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On new aging classes based on the reversed mean residual life order*

by

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Abstract: In this paper, we introduce new concepts of aging for lifetime distributions. The main idea is the comparison between reversed mean residual life of the random variables X and $(X - t \mid X > t)$, for all $t > 0$. Firstly, with some examples, we highlight the role of the proposed aging classes in reliability and life testing. Then, we try to find the connection between the new aging classes and other aging classes well-known from the literature. Reliability properties of the new classes are studied. Formally, we derive the preservation property under monotonic transformations. Some other characterization and implications are studied.

Keywords: reversed mean residual life, stochastic order, aging class, preservation, characterizations, IRMR.

1. Introduction and definitions

Stochastic orderings have long been a topic of interest in the reliability theory, economics, actuarial sciences, survival analysis and many other branches of statistics. Because the accurate distribution of the life of an element or a system is often unavailable practically, nonparametric aging properties are quite useful for modeling aging or wear-out processes, and for constructing maintenance policies. Such aging classes are derived via several notions of comparison between random variables. In the context of lifetime distributions, some stochastic orderings of probability distributions have been used to give characterizations and new definitions of aging classes. By aging we mean the phenomenon whereby an older system has a shorter remaining lifetime, in some statistical sense, than a younger one, Bryson and Siddiqui (1969). The main goal of this work is to provide another concept of aging for lifetime distributions along with some reliability properties of this concept. The new aging concept is based on stochastic comparison of new lifetime unit and the used one, according to the reversed mean residual life order.

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Let X be a non-negative random variable (r.v.) with an absolutely continuous distribution function (DF) F and a probability density function (pdf) f. Consider the random variable $X_t = (X - t \mid X > t),\ t < u_X = \sup\{x \in R : F(x) < 1\}$, which represents the additional lifetime of a device after time t, provided that it has survived up to t. This random variable is called the residual life of X, and the value of its survival function at point x is

$$\overline{F}_t(x) = \frac{\overline{F}(t+x)}{\overline{F}(t)}, \quad x \in (0, u_X - t). \tag{1}$$

In parallel, the random variable $X_{(t)} = (t - X \mid X \le t),\ t > l_X = \inf\{x \in R : F(x) > 0\}$, its DF being given by

$$F_{(t)}(x) = 1 - \frac{F(t-x)}{F(t)}, \quad x \in (0, t - l_X), \tag{2}$$

is called the reversed residual life of X at point t. The random variable $X_{(t)}$ determines the time elapsed since, for example, the failure of a device or a lifetime of the system under the condition that the failure has occurred before the time t. The reversed mean residual lifetime (RMR) of X is given by

$$m(t) = E(t - X \mid X \le t) = \frac{\int_0^t F(x)dx}{F(t)}, \quad t > 0 \tag{3}$$

and the reversed hazard rate (RH) of X is given by

$$r(t) = \frac{f(t)}{F(t)}, \quad t > 0. \tag{4}$$

The RMR function has recently received much attention from many authors (see Nanda et al., 2003; Kayid and Ahmad, 2004; Poursaeed and Nemotollahi, 2010; Eryilmaz, 2010). For some useful descriptions and applications of the RMR function we refer the reader to Asadi and Berred (2012). We mention here that the reversed mean residual life is also called the mean past lifetime or the mean inactivity time. Let X_1 and X_2 be two non-negative r.v's with DFs F_1 and F_2, pdfs f_1 and f_2, RH functions r_1 and r_2, RMR functions m_1 and m_2, respectively. Let $S_{X_i} = (l_i, u_i)$ be the support of X_i, where $l_i = \inf\{x : F_i(x) > 0\}$ and $u_i = \sup\{x : F_i(x) < 1\}$, for $i = 1, 2$. The following stochastic orders are defined according to Nanda et al. (2003) and Shaked and Shanthikumar (2007):

(i). Reversed hazard rate order ($X_1 \le_{RH} X_2$), if $r_1(x) \le r_2(x)$, for all $x > 0$.

(ii). Reversed mean residual life order ($X_1 \le_{RMR} X_2$), if $m_1(x) \ge m_2(x)$, for all $x > 0$.

