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Stability analysis of variational inequalities for bang-singular-bang controls*

by

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Abstract: The paper is related to parameter dependent optimal control problems for control-affine systems. The case of scalar reference control with bang-singular-bang structure is considered. The analysis starts from a variational inequality (VI) formulation of Pontryagin's Maximum Principle. In a first step, under appropriate higher-order sufficient optimality conditions, the existence of solutions for the linearized problem (LVI) is proven. In a second step, for a certain class of right-hand side perturbation, it is shown that the controls from LVI have bang-singular-bang structure and, in L_1 topology, depend Lipschitz continuously on the data. Applying finally a common fixed-point approach to VI, the results are brought together to obtain existence and structural stability results for extremals of the original control problem under parameter perturbation.

Keywords: parametric optimal control problems, bang-singular control structure, approximation of extremals

1. Introduction

The paper is concerned with optimal control problems in Mayer form when the system is control-affine, and the data functions smoothly depend on a real parameter. Pontryagin's Maximum Principle will be interpreted as a variational inequality, and its stability under parameter perturbation is investigated. We consider the particular case when the control function is subject to two-sided bound constraints. It will be further assumed that, for the reference parameter zero, the control has bang-singular-bang structure, i.e., it achieves its extremal values on certain subintervals to the left and right ends of the time interval, and takes "singular" values from the interior of the control set in the remaining part. The investigation makes essential use of preliminary work on singular controls

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in Goh (1966a, b), Dmitruk (2008), Poggiolini and Stefani (2008, 2011), and of contributions to Lipschitz stability in optimal control theory in Dontchev and Malanowski (2000), Malanowski (1995, 2001), and of stability, respectively metric regularity results for variational inequalities in Robinson (1980), Dontchev (1995), Dontchev and Rockafellar (1996), see further Dontchev et al. (2005, 2006, 2009). Generically, in the above situation, the optimal control is discontinuous and will continuously depend on the parameter in L_p topology for $p < \infty$ only; see Felgenhauer (2001). For the bang–bang case, Lipschitz stability results have been obtained in L_1 sense in Felgenhauer (2003). Further investigation of stability of bang–bang switching points from Felgenhauer et al. (2009), Felgenhauer (2010, 2013) confirm the related estimates. In the case of bang-singular-bang controls, stability results for the singular feedback control law are provided in Vossen (2005, 2010). Moreover, bang-singular junction points perturbations have been analyzed by shooting type methods in Maurer (1976), Oberle (1979), Vossen (2010) but without providing general theoretical results. An alternative approach for stability investigation consists in applying synthesis methods as, e.g., in Ledzewicz and Schättler (2007). In Felgenhauer (2012), the structural stability for bang-singular junction in case of one singular arc of order one was obtained under rather mild assumptions including the strong Legendre condition but without second-order sufficient optimality conditions. Instead, it had to be supposed that the perturbed system of first-order necessary conditions had a solution.

The recent development of second-order conditions for the problem class in Aronna et al. (2012a) now has allowed for completing the analysis. In this paper, the authors combine the theoretical foundations with the shooting method and prove its well-posedness for a quite general problem setting including terminal boundary constraints and vector-valued controls. As a particular result, the local structural stability together with piecewise L_∞ error estimates for the control are obtained in a transformed problem formulation. However, the second-order condition used is clearly stronger than the condition originally derived in Aronna et al. (2012a) in that the critical cone does not include now any restriction on the solution structure. The present paper also makes essential use of the result from Aronna et al. (2012a) but is independent of Aronna et al. (2012b). We will provide local Lipschitz continuity of extremals in L_1 based topology together with structural stability results for the original problem as well as for its linearized version. The assumptions on the reference solution herein include structural properties and second-order conditions in the spirit of Aronna et al. (2012a). Compared to the coercivity condition from Aronna et al. (2012a), the cone of variations of the control, respectively its primitive is only slightly modified to meet stability needs. Moreover, from a methodological point of view, the variational approach proposed in the present paper is clearly independent from the shooting formulation and related transformations as they have been utilized in Aronna et al. (2012b). For future research on general approximation approaches, the structural stability results obtained for the linearized variational inequalities may be of additional interest.

The paper starts with characterizing the extremals by Pontryagin's Maximum Principle which, for the problems under consideration, holds in normal form, and is equivalent to the first-order necessary optimality conditions. The conditions will be written in the form of a variational inequality (VI). Next, its linearization (LVI) is obtained and analyzed. Two points are crucial in the stability analysis: first, the weakness of coercivity and stability properties of the linearized VI do not allow to directly apply standard results from (strong, or, respectively metric) regularity theory as developed in Robinson (1980), Dontchev (1995), Dontchev and Rockafellar (1996), Dontchev and Lewis (2005), Dontchev et al. (2006) etc. (see also Dontchev and Veliov, 2009, for further references). However, an a-posteriori structural investigation of control components in the spirit of Felgenhauer (2012) shows that L_1 control stability holds for LVI in case that the included *rhs* terms are of appropriate regularity. Returning then to the roots of the stability proof by S. M. Robinson (1980), it turns out that the perturbed extremals are fixed points of the solution map for LVI with right-hand sides of the required type: this is the second important feature in proving L_1 Lipschitz stability of bang-singular-bang controls under parameter perturbation.

Plan of the paper. In Section 2, the problem and variational inequalities are formulated. The linearized VI will be analyzed in detail in Section 3 by means of Goh's transformation adapted to the state-adjoint system, and by suitable structural analysis techniques. The final L_1 stability result is proven in Section 4. In the concluding Section 5, an example is provided, and summarizing remarks are given. Finally, the Appendix contains some auxiliary material and calculations.

Notations. Let $\mathbb{R}^n$ be the Euclidean vector space with norm $|\cdot|$, and scalar product written as $a \cdot b = a^T b$. Superscript T is generally used for transposition of matrices, respectively vectors. The Lebesgue space of order p of vector-valued functions on $[0,1]$ is denoted by $L_p(0,1;\mathbb{R}^k)$. $W_p^l(0,1;\mathbb{R}^k)$ is the related Sobolev space, and norms are given as $\|\cdot\|_p$ and $\|\cdot\|_{l,p}$, $(1 \le p \le \infty,\ l \ge 1)$, respectively. For the scalar product in L_2 write $(\cdot,\cdot)$. According to Riesz' Theorem, the dual space to L_2 can be identified with L_2, i.e. $(L_2)^* \doteq L_2$. In places, where the duality pairing between a Banach space V and its dual V^* is needed, we use $\langle \cdot,\cdot \rangle$. Let A be a continuous, linear mapping from the Banach space V_1 to the Banach space V_2. Then $A^* : V_2^* \to V_1^*$ denotes the adjoint operator (or: transpose) to A satisfying $\langle A^* v, w \rangle = \langle v, Aw \rangle$ for all $v \in V_2^*$, $w \in V_1$. The symbols ∇_x, ∇_x^2 denote (partial) gradients or Jacobians, and Hessians respectively. For functions $f, g : \mathbb{R}^k \to \mathbb{R}^k$, Lie brackets are given by $[f,g] = \nabla_x g\, f - \nabla_x f\, g$. Finally, we add some abbreviations for particular variable spaces, the control set, and sets of right-hand side perturbations:

$$
\begin{aligned}
&X = W_1^1 \times W_1^1 \times L_1 \times L_\infty, \qquad Y = L_1 \times \mathbb{R}^n \times L_1 \times \mathbb{R}^n \times L_\infty \times L_1 \times L_1,\\
&U = \{ w \in L_\infty :\ 0 \le w(t) \le 1\ \ a.e.\,\text{on } [0,1] \} \subset L_1,\\
&D = L_1 \times \mathbb{R}^n \times L_1 \times \mathbb{R}^n \times L_\infty, \qquad \hat{D} = L_2 \times \mathbb{R}^n \times L_2 \times \mathbb{R}^n \times W_2^1.
\end{aligned}
$$

2. The problem and the VI solution characterization

2.1. The problem and the structural assumption

Consider the following parameter dependent optimal control problem with scalar-valued, bounded optimal control entering the state system linearly:

$$\textbf{(CP}_p\textbf{)} \qquad \text{minimize} \quad J_p(x,u) \;:=\; k(x(1),p) \tag{1}$$

subject to

$$\dot{x}(t) \;=\; f(x(t),p) \;+\; g(x(t),p)\,u(t) \quad \text{a.e. in } [0,1], \tag{2}$$

$$x(0) \;=\; a(p), \tag{3}$$

$$0 \;\le\; u(t) \;\le\; 1, \quad \text{a.e. in } [0,1]\,, \tag{4}$$

$$x \in W^1_\infty(0,1;\mathbb{R}^n), \;\; u \in L_\infty(0,1;\mathbb{R}). \tag{5}$$

For simplicity, assume that p is a real parameter lying in a neighborhood Π of $p_0 = 0$. Further, suppose that (x^0, u^0) is a reference solution of (CP_0). Concerning smoothness of input data, the following will be required:

(H0) There exists an open set $Z \subset \mathbb{R}^n$ such that $x^0(t) \in Z$ for almost all $t \in [0,1]$, and the functions k, f, g and a are smooth on $Z \times \Pi$, respectively Π. In particular, they are three times continuously differentiable w.r.t. x, and the related partial derivatives up to order three depend Lipschitz continuously on all their arguments.

REMARK 1 *The constraints for u can be replaced by more general bounds $\underline{u} \le u(t) \le \bar{u}$ with given $\underline{u} < \bar{u}$. Defining the new control $u'(t) := (u(t)-\underline{u})/(\bar{u}-\underline{u})$, and transforming f and g accordingly, make it possible to apply the following results nearly unchanged to arbitrary two-sided control bounds.*

Define the Lagrange functional $L:\, W^1_2 \times L_2 \times L_2 \times L_2 \times \Pi \to \mathbb{R}$ with vector-valued adjoint function $\lambda\colon [0,1] \to \mathbb{R}^n$ and multiplier $\mu\colon [0,1] \to \mathbb{R}^2$, $\mu_1, \mu_2 \ge 0$, by

$$\begin{aligned} L(x,u,\lambda,\mu,p) \;\;=\;\; & k(x(1),p) - (\lambda, \dot{x} - f(x,p) - g(x,p)u) \\ & \qquad\qquad -(\mu_1, u) + (\mu_2, u-1), \end{aligned}$$

and the pre-Hamilton function (or: control Hamiltonian) $H : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \Pi \to \mathbb{R}$ by

$$H(x,u,\lambda,p) \;=\; \lambda^T f(x,p) \;+\; \lambda^T g(x,p)\,u.$$

Then, for the given normal case, Pontryagin's Maximum Principle can be expressed in the form of a variational inequality

$$\textbf{(VI}_p\textbf{)} \qquad \begin{aligned} &\dot{x} - f(x,p) - g(x,p)u \,=\, 0, \quad x(0) - a(p) \,=\, 0, &(6)\\ &\dot{\lambda} + \nabla_x H(x,u,\lambda,p) \,=\, 0, \quad \lambda(1) - \nabla_x k(x(1),p) \,=\, 0, &(7)\\ &\lambda^T g(x,p) \,-\, \mu_1 + \mu_2 \,=\, 0, &(8)\\ &-u \,\in\, N_+(\mu_1), \qquad u-1 \,\in\, N_+(\mu_2) &(9)\end{aligned}$$

for almost every $t \in [0,1]$. The set N_+ stands for the normal cone to $\mathbb{R}_+$,

$$N_+(\bar{\mu}) \,=\, \begin{cases} \{\rho \in \mathbb{R}_- : \, \rho^T(\mu - \bar{\mu}) \le 0 \quad \forall \mu \in \mathbb{R}_+\} & \text{if } \bar{\mu} \in \mathbb{R}_+,\\ \emptyset & \text{if } \bar{\mu} \notin \mathbb{R}_+. \end{cases}$$

Notice that, for problem (CP$_0$), the adjoint and multiplier functions $\lambda^0, \mu_1^0, \mu_2^0$ associated to (x^0, u^0) are uniquely determined by (VI$_0$) and belong to W^1_∞.

Conditions (8), (9) can be equivalently written in terms of the switching function $\sigma(\cdot) \,=\, \lambda(\cdot)^T g(x(\cdot),p)$ and its positive, respectively negative parts $[\sigma]_\pm$: for $\mu_1 = [\sigma]_+$, $\mu_2 = [\sigma]_-$, the relations read as

$$\sigma \,=\, [\sigma]_+ - [\sigma]_-, \quad 0 \le u \le 1, \quad u\,[\sigma]_+ \,=\, 0,\ (u-1)\,[\sigma]_- \,=\, 0 \quad \text{a.e. in } [0,1].$$

It should be noticed that, under the given smoothness assumptions, for $(x^0,u^0) \in W^1_\infty \times L_\infty$ we have $\sigma^0 = (\lambda^0)^T g^0(x^0) \in C^1$, and $\dot{\sigma}^0 = (\lambda^0)^T[f^0, g^0]$ belongs to W^1_∞. Moreover,

$$\ddot{\sigma}^0 \,=\, P - u^0 R, \quad P = (\lambda^0)^T\left[f^0,[f^0,g^0]\right], \;\; R = -(\lambda^0)^T\left[g^0,[f^0,g^0]\right]. \quad (10)$$

(Here and in the following, $[\cdot,\cdot]$ stands for the Lie brackets).

Finally, the investigation will be restricted to bang-singular-bang controls u^0 satisfying

(H1) (strict structural assumption)
The function u^0 is of *strict* bang-singular-bang structure, i.e., there exist points t_1, t_2 with $0 < t_1 < t_2 < 1$ and a positive constant m such that $u^0 \equiv 0$ a.e. on $[0,t_1)$, $u^0 \equiv \bar{u} \in \{0,1\}$ on $(t_2,1]$, and $m < u^0(t) < 1 - m$ a.e. on (t_1,t_2). Moreover, $\sigma^0 \neq 0$ on $[0,t_1) \cup (t_2,1]$.

Given $\beta > 0$, define

$$J_\beta^+ \,:=\, \{\, t \in [0,1] : \, \sigma^0(t) \,>\, \beta \,\}, \quad J_\beta^- \,:=\, \{\, t \in [0,1] : \, \sigma^0(t) \,<\, -\beta \,\} \quad (11)$$

and $J_\beta = J_\beta^+ \cup J_\beta^-$. Condition (H1), in particular, ensures that, for sufficiently small β, the set $[0,1] \backslash J_\beta$ is a closed interval I_β such that $[t_1,t_2] \subset I_\beta \subset (0,1)$.

2.2. Variational inequality and its linearization

We start this section by reformulating (VI$_p$):

Let $p \in \Pi$ and $\xi = (x,\lambda,u,\mu) \in X \,=\, W^1_1 \times W^1_1 \times L_1 \times L_\infty$ be the inputs in the left-hand sides of the system abbreviated by $(-\psi(p,\xi))$, and

$$U = \{ w \in L_\infty : \, 0 \le w(t) \le 1 \;\; a.e. \text{ on } [0,1]\} \,\subset\, L_1.$$

As before, μ stands for a vector valued function with components μ_1, μ_2 : $[0,1] \to \mathbb{R}$.
Obviously, $\psi(p,\xi) \in Y = L_1 \times \mathbb{R}^n \times L_1 \times \mathbb{R}^n \times L_\infty \times L_1 \times L_1$. Further define

$$\mathcal{K} = \{\nu \in L_\infty : \nu(t) \geq 0 \quad \text{for a.e. } t \in [0,1]\},$$
$$N_{\mathcal{K}}(\nu) = \begin{cases} \{\phi \in L_\infty^* : \langle \phi, \nu' - \nu \rangle \leq 0 \ \forall \nu' \in \mathcal{K}\} & \text{if } \nu \in \mathcal{K} \\ \emptyset & \text{otherwise,} \end{cases}$$

and the set-valued map $\mathcal{F} : X \to 2^Y$,

$$\mathcal{F}(\xi) = \{0\} \times \cdots \times \{0\} \times (N_{\mathcal{K}}(\mu_1) \cap L_1) \times (N_{\mathcal{K}}(\mu_2) \cap L_1)\,.$$

Then (VI_p) can be written in abstract setting as

$$0 \in \psi(p,\xi) + \mathcal{F}(\xi). \tag{12}$$

For $p = 0$, $\xi^0 = (x^0, \lambda^0, u^0, \mu^0) \in X$ is a solution.

