Stabilization of linear systems in random horizon via control[*]

by

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Abstract: The control problem with random horizon at finite number of events is investigated in this paper, where the general aim of control is the stabilization (in mean square sense) of linear system at minimum cost. This problem is reduced to the task of optimal control with established finite horizon. Moreover, the differences between stabilization with fixed and random horizons are also given. To illustrate those differences a numerical example is included.

Keywords: optimal control, linear quadratic control, random horizon, stability

1. Introduction

The problem of optimal control for linear and non–linear systems has been given a lot of much attention for a long time (see e.g., Aoki, 1967; Bellman, 1961; Feldbaum, 196); Fleming and Rishel, 1975; Zabczyk, 1996). For discrete time systems, many problems (e.g. adaptive control, approximation, quadratic control, stabilization, identification, active learning etc.) have been studied by researchers during the past few years (see e.g. Banek and Kozłowski, 2005A, 2006; Bubnicki, 2000; Dong and Mei, 2009; Fleming and Rishel 1975; Harris and Rishel, 1986; Kozin, 1972; Liptser and Shiryaev, 1978; Rishel, 1985; Runggaldier, 1998; Saridis, 1995). Each of the above tasks consists in optimization of performance criterion for fixed finite or infinite time interval. A lot of results have been presented for the established horizon, with solutions of the above tasks obtained using the iterative technique (see, e.g., Bubnicki, 2000; Xu, 2011) or dynamic programming with approximation (see, e.g., Aoki, 1967; Banek and Kozłowski, 2006; Bellman, 1961; Chena, Edgarb and Manousiouthakisa, 2004; Fleming and Rishel, 1975; Feldbaum, 1965; Zabczyk, 1996). The question arises how to design the controller (control law) when the horizon of control is unknown or random?

The problems with random horizon are less noticeable in the literature of the subject and they can be divided into two types. The first type is based on classical optimal stopping of decision process (optimal stopping control, see Benht and Reikvam, 2004; Boetius and Kohlmann, 1998; Karatzas and Shreve, 1984, 1985; Shiryaev, 1978), which is modeled by a Snell envelope construction and stopping moment definition. We could also solve this problem by changing stopping into control, which is designed by an auxiliary task (see Banek and Kozłowski, 2005B). The second type is based on the assumption that the horizon does not depend on the behavior of the system (the system state does not influence the horizon). We obtain such a situation when we define the control horizon as, e.g., the number of losses or number of requests. In such a case we design the horizon by a random variable independent of state. The solving is based on the construction of a substitute task, where the functionals of losses and heredities are modified.

This paper presents a general stabilization problem of linear system, which is controlled. This stabilization must be realized in a random horizon which is independent of system state. The horizon of control is modeled by a random variable. The problem mentioned is replaced by an auxiliary task of control with finite horizon. In both cases the aim of control is the same but has different forms. The above problem is adapted to linear systems.

The organization of article is as follows. Section 2 introduces the Lyapunov stability. Section 3 presents task of optimal control with random horizon. Optimal controls of linear systems with quadratic criterion for deterministic and random horizons are given in Sections 4 and 5, respectively. Basic results (optimal control for stabilization of linear systems) and an illustrative example are provided in Section 6.

2. Stability

The common stability properties of stochastic systems that have been studied in the literature have generally been related to Lyapunov stability (see e.g. Bolzern, Colaneri and Nikolao, 2008; Hoagg and Bernstein, 2007; Kozin, 1972). Recognizing that stability in the Lyapunov sense is merely a uniform convergence with respect to initial conditions, the various concepts of stability for stochastic systems can be immediately defined by invoking one of usual models of convergence of probability theory. This means convergence in probability, convergence in the mean and almost sure convergence. Let $y_t \in R^n$ denote the system state at time t (with initial state $y_0 \in R^n$ at time 0) and $\|y\|$ denote a norm, such as an absolute value or quadratic norm.

DEFINITION 1 (Lyapunov Stability). *The system is stable if for given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for $\|y_0\| < \delta$, there is*

$$\sup_{t \geq t_0} \|y_t\| < \varepsilon.$$

Here, t_0 is called stabilization time. For a deterministic system we can calculate this time exactly.