(iii). Increasing concave order ($X_1 \le_{ICV} X_2$), if $\int_0^x F_1(u)du \ge \int_0^x F_2(u)du$, for all $x > 0$.

(iv). Usual stochastic order ($X_1 \le_{ST} X_2$), if $F_1(x) \ge F_2(x)$, for all $x > 0$.

Note that, $X_1 \le_{RH} X_2$ implies $X_1 \le_{ST} X_2$. Furthermore,

$$X_1 \le_{RH} X_2 \Rightarrow X_1 \le_{RMR} X_2 \Rightarrow X_1 \le_{ICV} X_2.$$

In the literature, there are various nontions of aging for the non-negative r.v. X, which are based on stochastic comparison with respect to different partial stochastic order between X and its residual at t, i.e. $X_t = (X - t \mid X > t)$, for all $t > 0$. For example, the new better (worse) than used, NBU (NWU) class is equivalent to saying that $X \geq_{ST} (\leq_{ST}) X_t$, for all $t > 0$. Also, the non-negative r.v. X is said to have NBU(2) (NWU(2)) aging property if $X \geq_{ICV} (\leq_{ICV}) X_t$, for all $t > 0$ (see Shaked and Shanthikumar, 2007). The NBU class, introduced by Bryson and Siddiqui (1969) and independently by Marshall and Proschan (1972), has grown to become one of the most studied classes of life distributions. Several extensions of the NBU class have been proposed in the literature (see Hollander et al., 1986; Kayid, 2007). The notions of NBU(2) and NWU(2) are defined in Deshpande et al. (1986). For a recent development of aging concepts we refer the reader to Pellerey and Petakos (2002), Knopik (2006), Elbatal, (2007), Li and Xu (2007, 2008), Ahmad et al. (2005) and Kayid et al. (2011). Also, for recent developments on stochastic comparisons and modeling of random variables with respect to the reversed mean residual life we refer the readers to Nanda et al. (2006), Zhang and Cheng (2010) and Izadkhah and Kayid (2013).

The rest of the paper is organized as follows. In Section 2, we discuss the motivation for our study and bring the main definition of the new aging class. In this section we provide some examples of some statistical distributions to possess the new aging property. In Section 3, we obtain a few connections between the new aging concept and other well-known concepts of aging mentioned earlier and also a characterization for the exponential distribution. In Section 4, we discuss the preservation property under monotonic transformation and further characterization and implication. Finally in Section 5, we conclude the paper with some further remarks on current or/and future research.

2. Motivation and main definition

In this section, we provide the main definition of the new classes of life distributions. Some examples are given for the sake of description of the new aging classes. Let X be life length of some unit. Then $X_t = (X - t \mid X > t)$ is the life length of the used unit of age t. We are going to compare X to X_t, for all $t > 0$, based on the reversed mean residual life function. This means that the expectation of the inactivity time of a new lower (higher) unit is than the expectation of the inactivity time of a used unit. Formally, we have the following definition.

Definition 2.1 *The non-negative random variable X is said to be new better (worse) than used in reversed mean residual life if $X \geq_{RMR} (\leq_{RMR})(X - t \mid X > t)$, for all $t > 0$, and we say X is NBU_{RMR} (NWU_{RMR}).*

In practical situations, because the accurate distribution of lifetime of systems or components is unknown, the new proposed notion can represent the applicability of that system or component based on the concept of inactivity time. In the following result we give an equivalent condition for a distribution to follow the new aging concept.

Proposition 2.1 *Let X have the DF F. Then, X is NBU_{RMR} (NWU_{RMR}), if and only if,*

$$g(x,t) = \frac{\int_0^x F(u)\ du}{\int_0^x [F(t+u) - F(t)]\ du} \tag{5}$$

is non-decreasing (non-increasing) in x, for all $t \in [0, \infty)$.

