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Further results on the equivalence to Smith form of multivariate polynomial matrices*

by

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Abstract: Multivariate polynomial matrices arise from the treatment of linear systems of partial differential equations, delay-differential equations or multidimensional discrete equations. In this paper we generalize some of the results obtained for the equivalence to the Smith normal form for a class of multivariate polynomial matrices.

Keywords: linear functional systems, multivariate polynomial matrices, unimodular equivalence, smith form, Gröbner bases

1. Introduction

In the polynomial approach, pioneered by Rosenbrock (1970), matrices over $\mathbb{R}[s]$, $s \equiv d/dt$ are used to represent linear systems of ordinary differential equations. This ring is a principal ideal domain with the Euclidean division property thus allowing for the establishment of canonical forms, e.g., the Smith normal form. The theory of such systems can be regarded as more or less complete. For more general linear functional systems e.g. partial differential systems or delay-differential systems, the resulting system matrices are multivariate. Multivariate polynomial rings are not principal ideal rings and do not have a Euclidean division. However, these rings admit Gröbner basis computations. Despite its importance in single variable matrix theory, the Smith normal form for multivariate polynomial matrices has received relatively little attention. The few exceptions are Frost and Storey (1979), Frost and Boudellioua (1986), Lee and Zak (1983), Lin et al. (2006), and Boudellioua and Quadrat (2010). The computations involved in the reduction of a given square matrix to its equivalent Smith form have been set out in Boudellioua (2012) using Maple. One of the motivations of transforming a multivariate polynomial matrix to its Smith form is to be able to reduce a system of linear functional equations to a system containing fewer equations and unknowns. The reduction involved must, of course,

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preserve relevant system properties from the systems theory point of view. The reduced equivalent representation simplifies in general the study of such systems. The results on the reduction to Smith form obtained by the authors mentioned earlier deal with the case of square matrices. In this paper we extend the results obtained by Lin et al. (2006) and Boudellioua and Quadrat (2010) to rectangular matrices. We will consider the case when the reduced matrix corresponds to a system which consists of only one equation with one unknown. This is equivalent to reducing the given multivariate polynomial matrix to a Smith form where all the diagonal elements are 1's, except the last one. In what follows, let $D = K[x_1, \ldots, x_n]$ denote a commutative multivariate polynomial ring with indeterminates $x_1, \ldots, x_n$ over an arbitrary but fixed field K.

2. Definitions

Definition 1. *Let $T \in D^{q\times p}, p > q$, the Smith form of T is given by*

$$S = \begin{pmatrix} diag\{\Phi_i\} & 0 \end{pmatrix} \tag{1}$$

where

$$\Phi_i = \begin{cases} \alpha_i/\alpha_{i-1}, & 1 \le i \le r \\ 0, & r < i \le q, \end{cases} \tag{2}$$

r is the normal rank of T, $\alpha_0 \equiv 1$, α_i is the gcd of all the $i \times i$ minors of T and Φ_i's satisfy the divisibility property

$$\Phi_1|\Phi_2|\ldots|\Phi_r. \tag{3}$$

Definition 2. *The general linear group $GL_p(D)$ is defined by*

$$GL_p(D) = \left\{M \in D^{p\times p} \;\;|\;\; \exists N \in D^{p\times p} : MN = NM = I_p\right\}. \tag{4}$$

An element $M \in GL_p(D)$ is called a unimodular matrix. It follows that M is unimodular if and only if $|M| \in \mathbb{K}\backslash\{0\}$.

Definition 3. *Let T_1 and T_2 denote two matrices in $D^{q\times p}$; then T_1 and T_2 are said to be (unimodular) equivalent if there exist two matrices $M \in GL_q(D)$ and $N \in GL_p(D)$ such that*

$$T_2 = MT_1N. \tag{5}$$

Unimodular equivalence has been shown to exhibit fundamental algebraic properties amongst its invariants. In particular, it preserves the zero structure of the original matrix which is captured by the determinantal ideals of the matrix. In fact, for the case when $D = K[x_1]$, it is well known that every matrix with elements in D is equivalent to its Smith form. However, this result is not valid for the case when $D = K[x_1, \ldots, x_n], n > 1$.

