-280.

Feng, Y. H. (1999b) Convergence theorems for fuzzy random variables and fuzzy martingales. *Fuzzy Sets and Systems* **103**, 435-441.

Feng, Y. H. (2000a) Mean-square Riemann-Stieltjes integrals of fuzzy stochastic processes and their applications. *Fuzzy Sets and Systems* **110**, 27-41.

Feng, Y. H. (2000b) Fuzzy stochastic differential systems. *Fuzzy Sets and Systems* **115**, 351-363.

Feng, Y. H. (2001) Sum of independent fuzzy random variables. *Fuzzy Sets and Systems* **123**, 11-18.

Feng, Y. H. (2003) The solutions of linear fuzzy stochastic differential systems. *Fuzzy Sets and Systems* **140**, 541-554.

Kim, J. H. (2005) On fuzzy stochastic differential equations. *Journal of Korean Mathematical Society* **42** (1), 153-169.

Lakshmikantham, V. (2000) Uncertain systems and fuzzy differential equations. *Journal Mathematical Analysis and Applications* **251**, 805-817.

Lakshmikantham, V. (2005) Set Differential Equations Versus Fuzzy Differential Equations. *Applied Mathematics and Computation* **164**, 277-294.

Lakshmikantham, V., Bhaskar, T. G. and Devi, J. V. (2006) *Theory of Set Differential Equations in Metric Spaces*. Cambridge Scientific Publishers, UK.

Lakshmikantham, V. and Leela, S. (1969) *Differential and Integral Inequalities*, Vol. I. Academic Press, New York.

Lakshmikantham, V. and Mohapatra, R. (2003) *Theory of Fuzzy Differential Equations and Inclusions*. Taylor and Francis, London.

Malinowski, M. T. and Michta, M. (2011) Stochastic fuzzy differential equations with an application. *Kybernetika* **47**, 123-143.

Malinowski, M. T. and Michta, M. (2010) Fuzzy stochastic integral equations. *Dynamic Systems and Applications* **19**, 473-494.

Michta, M. (2011) On set-valued stochastic integrals and fuzzy stochastic equations. *Fuzzy Sets and Systems* **177**, 1-19.

Ovsanikov, D. A. (1980) *Mathematical Methods for Sheaf-Control*. Publisher of Leningrad University, Leningrad.

Phu, N. D., Quang, L. T. and Tung, T. T. (2008) Stability criteria for set control differential equations. *Nonlinear Analysis TMA* **69**, 3715-3721.

Phu, N. D. and Tung, T. T. (2005) Sheaf-optimal control problem in fuzzy type. *Journal of Science and Technology Development* **8** (12), 5-11.

Phu, N. D. and Tung, T. T. (2006a) The comparison of sheaf-solutions in fuzzy control problems. *Journal of Science and Technology Development* **9** (2), 5-10.

Phu, N. D. and Tung, T. T. (2006b) Some properties of sheaf-solutions of sheaf fuzzy control problems. *Electronic Journal of Differential Equations* **108**, 1-8.

Phu, N. D. and Tung, T. T. (2007) Some results on sheaf-solutions of sheaf set control problems. *Nonlinear Analysis TMA* **67**, 1309-1315.

Puri, M.L. and Ralescu, D.A. (1986) Fuzzy random variables. *Journal Mathematical Analysis and Applications* **114**, 409-422.

Tolstonogov, A. (2000) *Differential Inclusions in a Banach Space*. Kluwer Academic Publishers, Dordrecht.