Proof. From Proposition 2.1 of Kayid and Ahmad (2004), we have $X \geq_{RMR}$ $(\leq_{RMR})(X - t \mid X > t)$, for all $t > 0$, if and only if,

$$\frac{\int_0^x F(u)\ du}{\int_0^x F_t(u)\ du} \tag{6}$$

is non-decreasing (non-increasing) in x, for all $t > 0$, where F_t is the DF of X_t which its survival function was given in (1). We have, for all $t > 0$

$$\begin{aligned} F_t(x) &= 1 - \overline{F}_t(x) \\ &= \frac{\overline{F}(t) - \overline{F}(t+x)}{\overline{F}(t)} \\ &= \frac{F(t+x) - F(t)}{\overline{F}(t)}, \quad x \geq 0. \end{aligned} \tag{7}$$

Therefore, X is NBU_{RMR} (NWU_{RMR}), if and only if,

$$\frac{\overline{F}(t) \int_0^x F(u)\ du}{\int_0^x [F(t+u) - F(t)]\ du} \tag{8}$$

is non-decreasing (non-increasing) in x, for all $t > 0$, which is equivalent to the desired condition. ■

Below we show in some examples that NBU_{RMR} and NWU_{RMR} classes are not empty.

Example 2.1 Let X have exponential distribution with the DF $F(x) = 1 - \exp(-\lambda x)$, $x \geq 0$ and $\lambda > 0$. For all $t \geq 0$, we have

$$\begin{aligned} \int_0^x [F(t+u) - F(t)]\ du &= \int_0^x [\exp(-\lambda t) - \exp(-\lambda(t+u))]\ du \\ &= \exp(-\lambda t) \int_0^x [1 - \exp(-\lambda u)]\ du \\ &= \exp(-\lambda t) \int_0^x F(u)\ du, \quad x \geq 0. \end{aligned} \tag{9}$$

Thus, in view of Proposition 2.1 we get $g(x,t) = \exp(\lambda t)$ which is constant in x, and hence X is NBU_{RMR} and also X is NWU_{RMR}. That is, analogous to other notions of aging from the literature, the exponential distribution satisfies the no-aging property.

Example 2.2 Suppose that X follows the power distribution with DF

$$F(x) = \left(\frac{x}{b}\right)^k, \quad 0 \leq x \leq b, \tag{10}$$

where k is a positive integer and $b > 0$. Then, according to Proposition 2.1, for all $x \geq 0, t \geq 0$,

$$\begin{aligned} g(x,t) &= \frac{\int_0^x F(u)\ du}{\int_0^x [F(t+u) - F(t)]\ du} \\ &= \frac{\int_0^x u^k\ du}{\int_0^x [(t+u)^k - t^k]\ du} \\ &= \frac{x^{k+1}}{(t+x)^{k+1} - (k+1)xt^k - t^{k+1}} \\ &= \frac{1}{P_k\left(\frac{t}{x}\right)}, \end{aligned} \tag{11}$$

where $P_k(y) = (1+y)^{k+1} - (k+1)y^k - y^{k+1}$, for all $y > 0$. We get

$$\begin{aligned} \frac{\partial}{\partial y} P_k(y) &= (k+1)(1+y)^k - k(k+1)y^{k-1} - (k+1)y^k \\ &= (k+1)[(1+y)^k - ky^{k-1} - y^k], \end{aligned} \tag{12}$$

which is non-negative for all $y > 0$. So $g(x,t)$ is non-decreasing in x and hence using Proposition 2.1 we have X is NBU_{RMR}.

3. Basic properties

In this section, we first give a characterization of the exponential distribution. Then, some relationships between the new aging classes and some other known aging classes are derived. In Example 2.1 we showed that the exponential distribution is simultaneously in the NBU_{RMR} class and in its dual class NWU_{RMR}. Ordinarily, for some well-known aging classes a random variable will have an aging property and the dual of it, at the same time, if and only if, that random variable is exponential. For example, X is NBU and also NWU, if and only if, it is exponential. As we demonstrate in the following, this fact is also true in the case of our aging classes.