EXAMPLE 1. *Consider the matrix over* $D = \mathbb{R}[s]$,

$$L(s) = \begin{pmatrix} s-1 & s \\ 0 & s+1 \end{pmatrix}, \qquad s \equiv d/dt.$$

Here we have,

$$\begin{aligned} \alpha_0 &= 1, \\ \alpha_1 &= gcd(s-1, s, 0, s+1) = 1, \\ \alpha_2 &= (s-1)(s+1) = s^2 - 1, \\ \Phi_1 &= \alpha_1/\alpha_0 = 1, \\ \Phi_2 &= \alpha_2/\alpha_1 = s^2 - 1. \end{aligned}$$

It can be easily shown here, using elementary row and column operations on $L(s)$ *that:*

$$L(s) = \begin{pmatrix} s-1 & s \\ 0 & s+1 \end{pmatrix} \sim S(s) = \begin{pmatrix} 1 & 0 \\ 0 & s^2-1. \end{pmatrix}$$

Now Consider the matrix over $D = \mathbb{R}[s, z]$ *as given by Lee and Zak (1983),*

$$T(s,z) = \begin{pmatrix} s & z+1 \\ z^2 & s \end{pmatrix}, \qquad s \equiv \partial/\partial t, z \equiv \partial/\partial x.$$

In this case,

$$\begin{aligned} \alpha_0 &= 1, \\ \alpha_1 &= gcd(s, z+1, z^2, s) = 1, \\ \alpha_2 &= s^2 - z^2(z+1), \\ \Phi_1 &= \alpha_1/\alpha_0 = 1, \\ \Phi_2 &= \alpha_2/\alpha_1 = s^2 - z^2(z+1). \end{aligned}$$

Unlike the previous example, it can be shown here that

$$T(s,z) = \begin{pmatrix} s & z+1 \\ z^2 & s \end{pmatrix} \not\sim S(s,z) = \begin{pmatrix} 1 & 0 \\ 0 & s^2 - z^2(z+1). \end{pmatrix}$$

This example shows that despite the fact that the first order minors of $T(s,z)$ generate the unit ideal D, $T(s,z)$ is not equivalent to its Smith normal form.

3. Reduction to Smith form by unimodular equivalence

The aim of the reduction is to simplify linear functional systems in the sense of finding an equivalent presentation which contains only one equation in one unknown. This generally makes it easier to study the structural properties of the linear functional system and in some cases can be used to compute its closed-form solutions. This reduction also finds applications in numerical analysis. The objective of the equivalence transformation applied to the matrix is to produce an identity matrix of appropriate size at the top left corner of the original matrix. The problem of reduction of a multivariate polynomial matrix to its

Smith form form was first studied by Frost and Storey (1979) but their results yielded only necessary and not sufficient conditions. Frost and Boudellioua (1986) later obtained for a class of square matrices necessary and sufficient conditions. Their result is given in the following:

Theorem 1 (Frost and Boudellioua, 1986). *Let $D = \mathbb{R}[s,z]$ and $T \in D^{q\times q}$, with full row rank, then T is equivalent to the Smith form*

$$S = \begin{pmatrix} I_{q-1} & 0 \\ 0 & |T| \end{pmatrix} \tag{6}$$

if and only if there exists a vector $U \in D^q$ which admits a left inverse in D such that the matrix $\begin{pmatrix} T & U \end{pmatrix}$ has a right inverse over D.

Lin et al. (2006) generalized this result to the case when $D = \mathbb{R}[z_1,\ldots,z_n]$, $n > 1$.

Theorem 2 (Lin et al., 2006). *Let $D = \mathbb{R}[z_1,\ldots,z_n]$ and $T \in D^{q\times q}$, with full row rank, then T is equivalent to the Smith form*

$$S = \begin{pmatrix} I_{q-1} & 0 \\ 0 & |T| \end{pmatrix} \tag{7}$$

if and only if there exist a vector $U \in D^q$ which admits a left inverse in D such that the matrix $\begin{pmatrix} T & U \end{pmatrix}$ has a right inverse over D.