DEFINITION 2 (Asymptotic Lyapunov Stability). *The system is asymptotically stable, if it is stable and if there exists $\delta > 0$ such that for $\|y_0\| < \delta$ there is*

$$\lim_{t \to \infty} \|y_t\| = 0.$$

DEFINITION 3. *The system is BIBO (Bounded-Input Bounded-Output) stable if to any bounded input corresponds a bounded output.*

It should be noted that BIBO stability is a "weak" stability. It is sufficient to consider the system, which for unit input signal (stroke, jump) responds with a sinusoidal signal. Of course, this system is BIBO stable, but in Lyapunov sense it is not. The stability, asymptotic stability, time stabilization can be extended to stochastic case in a similar way. By analogy we define

DEFINITION 4. *A stochastic system is stable if for given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for $\|y_0\| < \delta$, it follows that*

$$\sup_{t \geq \tau} E \|y_t\| < \varepsilon.$$

For a stochastic system the stabilization time τ depends on internal dynamics, disturbances, knowledge of system parameters. Thus, we see that stabilization time is a random variable.

DEFINITION 5 (Mean Square Stability). *The system is mean square stable (MSS) if*

$$\lim_{t \to \infty} E \|y_t\|^2 = 0$$

for any $y_0 \in R^n$ and $\|y_0\| < \infty$.

3. Problem formulation

The stabilization problem of linear or non-linear systems has attracted the interest of an increasing number of authors in the last years (see e.g. Abouzaid, Achhab and Wertz, 2011; Bolzern, Colaneri and Nikolao, 2008; Dong and Mei, 2009; Hoagg and Bernstein, 2007; Phat and Nam, 2007; Tian and Xie, 2007). The Lyapunov stability suffices for practical applications, and so there are a lot of results for this stability. However, in some practical applications the behavior of the system over fixed finite or random time intervals is important (see e.g. Kozłowski, 2010, 2011; Liu and Sun, 2007).

Let us consider the stability problem of a linear system at random horizon. In this task stabilization means MSS, and additionally we must stabilize the linear system at minimum cost. The objective function determines total costs, i.e. the sum of control costs and costs associated with instability of the system.

This total cost is called composite costs function (CCF). Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Suppose that $w_1, w_2, ...$ are independent n-dimensional random vectors on this space, with normal $N(0, I_n)$ distribution, and let τ be a random horizon with the same discrete distribution $P(d\tau)$. We assume that all the above mentioned objects are stochastically independent and an initial state $\|y_0\| < \infty$.

We will consider the stabilization problem via control for a stochastic linear system with state equation

$$y_{i+1} = Ay_i + B\xi - Cu_i + \sigma w_{i+1} \tag{1}$$

where $i = 0, ..., N - 1$, $y_i \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{n \times l}$ and $\sigma \in \mathbb{R}^{n \times n}$. On $(\Omega, \mathcal{F}, P)$ we define a family of sub-σ-fields $F_i = \sigma\{y_i : i = 0, 1, ..., j\}$. Below we assume that the parameters of linear system $\xi \in \mathbb{R}^k$ are known, $\|A\| < \infty$, $\|B\|\|\xi\| < \infty$, $\|C\| < \infty$, $\|\sigma\| < \infty$, where $\|\cdot\|$ for matrices A, B, C, σ denotes a matrix norm $\|A\| = \max_{\|x\| \leq 1} \|Ax\|$ (system (1) is BIBO stable). A $\mathbb{Y}_j$-measurable vector $u_j \in \mathbb{R}^l$ will be called a control action, and $u = (u_0, u_1, ...)$ an admissible control. The class of admissible controls is denoted by U. To specify the aim of control, we introduce a cost of control at time i as $u_i^T R u_i$ and a heredity function $y_\tau^T Q y_\tau$ as losses associated with system instability. The random variable τ represents the horizon of control and has discrete distribution $P(\tau = i) = p_i$, where $0 \leq p_i \leq 1$ for $i \geq 0$ and $\sum_{i=0}^{N} p_i = 1$. We assume that the system (1) is controlled until random time τ and after this time the system remains in the terminal state ($y_t = y_\tau$ for $t \geq \tau$). We put $u_{-1}^T R u_{-1} = 0$ or $u_{-1} = col(0, 0, ..., 0)$ and define the objective function as

$$J(u) = E\left[\sum_{i=-1}^{\tau-1} u_i^T R u_i + y_\tau^T Q y_\tau\right]. \tag{2}$$

At any time $0 \leq j < \tau$, which is not equal to horizon of control, we take control u_j, and at time τ we do not take control but only calculate the value of the heredity function. If the linear system (1) can be stopped at time $\tau = 0$, then we calculate only the value of heredity function.