Notice that $\mathcal{F}$ has closed graph. The function ψ is Fréchet differentiable w.r.t. ξ and, together with its derivative ψ', continuous around $(0, \xi^0)$. Further, ψ is Lipschitz continuous w.r.t. p uniformly in ξ close to ξ^0.
The linearization of (VI_p) at $p = 0$ w.r.t. ξ near ξ^0 can be written as

($\mathbf{LVI}_\delta$) $\qquad \bar{\delta} \in T(\xi) + \mathcal{F}(\xi)$

where we allow for some right-hand side perturbation $\bar{\delta}$. The operator $T = T(\xi)$ herein denotes

$$T(\xi) = \psi(p_0,\xi_0) + \psi'(p_0,\xi_0)(\xi - \xi_0)$$
$$= - \begin{pmatrix} \dot{z} - Az - Bw + Bu^0 \\ z(0) \\ \dot{q} + A^T q + Q_{11} z + Q_{12} w - Q_{12} u^0 \\ q(1) - K\, z(1) \\ B^T q + Q_{21} z - \nu_1 + \nu_2 + \mu_1^0 - \mu_2^0 \\ -w \\ w - 1 \end{pmatrix}, \tag{13}$$

$$A = \nabla_x (f^0 + g^0 u^0),\ B = \nabla_u (f^0 + g^0 u^0) = g^0,\ K = \nabla_x^2 k^0,$$
$$Q = \nabla^2_{(x,u)} H^0,\ \ Q_{11} = \nabla^2_{xx} H^0,\ \ Q_{12} = \nabla^2_{xu} H^0,$$

and $(\xi - \xi_0) = (z, q, w - u^0, \nu - \mu^0)$. Here and in the following, the superscript "0" says that the functions are evaluated at $p = 0$ along ξ^0.
Due to the linearity of the last two components of ψ, it is possible to restrict (LVI_δ) to $\bar{\delta} \in Y$ of the form $\bar{\delta} = (\delta_1, \ldots, \delta_5, 0, 0)$, or shorter, $\bar{\delta} = (\delta, 0)$. By Λ denote the solution operator for (LVI_δ), i.e.,

$$\Lambda(\delta) = \{\xi \in X : \bar{\delta} \in T(\xi) + \mathcal{F}(\xi),\ \ \bar{\delta} = (\delta, 0)\}. \tag{14}$$

Existence and stability results for variational inequalities of type (12) have been widely considered; we only mention here the seminal paper of S. M. Robinson (1980) on strongly regular generalized equations, and generalizations given by A. Dontchev and T. Rockafellar (1996). Further, we refer to Dontchev and Lewis (2005), Dontchev et al. (2006), Dontchev and Veliov (2009) for applications of metric regularity results. The investigations utilize the close relation between Lipschitz stability properties of solutions to (VI$_p$) w.r.t. the parameter p, and the Lipschitz continuity of Λ w.r.t. *rhs* perturbations δ (see, e.g., Dontchev, 1995, Dontchev and Malanowski, 2000).

For the given problem, let us start with the fixed point approach from Robinson (1980): Define $\bar\gamma(p,\xi) := T(\xi) - \psi(p,\xi)$. Then the element $\xi \in X$ solves (VI$_p$) if and only if it is a solution of (LVI$_\delta$) corresponding to $\bar\delta = (\delta, 0) = \bar\gamma(p,\xi)$. (Notice that the last two components in $\bar\gamma$ vanish due to the linearity of the related components of ψ.) The formulation can be further reduced to a fixed point problem in terms of the control component. To this aim, let $p \in \Pi$ and $u \in U$ be arbitrarily given: if $|p| \le \epsilon_0$, $\|u - u^0\|_1 \le \epsilon_1$ and ϵ_0, ϵ_1 are sufficiently small, one can find $x = x(\cdot, u, p)$, $\lambda = \lambda(\cdot, u, p)$ from (6), (7) and $\sigma = \sigma(\cdot, u, p) = \lambda(\cdot, u, p)^T g(x(\cdot, u, p), p)$. Obviously, $x, \lambda \in W^1_\infty$ and $\sigma \in W^2_\infty$. Setting $\mu_1 = \mu_1(\cdot, u, p) = [\sigma(\cdot, u, p)]_+$, $\mu_2 = [\sigma]_-$, we further have $\mu_1, \mu_2 \in W^1_\infty$. For the resulting $(x, \lambda, u, \mu) = \xi(u, p)$, the term $\bar\gamma = \bar\gamma(p, \xi)$ has the following components:

$$\begin{aligned}
\bar\gamma_1 &= \dot x^0 - \dot x + A(x - x^0) + B(u - u^0), \qquad \bar\gamma_2 = x^0(0) - x(0),\\
\bar\gamma_3 &= \dot\lambda^0 - \dot\lambda - A^T(\lambda - \lambda^0) - Q_{11}(x - x^0) - Q_{12}(u - u^0),\\
&\qquad\qquad \bar\gamma_4 = \lambda^0(1) - \lambda(1) + K\,(x(1) - x^0(1)),\\
\bar\gamma_5 &= \sigma - \sigma^0 - B^T(\lambda - \lambda^0) - Q_{21}(x - x^0), \qquad \bar\gamma_6 = \bar\gamma_7 = 0.
\end{aligned} \tag{15}$$

With $\bar\gamma' = (\bar\gamma_1, \ldots, \bar\gamma_5)$ define

$$\gamma(p,u) := \bar\gamma'(p, \xi(u,p)), \qquad \bar\gamma(p,\xi) = T(\xi) - \psi(p,\xi). \tag{16}$$

If we define by $\Lambda_u(\delta) \subset L_1$ the set of u-components of $\xi \in \Lambda(\delta)$ and further, set

$$\Phi_p(u) := \Lambda_u(\gamma(p,u)), \tag{17}$$

then the following fixed point characterization is obtained:

LEMMA 2.1 *The element $\xi = \xi(u,p) \in X$ solves* (VI$_p$) *if and only if $u \in \Phi_p(u)$.*

For the further analysis of (LVI$_\delta$) it will be useful to reduce the formulation to the control component $v := w - u^0$ and multiplier $\nu = (\nu_1, \nu_2)$ as main unknown variables. Following the approach from Dontchev and Malanowski (2000), the solution (z, q) of the linearized state-adjoint system from (LVI$_\delta$), (13), i.e.

$$\begin{aligned}
\dot z - Az - Bv &= -\delta_1, \quad z(0) = -\delta_2,\\
\dot q + A^T q + Q_{11} z + Q_{12} v &= -\delta_3, \quad q(1) = K\,z(1) - \delta_4,
\end{aligned} \tag{18}$$

can be represented as

$$z(t) \,=\, (Sv)(t) \,+\, z_\delta^{part}(t), \qquad q(t) \,=\, (\tilde{S}v)(t) \,+\, q_\delta^{part}(t) \tag{19}$$

where $S, \tilde{S}$ do not depend on δ (see Appendix 6.1). In analogy to Dontchev and Malanowski (2000), we can substitute the formulas into (13) and find a reduced version of (LVI$_\delta$) in terms of the unknown control u. In particular, using (69) – (70) from Appendix 6.1 we get

$$B^T q \,+\, Q_{21} z + \delta_5 \,=\, C\, v \,+\, r(\delta) \tag{20}$$

with

$$r(\delta) \,:=\, B^T q_\delta^{part} + Q_{21} z_\delta^{part} + \delta_5 \tag{21}$$

and further, by abbreviating $\hat{S}v = (Sv)(1)$,

$$C\, v \,:=\, \hat{S}^* K\, \hat{S} v \,+\, (S^* Q_{11} S + S^* Q_{12} + Q_{21} S)\, v. \tag{22}$$

If we denote

$$D \,=\, L_1 \times \mathbb{R}^n \times L_1 \times \mathbb{R}^n \times L_\infty, \qquad \hat{D} \,=\, L_2 \times \mathbb{R}^n \times L_2 \times \mathbb{R}^n \times W_2^1, \tag{23}$$

then r is linear and continuous as a mapping from D to L_∞, or from $\hat{D}$ to W_2^1. The mapping C is a linear self-adjoint operator on L_2 with $Cv \in W_2^1 \subset L_\infty$ for all $v \in L_2$. Summing up, the inclusion (LVI$_\delta$) transforms into

(LVI$_\delta^{red}$)
$$\begin{array}{l} C\, v \,-\, \nu_1 + \nu_2 + \sigma^0 + r(\delta) \,=\, 0, \\ -v - u^0 \,\in\, N_+(\nu_1), \quad v + u^0 - 1 \,\in\, N_+(\nu_2) \end{array}$$

for almost every $t \in [0, 1]$.

LEMMA 2.2 *For given δ, let $(z, q, v, \nu) \in L_\infty \times L_\infty \times L_2 \times L_\infty$ be a solution of* (LVI$_\delta$) *with $\nu = (\nu_1, \nu_2)$ satisfying $\|\nu_1 - \mu_1^0\|_\infty + \|\nu_2 - \mu_2^0\|_\infty < \beta/2$. Then v solves the variational inequality:*

(LVI') *find* $v \in W_\beta :\quad \big(C\, v + \sigma^0 + r(\delta),\, v' - v\big) \,\geq\, 0 \quad \forall\, v' \in W_\beta$

on $W_\beta = \{v' \in L_2(0,1;\mathbb{R}) :\, v' = 0$ *a.e. on* $J_\beta,\ 0 \leq v' + u^0 \leq 1$ *a.e. on* $[0,1]\}$.

Proof. Denote $w = v + u^0$, $w' = v' + u^0$. ¿From $w \in -N_+(\nu_1)$, $w - 1 \in N_+(\nu_2)$ and $v' \in W_\beta$ deduce

$$\begin{array}{rcl} \big(C\, v + \sigma^0 + r(\delta),\, v' - v\big) & = & (\nu_1, w' - w) - (\nu_2, (w' - 1) - (w - 1)) \\ & = & (\nu_1, w') + (\nu_2, 1 - w') \,\geq\, 0. \end{array}$$

Notice that a.e. on J_β we have $\mu_1^0 + \mu_2^0 = |\sigma^0| > \beta > 0$. By the assumption of the Lemma, $\nu_1(t) \geq \beta/2 > 0$ follows on J_β^+, and $\nu_2(t) \geq \beta/2 > 0$ on J_β^-, respectively: thus, $v(t) = 0$ on J_β, or $v \in W_\beta$ by complementarity. ∎

3. The linearized VI problem

3.1. Goh transformation. Monotonicity

The investigation of (LVI_δ) will start with a variable substitution which for the linearized state equation is well-known as Goh transformation. It will be also adapted to the linearized adjoint equation as follows:

Define the function $y\,:\,[0,1]\to\mathbb{R}$ by

$$y(t) \,=\, \int_0^t v(s)\,ds. \tag{24}$$

The integral is well-defined for all $v\in L_1$. Moreover, $y(0)=0$, and we will denote $y(1)=:h$. Now introduce $\zeta := z - B\,y$, $\eta := q + Q_{12}y$. In terms of these functions, the first equations in (LVI_δ) can be reformulated as follows:

$$\begin{array}{rclrcl} \dot\zeta - A\zeta - B_1 y &=& -\delta_1, & \zeta(0) &=& -\delta_2, \\ \dot\eta + A^T\eta + Q_{11}\zeta + M^T y &=& -\delta_3, & \eta(1) &=& K\,\zeta(1) + W\,h - \delta_4, \end{array} \tag{25}$$

where the matrices are given by

$$B_1 = AB - \dot B, \quad M \,=\, B^T Q_{11} - Q_{21}A - \dot Q_{21}, \quad W \,=\, Q_{12}(1) + K\,B(1).$$

Remark 1 *The notations follow mainly* Aronna et al. (2012a) *where similar calculations occur in transforming the second variation of the Lagrangian. Alternatively, the matrices can be expressed via Lie brackets as*

$$B_1 = -\left[f^0, g^0\right], \qquad M \,=\, -(\lambda^0)^T\nabla_x\left[f^0, g^0\right].$$

Notice further that

$$B^T\eta \,+\, Q_{21}\zeta \;=\; B^T q \,+\, Q_{21}z \,+\, (B^T Q_{12} -\, Q_{21}B)y.$$

In the given case of scalar controls, the last term vanishes so that almost everywhere on $[0,1]$ from (LVI_δ) we obtain

$$\begin{array}{c} B^T\eta \,+\, Q_{21}\zeta \,-\, \nu_1 + \nu_2 + +\mu_1^0 - \,\mu_2^0 \,+ \delta_5 = \,0, \\ -v - u^0 \,\in\, N_+(\nu_1), \;\; v + u^0 - 1 \,\in\, N_+(\nu_2). \end{array} \tag{26}$$

In analogy to (19), one can use solution operators S_1 and $\tilde S_1$ for (25) to eliminate ζ and η from the first equation in (26). We obtain

$$\zeta \,=\, S_1 y \,+\, \zeta_\delta^{part}, \qquad \eta \,=\, \tilde S_1 y \,+\, \tilde W h \,+\, \eta_\delta^{part},$$

together with related boundary conditions,

$$\zeta(0) \,=\, \zeta_\delta^{part}(0), \qquad \eta(1) \,=\, K\hat S_1 y \,+\, Wh \,+\, \eta_\delta^{part}(1), \quad \hat S_1 y \,:=\, (S_1 y)\,(1),$$

cf. (25). Notice that $\zeta_\delta^{part} = z_\delta^{part}$, $\eta_\delta^{part} = q_\delta^{part}$ coincide with the δ-dependent terms from (19). For details see the Appendix.

Inserting the above expressions into the first part of (26) yields

$$C\,v \;=\; B^T(\tilde S_1 y + \tilde W h) \;+\; Q_{21} S_1 y \;=:\; \hat C\, y \tag{27}$$

a.e. on $[0,1]$. As far as C, as an operator on L_2, is self-adjoint, we have

$$(C\,v, \tilde v\,) \;=\; \left(\hat C\, y, \dot{\tilde y}\right) \;=\; \left(\dot y, \hat C\,\tilde y\,\right) \tag{28}$$

for all $y, \tilde y \in W_2^1$ satisfying $y(t) = \int_0^t v(s)ds$, $\tilde y(t) = \int_0^t \tilde v(s)ds$ and $y(1) = h$, $\tilde y(1) = \tilde h$, respectively.

Next we apply to (28) an integration by parts as known from the transformation of the second variation of $J = J_0(x,u)$ in Goh (1966b), Dmitruk (2008), Aronna et al. (2012a), or Poggiolini and Stefani (2008). It will be shown that the result yields a new symmetric formulation for C such that $(C\,v, v\,)$ is equivalent to the quadratic form $\Omega_{\mathcal P}$ in Aronna et al. (2012a).
Let v belong to L_2. Then, by (27) we have $\hat C\, y \in W_2^1$ and further,

$$(C\,v, \tilde v\,) \;=\; \left(\hat C\, y, \dot{\tilde y}\right) \;=\; \hat C\, y \cdot \tilde y\Big|_{t=1} \;-\; \left(\frac{d}{dt}(\hat C\, y), \tilde y\,\right). \tag{29}$$

In terms of $\zeta_0 = S_1 y$, $\eta_0 = \tilde S_1 y + \tilde W h$ one can write

$$\begin{aligned} \frac{d}{dt}\hat C\, y \;&=\; \frac{d}{dt}\left(B^T\eta_0 + Q_{21}\zeta_0\right) \\ &=\; B^T(-A^T\eta_0 - Q_{11}\zeta_0 - M^T y) \;+\; \dot B^T\eta_0 \\ &\qquad +Q_{21}(A\zeta_0 + B_1 y) \;+\; \dot Q_{21}\zeta_0 \\ &=\; -B_1^T\eta_0 \;-\; M\zeta_0 \;-\; R\,y, \end{aligned}$$

where $R \;=\; B^T M^T - Q_{21}B_1$ can be equally expressed as

$$R = B^T Q_{11} B - B_1^T Q_{12} - Q_{21} B_1 - \frac{d}{dt}(B^T Q_{12}) \;=\; -(\lambda^0)^T\;\left[g^0, [f^0, g^0]\right], \tag{30}$$

see (10). Further, by direct calculation we find

$$\left(\hat C\, y, \dot{\tilde y}\right) \;=\; c_1(y, \tilde y) \;+\; \left(C^G y,\, \tilde y\right) \tag{31}$$

where c_1 and C^G are the related boundary and the integral forms, i.e,

$$\begin{aligned} c_1(y,\tilde y) \;&=\; (\hat S_1\tilde y + B(1)\tilde h)^T K\,(\hat S_1 y + B(1)h) \;+\; h\,\left(B(1)^T Q_{12}(1)\right)\tilde h \\ &\qquad +\; \tilde h Q_{21}(1)(\hat S_1 y) \;+\; (\hat S_1\tilde y)^T Q_{12}(1)\,h, \end{aligned} \tag{32}$$

$$\left(C^G y,\, \tilde y\right) \;=\; \left((S_1^* Q_{11} S_1 \;+\; S_1^* M^T \;+\; M\,S_1 \;+\; R) y,\, \tilde y\right).$$

Both quadratic forms are symmetric, and we define

$$\Omega(y,h) \;:=\; c_1(y,y) \;+\; (C^G y, y) \;=\; (C\,v, v\,)\,.$$

The following coercivity type condition was derived in Aronna et al. (2012a) and proven to be a sufficient condition for a strict Pontryagin minimum in (CP_0), see Theorem 5.5 therein. It will be used in the given context to obtain the monotonicity property of C as an operator from L_2 to $(L_2)^* \doteq L_2$. The formulation slightly differs from Aronna et al. (2012a) in that the subspace of feasible v is taken L_2 so that (y,h) belongs to $W_2^1 \times \mathbb{R}$ instead of $L_2 \times \mathbb{R}$, and further, it is given a stable formulation depending on certain threshold parameter β:

(H2) (strong second-order optimality condition)
There exist constants $\beta > 0,\ m > 0$ such that

$$\Omega(y,h) \;=\; (C\,v, v\,) \;\geq\; m\,\left(\|y\|_2^2 + |h|^2\right) \tag{33}$$

for all $(v,y,h) \in L_2 \times W_2^1 \times \mathbb{R}$ satisfying

$$y(t) \;=\; \int_0^t v(s)\,ds, \;\; y(1) \;=\; h, \quad v(t) \;=\; 0 \;\text{ a.e. } \;\text{on } J_\beta.$$

The condition says, in particular, that C as a linear symmetric operator on $W_\beta \subset L_2$ is strictly monotone (see Lemma 2.2 for notation).