Boudellioua and Quadrat (2010) gave similar conditions for a weaker type of equivalence between a square matrix and its Smith form using a module theoretic approach and showed that in this case the condition on the vector U is not necessary. In the case where the matrix is rectangular, the result given in Boudellioua and Quadrat (2010) does not lead necessarily to a Smith form.

Theorem 3 (Boudellioua and Quadrat, 2010). *Let $D = K[x_1,\ldots,x_n]$ be a commutative polynomial ring over a field K and $T \in D^{q\times p}$ a full row rank matrix. Then the following two assertions are equivalent:*

1. *There exists $U \in D^{q\times 1}$ which admits a left inverse over D such that the matrix*
$$P := \begin{pmatrix} T & U \end{pmatrix}$$
admits a right inverse over D.
2. *The matrix T is unimodular equivalent over D to the matrix:*
$$\tilde{T} = \begin{pmatrix} I_{q-1} & 0 \\ 0 & Q_2 \end{pmatrix} \tag{8}$$
where $Q_2 \in D^{1\times(p-q+1)}$.

It is clear that the matrix $\tilde{T}$ in (8) is not necessarily the Smith form of T. Before we generalize Theorem 2 to the case when the matrix T is not necessarily square, we first state the following result which is a statement of the positive answer of the Lin-Bose conjecture (see Lin and Bose, 2001). This theorem, which will be used later, is given by Fabiańska and Quadrat, 2007.

THEOREM 4 (Section 5 of Fabiańska and Quadrat, 2007). *Let $D = K[x_1, \ldots, x_n]$ be a commutative polynomial ring over a field K and $R \in D^{q\times p}$ a full row rank matrix. Then, the following two assertions are equivalent:*

1. *The ideal $I_q(R)$ generated by the $q \times q$ minors of R is principal, i.e. can be generated by the greatest common divisor Φ of these minors.*
2. *There exist $R' \in D^{q\times p}, R'' \in D^{q\times q}$, and $N \in GL_p(D)$ such that:*

$$R = R''R', \qquad \det(R'') = \Phi, \qquad R'N = \begin{pmatrix} I_q & 0 \end{pmatrix}. \tag{9}$$

Now we present the main result of this paper which is a consequence of Theorems 3 and 4.

THEOREM 5. *Let $D = K[x_1, \ldots, x_n]$ and $T \in D^{q\times p}$, $p > q$ with full row rank, then T is equivalent to the Smith form*

$$S = \begin{pmatrix} I_{q-1} & 0 & 0 \\ 0 & \Phi_q & 0 \end{pmatrix} \tag{10}$$

where $\Phi_q \in D$ is the gcd of the $q \times q$ minors of T, if and only if there exists a vector $U \in D^q$ which admits a left inverse in D such that the matrix $\begin{pmatrix} T & U \end{pmatrix}$ has a right inverse over D and the ideal generated by the $q \times q$ minors of T is principal.

Proof. Let $T \in D^{q\times p}$ and suppose that there exists a vector $U \in D^q$ which admits a left inverse in D, satisfying the given condition and that the ideal generated by the $q \times q$ minors of T is principal. Then, since U admits a left inverse in D, there exists a matrix $M_1 \in GL_q(D)$ such that $M_1U = E_q$, where E_q is the q-th column of I_q. It follows that

$$M_1 \begin{pmatrix} T & U \end{pmatrix} = \begin{pmatrix} T_1 & 0 \\ T_2 & 1 \end{pmatrix} \tag{11}$$

where $T_1 \in D^{(q-1)\times p}$ and $T_2 \in D^{1\times p}$ are given by

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = M_1T. \tag{12}$$

Now, since the matrix on the RHS of (11) admits a right inverse over D, it follows that T_1 also admits a right inverse over D, i.e. there exists a matrix $N_1 \in GL_p(D)$ such that