The main aim of optimal control is stabilization of system (1) at lowest cost, which is the sum of control costs and stability losses. Then, the task is to find

$$\inf_{u \in U} J(u) \tag{3}$$

and to determine a sequence of admissible control $u^* = (u_0^*, ..., u_{\tau-1}^*)$ for which the infimum is attained.

REMARK 1. *For $Q = I$ (identity matrix) we have the heredity functional as a value of MSS.*

REMARK 2. *Without loss of generality the above mentioned problem can be extended to the case when the vector ξ is unknown. This problem is called adaptive optimal control problem and the design of optimal control must take into account also the estimation of unknown parameters of linear system (1) (see Liptser and Shiryaev, 1978).*

4. Linear quadratic control

Let the linear system be described by state equation (1) and the quadratic criterion with deterministic horizon N be

$$\inf_{u \in U} E \left(\sum_{i=0}^{N-1} \left[u_i^T R_i u_i + y_i^T Q_i y_i \right] + y_N^T Q_i y_N \right). \tag{4}$$

The theorem below presents the optimal control of system (1) and the value of the composite costs function. The solution of task (4) for linear system (1) where $Q_i = 0$ and $B = 0$ (matrix of zeros) can be found in Kozłowski (2010).

THEOREM 1. *Let*

$$K_i = Q_i + A^T \left(K_{i+1} - K_{i+1}^T C \left(R_i + C^T K_{i+1} C \right)^{-1} C^T K_{i+1} \right) A \tag{5}$$

$$L_i = A^T$$
$$\left(L_{i+1} + 2K_{i+1} B\xi - K_{i+1}^T C \left(R_i + C^T K_{i+1} C \right)^{-1} C^T \left(2K_{i+1} B\xi + L_{i+1} \right) \right) \tag{6}$$

$$M_i = \xi^T B^T K_{i+1} B\xi + tr \left(\sigma^T K_{i+1} \sigma \right) + \xi^T B^T L_{i+1} + M_{i+1}$$
$$- \frac{1}{4} \left(2K_{i+1} B\xi + L_{i+1} \right)^T C \left(R_i + C^T K_{i+1} C \right)^{-1} C^T \left(2K_{i+1} B\xi + L_{i+1} \right) \tag{7}$$

for $i = 0, 1, ..., N-1$ and $K_N = Q_N$, $L_N = 0$, $M_N = 0$. If $det \left(R_i + C^T K_{i+1} C \right) \neq 0$, then the optimal control is

$$u_i^* = \frac{1}{2} \left(R_i + C^T K_{i+1} C \right)^{-1} C^T \left(2K_{i+1} \left(Ay_i + B\xi \right) + L_{i+1} \right) \tag{8}$$

and

$$\inf_{u \in U} E \left\{ \sum_{i=0}^{N-1} \left[u_i^T R_i u_i + y_i^T Q_i y_i \right] + y_N^T Q_N y_N \right\} = W_0 \left(y_0 \right)$$

where

$$W_i \left(y_i \right) = y_i^T K_i y_i + y_i^T L_i + M_i. \tag{9}$$

Proof. First we define the Bellman functions which are non-negative (they are defined as a sum of quadratic forms). For the time N we have

$$W_N(y_N) = y_N^T Q_N y_N \tag{10}$$

and

$$W_i(y_i) = \min_{u_i} E\left\{ u_i^T R_i u_i + y_i^T Q_i y_i + W_{i+1}(y_{i+1}) \middle| F_i \right\} \tag{11}$$

for $j = 0, 1, ..., N - 1$. The value of the Bellman function in step N is given by (10) and in step $N - 1$ we have

$$W_{N-1}(y_{N-1}) =$$
$$\min_{u_{N-1}} E\left\{ u_{N-1}^T R_{N-1} u_{N-1} + y_{N-1}^T Q_{N-1} y_{N-1} + W_N(y_N) \middle| F_{N-1} \right\}$$
$$= \min_{u_{N-1}} \left\{ u_{N-1}^T \left(R_{N-1} + C^T Q_N C \right) u_{N-1} - 2 u_{N-1}^T C^T Q_N \left(A y_{N-1} + B\xi \right) \right.$$
$$\left. + \left(A y_{N-1} + B\xi \right)^T Q_N \left(A y_{N-1} + B\xi \right) + y_{N-1}^T Q_{N-1} y_{N-1} + tr\left(\sigma^T Q_N \sigma \right) \right\}$$