Under the structural assumption (H1), one can replace (33) by the weaker condition

$$\Omega(y,h) \;\geq\; m'\,\|y\|_2^2 \tag{34}$$

for some constant $m' > 0$. Indeed, the triple (v,y,h) is restricted in a way guaranteeing $y(t) \equiv h$ on $J_\beta \cap (t_2,1)$. Taking β sufficiently small, the latter interval has a length $l \geq \bar{l} = (1-t_2)/2 > 0$ so that $\|y\|_2^2 \geq \bar{l}\,h^2$ with $\bar{l}$ independent of (v,y,h). In this sense, for appropriately chosen constants, (33) and (34) are equivalent.

As it was pointed out in Aronna et al. (2012a), Dmitruk (2008), the inclusion

$$\{(y,h) \in W_2^1 \times \mathbb{R} : \, y(1) = h\} \subset \; L_2 \times \mathbb{R}$$

is dense. Since Ω is continuous on $L_2 \times \mathbb{R}$, (H2) further yields (33) to hold for all $(y,h) \in L_2 \times \mathbb{R}$ such that $y \equiv 0$ on $J_\beta \cap (0,t_1)$, $y \equiv h$ on $J_\beta \cap (t_2,1)$. This property can be used to show that

$$R(t) \;\geq\; m \;>\; 0 \qquad \text{for all } t \in I_\beta. \tag{35}$$

The proof follows an idea from Dontchev and Malanowski (2000): for τ being an interior point of I_β and ϵ a small positive number, set

$$y_\epsilon(t) = \begin{cases} 1 & \text{if} \quad \tau \leq t \leq \tau + \epsilon, \\ 0 & \qquad \text{otherwise.} \end{cases}$$

From (32) we obtain

$$\Omega(y_\epsilon, 0) = \int_\tau^{\tau+\epsilon} R(s)\, ds + \mathrm{o}(\epsilon).$$

Combining this estimate with $\Omega(y_\epsilon, 0) \geq m\|y_\epsilon\|_2^2 = m\epsilon$ and dividing by ϵ, the assertion follows for $\epsilon \searrow 0$.

REMARK 2 *The condition* $R = -(\lambda^0)^T \left[g^0, \left[f^0, g^0\right]\right] \geq m > 0$ *is known as the (strict) higher-order Legendre condition, see, e.g.* Krener (1977), Dmitruk (1977), Knobloch (1981), Poggiolini and Stefani (2005, 2008), Stefani (2008). *In particular, it ensures that* u^0 *has a singular arc of* order one *on* $[t_1, t_2]$ (*see* Krener, 1977, Knobloch, 1981, or Zelikin and Borisov, 1994) *such that* $u^0 = u_s(x^0, \lambda^0) = P\,/\,R$ *on* (t_1, t_2).

As a direct consequence of the monotonicity condition (H2) for C, we obtain

LEMMA 3.1 *Under Assumptions* (H1) *and* (H2), *for arbitrary* $\delta \in D$ *from* (23) *the variational inequality* (LVI') *has a unique solution* $v^\delta \in L_2(0, 1; \mathbb{R})$.

Proof. As it was mentioned above, the mapping $C : L_2 \to L_2$ is a linear self-adjoint operator. It is continuous due to (22). By (H2), C is strictly monotone on W_β. By its construction, the set $W_\beta \subset L_2$ is nonempty, bounded, closed and convex. From the existence theory for variational inequalities in separable, reflexive Banach spaces from Kinderlehrer and Stampacchia (1980) (see Theorem 1.7, ch. III therein) it follows that, for arbitrary $r(\delta) \in L_\infty \subset L_2$, the variational inequality (LVI') has a unique solution $v = v^\delta \in W_\beta$. ■

REMARK 3 *Due to the monotonicity and symmetry properties of* C *(respectively* c_1, C^G*) and the convexity of* W_β*, the solution* v^δ *of* (LVI') *solves the problem*

$$\min \frac{1}{2}\,(\,C\,v, v\,) + \left(\sigma^0 + r(\delta), v\,\right) \qquad s.t. \quad v \in W_\beta \subset L_2(0, 1; \mathbb{R}). \quad (36)$$

3.2. Existence result for (LVI$_\delta$)

In this section, existence and local stability of solutions for the linearized variational inequalities (LVI$_\delta$), respectively (LVI$_\delta^{red}$) will be proved.

THEOREM 1 *Let Assumptions* (H1) *and* (H2) *hold for* $p = 0$. *Then, there exists a neighborhood* $W_D \subset \hat{D}$ *of* $\delta^0 = 0$ *such that, for arbitrary* $\delta \in W_D$, *the variational inequality* (LVI$_\delta$) *has an unique solution* $\xi^\delta = (z^\delta, q^\delta, v^\delta, \nu^\delta) \in W_\infty^1 \times W_\infty^1 \times L_\infty \times (L_\infty^+)^2$. *Moreover, the components* $(z^\delta, q^\delta, \nu^\delta)$ *depend Lipschitz continuously on* $\delta \in W_D \subset \hat{D}$ *in the following sense:*

$$\|z^\delta - z^{\delta'}\|_2 + \|q^\delta - q^{\delta'}\|_2 + \|\nu^\delta - \nu^{\delta'}\|_\infty = O(\|\delta - \delta'\|_{\hat{D}}),$$
$$|\pi z^\delta - \pi z^{\delta'}| + |\pi q^\delta - \pi q^{\delta'}| = O(\|\delta - \delta'\|_{\hat{D}})$$

where $\pi : W_\infty^1 \to R^{2n}$ *denotes the boundary trace operator:* $\pi\phi = (\phi(0), \phi(1))$.

As a first auxiliary result, consider the solution stability for (LVI'):

LEMMA 3.2 *If the perturbations* δ *in* (LVI') *are restricted to* $\hat{D} \subset D$ *from* (23) *then the solutions* $v = v^\delta$, *respectively* $y = y^\delta$ *satisfy*

$$\|y^\delta - y^{\delta'}\|_2 + |h^\delta - h^{\delta'}| \,=\, O(\|\delta - \delta'\|_{\hat{D}}) \qquad \forall\, \delta, \delta' \in \hat{D}. \tag{37}$$

An analogous estimate holds for $v^\delta, v^{\delta'}$ *taken as elements of* H^{-1}.

Proof. Let v, v' be the solutions of (LVI') related to $r(\delta)$, respectively $r(\delta')$, and define y, y' by

$$y(t) \,=\, \int_0^t v(s)\, ds, \qquad y'(t) \,=\, \int_0^t v'(s)\, ds.$$

Then,

$$\begin{aligned}\left(\hat{C}(y - y'),\, v - v'\right) &\leq\; (r(\delta') - r(\delta),\, v - v') \\ &=\; (r(\delta') - r(\delta))|_{t=1} \cdot (h - h') - (\dot{r}(\delta') - \dot{r}(\delta), y - y') \\ &\leq\; c_r \|r(\delta) - r(\delta')\|_{1,2} \left(|h - h'| \,+\, \|y - y'\|_2\right).\end{aligned}$$

Notice that, due to (21), $\|r(\delta) - r(\delta')\|_{1,2} \leq c\|\delta - \delta'\|_{\hat{D}}$ for some constant $c > 0$. On the other hand, (H2) yields

$$\left(\hat{C}(y - y'),\, v - v'\right) \geq m \left(\|y - y'\|_2^2 + |h - h'|^2\right)$$

so that

$$\|y - y'\|_2 + |h - h'| \,\leq\, \frac{2c_r}{m} \|r(\delta) - r(\delta')\|_{1,2} = \mathrm{O}(\|\delta \,-\, \delta'\|_{\hat{D}}), \tag{38}$$

and hence the Lemma. ∎

The next lemma shows how (LVI') can be used to solve ($\mathrm{LVI}_\delta^{red}$):

LEMMA 3.3 *For* $p = 0$, *let the assumptions* (H1) *and* (H2) *be fulfilled. If* $\delta = (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5)$ *is sufficiently close to* $\delta^0 = 0$ *in* $\hat{D} = L_2 \times \mathbb{R}^n \times L_2 \times \mathbb{R}^n \times W_2^1$, *then there exists a unique multiplier pair* $\nu^\delta = (\nu_1^\delta, \nu_2^\delta) \in L_\infty(0, 1; \mathbb{R}_+^2)$ *such that* (v^δ, ν^δ) *(with* v^δ *chosen as in Lemma 3.1) solves* ($\mathrm{LVI}_\delta^{red}$). *In addition, the following estimate holds:*

$$\|\nu^\delta \,-\, \nu^{\delta'}\|_\infty =\; O(\|\delta - \delta'\|_{\hat{D}}).$$

Proof. By Lemma 3.1, the function v^δ solves (36) on W_β, or equivalently, on

$$W'_\beta = \{v \in L_2 : \;\; v = 0 \;\text{ on } J_\beta, \quad 0 \leq v + u^0 \leq 1 \;\text{ on } I_\beta\}.$$

Due to the linear independence of active constraints in this problem, one can find unique multiplier functions

$$\nu' \in L_2(J_\beta, \mathbb{R}), \quad \nu_1'', \nu_2'' \in L_2(I_\beta, \mathbb{R}_+)$$

such that the following first-order optimality condition is fulfilled for $v = v^\delta$:

$$C\,v^\delta + \sigma^0 + r(\delta) - \nu'\chi(J_\beta) - (\nu_1'' - \nu_2'')\chi(I_\beta) = 0 \in L_2.$$

(The notation $\chi = \chi(G)$ is used for the indicator function of a set G.)
Setting

$$\nu_1^\delta := \nu'\chi(J_\beta^+) + \nu_1''\chi(I_\beta), \quad \nu_2^\delta := -\nu'\chi(J_\beta^-) + \nu_2''\chi(I_\beta),$$

we obtain

$$\begin{aligned}\nu_1^\delta - \nu_2^\delta &= \hat{C}\,y^\delta + \sigma^0 + r(\delta)\\ &= \hat{C}\,y^0 + \sigma^0 + \mathrm{O}(\|\delta - \delta^0\|_{\hat{D}})\\ &= \mu_1^0 - \mu_2^0 + \mathrm{O}(\|\delta - \delta^0\|_{\hat{D}}),\end{aligned}$$

i.e. $(\nu_1^\delta - \nu_2^\delta)$ is close to $(\mu_1^0 - \mu_2^0)$ in L_∞ sense. In particular, for small perturbations $(\delta - \delta^0)$, on J_β^+ the multiplier $\nu_1^\delta = \nu' \approx \mu_1^0$ is strictly positive (and $\nu_2^\delta > 0$ on J_β^-) so that $\nu^\delta \geq 0$ on the whole interval $[0,1]$. Thus, the triple $(v^\delta, \nu_1^\delta, \nu_2^\delta) \in L_2 \times L_\infty \times L_\infty$ solves (LVI$_\delta^{red}$).
In order to verify the estimate for $(\nu^\delta - \nu^{\delta'})$ notice that, by $\nu_1^\delta(t)\cdot\nu_2^\delta(t) = 0$ a.e. on $[0,1]$ and the non-negativity of both ν_1^δ and ν_2^δ, we have

$$\nu_1^\delta = [\hat{C}\,y^\delta + \sigma^0 + r(\delta)]_+, \quad \nu_2^\delta = [\hat{C}\,y^\delta + \sigma^0 + r(\delta)]_-.$$

Using the relation $|[a]_\pm - [b]_\pm| \leq |a-b|$ for $a,b \in \mathbb{R}$, we see that

$$\left|\nu_{1,2}^\delta - \nu_{1,2}^{\delta'}\right| \leq \left|\hat{C}(y^\delta - y^{\delta'}) + r(\delta) - r(\delta')\right|$$

a.e. on $[0,1]$. By (27), $\hat{C}$ as a mapping from $(y,h) \in L_2 \times \mathbb{R}$ to $\hat{C}y \in L_\infty$ is Lipschitz continuous. The same is true for $r = r(\delta) \in L_\infty$ and $\delta \in \hat{D}$; see (21). Combined with the estimate (37) from Lemma 3.2, these facts yield the desired Lipschitz continuity of $\nu^\delta \in L_\infty$ w.r.t. $\delta \in \hat{D}$. ■

After these preliminaries we come to the proof of Theorem 1:

Proof. Given $\delta \in W_D \subset \hat{D} \subset D$, find the solution $v = v^\delta \in W_\beta$ of (LVI') (respectively (36)). Due to the definition of W_β, the solution is bounded so that $v^\delta \in L_\infty$. According to Lemma 3.3, one can construct associated multipliers $\nu_1^\delta, \nu_2^\delta \in L_\infty^+$. Further, find $y^\delta \in W_\infty^1 \subset L_\infty$ by integrating v^δ as in (24) and ζ^δ, $\eta^\delta \in W_2^1 \subset L_\infty$ from solving the system (25). Then, one can determine

$$z^\delta = \zeta^\delta + By^\delta, \quad q^\delta = \eta^\delta - Q_{12}y^\delta \tag{39}$$

as functions in W_2^1, and the resulting vector ξ^δ solves (LVI$_\delta$).
Assume for the moment that there exists $\hat{\xi}^\delta \neq \xi^\delta$ solving (LVI$_\delta$).

$$|C\hat{v}^\delta + r(\delta)| \leq |B^T\hat{\eta}^\delta + Q_{21}\hat{\zeta}^\delta| + |\delta_5| \leq \|B\|_\infty\|\hat{\eta}\|_\infty + \|Q_{21}\|_\infty\|\hat{\zeta}\|_\infty + \|\delta\|_D.$$

But $\hat{\zeta}, \hat{\eta}$ solve (25) so that, by Lemma 3.2,

$$\|\hat{\zeta}\|_\infty + \|\hat{\eta}\|_\infty \leq c(\|\hat{y}\|_2 + \|\delta\|_D) \leq \hat{c}\,\|\delta\|_{\hat{D}}.$$

By shrinking W_D, if necessary, the latter will become small enough to ensure $|C\hat{v}^\delta + r(\delta)| < \beta/2$, so that $(\hat{\nu}_1 - \hat{\nu}_2)$ is close to $(\mu_1^0 - \mu_2^0)$ in L_∞ sense. In particular, $\hat{\nu}_1 \geq \mu_1 - \beta/2 > 0$ on J_β^+, and $\hat{\nu}_2 < 0$ on J_β^-: thus, $\hat{v}^\delta \in W_\beta$ is a solution of (LVI'). By Lemma 3.1, it coincides with v^δ. Since the components $(z^\delta, q^\delta, \nu^\delta)$ are uniquely determined from v^δ by the linearized canonical system, respectively Lemma 3.3, we get $\hat{\xi}^\delta = \xi^\delta$.