$$T_1N_1 = \begin{pmatrix} I_{q-1} & 0 \end{pmatrix}. \tag{13}$$

Then,

$$\begin{pmatrix} T_1 & 0 \\ T_2 & 1 \end{pmatrix} \begin{pmatrix} N_1 & 0 \\ 0 & 1 \end{pmatrix} = \left(\begin{array}{cc|c} I_{q-1} & 0 & 0 \\ T_3 & T_4 & 1 \end{array}\right) \tag{14}$$

where $T_3 \in D^{1\times(q-1)}, T_4 \in D^{1\times(p-q+1)}$ and

$$\begin{pmatrix} T_3 & T_4 \end{pmatrix} = T_2 N_1. \tag{15}$$

It follows that

$$M_1 T N_1 = \begin{pmatrix} I_{q-1} & 0 \\ T_3 & T_4 \end{pmatrix}. \tag{16}$$

Premultiplying the matrix $M_1 T N_1$ in (16) by the unimodular matrix

$$M_2 = \begin{pmatrix} I_{q-1} & 0 \\ -T_3 & 1 \end{pmatrix} \tag{17}$$

yields the matrix

$$M_2 M_1 T N_1 = \begin{pmatrix} I_{q-1} & 0 \\ 0 & T_4 \end{pmatrix}. \tag{18}$$

Now since the ideal generated by the $q \times q$ minors of T is principal, by virtue of the Lin-Bose Theorem 4, there exists a matrix $N_2 \in GL_p(D)$ with

$$N_2 = \begin{pmatrix} I_q & 0 \\ 0 & \bar{N} \end{pmatrix} \tag{19}$$

such that

$$M_2 M_1 T N_1 N_2 = \begin{pmatrix} I_{q-1} & 0 & 0 \\ 0 & \Phi_q & 0 \end{pmatrix} \tag{20}$$

where $T_4 \bar{N} = \begin{pmatrix} \Phi_q & 0 \end{pmatrix}$.

Conversely, assume that $T \in D^{q\times p}$ is equivalent to the Smith form

$$S = \begin{pmatrix} I_{q-1} & 0 & 0 \\ 0 & \Phi_q & 0 \end{pmatrix}, \tag{21}$$

where $\Phi_q \in D$ is the gcd of all the i^{th} order minors of T. It follows that there exist unimodular matrices $M \in GL_q(D)$ and $N \in GL_p(D)$ such that $S = MTN$. Now, consider the vector $U = M^{-1}E_q$, where E_q is the q^{th} column of I_n, then

$$\begin{aligned} M \begin{pmatrix} T & U \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} &= M \begin{pmatrix} T & M^{-1}E_q \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} MTN & E_q \end{pmatrix} \\ &= \begin{pmatrix} I_{q-1} & 0 & 0 & 0 \\ 0 & \Phi_q & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} I_q & 0 \end{pmatrix} \end{aligned} \tag{22}$$

i.e. the matrix $\begin{pmatrix} T & U \end{pmatrix}$ has a right inverse over D. Clearly, the ideal of the $q \times q$ minors of S is generated by the unique polynomial $\Phi_q \in D$ and therefore the ideal generated by $q \times q$ minors of T is principal. ∎

The problem of finding a vector $U \in D$ when it exists such that the condition in Theorem 5 is satisfied is neither trivial nor random. On simple examples over a commutative polynomial ring $D = K[x_1, \ldots, x_n]$ with coefficients in a computable field K (e.g., $K = \mathbb{Q}$), one may take a generic vector $U \in D^q$ with a fixed total degree in the x_i's and compute the D-module $\mathrm{ext}^1_D(E, D) = D^{1\times q}/\left(D^{1\times(p+1)}\,(T \quad U)^T\right)$ by means of a Gröbner basis computation and check whether or not the D-module $\mathrm{ext}^1_D(E, D)$ vanishes on certain branches of the corresponding *tree of integrability conditions* (Pommaret and Quadrat, 2000) or on certain *obstructions to genericity* (i.e., constructible sets of the K-parameters of U) (Levandovskyy and Zerz, 2007). See Levandovskyy and Zerz (2007) for a survey explaining these techniques and their implementations in SINGULAR.