Thus, the optimal control is

$$u_{N-1}^* = \left(R_{N-1} + C^T Q_N C \right)^{-1} C^T Q_N \left(A y_{N-1} + B\xi \right)$$

and

$$W_{N-1}(y_{N-1}) =$$
$$- \left(A y_{N-1} + B\xi \right) Q_N^T C \left(R_{N-1} + C^T Q_N C \right)^{-1} C^T Q_N \left(A y_{N-1} + B\xi \right)$$
$$+ y_{N-1}^T Q_{N-1} y_{N-1} + \left(A y_{N-1} + B\xi \right)^T Q_N \left(A y_{N-1} + B\xi \right) + tr\left(\sigma^T Q_N \sigma \right)$$
$$= y_{N-1}^T \left(Q_{N-1} + A^T Q_N A - A^T Q_N^T C \left(R_{N-1} + C^T Q_N C \right)^{-1} C^T Q_N A \right) y_{N-1}$$
$$+ 2 y_{N-1}^T \left(A^T Q_N - A^T Q_N^T C \left(R_{N-1} + C^T Q_N C \right)^{-1} C^T Q_N \right) B\xi$$
$$+ tr\left(\sigma^T Q_N \sigma \right) + \xi^T B^T \left(Q_N - Q_N^T C \left(R_{N-1} + C^T Q_N C \right)^{-1} C^T Q_N \right) B\xi$$
$$= y_{N-1}^T K_{N-1} y_{N-1} + y_{N-1}^T L_{N-1} + M_{N-1}.$$

We assume that equations (5)–(7) and (9) are true for $i + 1$. From (11) and using the properties of conditional expectation we have

$$W_i(y_i) = \min_{u_i} E\left\{ u_i^T R_i u_i + y_i^T Q_i y_i + W_{i+1}(y_{i+1}) \middle| F_i \right\}$$
$$= \min_{u_i} \left\{ u_i^T \left(R_i + C^T K_{i+1} C \right) u_i - u_i^T C^T \left(2 K_{i+1} \left(A y_i + B\xi \right) + L_{i+1} \right) + y_i^T Q_i y_i \right.$$
$$\left. + \left(A y_i + B\xi \right)^T K_{i+1} \left(A y_i + B\xi \right) + \left(A y_i + B\xi \right)^T L_{i+1} + tr\left(\sigma^T K_{i+1} \sigma \right) + M_{i+1} \right\}.$$

Thus, the optimal control is

$$u_i^* = \frac{1}{2} \left(R_i + C^T K_{i+1} C \right)^{-1} C^T \left(2K_{i+1} \left(A y_i + B\xi \right) + L_{i+1} \right)$$

and finally

$$W_i \left(y_i \right) =$$

$$y_i^T \left(Q_i + A^T K_{i+1} A - A^T K_{i+1}^T C \left(R_i + C^T K_{i+1} C \right)^{-1} C^T K_{i+1} A \right) y_i$$

$$+ y_i^T \left(A^T L_{i+1} + 2 A^T K_{i+1} B\xi \right) -$$

$$y_i^T \left(A^T K_{i+1}^T C \left(R_i + C^T K_{i+1} C \right)^{-1} C^T \left(2K_{i+1} B\xi + L_{i+1} \right) \right)$$

$$+ \xi^T B^T K_{i+1} B\xi + tr \left(\sigma^T K_{i+1} \sigma \right) + \xi^T B^T L_{i+1} + M_{i+1}$$

$$- \frac{1}{4} \left(2K_{i+1} B\xi + L_{i+1} \right)^T C \left(R_i + C^T K_{i+1} C \right)^{-1} C^T \left(2K_{i+1} B\xi + L_{i+1} \right)$$