Theorem 3.1 *Let X be a continuous non-negative random variable such that $\int_0^\infty F(u)\ du = \infty$. Then X belongs to both NBU_{RMR} and NWU_{RMR} classes if and only if, X is exponential.*

Proof. The "if" part is evident from Example 2.1. Thus we prove the "only if" part. Suppose that X is in both NBU_{RMR} and NWU_{RMR} aging classes. Then, by Proposition 2.1, we should have $g(x,t) = c(t)$, for all $t \geq 0, x \geq 0$, where c is non-negative. Set $\Lambda(x)=\int_0^x F(u)\ du$. We can then get

$$\begin{aligned}
\lim_{x\to\infty} g(x,t) &= \lim_{x\to\infty} \frac{\int_0^x F(u)\ du}{\int_0^x [F(t+u)-F(t)]\ du} \\
&= \lim_{x\to\infty} \frac{1}{\left(\frac{\int_0^x F(t+u)\ du}{\int_0^x F(u)\ du} - \frac{xF(t)}{\int_0^x F(u)\ du}\right)} \\
&= \lim_{x\to\infty} \frac{1}{\left(\frac{\int_0^{t+x} F(u)\ du}{\int_0^x F(u)\ du} - \frac{xF(t)}{\int_0^x F(u)\ du}\right)} \\
&= \lim_{x\to\infty} \frac{1}{\left(\frac{\Lambda(t+x)}{\Lambda(x)} - \frac{xF(t)}{\Lambda(x)}\right)}.
\end{aligned} \tag{13}$$

Due to the assumption, $\Lambda(+\infty)=\infty$. Knowing that $\Lambda'(x)=F(x)$, and using L'Hospital's rule we can continue. Note that, for all $t\geq 0$,

$$\lim_{x\to\infty} \frac{xF(t)}{\Lambda(x)} = \lim_{x\to\infty} \frac{F(t)}{F(x)} = F(t), \tag{14}$$

and

$$\lim_{x\to\infty} \frac{\Lambda(t+x)}{\Lambda(x)} = \lim_{x\to\infty} \frac{F(t+x)}{F(x)} = 1. \tag{15}$$

Thus, in view of (13), it is concluded that

$$\lim_{x\to\infty} g(x,t) = \frac{1}{1-F(t)} = [\overline{F}(t)]^{-1}, \quad t\geq 0. \tag{16}$$

So we have $c(t)=[\overline{F}(t)]^{-1}$. Therefore, for all $x,t\geq 0$, we have the identity

$$\frac{\int_0^x F(u)\ du}{\int_0^x [F(t+u)-F(t)]\ du} = [\overline{F}(t)]^{-1}, \tag{17}$$

which holds if and only if

$$\int_0^x [\overline{F}(t+u)-\overline{F}(t)\overline{F}(u)]\ du = 0 \tag{18}$$

for all $x,t\geq 0$. By taking derivative with respect to x, we obtain

$$\overline{F}(t+x)-\overline{F}(t)\overline{F}(x)=0, \tag{19}$$

for all $x,t\geq 0$. This means that X admits the lack of memory property which is the unique characteristic of the exponential distribution among continuous distributions. ■

In the use of Theorem 3.1 it is noticeable that for a non-negative r.v. X for which $E(X) < \infty$, we know from probability theory that $\int_0^\infty F(u)\ du = \infty$. Before stating another result we recall two concepts of aging from the literature. The random variable X with RH function r and RMR function m is said to have $IRHR$ property if r is a non-decreasing function and X is said to have $IRMR$ property if m is a non-decreasing function (see Nanda et al., 2003). In the following, we give some relationships between the new aging classes and the other known aging classes.

THEOREM 3.2 *(i). If* X *is* NBU_{RMR} (NWU_{RMR})*, then* X *is* $NBU(2)$ $(NWU(2))$.
(ii). If X *is* NBU_{RMR}*, then* X *is* $IRMR$.
(iii). If X *is* $IRHR$*, then* X *is* NWU_{RMR}.

Proof (i). The proof of this assertion is trivial because the RMR order implies the ICV order (see, for example, Theorem 2.1 of Kayid and Ahmad, 2004).