Finally, let us check the Lipschitz properties of the solution $\xi = \xi^\delta$: Lemma 3.2 says that $y = y^\delta$ as a function of L_2 as well as $h = h^\delta$ depend Lipschitz continuously on $\delta \in \hat{D}$. Thus, the same is true for $\zeta^\delta, \eta^\delta \in W_2^1 \subset L_\infty$ by (25) and further, for $z^\delta, q^\delta \in L_2$ by construction (39). The estimate for the boundary values follows from

$$\begin{aligned} z^\delta(0) &= \zeta^\delta(0), & q^\delta(0) &= \eta^\delta(0),\\ z^\delta(1) &= \zeta^\delta(1) + B(1)h^\delta, & q^\delta(1) &= \eta^\delta(1) - Q_{12}(1)h^\delta. \end{aligned}$$

Using (26), we further obtain

$$\phi(\delta) := \nu_1^\delta - \nu_2^\delta = B^T\eta^\delta + Q_{21}\zeta^\delta + \sigma^0 + \delta_5,$$

i.e., this expression as a function in $W_2^1 \subset L_\infty$ is Lipschitz continuous w.r.t. $\delta \in \hat{D}$. Since $\nu_1^\delta = [\phi(\delta)]_+$, $\nu_2^\delta = [\phi(\delta)]_-$ as a consequence of the complementarity relations from (26), we end up with the desired estimates for $\nu_1, \nu_2 \in L_\infty$. ■

3.3. Solution structure

In Theorem 1 and Lemma 3.2, Lipschitz continuity properties for solutions of (LVI$_\delta$) are obtained. They include the continuity of $z, q, y \in L_2$ w.r.t. δ but do not cover corresponding estimates for the control v as an element of L_1; e.g. the control stability analysis requires further structural investigation answering, in particular, the question of a possible preservation of the bang-singular-bang behavior for $w := v + u^0$. For certain local solutions of (VI$_p$), a related first stability theorem was derived in Felgenhauer (2012).

Suppose the conditions (H1) and (H2) hold for the extremal (x^0, u^0, λ^0) of (CP$_0$). Taking the data from (LVI$_\delta^{red}$), set

$$\tilde{\sigma} := C\,v + \sigma^0 + r(\delta).$$

Then the reduced variational inequality is equivalent to

$$0 \leq w = v + u^0 \leq 1, \quad [\tilde{\sigma}]_+ w = 0, \quad [\tilde{\sigma}]_-(w-1) = 0, \tag{40}$$

i.e., $\tilde{\sigma}$ represents a switching function for $w = v + u^0$. By construction (20),

$$\tilde{\sigma} = \sigma^0 + B^T q + Q_{21} z + \delta_5 = \sigma^0 + B^T \eta + Q_{21}\zeta + \delta_5 \tag{41}$$

where (ζ, η) solves the linear system (25). Using Lemma 3.2 we obtain

$$\|\tilde{\sigma} - \sigma^0\|_\infty = \mathrm{O}(\|\delta\|_{\hat{D}}). \tag{42}$$

As long as $v \in L_\infty$ and $\delta_1, \delta_3 \in W_l^1$, $\delta_5 \in W_l^2$, the function $\tilde{\sigma}$ will belong to W_l^2, $1 \le l \le \infty$. Indeed,

$$\begin{aligned} \frac{d}{dt}(B^T q + Q_{21} z + \delta_5) &= -Mz - B_1^T q - r_1(\delta), \\ \frac{d^2}{dt^2}(B^T q + Q_{21} z + \delta_5) &= -M_1 z + B_2^T q + (B_1^T Q_{12} - MB)v \\ &\qquad + (M\delta_1 + B_1^T \delta_3) - \frac{d}{dt} r_1(\delta) \end{aligned}$$

with the abbreviations $B_2 = AB_1 - \dot{B}_1$, $M_1 = MA + \dot{M} - B_1^T Q_{11}$, and $r_1 = r_1(\delta) := B^T \delta_3 + Q_{21}\delta_1 - \dot{\delta}_5$. Notice that $B_1^T Q_{12} - MB = -R$ (see (30)) so that we finally arrive at

$$\dot{\tilde{\sigma}} = \dot{\sigma}^0 - Mz - B_1^T q - r_1, \tag{43}$$

$$\ddot{\tilde{\sigma}} = \ddot{\sigma}^0 - M_1 z + B_2^T q - R^T v + r_2 \tag{44}$$

with $r_2 = r_2(\delta) := M\delta_1 + B_1^T \delta_3 - \dot{r}_1$. Using the notations from (10) and relations (30), the second order derivative can be equally expressed as

$$\ddot{\tilde{\sigma}} = (P + B_2^T q - M_1 z + r_2) - w\,R =: \tilde{P} - w\,R. \tag{45}$$

On the set I_β, where $R \ge m > 0$ is guaranteed by (H2), define

$$\tilde{u}_s := \tilde{P}/R. \tag{46}$$

The first auxiliary result concerns a selection property for $w = w(t)$:

LEMMA 3.4 *Let* $(z, q, v, \nu) \in W_\infty^1 \times W_\infty^1 \times L_\infty \times L_\infty$ *be a solution of* (LVI$_\delta$) *for some* $\delta \in \hat{D}$ *satisfying the additional assumption* $r_1 \in W_2^1$, $r_2 \in L_2$. *Then* $\tilde{\sigma}$ *defined by* (41) *belongs to* W_2^2, *and*

$$\|\tilde{u}_s - u_s\|_{2,I_\beta} = O(\|\delta\|_{\hat{D}} + \|r_2\|_2). \tag{47}$$

Further, $w(t) \in \{0,\, 1,\, \tilde{u}_s(t)\}$ *for almost every* $t \in [0,1]$.

Proof. Relation (47) follows from (46) and the definition $u_s = P/R$ on the interval $I_\beta \supset [t_1, t_2]$:

$$|\tilde{u}_s - u_s| \le m^{-1} |B_2^T q - M_1 z + r_2|$$

together with Theorem 1 yields the desired estimate.

In order to prove the selection property for w, consider next the set $\tilde{I} = \{t \in [0,1] : \tilde{\sigma} = 0\}$. It will be shown that, almost everywhere on $\tilde{I}$, $\dot{\tilde{\sigma}}(t) = \ddot{\tilde{\sigma}}(t) = 0$.

Let $t \in \tilde{I}$ be a point where $\dot{\tilde{\sigma}}(t) \neq 0$ then $\tilde{\sigma} \neq 0$ in a neighborhood of t, i.e., t is a boundary point of an interval where $\tilde{\sigma} > 0$. Due to $\tilde{\sigma} \in W_2^2 \subset C^1$, the number of such points is at most countable so that they do not contribute to $meas \tilde{I}$. Now consider the second-order derivative of $\tilde{\sigma}$ on $\tilde{I}$: due to $\ddot{\tilde{\sigma}} \in L_2$, it is sufficient to show that the function vanishes in all its Lebesgue points on the set $\tilde{I}$ (see Rudin, 1987). If $\dot{\tilde{\sigma}}(t) = 0$ but $\ddot{\tilde{\sigma}}(t) \neq 0$, then it follows from

$$\dot{\tilde{\sigma}}(\tau) \;=\; \dot{\tilde{\sigma}}(t) \;+\; \int_t^\tau \ddot{\tilde{\sigma}}(s)\, ds \;=\; (\tau - t)\, \frac{\int_t^\tau \ddot{\tilde{\sigma}}(s)\, ds}{\tau - t}$$

that $\dot{\tilde{\sigma}} \neq 0$ for $\tau \neq t$ in a neighborhood of t. Again, t is a boundary point of an open interval where $\tilde{\sigma} \neq 0$, i.e. it belongs to some countable subset of $\tilde{I}$. Therefore, almost everywhere on $\tilde{I}$, the relation $\ddot{\tilde{\sigma}} = 0$ is valid. According to (45), $w(t)$ and $\tilde{u}_s(t)$ then coincide almost everywhere on $\tilde{I}$ whereas outside of this set $w = 0$ or $w = 1$ depending on the sign of $\tilde{\sigma}$. ∎

Next we will formulate the main structural result for (LVI$_\delta$). It will be shown that, for appropriately chosen small δ, the control component $w = v + u^0$ has the same principal bang-singular-bang structure as u^0. The idea of the proof follows the argumentation from Felgenhauer (2012). The difficulty in the present setting is the lack of an L_∞ estimate for z^δ and q^δ (replacing in a sense $(x^\delta - x^0)$ or $(\lambda^\delta - \lambda^0)$) whereas a crucial assumption in Theorem 3.1 from Felgenhauer (2012) was the restriction to extremals satisfying $\|x - x^0\|_\infty + \|\lambda - \lambda^0\|_\infty < \epsilon$ for certain small $\epsilon > 0$.

Theorem 2 *Let the assumptions* (H1) *and* (H2) *hold for* (CP$_0$). *Further, for given $\delta \in W_D \subset \hat{D}$ with the additional property $r_1(\delta), r_2(\delta) \in L_\infty$, let $\xi^\delta = (z^\delta, q^\delta, v^\delta, \nu^\delta)$ denote the solution of* (LVI$_\delta$) *with $v^\delta \in W_\beta$. Then there exists a constant $\bar{\epsilon} > 0$ such that, for all ϵ, δ satisfying*

$$\|\delta\|_{\hat{D}} + \|r_1(\delta)\|_\infty + \|r_2(\delta)\|_\infty < \epsilon \leq \bar{\epsilon}, \tag{48}$$

the function $w = v^\delta + u^0$ has bang-singular-bang structure in the following sense: there exist points $\tilde{t}_1 < \tilde{t}_2$ such that

$$w(t) \;=\; \begin{cases} 0 & on \;\; [0, \tilde{t}_1), \\ \tilde{u}_s(t) & on \;\; (\tilde{t}_1, \tilde{t}_2), \\ \bar{u} & on \;\; (\tilde{t}_2, 1], \end{cases}$$

and $\|w - u^0\|_1 + |\tilde{t}_1 - t_1| + |\tilde{t}_2 - t_2| = O(\epsilon)$.
Moreover, $0 < m/2 \leq \tilde{u}_s \leq 1 - m/2 < 1$ *a.e. on* $[\tilde{t}_1, \tilde{t}_2]$.

The proof requires a piecewise inspection of the sign of the switching function $\tilde{\sigma}$ (see Felgenhauer, 2012). Making a prediction for the possible value of w, we find the prospective switching function and correct the control if the sign changes. By $u^\pm$ abbreviate the assignments $u^+ \equiv 0$ or $u^- \equiv 1$. Then, for given $t_0 \in [0, 1]$, consider the system

$$\begin{aligned} \dot{x}^\pm &= f(x^\pm) + u^\pm g(x^\pm), & x^\pm(t_0) = x^0(t_0), \\ \dot{\lambda}^\pm &= -\nabla_x H(x^\pm, u^\pm, \lambda^\pm), & \lambda^\pm(t_0) = \lambda^0(t_0). \end{aligned} \tag{49}$$

(In this this part of the proof, $p = 0$ is fixed and thus will be omitted).
Under the smoothness assumptions on the data, there exists a constant $\Delta > 0$ such that, for each $t_0 \in [0,1]$, the systems have unique solutions defined at least on $[t_0 - \Delta, t_0 + \Delta]$. By $\sigma^\pm$ abbreviate $(\lambda^\pm)^T g(x^\pm)$; then

$$\dot\sigma^\pm = (\lambda^\pm)^T [f(x^\pm), g(x^\pm)], \quad \ddot\sigma^\pm = P^\pm - u^\pm R^\pm$$

in analogy to (10). If $t_0 \in (t_1 - \Delta/2, t_2 + \Delta/2)$ and $t \in (t_1, t_2)$ then

$$\begin{aligned} \ddot\sigma^\pm &= \ddot\sigma^0 + (P^\pm - P) - u^\pm(R^\pm - R) - (u^\pm - u^0)R \\ &= \pm R\,|u^\pm - u^0| + \mathrm{O}(|t - t_0|), \end{aligned}$$

see Lemma 6.1. By assumptions (H1), (H2), $R\,|u^\pm - u^0| \geq m^2$ on (t_1, t_2) so that the continuity of $\ddot\sigma^\pm$ yields

$$\pm\ddot\sigma^\pm \geq m^2/2 \tag{50}$$

for all $t \in (t_0 - \Delta/2, t_0 + \Delta/2)$ if only Δ is taken sufficiently small.
Similarly to (49), define $z^\pm, q^\pm$ and $\tilde\sigma^\pm$ as solutions of

$$\begin{array}{rcll} \dot z^\pm &=& Az^\pm + B(u^\pm - u^0) - \delta_1, & z^\pm(t_0) = z(t_0), \\ \dot q^\pm &=& -A^T q^\pm - Q_{11} z^\pm - Q_{12}(u^\pm - u^0) - \delta_3, & q^\pm(t_0) = q(t_0), \\ \tilde\sigma^\pm &=& \sigma^0 + B^T q^\pm + Q_{21} z^\pm + \delta_5. & \end{array} \tag{51}$$

To begin with, we will show that $w = v + u^0$ with $v = v^\delta$ from (LVI$_\delta$) has a left bang–arc with $w \equiv 0$ (and analogously, a right bang–arc with $w \equiv \bar u \in \{0,1\}$) if ϵ in (48) is taken sufficiently small (for comparison see Lemma 4.1, Felgenhauer, 2012):

LEMMA 3.5 *Let the Assumptions* (H1) *and* (H2) *hold. If* $\bar\epsilon > 0$ *is sufficiently small then, for all* δ *satisfying* (48)*, there exists a point* $\bar t_1$ *such that* $|\bar t_1 - t_1| = \mathrm{O}(\epsilon^{1/2})$*, and* $w \equiv 0$ *on* $[0, \bar t_1]$.

Proof. Starting from the left, we see that $\sigma^0 = \sigma^+ = P$ for all $t \leq t_1$, and $\ddot\sigma^+ > m^2/2$ on $I_\beta \cap [0, t_1]$ if only β is chosen sufficiently small. Thus, from the Taylor expansion of order two for σ^+ at $t = t_1$,

$$\sigma^0 = \sigma^+ > \rho > 0 \quad \forall\, t \in [0, t_1 - 2\sqrt{\rho/m^2}]$$

follows for $\rho \in (0, \beta)$. By Lemma 6.2, (i), for $t \leq t_1$ we obtain

$$\tilde\sigma^+ \geq \sigma^+ - |\tilde\sigma^+ - \sigma^+| > \rho - c'\epsilon > \rho/2$$

for all $t \leq t_1 - \mathrm{O}(\sqrt{\rho})$ in case $\beta > \rho \geq 2c'\epsilon$: thus, for sufficiently small ϵ, setting $\rho' = 2c'\epsilon$, $\bar t_1 = t_1 - \sqrt{8c'\epsilon/m^2}$ we have $\tilde\sigma^+ = \tilde\sigma > 0$, and $w \equiv 0$ at least on $[0, \bar t_1]$. ∎

The boundary points of the left and right end bang-arcs are given by

$$\tilde{t}_1 = \sup\{t > 0 :\ \tilde{\sigma} > 0 \quad \text{on} \quad [0,t]\}, \quad \tilde{t}_2 = \inf\{t < 1 :\ |\tilde{\sigma}| > 0 \quad \text{on} \quad [t,1]\},$$

and satisfy $0 < \tilde{t}_1 \leq 1,\ 0 \leq \tilde{t}_2 < 1$.

LEMMA 3.6 *Let the assumptions of Lemma* 3.5 *and Theorem* 2 *hold true. If* $\tilde{t}_1 < \tilde{t}_2$ *and* $(t', t'') \subset (\tilde{t}_1, \tilde{t}_2)$ *is an interval where either* $\tilde{\sigma} > 0$, *or* $\tilde{\sigma} < 0$, *then* $|t' - t''| = \mathrm{O}(\epsilon^2)$. *As a consequence,* $\|w - u^0\|_1 + \|z\|_\infty + \|q\|_\infty = \mathrm{O}(\epsilon)$.