$$= y_i^T K_i y_i + y_i^T L_i + M_i$$

what finishes the proof. ∎

5. Linear quadratic control problem with random horizon at finite number of events

This section presents a transformation of the LQC problem (2) - (3) with random horizon to a task with deterministic horizon. The obtained task is still the LQC problem but has a modified objective function. Using the definitions of conditional probability and conditional expectation the composite costs functional (2) can be presented as

$$J \left(u \right) = E \left[\sum_{i=-1}^{\tau-1} u_i^T R u_i + y_\tau^T Q y_\tau \right] = E \left(y_0^T Q y_0 \right) P \left(\tau = 0 \right) +$$

$$+ E \left(u_0^T R u_0 + y_1^T Q y_1 \right) P \left(\tau = 1 \right) + ... +$$

$$+ E \left(\sum_{i=0}^{N-1} u_i^T R u_i + y_N^T Q y_N \right) P \left(\tau = N \right)$$

$$= E \left[\sum_{i=0}^{N-1} u_i^T R u_i \sum_{j=i+1}^{N} P \left(\tau = j \right) + \sum_{i=0}^{N} y_i^T Q y_i P \left(\tau = i \right) \right]$$

$$= E \left[\sum_{i=0}^{N-1} u_i^T R u_i P \left(\tau > i \right) + \sum_{i=0}^{N} y_i^T Q y_i P \left(\tau = i \right) \right].$$

Finally, the above mentioned functional can be presented as

$$J \left(u \right) = E \sum_{j=0}^{N} \left(u_j^T R_j u_j + y_j^T Q_j y_j \right) \tag{12}$$

where

$$R_j = P\left(\tau > j\right)R \quad \text{and} \quad Q_j = P\left(\tau = j\right)Q \tag{13}$$

for $j = 0, 1, ..., N$. From distribution of random horizon τ we see that $P\left(\tau > N\right) = 0$ thus $R_N = [0]$ (matrix of zeroes). Therefore, we replace the task of optimal control with random horizon at finite number of elementary events (3) by the task of optimal control with finite horizon

$$\inf_{u \in U} E\left(\sum_{j=0}^{N-1}\left(u_j^T R_j u_j + y_j^T Q_j y_j\right) + y_N^T Q_N y_N\right). \tag{14}$$

The CCF value is the same but the design of optimal control for a task with established horizon is easier. Below we consider the auxiliary (replacement) task (14) to design the optimal control of system (1) with random horizon τ. From Theorem 1 we have

COROLLARY 1. *If* $\det\left(R_i + C^T K_{i+1} C\right) \neq 0$ *, then the optimal control of system (1) is*

$$u_i^* = \frac{1}{2}\left(R_i + C^T K_{i+1} C\right)^{-1} C^T\left(2K_{i+1}\left(Ay_i + B\xi\right) + L_{i+1}\right)$$

where $K_N = Q_N$, $L_N = 0$, $M_N = 0$ *and* K_i, L_i, M_i *for* $i = 0, 1, ..., N-1$ *are given by (5)–(7), respectively, but* R_i, Q_i *are given by (13).*

6. Stabilization of a linear system

Let us consider the case, where the BIBO system (1) functions in random horizon τ and the maim aim is stabilization. Thus, this system may be controlled only at times $0, 1, ..., \tau - 1$ and must be stabilized at lowest cost. The cost of control (energetic cost) in each step has a quadratic form $u_i^T R u_i$ but from the definition of stabilization (quadratic Lyapunov function) the value $\|y_\tau\|^2$ must be minimized. In case where horizon of control τ is random, the task

$$\inf_{u \in U} E\left\{\sum_{i=0}^{\tau-1} u_i^T R u_i + \|y_\tau\|^2\right\} \tag{15}$$

is replaced by the one with deterministic horizon

$$\inf_{u \in U} E\left(\sum_{j=0}^{N-1}\left(P\left(\tau > j\right) u_j^T R u_j + P\left(\tau = j\right)\|y_j\|^2\right) + P\left(\tau = N\right)\|y_N\|^2\right) \tag{16}$$

where $P\left(\tau = i\right) = p_i$, $0 \leq p_i \leq 1$ for $i \geq 0$ and $\sum_{i=0}^{N} p_i = 1$.