(ii). On using Proposition 2.1, X is NBU_{RMR} implies that $g(x,t)$ is non-decreasing in $x \geq 0$, for all $t \geq 0$. On the other hand, by setting $\Lambda(x) = \int_0^x F(u)\ du$, we have, for all $x, t \geq 0$,

$$g(x,t) = \frac{\Lambda(x)}{\Lambda(x+t) - xF(t)}, \tag{20}$$

or equivalently,

$$\frac{\Lambda(x+t)}{\Lambda(x)} = \frac{1}{g(x,t)} + \frac{xF(t)}{\Lambda(x)}. \tag{21}$$

We know that $\frac{x}{\Lambda(x)}$ is non-increasing because

$$\begin{aligned} \frac{d}{dx}\frac{\Lambda(x)}{x} &= \frac{xF(x) - \int_0^x F(u)\ du}{x^2} \\ &= \frac{\int_0^x [F(x) - F(u)]\ du}{x^2} \geq 0, \end{aligned} \tag{22}$$

for all $x \geq 0$. Thus, using the equality in (21), the ratio $\Lambda(t+x)/\Lambda(x)$ is non-increasing in x, for all $t \geq 0$. This is equivalent to saying that Λ is a log-concave function which means that $F(x)/\int_0^x F(u)\ du$ is non-increasing. So, it follows that X is $IRMR$.

(iii). Recall the result of Proposition 2.1. We know that, for all $x, t \geq 0$,

$$\begin{aligned} \frac{\partial}{\partial x} g(x,t) &= \frac{\partial}{\partial x}\left[\frac{\int_0^x F(u)\ du}{\int_0^x [F(t+u) - F(t)]\ du}\right] \\ &= \frac{\int_0^x [F(x)(F(t+u) - F(t)) - F(u)(F(t+x) - F(t))]\ du}{\left(\int_0^x [F(t+u) - F(t)]\ du\right)^2} \geq 0. \end{aligned} \tag{23}$$

Thus, X is NWU_{RMR}, if and only if, $\frac{\partial}{\partial x} g(x,t) \leq 0$, for all $x, t \geq 0$. That is, X is NWU_{RMR}, if and only if,

$$F(x)[F(t+u) - F(t)] \leq F(u)[F(t+x) - F(t)], \tag{24}$$

for all $u \leq x \in [0, \infty)$ and for all $t \geq 0$. The above inequality means that

$$\frac{F(t+x) - F(t)}{F(x)} \tag{25}$$

is non-decreasing in x, for all $t \geq 0$. Note that the fact that X is $IRHR$ implies that $F(t+x)/F(x)$ is non-decreasing in x, for all $t \geq 0$. So

$$\frac{F(t+x)}{F(x)} - \frac{F(t)}{F(x)} \tag{26}$$

is non-decreasing in x, for all $t \geq 0$. Hence X is NWU_{RMR}. ■

4. Further results

In this section, we discuss some implication, characterization and preservation property of the new classes. In the following, a characterization result for the new aging classes are given. Denote by $X_T = (X - T \mid X > T)$, the residual lifetime of X at random time T.

Theorem 4.1 *The random variable X is NBU_{RMR} (NWU_{RMR}), if and only if, $X \geq_{RMR} (\leq_{RMR}) X_T$, for all non-negative random variables T, independent from X.*

Proof. The "if" part is proved if we take T to be degenerate so that $P(T = t)$, for all $t \geq 0$, one at a time. Thus, we prove the "only if" part. First note that if X is NBU_{RMR} (NWU_{RMR}), then by Proposition 2.1, $E[g(x,T)]^{-1}$ is non-increasing (non-decreasing) in $x \geq 0$. On applying Proposition 2.1 from Kayid and Ahmad (2004), $X \geq_{RMR} (\leq_{RMR}) X_T$ if and only if,

$$\frac{\int_0^x P(X \leq u)\, du}{\int_0^x P(X_T \leq u)\, du} \tag{27}$$

is non-decreasing (non-increasing) in $x > 0$. Because X and T are independent, using the total probability formula we can write, for all $u \geq 0$,

$$P(X_T \leq u) = \int_0^\infty F_t(u)\, dG(t), \tag{28}$$

where G is the DF of T and F_t is the DF of $X_t = (X - t \mid X > t)$. Substituting (28) into (27) and using the Fubini Theorem, we can get, for all $x \geq 0$,

$$\begin{aligned}\frac{\int_0^x F(u)\,du}{\int_0^x \int_0^\infty F_t(u)\,dG(t)du} &= \frac{\int_0^x F(u)\,du}{\int_0^\infty \int_0^x F_t(u)\,dudG(t)}\\ &= \frac{1}{\int_0^\infty g(x,t)\,dG(t)}\\ &= \frac{1}{E[g(x,T)]^{-1}},\end{aligned} \tag{29}$$

which is non-decreasing (non-increasing) in $x \geq 0$. This concludes the proof. ■