Proof. Let $I' = (t', t'')$ be an interval where $\tilde{\sigma} > 0$. By Lemma 3.4, $\tilde{\sigma}$ coincides with $\tilde{\sigma}^+$ on I' where the latter is constructed with $t_0 = t'$. If we further assume that the interval is maximally extended, then $\dot{\tilde{\sigma}}(t') \geq 0$. Remember that

$$\begin{aligned} \ddot{\tilde{\sigma}}^+ - \ddot{\sigma}^+ \quad &=: \quad \phi_\sigma^+ + \psi_\sigma^+, \qquad (52)\\ \phi_\sigma^+ \quad &= \quad (P - P^+) + B_2^T(q^+ - q) + M_1(z^+ - z)\\ &\qquad\qquad \text{with } \|\phi_\sigma^+\|_{\infty,[t_0,t]} = \mathrm{O}(|t - t_0|),\\ \psi_\sigma^+ \quad &= \quad B_2^T q + M_1 z + r_2 \qquad \text{with } \|\psi_\sigma^+\|_2 = \mathrm{O}(\epsilon). \end{aligned}$$

Without loss of generality, let Δ be taken small enough to ensure

$$\ddot{\sigma}^+(s) \geq m^2/2, \qquad |\phi_\sigma^+(s)| < m^2/8 \quad \forall\, s \in (t', t' + \Delta),$$

see (50). Defining the set $\omega^+ = \{t \in [0,1] :\ |\psi_\sigma^+(t)| > m^2/8\}$, from

$$\|\psi_\sigma^+\|_2^2 = \int_0^1 (\psi_\sigma^+(s))^2 ds \;\geq\; \int_{\omega^+} (\psi_\sigma^+(s))^2 ds \;\geq\; \frac{m^4}{64}\, meas\, \omega^+$$

the estimate $meas\, \omega^+ = \mathrm{O}(\epsilon^2)$ is obtained.
Now consider the Taylor expansion for $\tilde{\sigma}^+$ at t': in case $t \in (t', t' + \Delta)$,

$$\begin{aligned} \tilde{\sigma}^+(t) \quad &= \quad \tilde{\sigma}^+(t') + (t - t')\dot{\tilde{\sigma}}^+(t') + \int_{t'}^t (t - s)\ddot{\tilde{\sigma}}^+(s)\, ds\\ &\geq \quad \int_{[t',t]\setminus\omega^+} (t - s)(m^2/4) ds + \int_{[t',t]\cap\omega^+} (t - s)\ddot{\tilde{\sigma}}^+(s)\, ds\\ &\geq \quad (m^2/8)(t - t')^2 - (t - t') \cdot \|\ddot{\tilde{\sigma}}^+\|_\infty\, meas(\omega^+ \cap [t', t]). \end{aligned}$$

At $t = t''$, the function $\tilde{\sigma} = \tilde{\sigma}^+$ vanishes so that

$$t'' - t' \leq c \cdot meas(\omega^+ \cap [t', t]) \;\leq\; c \cdot meas\, \omega^+ \;=\; \mathrm{O}(\epsilon^2). \qquad (53)$$

A similar estimate can be obtained for any interval $(\hat{t}', \hat{t}'') \subset (\tilde{t}_1, \tilde{t}_2)$ where $\tilde{\sigma} = \tilde{\sigma}^- < 0$.

The number of maximal subintervals of $(\tilde{t}_1, \tilde{t}_2)$ where $\tilde{\sigma} \in C \subseteq W_2^2$ is strictly positive, or strictly negative, is at most countable and the function vanishes at the interval ends. If the end points are appropriately enumerated,

$$|\tilde{\sigma}(t)| \neq 0 \quad \text{for } t'_k < t < t''_k, \qquad k \in \mathcal{N},$$

from (53) we get the estimate

$$\sum_{k\in\mathcal{N}} (t_k'' - t_k') \le c \sum_{k\in\mathcal{N}} meas(\omega^+ \cap [t_k', t_k'']) \le c \cdot meas\,\omega^+ = \mathrm{O}(\epsilon^2)$$

or finally, $\quad meas\{t \in (\tilde t_1, \tilde t_2) : \; \tilde\sigma(t) \ne 0\} \;=\; \mathrm{O}(\epsilon^2).$
Next, consider $\tilde u_s$ from (46):

$$|\tilde u_s - u_s| = R^{-1}|B_2^T q - M_1 z + r_2| \le \; m^{-1}(c\,(|z| + |q|) \;+\; |r_2|)$$

holds allover I_β. Thus, for $t_k' = \min\{t_k, \tilde t_k\}$, $t_k'' = \max\{t_k, \tilde t_k\}$, $\; k = 1, 2$, we obtain from Lemma 3.4

$$\begin{aligned}\|w - u^0\|_1 \;&=\; \int_{t_1'}^{t_1''} |w - u^0| dt + \int_{t_1''}^{t_2'} |\tilde u_s - u^0| dt + \int_{t_2'}^{t_2''} |w - u^0| dt + \mathrm{O}(\epsilon^2)\\ &=\; \int_{t_1''}^{t_2'} |\tilde u_s - u_s| dt \;+\; \mathrm{O}(\epsilon)\\ &\le\; m^{-1}(c(\|z\|_1 + \|q\|_1) + \|r_2\|_1) \;+\; \mathrm{O}(\epsilon) \;=\; \mathrm{O}(\epsilon).\end{aligned}$$

Finally, from (18) conclude that $\|z\|_\infty + \|q\|_\infty = \mathrm{O}(\epsilon)$. ∎

By the results of the previous lemma, from (52) we obtain the estimate

$$|\ddot{\tilde\sigma}^+ - \ddot\sigma^+| \;=\; \mathrm{O}(\epsilon \;+\; |t - t_0|)$$

for all t satisfying $|t - t_0| < \Delta$. If Δ and $\bar\epsilon$ are sufficiently small then it follows from (50) that, for given $t_0 \in (t_1 - \Delta/2, t_2 + \Delta/2)$,

$$\pm\ddot{\tilde\sigma}^\pm(t) \;\ge\; m^2/4 \qquad \forall\, t \;\; s.t. \;\; |t - t_0| < \Delta/2. \tag{54}$$

We finish the section with the proof of Theorem 2:

Proof. Consider $\hat t_0 = \min\{t_1, \tilde t_1\}$: by Lemma 3.5,

$$|t - \hat t_0| \;\le |t - t_1| \;+\; \max\{0, t_1 - \tilde t_1\} \;=\; |t - t_1| \;+\; \mathrm{O}(\sqrt{\epsilon}). \tag{55}$$

If σ^+, $\tilde\sigma^+$ are continuations of $\sigma^0, \tilde\sigma$ from $t_0 = \hat t_0$ then, due to Lemma 6.2,

$$|\dot{\tilde\sigma}^+ - \dot\sigma^+| \;=\; \mathrm{O}(\epsilon + |t - \hat t_0|^2) \;=\; \mathrm{O}(\epsilon + |t - t_1|^2).$$

Further, (54) ensures that $\ddot{\tilde\sigma}^+(t) \;\ge\; m^2/4$ at least on $(\hat t_1 - \Delta/2, \hat t_1 + \Delta/2)$. Applying now the Implicit Function Theorem to $\dot{\tilde\sigma}$, the following property is obtained: there exists a unique zero $\hat t_1$ of $\dot{\tilde\sigma}$ near t_1, and $\; |\hat t_1 - t_1| = \mathrm{O}(\epsilon)$. In addition, the point is a strict local minimizer of $\tilde\sigma^+$.

In analogy to Felgenhauer (2012), consider the following situations:

Case 1: $\tilde{\sigma}^+(\hat{t}_1) > 0$.

For $t \in [\hat{t}_1 - \Delta/4, \hat{t}_1 - \Delta/4]$,

$$\tilde{\sigma}^+(t) = \tilde{\sigma}^+(\hat{t}_1) + \int_{\hat{t}_1}^{t} (t-s)\ddot{\tilde{\sigma}}^+(s)\,ds > \frac{m^2}{8}\,(t-\hat{t}_1)^2 > 0.$$

Therefore, $w = u^+ = 0$ and $\tilde{\sigma} = \tilde{\sigma}^+$ at least for $t \in [0, \hat{t}_1 + \Delta/4]$ if ϵ is taken sufficiently small. In particular, it may be assumed that $\hat{t} = \hat{t}_1 + \Delta/4 > t_1$. Independently of ϵ, the above estimate yields

$$\tilde{\sigma}^+(\hat{t}) > \frac{m^2\Delta^2}{128}$$

in contradiction to $|\tilde{\sigma}|$ =O(ϵ) on (t_1, t_2) for all $\epsilon < \bar{\epsilon}$ (see (42)).

Case 2: $\tilde{\sigma}^+(\hat{t}_1) < 0$.

In this case, the function $\tilde{\sigma}^+$ has a zero point $t_1' < \hat{t}_1$ which is locally unique on $(t_1 - \Delta/2, \hat{t}_1)$, and the Taylor expansion allows for deriving the estimate $|t_1' - \hat{t}_1|$ =O$(\sqrt{\epsilon})$. In particular, suppose $|t_1' - t_1| < \Delta/8$.

The point t_1' is a control switching point with $\dot{\sigma}(t_1') < 0$ so that near t_1'

$$\tilde{\sigma}(t) = \begin{cases} \tilde{\sigma}^+(t) & \text{if} \quad t < t_1', \\ \tilde{\sigma}^-(t) & \text{if} \quad t > t_1'. \end{cases}$$

Consider $t \in (t_1', t_1' + \Delta/4)$: due to (50), the function $\tilde{\sigma} = \tilde{\sigma}^-$ continued from $t_0 = t_1'$ is strictly concave and

$$\tilde{\sigma}(t) = \tilde{\sigma}(t_1') + (t - t_1')\dot{\tilde{\sigma}}(t_1') + \int_{t_1'}^{t} (t-s)\ddot{\tilde{\sigma}}^-(s)\,ds < -\frac{m^2}{8}(t-t_1')^2.$$

As in Case 1, for appropriately chosen t we end up with a contradiction.

The only remaining case is $\tilde{\sigma}^+(\hat{t}_1) = 0$. In this situation, $\hat{t}_1$ coincides with the end point $\tilde{t}_1$ of the bang-arc.

Analogously, one can carry out the analysis starting from the right and obtain

$$\tilde{\sigma}(\tilde{t}_k) = \dot{\tilde{\sigma}}(\tilde{t}_k) = 0, \quad k = 1, 2.$$

It remains to prove that $\tilde{\sigma} \equiv 0$ on $[\tilde{t}_1, \tilde{t}_2]$:

Assume that $(t', t'') \subset (\tilde{t}_1, \tilde{t}_2)$ is a (maximally extended) interval where $\tilde{\sigma} > 0$, and $w \equiv 0$. Then the function $\tilde{\sigma}$ attains its (positive) maximum at certain $\bar{t} \in (t', t'')$. Denoting by $\tilde{\sigma}^+$ the continuation of $\tilde{\sigma}$ from $t_0 = \bar{t}$, we have $\tilde{\sigma} = \tilde{\sigma}^+$ on the interval. In particular, it follows from (54) that $\ddot{\tilde{\sigma}}(\bar{t}) > 0$ in contradiction to the local maximum property of the point.

Similarly, $\tilde{\sigma} < 0$ on $[\tilde{t}_1, \tilde{t}_2]$ can be excluded. ∎

4. The L_1 stability result

The main result of the present paper consists in the following

Theorem 3 *Let the assumptions* (H1) *and* (H2) *hold for* (CP$_0$). *Then there exist positive constants* ϵ_0, ϵ_1 *such that, for all* $p \in \Pi$ *with* $|p| < \epsilon_0$*, the following statements are true:*

(i) *on the set* $W_1 = \{\xi = (x, \lambda, u, \mu) \in X : \|u - u^0\|_1 < \epsilon_1\}$ *(with* $X = W_1^1 \times W_1^1 \times L_1 \times L_\infty$*), the variational inequality* (VI$_p$) *has a unique solution* $\xi(p) = (x(p), \lambda(p), u(p), \mu(p))$*,*

(ii) *on the set* W_1*, the solutions depend Lipschitz continuously on p, i.e.* $\|\xi(p') - \xi(p)\|_X = \mathrm{O}(|p' - p|)$ *for all* $p', p \in \Pi$ *such that* $\max\{|p'|, |p|\} \leq \epsilon_0$.

As it was pointed out in Section 2.2, the solutions of (VI$_p$) can be characterized as fixed points of the mapping Φ_p from (17), i.e.,

$$\Phi_p(u) \;:=\; \Lambda_u(\gamma(p, u)) \tag{56}$$

where Λ_u denotes the control component of the solution for (LVI$_\delta$), and the right-hand side $\gamma = \gamma(p, u)$ is defined by (16). Estimates for γ, depending,on ϵ_0, ϵ_1 and u, are provided in Appendix 6.3. Together with Theorem 1, they ensure in particular, existence and uniqueness of a solution $w = \Phi_p(u)$ for (LVI$_\gamma$) if only ϵ_0 and ϵ_1 are sufficiently small. Following the scheme of Robinson (1980), the proof of Theorem 3 requires showing, first, the uniform strict contractivity of Φ_p on $U_1 = \{w \in U : \|w - u^0\| \leq \epsilon_1\}$, and, second, its Lipschitz continuity w.r.t. p.

Lemma 4.1 *There exist positive constants* $\bar{\epsilon}_0, \bar{\epsilon}_1$, l_0 *and* $\bar{l} < 1$ *such that, for* $|p| < \epsilon_0 \leq \bar{\epsilon}_0$*, and,* $\epsilon_1 \leq \bar{\epsilon}_1$*, the mapping* Φ_p *as a mapping from* U_1 *to* L_1 *is Lipschitz continuous with a Lipschitz constant* $l \leq l_0(\epsilon_0 + \epsilon_1) \leq \bar{l} < 1$.

Proof. Let ϵ_0 and ϵ_1 be sufficiently small so that, for arbitrary $u, u' \in U_1$, $\Phi_p(u) = w$ and $\Phi_p(u') = w'$ are well-defined. Due to Theorem 2, each of the functions w, w' has bang-singular structure, i.e. there exist points $\tilde{t}_i, \tilde{t}_i'$, $i = 1, 2$, such that

$$w(t) \;=\; \begin{cases} 0 & \text{if} \quad 0 < t < \tilde{t}_1, \\ \tilde{u}_s(t) & \text{if} \quad \tilde{t}_1 < t < \tilde{t}_2, \\ \bar{u} & \text{if} \quad \tilde{t}_2 < t < 1, \end{cases} \qquad w'(t) \;=\; \begin{cases} 0 & \text{if} \quad 0 < t < \tilde{t}_1', \\ \tilde{u}_s'(t) & \text{if} \quad \tilde{t}_1' < t < \tilde{t}_2', \\ \bar{u} & \text{if} \quad \tilde{t}_2' < t < 1. \end{cases}$$

The control values further satisfy $\bar{u} = u^0(t_2+)$ and

$$0 < \frac{m}{2} \leq \tilde{u}_s, \tilde{u}_s' \leq 1 - \frac{m}{2} \;<\; 1.$$

On the singular arcs,

$$R(\tilde{u}_s - u_s^0) = B_2^T q - M_1 z + r_2, \quad R(\tilde{u}_s' - u_s^0) = B_2^T q' - M_1 z' + r_2', \tag{57}$$

where (z, q) are the state-adjoint solution components of (LVI_γ) for $\gamma = \gamma(p, u)$, and $r_2 = \bar{r}_2(p, u)$ (for the control input u' the corresponding data are (z', q') and r_2'). Notice that, by Theorem 2, $\tilde{u}_s$, $\tilde{u}_s'$ as well as u_s^0 are defined at least on some open interval $I_\beta = \{t \in [0, 1] : |\sigma^0| < \beta\}$ covering the singular arcs of u^0, w and w' in case when $\epsilon_{0,1}$ are sufficiently small.