THEOREM 2. *Let* $R_j = P(\tau > j) R$, $Q_j = P(\tau = j) I$ *where* I *is an identity matrix, the matrices* K_i, L_i, M_i *for* $i = 0, 1, ..., N-1$ *are given by (5) - (7) and* $K_N = P(\tau = N) I$, $L_N = 0$, $M_N = 0$. *If* $\|A\| < \infty$, $\|B\xi\| < \infty$, $\|C\| < \infty$, $tr(\sigma^T \sigma) < \infty$ *and* $det(R_i + C^T K_{i+1} C) \neq 0$, *then the optimal control to stabilize system (1) is*

$$u_i^* = \frac{1}{2}(R_i + C^T K_{i+1} C)^{-1} C^T (2K_{i+1}(Ay_i + B\xi) + L_{i+1}). \tag{17}$$

The value of the composite costs function is

$$\inf_{u \in U} E \left(\sum_{j=0}^{N-1} \left(u_j^T R_j u_j + P(\tau = j) \|y_j\|^2 \right) + P(\tau = N) \|y_N\|^2 \right)$$
$$= y_0^T K_0 y_0 + y_0^T L_0 + M_0. \tag{18}$$

Proof. Results from direct application of Theorem 1. ∎

COROLLARY 2. *The above formulas can be used for the classical linear quadratic control with established horizon. Let the system be described by (1) and the task have the form*

$$\inf_{u \in U} E \left\{ \sum_{i=0}^{N-1} u_i^T R u_i + \|y_N\|^2 \right\}. \tag{19}$$

In this case we put $P(\tau = N) = 1$ *and* $P(\tau = i) = 0$ *for* $i = 0, ..., N-1$, *then the optimal control has the form (17), where* $R_i = R$ *and* $Q_i = [0]$ *(matrix of zeros) for* $i = 0, ..., N-1$, *and* $K_N = I$ *(identity matrix). The value of performance criterion is*

$$\inf_{u \in U} E \left\{ \sum_{i=0}^{N-1} u_i^T R u_i + \|y_N\|^2 \right\} = y_0^T K_0 y_0 + y_0^T L_0 + M_0$$

where K_0, L_0, M_0 *are given by (5) - (7).*

COROLLARY 3. *Let the linear system be described by the state equation*

$$y_{i+1} = y_i - C u_i + \sigma w_{i+1} \tag{20}$$

and the performance criterion with random horizon have the form (15), which we decompose to auxiliary task (16), where $R_i = P(\tau > i) R$ *and* $Q_i = P(\tau = i) I$ *for* $i = 0, 1, 2, ..., N$. *Using (17) and (5) - (7) we have: if* $det(R_i + C^T G_{i+1} C) \neq 0$ *for* $i = 0, 1, ..., N-1$ *then the optimal control of system (20) is*

$$u_i^* = \left[R_i + C^T K_{i+1} C \right]^{-1} C^T K_{i+1} y_i$$

where

$$K_i = Q_i + K_{i+1} - K_{i+1}^T C \left[R_i + C^T K_{i+1} C \right]^{-1} C^T K_{i+1} \ and \ K_N = P(\tau = i) I$$

and the value of performance criterion of auxiliary task (16) is

$$\inf_{u \in U} E \left\{ \sum_{i=0}^{N-1} \left[u_i^T R_i u_i + P(\tau = i) \|y_i\|^2 \right] + P(\tau = N) \|y_N\|^2 \right\}$$

$$= y_0^T K_0 y_0 + \sum_{j=1}^{N} tr\left(\sigma^T K_j \sigma \right).$$

REMARK 3. *In present case the external stopping time of stabilization of linear system (1) can be zero ($\tau = 0$) (it is the case, where system is disturbed and can not be stabilized). From the mathematical point of view this is correct, while the theory of control says that if we stabilize we do act. In this case it is sufficient to put $P(\tau = 0) = p_0 = 0$ which means that we surely undertake an action to stabilize the system at time $\tau = 0$.*

EXAMPLE 1. *Let us stabilize a linear system with state equation (20) for random and fixed time. Let us assume*

$$R = \left[\begin{array}{cc} 0.27 & 0.03 \\ 0.03 & 0.35 \end{array} \right], \quad C = \left[\begin{array}{cc} 1.3 & -0.2 \\ -0.4 & 2.1 \end{array} \right], \quad \sigma = \left[\begin{array}{cc} 1.2 & -0.3 \\ 0.2 & 0.9 \end{array} \right]$$