EXAMPLE 2. *Consider the system of linear delay-differential equations*

$$T\psi(t) = 0 \tag{23}$$

where $\psi(t) = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ \psi_3(t) \\ \psi_4(t) \end{pmatrix}$ *and the system matrix T is given by*

$$T = \begin{pmatrix} 2d\sigma^2 + \sigma^3 + \sigma^2 + 1 & d\sigma^2 - d\sigma + d & 2d\sigma + \sigma^2 & d\sigma^2 + d\sigma + d + \sigma^2 \\ 2d\sigma + \sigma^2 + \sigma & d\sigma - d & 2d + \sigma & d\sigma + d + \sigma \\ 2d^2\sigma + d\sigma^2 + d\sigma + \sigma & d^2\sigma - d^2 - 1 & 2d^2 + d\sigma + 1 & d^2\sigma + d^2 + d\sigma \end{pmatrix} \tag{24}$$

where $D = \mathbb{R}\,[d, \sigma]$, $d\,f(t) = \dot{f}(t)$, $\sigma f(t) = f(t-h)$ *and* $h \in \mathbb{R}^+$. *Consider* $U = (\sigma \quad 1 \quad d)^T \in D^3$ *and* $P = (T \quad U) \in D^{3\times 5}$. *Using the package* OREMODULES *in Maple (see Chyzak et al., 2007), we can check that P admits a right inverse over D. Also using Gröbner bases, we can verify that the ideal of the 3×3 minors of T is generated by the polynomial $d + \sigma$. Now we can compute a minimal parametrization, $Q_m \in D^{5\times 2}$ of P (see Chyzak and Robertz, 2005), where*

$$Q_m = \begin{pmatrix} Q_1^T & Q_2^T \end{pmatrix}^T, \qquad PQ_m = 0,$$

and

$$Q_m = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \left(\begin{array}{cc} -d & 0 \\ 0 & 1 \\ d\sigma & 1 \\ 1 & -1 \\ \hline -d - \sigma & 0 \end{array}\right). \tag{25}$$

Computing the SyzygyModule $F \in D^{2\times 4}$ of Q_1, i.e. $FQ_1 = 0$ gives

$$F = \begin{pmatrix} \sigma & -1 & 1 & 0 \\ 1 & d & 0 & d \end{pmatrix} \tag{26}$$

where $Q_3 \in D^{4\times 2}$ *is a right inverse of* F*, i.e.,*

$$Q_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & -\sigma \\ 0 & 0 \end{pmatrix}. \tag{27}$$

Thus, the matrix $N \in GL_4(D)$ *is given by*

$$N = \begin{pmatrix} 0 & 1 & -d & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -\sigma & d\sigma & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \tag{28}$$

and the matrix $M = \begin{pmatrix} TQ_3 & U \end{pmatrix}^{-1} \in GL_3(D)$ *is given by*

$$M = \begin{pmatrix} 0 & -d & 1 \\ 1 & -\sigma & 0 \\ -\sigma & 2d^2 + \sigma^2 + d\sigma + 1 & -2d - \sigma \end{pmatrix}. \tag{29}$$

Now it can be easily verified that the matrix MTN *yields the Smith form:*

$$S = \begin{pmatrix} I_2 & 0 & 0 \\ 0 & d + \sigma & 0 \end{pmatrix}. \tag{30}$$

and it follows that the system in $(23, 24)$ *is equivalent to the following simple delay-differential equation:*

$$\dot{x}(t) + x(t - h) = 0. \tag{31}$$

Conclusions

We have presented a constructive result for the reduction to the Smith form of a class of rectangular multivariate polynomial matrices. In particular, we have given necessary and sufficient conditions under which a matrix can be reduced by unimodular equivalence to a Smith form that corresponds to the reduction of a linear functional system to a single equation with only one unknown. Furthermore, we have shown that the result can be implemented using modern symbolic computation software.

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