Denote by X_e the equilibrium random variable associated with non-negative r.v. X for which $E(X) < \infty$. The r.v. X_e has pdf $f_e(x) = \overline{F}(x)/E(X),\ x \geq 0$. The random variable X_e is the characteristic of an old age used lifetime unit (see Ahmad and Mugdadi, 2010). We have the following result:

Theorem 4.2 *Let X be NBU_{RMR} (NWU_{RMR}). Then $X \geq_{RMR} (\leq_{RMR}) X_e$.*

Proof. By Proposition 2.1 of Kayid and Ahmad (2004), $X \geq_{RMR} (\leq_{RMR}) X_e$, if and only if

$$\frac{\int_0^x F(u)\,du}{\int_0^x \int_0^u \overline{F}(y)\,dydu} \tag{30}$$

is non-decreasing (non-increasing) in $x \geq 0$, which means that, for all $x \geq 0$,

$$\int_0^x \left[F(x)\int_0^u \overline{F}(y)dy - F(u)\int_0^x \overline{F}(y)dy\right]\,du \geq (\leq)\ 0. \tag{31}$$

On the other hand, from the proof of Theorem 3.2(iii), one can see that X is NBU_{RMR} (NWU_{RMR}), if and only if,

$$\int_0^x [F(x)(\overline{F}(t) - \overline{F}(t+u)) - F(u)(\overline{F}(t) - \overline{F}(t+x))]\,du \geq (\leq)\ 0, \tag{32}$$

for all $x, t \geq 0$. By integrating the above inequalities with respect to $t \in (0, \infty)$, using the Fubini Theorem and knowing that

$$\int_0^\infty [\overline{F}(t) - \overline{F}(t+x)]\,dt = \int_0^x \overline{F}(y)\,dy, \tag{33}$$

we arrive at the inequalities given in (31) and hence the result is proved. ■

The following result will develop the preservation property of the NBU_{RMR} class under monotone transformations.

Theorem 4.3 *Let ϕ be a non-negative, strictly increasing and concave function such that $\phi(0) = 0$. Then*

$$X \in NBU_{RMR} \ \Rightarrow\ \phi(X) \in NBU_{RMR}.$$

Proof. Suppose that X is NBU_{RMR}. Then, for all $t \geq 0$, $X \geq_{RMR} X_t$. According to Theorem 3.1 of Li and Xu (2006), it follows that $\phi(X) \geq_{RMR} \phi(X_t)$, for all $t \geq 0$. Because ϕ is concave, then it is sub-additive, i.e. for each $x, y \geq 0$, $\phi(x+y) \leq \phi(x)+\phi(y)$. If for a given value $t > 0$, it holds that $X > t$, then from the sub-additivity property of ϕ, we have $\phi(X) \leq \phi(X-t)+\phi(t)$. In addition, because ϕ is strictly increasing, $X > t$ is equivalent to $\phi(X) > \phi(t)$. Therefore,

$$\begin{aligned} \phi(X) &\geq {}_{RMR}\phi(X_t) \\ &= [\phi(X-t) \mid X > t] \\ &= [\phi(X-t) \mid \phi(X) > \phi(t)] \\ &\geq {}_{RMR}[\phi(X)-\phi(t) \mid \phi(X) > \phi(t)] = (\phi(X))_{t'}, \end{aligned} \tag{34}$$

where $t' = \phi(t)$ is non-negative. Consequently, for all $t' \geq 0$, it holds that $\phi(X) \geq_{RMR} (\phi(X) - t' \mid \phi(X) > t')$. That is, $\phi(X)$ is NBU_{RMR}. ■

5. Conclusion and future research

Two new aging properties were introduced in this paper using the concept of the reversed mean residual life order. Various properties of the new aging classes were studied. We showed that the exponential distribution admits no-aging property according to the new proposed aging notions. The study of the new aging classes can be extended to discuss other preservation properties of NBU_{RMR} and NWU_{RMR} aging classes under mixture, order statistics and convolution. Furthermore, one can develop a test of exponentiallity against alternatives that belong to NBU_{RMR} class, but are not exponential.

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