We must stabilize the system (20), which is at an initial point $y_0 = \left(\begin{array}{c} 90 \\ -36 \end{array} \right)$. The random horizon has a Bernoulli distribution

$$P(\tau = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for $k = 0, 1, 2, ..., 10$. The Bellman's function $W_j(y_j) = y_j^T K_j y_j + \sum_{i=j+1}^{10} tr\left(\sigma^T K_i \sigma \right)$ represents a CCF at time j. For $p = 0.56$ and $p = 1$ we have Table 1 presents a possible trajectory of system states y_j, optimal controls u_j^* and values of Bellman's function. It can be seen that until possible time $[E\tau]$ we try to stabilize the system, but after the moment $[E\tau]$ we suppress (level) external disturbances by control (the symbol $[]$ means the integer part). In case with random horizon the control values are higher at the beginning and successively decrease in following moments until time $[E\tau]$, whereas in case with fixed horizon the control values are evenly distributed.

Fig. 1 shows dependencies between the composite costs function and probability of success p. We can see that with increasing p the expected horizon of control is being extended and the energetic costs are more spread over time, therefore we have lower costs during system stabilization.

EXAMPLE 2. *To uncover the additional differences between the control system with random and deterministic horizons, we consider the problem of stability of linear system (20) without cost of controls.*
Thus, in case with random horizon the task has the form

$$\inf_{u \in U} E \|y_\tau\|^2 \tag{21}$$

j	$p = 0.56$					$p = 1$				
	$W_j(y_j)$	y_j		u_j		$W_j(y_j)$	y_j		u_j	
0	341.82	90	-36	18.36	-2.73	127.13	90	-36	6.75	-0.44
1	240.20	64.49	-23.12	16.84	-2.09	115.36	81.35	-32.85	6.76	-0.46
2	134.11	40.18	-13.01	15.58	-2.10	110.29	75.05	-30.36	6.99	-0.48
3	69.316	20.34	-2.60	9.65	0.23	97.15	65.77	-25.73	6.99	-0.43
4	16.17	7.63	-0.43	4.14	0.38	86.09	57.31	-22.29	7.09	-0.43
5	2.08	1.51	-0.33	0.88	-0.01	76.31	49.36	-19.69	7.28	-0.51
6	0.89	0.53	-0.62	0.31	-0.22	68.92	42.03	-16.43	7.69	-0.51
7	0.56	-1.19	0.32	-0.75	0.02	55.65	32.71	-12.13	7.88	-0.46
8	0.22	0.95	-0.94	0.59	-0.32	42.22	23.31	-7.71	8.23	-0.32
9	0.01	0.31	0.41	0.22	0.21	24.67	12.63	-2.69	8.33	0.16
10	0.01	1.31	0.84	0	0	1.08	0.99	-0.32	0	0

Table 1. The CCF, states and control values for random and fixed times.

which we reduce to the form

$$\inf_{u \in U} E \left(\sum_{j=0}^{N} P(\tau = j) \|y_j\|^2 \right). \tag{22}$$

The optimal control is

$$u_i^* = \left(C^T K_{i+1} C \right)^{-1} C^T K_{i+1} y_i \tag{23}$$

where $K_N = P(\tau = N) I$ and

$$K_i = P(\tau = i) I + K_{i+1} - K_{i+1}^T C \left(C^T K_{i+1} C \right)^{-1} C^T K_{i+1} \tag{24}$$

for $i = 0, 1, ..., N-1$ and I is identity matrix. The value of performance criterion (22) is $y_0^T K_0 y_0 + \sum_{j=1}^{N} tr\left(\sigma^T K_j \sigma \right)$.

In case with deterministic (fixed) horizon the task has the form

$$\inf_{u \in U} E \|y_N\|^2 \tag{25}$$

where the optimal control is also given by (23), where $K_N = I$ and

$$K_i = K_{i+1} - K_{i+1}^T C \left(C^T K_{i+1} C \right)^{-1} C^T K_{i+1} \tag{26}$$

for $i = 0, 1, ..., N-1$ and I is identity matrix. The value of performance criterion (25) is $y_0^T K_0 y_0 + \sum_{j=1}^{N} tr\left(\sigma^T K_j \sigma \right)$.