In a first step, estimate $\|y' - y\|_1$ for $y(t) = \int_0^t (w(s) - u^0(s))ds$ and similarly defined y': as a function on $[0, 1]$, $\phi = y' - y$ satisfies $\|\phi\|_1 \le \|\phi\|_2 \le \|\phi\|_\infty$. From Lemma 6.3 and the proof of Lemma 3.2 we therefore get

$$\begin{aligned} \|y - y'\|_1 + |h - h'| &\le \|y - y'\|_2 + |h - h'| \\ &\le c\,\|r(\gamma') - r(\gamma)\|_{1,2} = \mathrm{O}(\epsilon_0 + \epsilon_1)\,\|u' - u\|_1. \end{aligned} \tag{58}$$

Secondly, we consider $(u_s' - \tilde{u}_s)$: by the representation (57),

$$m\|\tilde{u}_s' - \tilde{u}_s\|_{1,I_\beta} \le \|z' - z\|_1 + \|q' - q\|_1 + \|r_2(p, u') - r_2(p, u)\|_1.$$

Using $z = \zeta + By$, $q = \eta - Q_{12}y$ and the BVP (25) for (ζ, η), we derive

$$\begin{aligned} \|z' - z\|_1 + \|q' - q\|_1 &\le \|\zeta' - \zeta\|_\infty + \|\eta' - \eta\|_\infty + c'\|y' - y\|_1 \\ &\le c\left(\|\gamma' - \gamma\|_D + \|y' - y\|_1\right). \end{aligned}$$

Thus, the Lipschitz estimates for γ and r_2 from Appendix 6.3, together with (58), allow to deduce

$$\|\tilde{u}_s' - \tilde{u}_s\|_{1,I_\beta} = \mathrm{O}(\epsilon_0 + \epsilon_1)\,\|u' - u\|_1.$$

Finally, consider $\|w' - w\|_1$: denoting by $\tilde{I}_1$ and $\tilde{I}_2$ the intervals between $\tilde{t}_1$ and $\tilde{t}_1'$ (respectively $\tilde{t}_2$ and $\tilde{t}_2'$), and by $\tilde{I}_s$ the intersection $(\tilde{t}_1, \tilde{t}_2) \cap (\tilde{t}_1', \tilde{t}_2')$, we have

$$\begin{aligned} \|w' - w\|_1 &= \int_0^1 |w'(t) - w(t)|\,dt \le \int_{\tilde{I}_1 \cup \tilde{I}_2} dt + \int_{\tilde{I}_s} |\tilde{u}_s' - \tilde{u}_s|\,dt \\ &\le |\tilde{t}_1' - \tilde{t}_1| + |\tilde{t}_2' - \tilde{t}_2| + \|\tilde{u}_s' - \tilde{u}_s\|_{1,I_\beta}. \end{aligned} \tag{59}$$

In order to estimate $|\tilde{t}_1' - \tilde{t}_1| + |\tilde{t}_2' - \tilde{t}_2|$, remember that $(y' - y)$ is given by

$$y'(t) - y(t) = \int_0^t (w'(\tau) - w(\tau))\,d\tau = (h' - h) - \int_t^1 (w'(\tau) - w(\tau))\,d\tau.$$

In case of $\tilde{t}_1 \le \tilde{t}_1'$, we get from the first expression for $t \in \tilde{I}_s$ the relation

$$\begin{aligned} |y'(t) - y(t)| &\ge \left| \int_{\tilde{t}_1}^{\tilde{t}_1'} \tilde{u}_s(\tau)d\tau \right| - \int_{\tilde{t}_1'}^t |\tilde{u}_s'(\tau) - \tilde{u}_s(\tau)|d\tau \\ &\ge \frac{m}{2}|\tilde{t}_1' - \tilde{t}_1| - \|\tilde{u}_s' - \tilde{u}_s\|_{1,I_\beta} \end{aligned}$$

(and the same final estimate holds in case of $\tilde{t}_1 > \tilde{t}'_1$, too). Analogously, from the second expression for $(y'-y)$ and $t \in \tilde{I}_s$ deduce

$$|y'(t)-y(t)| \geq \frac{m}{2}|\tilde{t}'_2-\tilde{t}_2| - \|\tilde{u}'_s - \tilde{u}_s\|_{1,I_\beta} - |h'-h|.$$

Notice that the length of $\tilde{I}_s$ is close to $|t_2-t_1| \gg (\epsilon_0+\epsilon_1)$ so that we obtain

$$\begin{aligned} \|y'-y\|_1 &\geq c_1|\tilde{t}'_1-\tilde{t}_1| - \|\tilde{u}'_s - \tilde{u}_s\|_{1,I_\beta}, \\ \|y'-y\|_1 &\geq c_2|\tilde{t}'_2-\tilde{t}_2| - |h'-h| - \|\tilde{u}'_s - \tilde{u}_s\|_{1,I_\beta}, \end{aligned}$$

or

$$\begin{aligned} |\tilde{t}'_1-\tilde{t}_1| + |\tilde{t}'_2-\tilde{t}_2| &\leq c\left(\|y'-y\|_1 + |h'-h| + \|\tilde{u}'_s - \tilde{u}_s\|_{1,I_\beta}\right) \qquad (60) \\ &= \mathrm{O}(\epsilon_0+\epsilon_1)\,\|u'-u\|_1. \end{aligned}$$

Inserting the result into (59) yields $\|w'-w\|_1 = \mathrm{O}(\epsilon_0+\epsilon_1)\|u'-u\|_1$ and hence the lemma. ∎

Besides the estimates for fixed p and varying u as considered above, continuity results for γ and Φ_p w.r.t. the parameter input p will be useful:

LEMMA 4.2 *Let $\epsilon_{0,1}$ be sufficiently small. Then, for all $u \in U_1$ and $p, \hat{p}$ such that $|p| < \epsilon_0$, $|\hat{p}| < \epsilon_0$,*

$$\|\gamma(p,u)-\gamma(\hat{p},u)\|_{\hat{D}} + \|\Phi_p(u)-\Phi_{\hat{p}}(u)\|_1 = \mathrm{O}(|p-\hat{p}|).$$

Proof. The proof repeats in essence the steps of the previous one:
Denote $w = \Phi_p(u)$, $\hat{w} = \Phi_{\hat{p}}(u)$. According to the results of Theorem 2, both control functions have bang-singular-bang structure given by junction points $\tilde{t}_{1,2}$ and $\hat{t}_{1,2}$, and singular control functions $\tilde{u}_s$ or $\hat{u}_s$, respectively. For the related integrated controls $y, \hat{y}$, in analogy to the previous proof the following estimate is obtained from Lemmas 6.3, 6.4, Appendix 6.3:

$$\|\hat{y}-y\|_1 + |\hat{h}-h| \leq \hat{c}_y|\hat{p}-p|.$$

Using further (60), we see that

$$|\tilde{t}_1-\hat{t}_1| + |\tilde{t}_2-\hat{t}_2| \leq \|\hat{y}-y\|_1 + |\hat{h}-h| + \|\hat{u}_s - \tilde{u}_s\|_{1,I_\beta},$$

where the control difference is given by

$$\hat{u}_s - \tilde{u}_s = R^{-1}(B_2^T(\hat{q}-q) - M_1(\hat{z}-z) + \hat{r}_2 - r_2).$$

Remembering the construction of r_2 and the functions $x = x(\cdot,u,p), \hat{x} = x(\cdot,u,\hat{p})$ etc., standard estimates for the canonical equations yield

$$\|\hat{r}_2-r_2\|_1 = \mathrm{O}(|\hat{p}-p|),$$

and similar results for $z, \hat{z}, q, \hat{q}$:

$$\|\hat{z} - z\|_1 + \|\hat{q} - q\|_1 \le c(\|\hat{\gamma} - \gamma\|_D + \|\hat{y} - y\|_1) = \mathrm{O}(|\hat{p} - p|).$$

Combining the particular results allows for finishing the proof with

$$\begin{aligned} \|\hat{w} - w\|_1 &\le |\tilde{t}_1 - \hat{t}_1| + |\tilde{t}_2 - \hat{t}_2| + \|\hat{u}_s - \tilde{u}_s\|_{1,I_\beta} \\ &\le \|\hat{y} - y\|_1 + |\hat{h} - h| + \|\hat{\gamma} - \gamma\|_D = \mathrm{O}(|\hat{p} - p|). \end{aligned}$$ ■

After these preliminaries, the proof of Theorem 3 can be completed:

Proof. (Main steps are adopted from the proof of Theorem 2.1, Robinson, 1980.) By Lemma 4.1, $\Phi_p(u)$ is a contractive map on $U_1 = \{w \in U : \|w - u^0\| \le \epsilon_1\}$ if $|p| < \epsilon_0$. Therefore, the existence of a fixed point follows by Banach's Theorem if only Φ_p is a self-map from U_1 to U_1. In a first step, consider $\Phi_p(u^0)$: by the definition of the map and by Lemma 4.2,

$$\|\Phi_p(u^0) - u^0\|_1 = \|\Lambda(\gamma(p, u^0)) - \Lambda(\gamma(0, u^0))\|_1 \le c_0|p|$$

for some constant c_0 independent of p. Consequently,

$$\begin{aligned} \|\Phi_p(u) - u^0\|_1 &\le \|\Phi_p(u) - \Phi_p(u^0)\|_1 + \|\Phi_p(u^0) - u^0\|_1 \\ &\le \bar{l}\|u - u^0\|_1 + c_0|p| \le \bar{l}\epsilon_1 + c_0\epsilon_0 < \epsilon_1 \end{aligned}$$

if only $\epsilon_0 < c_0^{-1}(1 - \bar{l})\epsilon_1$. Consequently, for appropriately chosen ϵ_0, ϵ_1, the mapping Φ_p has a unique fixed point $u = u(p)$ on U_1. Finding further $x(p) = x(\cdot, u(p)), \lambda(p) = \lambda(\cdot, u(p))$ from the canonical equations, $\sigma(p) = \lambda(p)^T g(x(p), p)$, and $\mu_1(p), \mu_2(p)$ as the positive, respecitvely negative part of $\sigma(p)$, a solution $\xi = \xi(p)$ of (VI$_p$) is obtained.

The Lipschitz continuity of $u = u(p)$ follows from the classical result of S. M. Robinson (1980):

$$\|u(p) - u(q)\|_1 \le (1 - \bar{l})^{-1}\|\Phi_p(u(q)) - \Phi_q(u(q))\|_1 = \mathrm{O}(|p - q|).$$

Thus, the desired estimate for $\|\xi(p) - \xi(q)\|_X$ is directly obtained from the construction of $(x, \lambda, \mu) = (x(p), \lambda(p), \mu(p))$. ■

5. Example and conclusion

The following example is chosen to illustrate how the coercivity assumption (H2) can be verified for problems of type (CP). In a preliminary step, an extremal with bang-singular-bang control structure satisfying (H1) has to be constructed. Consider

$$\textbf{(P1}^\rho\textbf{)} \quad \min \quad J(x, u) := k(x(T), \rho) = x_3(T) + \frac{\rho}{2}(x_1^2(T) + x_2^2(T)) \qquad (61)$$

subject to

$$\dot{x}_1 = x_2 + u, \quad \dot{x}_2 = -u, \quad \dot{x}_3 = \frac{1}{2}x_1^2, \tag{62}$$

$$x(0) = (a,0,0)^T, \qquad |u(t)| \leq 1 \quad \text{a.e. in } [0,T] \tag{63}$$

for given a, $\rho > 0$ and $T > 0$.
This problem represents a relaxation of problem (P1) in Felgenhauer (2012) where additional end-point constraints $x_1(T) = 0$, $x_2(T) = 0$ are included. Via synthesis techniques it was shown that, for $a \in (1/2, 2)$ and sufficiently large T, there always exists a unique and locally stable bang-singular-bang extremal with the properties (H1). Moreover, along the solution (x^0, u^0, λ^0), $R = 1 > 0$ is fulfilled. The construction can be repeated for the following perturbed problem,

$$\textbf{(P2}_\xi\textbf{)} \quad \min \quad \tilde{J}_\xi(x,u) := \tilde{k}(x(T),\xi,\rho) = x_3(T) + \frac{\rho}{2}(\xi_1^2 + \xi_2^2) \tag{64}$$

subject to (62), (63) and $x_1(T) = \xi_1$, $x_2(T) = \xi_2$, if only $|\xi|$ is sufficiently small.
As the calculations from Felgenhauer (2012) show, the related objective function values, as well as the switching points, will be differentiable functions of the input ξ at least on a certain ball $B_{\bar{r}} = \{\xi \in \mathbb{R}^2 : \ |\xi| \leq \bar{r}\}$. Let (x^ξ, u^ξ) and λ^ξ denote the related extremal and associated adjoint function. The objective function value $\tilde{k}(x^\xi(T),\xi,\rho)$ will attain its minimum on the ball $B_{\bar{r}}$ at a certain point ξ^* where $(x^\xi, u^\xi) = (x^*, u^*)$. Now, the inequality

$$\tilde{k}(x^*(T),\xi^*,\rho) = x_3^*(T) + \frac{\rho}{2}|\xi^*|^2 \leq \tilde{k}(x^0(T),0,\rho) = x_3^0(T)$$

shows $|\xi^*|^2 = \mathrm{O}(\rho^{-1})$: thus, ξ^* is an inner point of $B_{\bar{r}}$ if only ρ is sufficiently large. In this case, the following holds true for the derivatives calculated by the chain rule (see Fiacco and McCormick, 1968, Fiacco, 1983):

$$\begin{aligned} 0 &= \frac{\partial}{\partial \xi_i}\tilde{k}(x^\xi(T),\xi,\rho)\Big|_{(x^*,\xi^*)} \\ &= \frac{\partial}{\partial \xi_i}\tilde{k}(x^*(T),\xi,\rho)\Big|_{\xi=\xi^*} + \nabla_{x(T)}\tilde{k}(x^\xi(T),\xi^*,\rho)\Big|_{x=x^*} \cdot \frac{\partial}{\partial \xi_i}x^*(T) \\ &= \rho\,\xi_i^* - \lambda_i^*(T), \qquad i = 1,2. \end{aligned}$$

The last relation yields additional terminal transversality conditions for λ^* showing that (x^*, u^*) is a bang-singular-bang extremal for $(\mathrm{P1}^\rho)$. The triple (x^*, u^*, λ^*) satisfies all conditions of Pontryagin's maximum principle together with the requirements (H1).

It remains to check condition (H2). To this aim, consider the system (25) related to (P1$^\rho$) for $\delta = 0$, i.e

$$\begin{array}{lllllll} \dot\zeta_1 &=& \zeta_2 - y, & \dot\zeta_2 &=& 0, & \zeta_1(0) = \zeta_2(0) = 0, \\ \dot\zeta_3 &=& x_1^*(\zeta_1 + y), & \zeta_3(0) &=& 0, & \\ \dot\eta_1 &=& -\zeta_1 - y, & \eta_1(T) &=& \rho(\zeta_1(T) + h), & \\ \dot\eta_2 &=& -\eta_1, & \eta_2(T) &=& \rho(\zeta_2(T) - h), & \\ \dot\eta_3 &=& 0, & \eta_3(T) &=& 0. & \end{array} \tag{65}$$

It is easy to see that $\zeta_2 \equiv 0$, $\eta_3 \equiv 0$ on $[0,T]$. In order to evaluate $\Omega(y,h) = (Cv,v)$, we use

$$\hat{C}y \,=\, B^T\eta \,=\, \eta_1 - \eta_2, \qquad \frac{d}{dt}\hat{C}y \,=\, -\zeta_1 + \eta_1 - y,$$

and from (29) directly get

$$\begin{aligned} \Omega(y,h) &= \hat{C}\,y \cdot y\Big|_{t=T} - \left(\frac{d}{dt}(\hat{C}\,y), y\right) \\ &= (\eta_1(T) - \eta_2(T))h - \int_0^T (\eta_1(s) - \zeta_1(s) - y(s)) \cdot y(s)\,ds \\ &= \int_0^T y^2(s)\,ds - \int_0^T (\dot\eta_1(s) - \dot\zeta_1(s))\zeta_1(s)\,ds \\ &\quad + (\eta_1(T) - \eta_2(T))h + (\eta_1(T) - \zeta_1(T))\zeta_1(T) \\ &= \int_0^T (y^2(s) + \zeta_1^2(s))\,ds + \rho h(\zeta_1(T) + 2h) + \rho\zeta_1(T)\,(\zeta_1(T) + h) - \zeta_1^2(T) \\ &= \int_0^T (y^2(s) + \zeta_1^2(s))\,ds + 2\rho h^2 + 2h\rho\zeta_1(T) + (\rho - 1)\zeta_1^2(T). \end{aligned}$$

Taking into account the estimate

$$|2h\rho\zeta_1(T)| \,\le\, (\rho - 1)\zeta_1^2(T) + \frac{\rho^2}{\rho - 1}h^2$$

valid for all $\rho > 1$, finally obtain

$$\Omega(y,h) \,\ge\, c(\rho)\,h^2 + \int_0^T y^2(s)\,ds$$

with some $c = c(\rho) > 0$ if only $\rho^2 > 2$. Therefore, condition (H2) is fulfilled for the extremal (x^*, u^*) if the penalty parameter ρ is sufficiently large.

Conclusion

In the paper, bang-singular-bang optimal controls appearing in parameter dependent problems of type (CP$_p$) have been considered. The analysis so far was restricted to the case of scalar-valued control and an initial value problem for the vector-valued state function. Although further extensions seem to be quite natural (and partly were successfully implemented in Aronna et al., 2012b, already), the formulation of appropriate regularity or non-degeneracy assumptions on boundary constraints for both the original and the linearized variational inequalities is open yet.

The given results include local existence, uniqueness and stability for extremals of the problem under parameter perturbation. In particular, the structure of the control components remains to be stable for small parameter values in a certain L_1 neighborhood of the reference control, and L_1 error estimates are provided. Besides the system of first-order optimality conditions given as a variational inequality VI, structural stability was proved for solutions of the linearized system LVI under right-hand side perturbation δ restricted to a subset of $\hat{D}$. The additional assumptions made in Theorem 2 on the auxiliary perturbation terms $r_1(\delta), r_2(\delta)$ look rather technical, but are naturally fulfilled for the linearization errors γ, (15) and (17), which are to be inserted into LVI for proving Theorem 3.

The solution behavior of LVI, however, can be considered also in more general setting as the following statement shows. Here, the assumptions on the right-hand side vector δ are weakened and, in the result, the control structure then is stable only up to sets of small measure:

THEOREM 4 *Let the assumptions* (H1) *and* (H2) *hold for* (CP_0). *Further, for given* $\delta \in \hat{D}$ *with the additional property* $r_1(\delta),\, r_2(\delta) \in L_\infty$, *let* $v = v^\delta$ *be a solution of* $\mathrm{LVI}_\delta^{red}$. *Then there exist constants* $m_r, \bar{\epsilon} > 0$ *such that, for all* ϵ, δ *satisfying*

$$\|\delta\|_{\hat{D}} + \|r_1(\delta)\|_\infty + \|r_2(\delta)\|_2 < \epsilon \leq \bar{\epsilon}, \tag{66}$$

together with $\|r_2(\delta)\|_\infty < m_r$, *the function* $w = v^\delta + u^0$ *has approximate bang-singular-bang structure in the following sense:*
There exist points $\tilde{t}_1 < \tilde{t}_2$ *and a set* ω^* *of measure* $O(\epsilon)$ *such that*

$$w(t) = \begin{cases} 0 & on \quad [0, \tilde{t}_1), \\ \tilde{u}_s(t) & on \quad (\tilde{t}_1, \tilde{t}_2)\backslash\omega^*, \\ \bar{u} & on \quad (\tilde{t}_2, 1], \end{cases}$$

$$\textit{and} \quad \|w - u^0\|_1 + |\tilde{t}_1 - t_1| + |\tilde{t}_2 - t_2| = O(\epsilon).$$

(The proof of the theorem will appear elsewhere.) The further investigation of the linearized problems and their discretizations could be useful for designing approximation methods not requiring a-priori knowlegde of the control structure.

6. Appendix

6.1. Solution operators

Let $\Psi = \Psi(t,s)$ denote the fundamental matrix solution for the linearized state equation, i.e.

$$\frac{d}{dt}\Psi(t,s) - A(t)\Psi(t,s) = 0, \;\; \frac{d}{ds}\Psi^T(t,s) + A^T(s)\Psi^T(t,s) = 0, \;\; \Psi(t,t) = I.$$

Then functions z, q from (LVI$_\delta$), respectively (18) can be expressed as

$$\begin{aligned} z(t) &= -\Psi(t,0)\delta_2 + \int_0^t \Psi(t,s)(B(s)v(s) - \delta_1(s))\, ds \\ &=: (S\,v)(t) + z_\delta^{part}(t), \qquad (67) \\ q(t) &= \int_1^t \Psi^T(s,t)\,(Q_{11}(s)\cdot(Sv)(s) + Q_{12}(s)v(s) - \delta_3(s))\, ds \\ &\quad - \Psi^T(1,t)(Kz(1) - \delta_4) + \int_1^t \Psi^T(s,t)Q_{11}(s)z_\delta^{part}(s)ds \\ &=: (\tilde{S}v)(t) + q_\delta^{part}(t). \qquad (68) \end{aligned}$$

Let us consider the term $(B^Tq + Q_{21}z)$ from (13): the above formulas yield

$$B^Tq + Q_{21}z = (B^T\tilde{S} + Q_{21}S)\,v + (B^Tq_\delta^{part} + Q_{21}z_\delta^{part}), \qquad (69)$$

and further, for arbitrarily given $w \in L_2$,

$$\begin{aligned} (w,(B^T\tilde{S} + Q_{21}S)v) &= (Sw, Q_{11}Sv) + (Sw, Q_{12}v) \\ &\quad + (w, Q_{21}Sv) + ((Sw)(1))^T\,K\,((Sv)(1))\,. \end{aligned} \qquad (70)$$

Analogously to (67)–(68), the solution of (25) takes the form

$$\zeta = S_1y + \zeta_\delta^{part}, \qquad \eta = \tilde{S}_1y + \tilde{W}h + \eta_\delta^{part}$$

where the maps operating on (y,h) are defined by

$$\begin{aligned} (S_1y)(t) &:= \int_0^t \Psi(t,s)B_1(s)y(s)\, ds, \\ (\tilde{S}_1y)(t) &:= \int_t^1 \Psi^T(s,t)[Q_{11}(s)\cdot(S_1y)(s) + M^T(s)y(s)]\, ds + \Psi^T(0,t)K\,\hat{S}_1y, \\ (\tilde{W}h)(t) &:= \Psi^T(0,t)W\,h. \end{aligned}$$

The formulas show that, in case of $y \in L_2$, $\delta \in D$, the functions η and ζ belong to W_1^1, and $\zeta_\delta^{part}, \eta_\delta^{part}$ depend Lipschitz continuously on δ.

6.2. Auxiliary estimates

In the following it will be assumed that ϵ is a common bound for $\|\delta\|_{\hat{D}}$ and $\|r_1(\delta)\|_\infty$. There exist constants $\bar{\epsilon} > 0$ and $\Delta > 0$ such that, for arbitrary $t_0 \in [0,1]$, (x^0, λ^0) can be continued by $(x^\pm, \lambda^\pm)$ at least to $[t_0 - \Delta, t_0 + \Delta]$. If, in addition, $\epsilon < \bar{\epsilon}$, then there exist similar continuations $(z^\pm, q^\pm)$ for (z, q) on $[t_0 - \Delta, t_0 + \Delta]$, too. Without loss of generality, assume $\Delta \ll 1$.

Lemma 6.1 *Let $x^\pm, \lambda^\pm, z^\pm$ and $q^\pm$ be defined by* (49), (51). *Then,*

(i) *for* $0 \le t_0 \le 1$, $t_0 - \Delta \le t \le t_0 + \Delta$*:* $|x^\pm - x^0| + |\lambda^\pm - \lambda^0| = \mathrm{O}(|t - t_0|)$, *and similarly,* $|z^\pm - z| + |q^\pm - q| = \mathrm{O}(|t - t_0|)$,

(ii) *for* $t_0 = 0$, $0 \le t \le t_1 + \Delta$*:* $|x^+ - x^0| + |\lambda^+ - \lambda^0| = \mathrm{O}(\max\{0, t - t_1\})$, *and* $|z^+| + |q^+| = \mathrm{O}(\epsilon + \max\{0, t - t_1\})$,

(iii) *for* $0 \le t_0 \le 1$, $t_0 - \Delta \le t \le t_0 + \Delta$*:*
$|z^\pm - x^\pm + x^0| + |q^\pm - \lambda^\pm + \lambda^0| = \mathrm{O}\left(\epsilon + |t - t_0|^2\right)$.

Lemma 6.2 *For* $\sigma^\pm = (\lambda^\pm)^T g(x^\pm)$, $\sigma^0 = (\lambda^0)^T g(x^0)$ *and* $\tilde{\sigma}^\pm$ *constructed by* (41), *i.e.*

$$\tilde{\sigma}^\pm = \sigma^0 + B^T q^\pm + Q_{21} z^\pm + \tilde{\delta}_5$$

(with data as in Lemma 6.1), the following estimates hold:

(i) *for* $t_0 = 0$, $t \le t_1 + \Delta$*:* $|\tilde{\sigma}^+ - \sigma^+| = \mathrm{O}(\epsilon + \max\{0, t - t_1\})$,

(ii) *for* $0 \le t_0 \le 1$, $|t - t_0| \le \Delta$*:* $|\dot{\tilde{\sigma}}^+ - \dot{\sigma}^+| = \mathrm{O}(\epsilon + |t - t_0|^2)$.

Proof. Consider the values of the switching functions $\tilde{\sigma}^+$ and σ^+ at t: for appropriate constants c, c' etc.,

$$\begin{aligned}
|\tilde{\sigma}^+ - \sigma^+| &\le |(\lambda^0)^T g(x^0) - (\lambda^+)^T g(x^+)| + |B^T q^+ + Q_{21} z^+ + \tilde{\delta}_5| \\
&\le c'(|x^+ - x^0| + |\lambda^+ - \lambda^0|) + c''(|z^+| + |q^+|) + |\tilde{\delta}_5| \\
&\le c\,(\epsilon + \max\{0, t - t_1\})
\end{aligned}$$

showing part (i) of the lemma. For the time-derivatives obtain

$$\begin{aligned}
\dot{\tilde{\sigma}}^+ - \dot{\sigma}^0 &= -B_1^T q^+ - M z^+ - r_1, \\
\dot{\sigma}^+ - \dot{\sigma}^0 &= (\lambda^+ - \lambda^0)^T [f^0, g^0] + (\lambda^+)^T \left([f^+, g^+] - [f^0, g^0]\right) \\
&= -B_1^T(\lambda^+ - \lambda^0) - M(x^+ - x^0) + \mathrm{O}(|x^+ - x^0|^2 + |\lambda^+ - \lambda^0|^2)
\end{aligned}$$

due to $B_1 = -[f^0, g^0]$ and $M = -(\lambda^0)^T \nabla [f^0, g^0]$. Therefore,

$$\begin{aligned}
\dot{\tilde{\sigma}}^+ - \dot{\sigma}^+ &= -B_1^T(q^+ - \lambda^+ + \lambda^0) - M(z^+ - \lambda^+ + \lambda^0) - r_1 + \mathrm{O}(|t - t_0|^2), \\
|\dot{\tilde{\sigma}}^+ - \dot{\sigma}^+| &= \mathrm{O}(\epsilon + |t - t_0|^2).
\end{aligned}$$

■

6.3. *Rhs* estimates related to the fixed point problem

LEMMA 6.3 *Let $\gamma = \gamma(p, u)$ be given by (16). Then, for $|p| \le \epsilon_0$ and $u, u' \in U_1 = \{w \in U : \|w - u^0\|_1 \le \epsilon_1\}$, the following estimates hold:*

$$\begin{aligned}
\|\gamma(p,u)\|_D &= \mathrm{O}\left(|p| + \|u-u^0\|_1^2\right), \\
\|\gamma(p,u)\|_{\hat{D}} &= \mathrm{O}\left(|p| + \|u-u^0\|_1^{3/2}\right), \\
\|\gamma(p,u') - \gamma(p,u)\|_D &= \mathrm{O}(\epsilon_0 + \epsilon_1)\,\|u'-u\|_1.
\end{aligned}$$

Proof. Let $x = x(u)$, $\lambda = \lambda(u)$ and $\sigma = \sigma(u) = \lambda(u)^T g(x(u))$ be determined by (6), (7). Standard estimates yield

$$\|x - x^0\|_{1,1} + \|\lambda - \lambda^0\|_{1,1} + \|\sigma - \sigma^0\|_{2,1} = \mathrm{O}(|p| + \|u-u^0\|_1).$$

Consider now the components of γ: a.e. on $[0,1]$,

$$\begin{aligned}
\gamma_1 &= \dot{x}^0 - \dot{x} + A(x-x^0) + B(u-u^0) \\
&= f(x^0,0) + u^0 g(x^0,0) - f(x,0) - u^0 g(x,0) + \nabla(f^0 + u^0 g^0)(x-x^0) \\
&\quad + f(x,0) - f(x,p) + u^0(g(x,0) - g(x,p)) - (u-u^0)(g(x,p) - g(x^0,0)), \\
|\gamma_1| &= \mathrm{O}(|p| + |x-x^0|^2) + |u-u^0|\,\mathrm{O}(|p| + |x-x^0|),
\end{aligned}$$

analogously

$$\begin{aligned}
|\gamma_3| &= \mathrm{O}(|p| + |x-x^0|^2 + |\lambda-\lambda^0|^2) + |u-u^0|\,\mathrm{O}(|x-x^0| + |\lambda-\lambda^0|), \\
|\gamma_2| &= |a(0) - a(p)| = \mathrm{O}(|p|), \qquad |\gamma_4| = \mathrm{O}(|p| + |x(1) - x^0(1)|^2).
\end{aligned}$$

For the last component γ_5 and its time derivative $\dot{\gamma}_5$ obtain

$$\begin{aligned}
\gamma_5 &= \sigma - \sigma^0 - B^T(\lambda - \lambda^0) - Q_{21}(x-x^0) \\
&= \lambda^T(g(x,p) - g(x^0,0)) - (\lambda^0)^T \nabla g^0 (x-x^0) \\
&= \lambda^T(g(x,p) - g(x,0)) + (\lambda-\lambda^0)^T(g(x,0) - g^0) \\
&\quad + (\lambda^0)^T(g(x,0) - g^0 - \nabla g^0(x-x^0)), \\
|\gamma_5| &= \mathrm{O}(|p| + |x-x^0|^2 + |\lambda-\lambda^0|^2),
\end{aligned}$$

$$\begin{aligned}
\dot{\gamma}_5 &= \dot{\lambda}^T(g(x,p) - g^0) + \lambda^T(\nabla g(x,p)\dot{x} - \nabla g(x,0))\dot{x}^0) - (\dot{\lambda}^0)^T \nabla g^0(x-x^0) \\
&\quad -(\lambda^0)^T(\nabla g^0(\dot{x} - \dot{x}^0) + (x-x^0)^T \nabla^2 g^0 \dot{x}^0) \\
&= \dot{\lambda}^T(g(x,p) - g(x,0)) + \lambda^T(\nabla g(x,p) - \nabla g(x,0))\dot{x} \\
&\quad +(\dot{\lambda} - \dot{\lambda}^0)^T(g(x,0) - g^0) + (\lambda-\lambda^0)^T(\nabla g(x,0)\dot{x} - \nabla g^0 \dot{x}^0) \\
&\quad +(\dot{\lambda}^0)^T(g(x,0) - g^0 - \nabla g^0(x-x^0)) + (\lambda^0)^T(\nabla g(x,0) - \nabla g^0)(\dot{x} - \dot{x}^0) \\
&\quad + (\lambda^0)^T(\nabla g(x,0) - \nabla g^0 - \nabla^2 g^0(x-x^0))\dot{x}^0
\end{aligned}$$

so that

$$|\dot{\gamma}_5| = \mathrm{O}(|p| + |x-x^0|^2 + |\lambda-\lambda^0|^2) + |u-u^0|\,\mathrm{O}(|x-x^0| + |\lambda-\lambda^0|). \quad (71)$$

Summing up the results, for $|p| \le \epsilon_0$ and $\|u - u^0\|_1 \le \epsilon_1$, we get the relations

$$\begin{aligned} \|\gamma_1\|_\infty + \|\gamma_3\|_\infty + \|\gamma_5\|_{1,\infty} &= \mathrm{O}(\epsilon_0 + \epsilon_1), \qquad |\gamma_2| \,=\, \mathrm{O}(\epsilon_0), \\ \|\gamma_1\|_1 + \|\gamma_3\|_1 + |\gamma_4| + \|\gamma_5\|_{1,1} &= \mathrm{O}(\epsilon_0 + \epsilon_1^2). \end{aligned} \tag{72}$$

For estimating $\dot\gamma_5$ we make use of $u, u^0 \in U$ so that $|u - u^0| \le 1$ almost everywhere on $[0,1]$ and consequently, $\|u - u^0\|_2^2 \le \|u - u^0\|_1$. Therefore,

$$\|\dot\gamma_5\|_2 \,=\, \mathrm{O}(\epsilon_0 + \epsilon_1^2) \,+\, \mathrm{O}(\epsilon_0 + \epsilon_1)\|u - u^0\|_2 = \mathrm{O}(\epsilon_0 + \epsilon_1^{3/2}). \tag{73}$$