We see that the formulas, which determine the values of performance criterion (22) and (25) and controls, are the same. The matrices K_i for the above task are different!
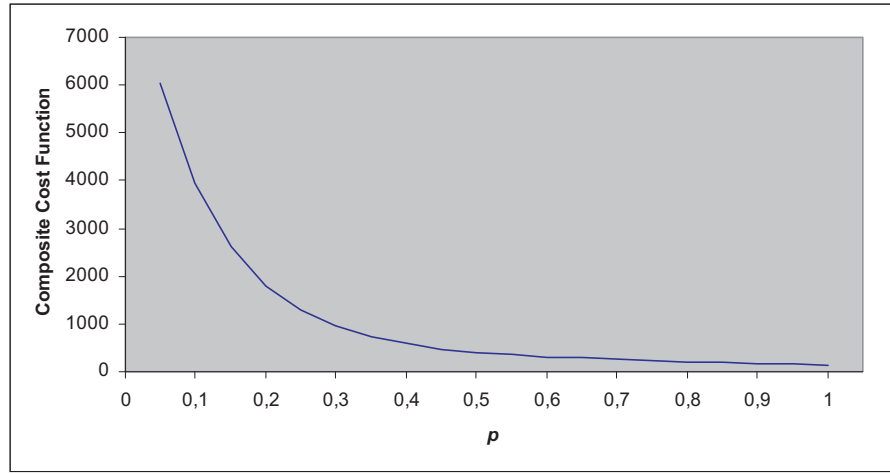
Figure 1. The relationship between CCF and probability p

EXAMPLE 3. *If $y_i, u_i, w_i, C, \sigma \in R$, then for the case with deterministic horizon of control from (26) we have $K_0 = K_1 = ... = K_{N-1} = 0$ and $K_N = 1$, while the optimal control is $u_0 = u_1 = ... = u_{N-2} = 0$, $u_{N-1}^* = \frac{y_{N-1}}{C}$ thus*

$$\inf_{u \in U} E \|y_N\|^2 = \sigma^2.$$

Thus we see, that the system evolves freely (without controls) in steps $0, 1, ...N-2$, and we must take control at time $N-1$ only.

In case with random horizon we act completely differently, from (24) we have $K_i = P(\tau = i)$ and $u_i^* = \frac{y_i}{C}$ for $i = 0, 1, ..., N-1$. For one-dimensional case we see that control does not depend on distribution of control horizon. The value of performance criterion is

$$
\begin{aligned}
\inf_{u \in U} E \left(\sum_{j=0}^{N} P(\tau = j) \|y_j\|^2 \right) &= P(\tau = 0) \|y_0\|^2 + \sigma^2 \sum_{j=1}^{N} P(\tau = j) \\
&= \sigma^2 + P(\tau = 0) \left(\|y_0\|^2 - \sigma^2 \right).
\end{aligned}
$$

REMARK 4. *The one-dimensional case for system (20) showed the significant differences of control. For a fixed horizon we take control only in the penultimate step $N-1$, the system evolves freely until time $N-2$. For a random horizon we take control from time 0 until time $\tau - 1$ (all the time we must stabilize the system; we are aiming at a target, wich is the origin of coordinates). The control at time 0 depends on the initial state y_0, while the controls in subsequent moments depend on system states and eliminate external disturbances.*

7.   Conclusion

In this article, the stabilization problem of stochastic discrete-time linear system for random horizon via control was presented. The random horizon was modeled by a random variable at finite number of elementary events. The described problem was reduced to optimal control task with finite horizon. The aims of control for primary and substitute tasks are the same. Solution of auxiliary task gives the optimal control laws to stabilize the linear system. Additionally, a simple example shows that the optimal controls for stochastic system stabilization in random and fixed time intervals are different. Thus, to design stabilization of a linear system for random time we cannot directly refer to the task with established horizon, we must necessarily modify the composite costs function. The considered stochastic system has known parameters, but the obtained results could be extended to the task of stabilizing a system with unknown parameters.

The extension of the described problem can be used, for example, to system identification, image recognition, perfect tracking etc. in random time interval.

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