It remains to show the Lipschitz property for $\gamma = \gamma(p, u)$ defined by (15):

$$\begin{aligned} \bar\gamma_1(p, u') - \bar\gamma_1(p, u) &= \dot x - \dot x' + A(x' - x) + B(u' - u) \\ &= f(x, p) + u^0 g(x, p) - g(x', p) - u^0 g(x', p) \\ &\quad +\nabla(f^0 + u^0 g^0)(x' - x) \,+\, (u' - u)g^0 \\ &\quad +(u - u^0)g(x, p) - (u' - u^0)g(x', p) \\ &= \mathrm{O}(\epsilon_0 + \epsilon_1)|x' - x| + (u - u^0)(g(x, p) - g(x', p)) \\ &\quad -(u' - u)(g(x', p) - g^0) \\ &= \mathrm{O}(\epsilon_0 + \epsilon_1)(|x' - x| + |u' - u|), \end{aligned}$$

analogously obtain

$$\begin{aligned} \bar\gamma_3(p, u') - \bar\gamma_3(p, u) &= \dot\lambda - \dot\lambda' - A^T(\lambda' - \lambda) - Q_{11}(x' - x) - Q_{12}(u' - u) \\ &= \mathrm{O}(\epsilon_0 + \epsilon_1)(|x' - x| + |\lambda' - \lambda| + |u' - u|). \end{aligned}$$

The last function term gives

$$\begin{aligned} \bar\gamma_5(p, u') - \bar\gamma_5(p, u) &= \sigma' - \sigma - B^T(\lambda' - \lambda) - Q_{21}(x' - x) \\ &= (\lambda')^T g(x', p) - \lambda^T g(x, p) - (\lambda' - \lambda)^T g^0 \\ &\quad -(\lambda^0)^T \nabla g^0 (x' - x) \\ &= (\lambda' - \lambda)(g(x', p) - g^0) + (\lambda - \lambda^0)^T(g(x', p) - g(x, p)) \\ &\quad +(\lambda^0)^T(g(x', p) - g(x, p) - \nabla g^0(x' - x)) \\ &= \mathrm{O}(\epsilon_0 + \epsilon_1)(|x' - x| + |\lambda' - \lambda| + |u' - u|). \end{aligned}$$

The functions x, λ (and x', λ', respectively) are given as solutions of the state and adjoint equations (6), (7) for given u (or u' respectively) and fixed parameter value p. In L_∞, their differences are bounded by $\mathrm{O}(\|u' - u\|_1)$. Together with the above formulas, the estimates for γ_1, γ_3 and γ_5 immediately follow. Direct estimates for the boundary terms γ_2, γ_4 complete the proof. ■

Lemma 6.4 *Suppose the assumptions of the previous lemma hold. According to 43, 44, let* $\bar r(p, u) = r(\gamma(p, u))$, $\bar r_{1,2}(p, u) = r_{1,2}(\gamma(p, u))$ *denote*

$$\begin{aligned} r(\gamma) &= B^T q_\gamma^{part} + Q_{21} z_\gamma^{part} + \gamma_5, \\ r_1(\gamma) &= B^T \gamma_3 + Q_{21}\gamma_1 - \dot\gamma_5, \\ r_2(\gamma) &= B_1^T \gamma_3 + M\gamma_1 - \dot r_1(\gamma). \end{aligned}$$

Then $\bar r$, $\bar r_1 \in W^1_\infty$ *and* $\bar r_2 \in L_\infty$ *with*

$$\|\bar r\|_{1,\infty} + \|\bar r_1\|_\infty + \|\bar r_2\|_\infty = \mathrm{O}(\epsilon_0 + \epsilon_1).$$

Moreover, r, $\bar r_1$ *and* $\bar r_2$ *are lipschitzian w.r.t.* u *in the following sense:*

$$\|\bar r(p,u') - \bar r(p,u)\|_{1,\infty} = \mathrm{O}(\epsilon_0 + \epsilon_1)\,\|u'-u\|_1,$$
$$\|\bar r_1(p,u') - \bar r_1(p,u)\|_\infty + \|\bar r_2(p,u') - \bar r_2(p,u)\|_1 = \mathrm{O}(\epsilon_0 + \epsilon_1)\,\|u'-u\|_1.$$

Proof. The analysis starts with

$$r(\gamma) = B^T q^{part}_\gamma + Q_{21} z^{part}_\gamma + \gamma_5, \tag{74}$$

where the functions z^{part}, q^{part} solve (18) for $v = 0$, $\delta = \gamma(p,u)$. As functions in L_∞, they are Lipschitz continuous w.r.t. γ so that

$$\|r(\gamma') - r(\gamma)\|_\infty = \mathrm{O}(\|\gamma' - \gamma\|_D + \|\gamma_5' - \gamma_5\|_\infty).$$

The estimates from Lemma 6.3 therefore yield

$$\|\bar r(p,u') - \bar r(p,u)\|_\infty = \mathrm{O}(\epsilon_0 + \epsilon_1)\,\|u'-u\|_1.$$

Next we consider $r_{1,2} = \bar r_{1,2}(p,u)$ (see 43, 44):

$$\begin{aligned}
r_1 &= B^T\gamma_3 + Q_{21}\gamma_1 - \dot\gamma_5\\
&= B^T(\dot\lambda^0 - \dot\lambda - A^T(\lambda-\lambda^0) - Q_{11}(x-x^0) - Q_{12}(u-u^0))\\
&\quad + Q_{21}(\dot x^0 - \dot x + A(x-x^0) + B(u-u^0))\\
&\quad - \dot\sigma + \dot\sigma^0 + B^T(\dot\lambda - \dot\lambda^0) + Q_{21}(\dot x - \dot x^0) + \dot B^T(\lambda-\lambda^0) + \dot Q_{21}(x-x^0)\\
&= \dot\sigma^0 - \dot\sigma - M(x-x^0) - B_1^T(\lambda-\lambda^0)
\end{aligned}$$

with W^1_∞ matrix functions B_1 and M (see Remark 1, in Section 3.1). Thus, $r_1 \in W^1_\infty$.
Analogously, we obtain

$$r_2 = \ddot\sigma - \ddot\sigma^0 + M_1(x-x^0) - B_2^T(\lambda-\lambda^0) + R(u-u^0)$$

where $R = -(\lambda^0)^T\left[g^0,[f^0,g^0]\right] \in W^1_\infty$, and M_1, $B_2 \in L_\infty$. Indeed, by direct calculation one can find the following representations for B_2 and M_1:

$$\begin{aligned}
B_2 &= \left[f^0[f^0,g^0]\right] + u^0\left[g^0,[f^0,g^0]\right],\\
M_1 &= -(\lambda^0)^T\nabla_x\left(\left[f^0,[f^0,g^0]\right] + u^0\left[g^0,[f^0,g^0]\right]\right).
\end{aligned}$$

Summing up, we obtain the estimate $\|r_1\|_\infty + \|r_2\|_\infty = \mathrm{O}(\epsilon_0 + \epsilon_1)$.
Returning now to the estimate for $r = r(\gamma)$: differentiating (74) leads to

$$\dot r(\gamma) = B_1^T q^{part}_\gamma - M z^{part}_\gamma - r_1(\gamma)$$

and, together with the former estimates, $\|\bar r\|_{1,\infty} = \mathrm{O}(\epsilon_0 + \epsilon_1)$ follows.

It remains to consider the Lipschitz properties of $\bar r_1, \bar r_2$ and $\dot{\bar r}$ w.r.t. u:

$$\begin{aligned}
r_1(\gamma') - r_1(\gamma) &= \dot\sigma - \dot\sigma' - M(x'-x) - B_1^T(\lambda'-\lambda) \\
&= \lambda^T[f,g] - (\lambda')^T[f',g'] + (\lambda^0)^T\nabla[f^0,g^0](x'-x) \\
&\quad +(\lambda'-\lambda)[f^0,g^0] \\
&= -(\lambda'-\lambda)^T\left([f',g'] - [f^0,g^0]\right) \\
&\quad -\lambda^T\left([f',g'] - [f,g]\right) + (\lambda^0)^T\nabla[f^0,g^0](x'-x) \\
&= \mathrm{O}(\epsilon_0+\epsilon_1)\left(|x'-x| + |\lambda'-\lambda|\right),
\end{aligned}$$

$$\begin{aligned}
r_2(\gamma') - r_2(\gamma) &= \ddot\sigma' - \ddot\sigma + M_1(x'-x) - B_2^T(\lambda'-\lambda) + R(u'-u) \\
&= (\hat P' - u^0\hat R' - \hat P + u^0\hat R) + (R - \hat R')(u'-u) \\
&\quad +(\hat R - \hat R')(u-u^0) + M_1(x'-x) - B_2^T(\lambda'-\lambda)
\end{aligned}$$

where $\hat P = \lambda^T\left[f,[f,g]\right]$, $\hat R = \lambda^T\left[g,[f,g]\right]$ denote the data corresponding to $(x(u),\lambda(u))$, and the prime is used for $(x',\lambda') = (x(u'),\lambda(u'))$. Using the representations found for M_1 and B_2, we see that

$$\begin{aligned}
&B_2^T(\lambda'-\lambda) - M_1(x'-x) = \\
&\qquad (\lambda^0)^T\nabla_x\left[f^0,[f^0,g^0]\right](x'-x) + (\lambda'-\lambda)^T\left[f^0,[f^0,g^0]\right] \\
&\qquad +u^0\left((\lambda^0)^T\nabla_x\left[g^0,[f^0,g^0]\right](x'-x) + (\lambda'-\lambda)^T\left[g^0,[f^0,g^0]\right]\right)
\end{aligned}$$

approximates $(\hat P' - \hat P - u^0\hat R' + u^0\hat R)$ up to $\mathrm{O}(\epsilon_0+\epsilon_1)\left(|x'-x| + |\lambda'-\lambda|\right)$, so that we obtain

$$\|r_2(\gamma') - r_2(\gamma)\|_1 = \mathrm{O}(\epsilon_0+\epsilon_1)\|u'-u\|_1. \qquad \blacksquare$$

References

Agrachev, A.A. and Sachkov, Yu.L. (2004) *Control Theory from the Geometric Viewpoint. Encyclopaedia of Mathematical Sciences **87**. Control Theory and Optimization*, II. Springer, Berlin.

Aronna, S.M., Bonnans, J.F., Dmitruk, A.V. and Lotito, P.A. (2012a) Quadratic order conditions for bang-singular extremals. *Numer. Algebra, Contr. and Optim.* **2** (3), 511–546.

Aronna, S.M., Bonnans, J.F. and Martinon, P. (2012b) A shooting algorithm for optimal control problems with singular arcs. *J. Optim. Theory Appl.*, doi 10/10077s109577-012-0254-8 (electronically), 41 pp.

Dmitruk, A.V. (1977) Quadratic conditions for a weak minimum for singular regimes in optimal control problems. *Soviet Math. Doklady* **18**(2).

Dmitruk, A.V. (2008) Jacobi type conditions for singular extremals. *Control & Cybernetics*, **37** (2), 285-306.

Dontchev, A.L. (1995) Characterization of Lipschitz stability in optimization, In: R. Luccheti and J. Revalski, eds., *Recent Developments in Well-posed Variational Problems*, Kluwer, Dordrecht, 95-116.

DONTCHEV, A.L. and LEWIS, A. (2005) Perturbations and metric regularity. *Set-Valued Anal.* **13**, 417-438.

DONTCHEV, A.L. and MALANOWSKI, K. (2000) A characterization of Lipschitzian stability in optimal control. In: A. Ioffe, S. Reich and I. Shafri, eds. *Calculus of variations and optimal control, Haifa 1998*. Chapman & Hall/CRC, Boca Raton, FL., 62-76.

DONTCHEV, A.L., QUINCAMPOIX, M. and ZLATEVA, N. (2006) Aubin criterion for metric regularity. *J. Convex Anal.* **13**, 281-297.

DONTCHEV, A.L. and ROCKAFELLAR, R.T. (1996) Characterization of strong regularity for variational inequalities over polyhedral sets. *SIAM J. Optimization* **6**, 1087-1105.

DONTCHEV, A.L. and VELIOV, V.M. (2009) Metric regularity under approximations. *Control and Cybernetics* **38**, 1283-1303.

FELGENHAUER, U. (2001) Structural properties and approximation of optimal controls. *Nonlinear Analysis* **47**(3), 1869-1880.

FELGENHAUER U. (2003) On stability of bang–bang type controls. *SIAM J. Control Optim.* **41**(6), 1843-1867.

FELGENHAUER U. (2010) Directional sensitivity differentials for parametric bang-bang control problems. In: I. Lirkov et al., eds., *Large-Scale Scientific Computing, Lecture Notes Comp. Sci.* **5910**, Springer, Berlin, 264-271.

FELGENHAUER U. (2012) Structural stability investigation of bang-singular-bang optimal controls. *J. Optim. Theory Appl.* **152**(3), 605-631.

FELGENHAUER U. (2013) Note on local quadratic growth estimates in bang-bang optimal control problems. *Optimization*, doi 10.1080/02331934.2013.773000 (electronically), 17 pp.

FELGENHAUER U., POGGIOLINI, L. and STEFANI, G. (2009) Optimality and stability result for bang-bang optimal controls with simple and double switch behavior. *Control & Cybernetics* **38**(4B), 1305-1325.

FIACCO, A. V. (1983) *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming.* Academic Press, New York.

FIACCO, A.V. and MCCORMICK, G.P. (1968) *Nonlinear Programming: Sequential Unconstrained Minimization Techniques.* John Wiley & Sons, New York.

GOH, B.S. (1966a) The second variation for the singular Bolza problem. *SIAM J. Control* **4** (2), 309–325.

GOH, B.S. (1966b) Necessary conditions for singular extremals involving multiple control variables. *SIAM J. Control* **4** (4), 716-731.

KINDERLEHRER, D. and STAMPACCHIA, G. (1980) *An Introduction to Variational Inequalities and their Applications.* Academic Press, New York.

KNOBLOCH, H.W. (1981) *Higher Order Necessary Conditions in Optimal Control Theory. Lecture Notes Contr. Inf. Sci.* **34**, Springer, Berlin.

KRENER, A.J. (1977) The high order maximum principle and its application to singular extremals. *SIAM J. Control Optim.* **15**(2), 256-292.

LEDZEWICZ, U. and SCHÄTTLER, H. (2007) Antiangiogenic therapy in cancer treatment as an optimal control problem. *SIAM J. Control Optim.*

46(3), 1052-1079.

Malanowski, K. (1995) Stability and sensitivity analysis of solutions to nonlinear optimal control problems. *Appl. Math. Optim.* **32**, 111-141.

Malanowski, K. (2001) *Stability and Sensitivity Analysis for Optimal Control Problems with Control-State Constraints. Dissertationes Mathematicae*, Polska Akademia Nauk, Instytut Matematyczny, Warszawa.

Maurer, H. (1976) Numerical solution of singular control problems using multiple shooting techniques. *J. Optim. Theory Appl.* **18**(2), 235-257.

Oberle, H.J. (1979) Numerical computation of singular control problems with application to optimal heating and cooling by solar energy. *Appl. Math. Optim.* **5**, 297-314.

Poggiolini L. and Stefani G. (2005) Minimum time optimality for a bang-singular arc: second-order sufficient conditions. In: *Decision and Control, 2005, and 2005 European Control Conference, CDC-ECC'05, 44th IEEE Conference on, Sevilla (Spain)*. IEEE, 1433-1438.

Poggiolini L. and Stefani G. (2008) Sufficient optimality conditions for a bang-singular extremal in the minimum time problem. *Control & Cybernetics* **37** (2), 469-490.

Poggiolini L. and Stefani G. (2011) Bang-singular-bang extremals: Sufficient optimality conditions. *J. Dyn. Contr. Systems* **17**(4), 469-514.

Robinson, S.M. (1980) Strongly regular generalized equations. *Math. Oper. Res.* **5**, 43–62

Rudin, W. (1987) *Real and Complex Analysis*. McGraw-Hill, New York.

Stefani, G. (2008) Strong optimality of singular trajectories. In: F. Ancona et al., eds., *Geometric Control and Nonsmooth Analysis*, *Series Adv. Math. Appl. Sci.*, **76**, World Sci. Publ., Hackensack, New Jersey, 300-327.

Vossen, G. (2005) *Numerische Lösungsmethoden, hinreichende Optimalitätsbedingungen und Sensitivitätsanalyse für optimale bang-bang und singuläre Steuerungen.* PhD Thesis, Faculty of Mathematics and Science, University Münster (Germany).

Vossen, G. (2010) Switching time optimization for bang-bang and singular controls. *J. Optim. Theory Appl.* **144**, 409-429.

Zelikin, V.F. and Borisov, V.F. (1994) *Theory of Chattering Control.* Birkhäuser, Boston.