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Φ-$\alpha(\cdot)$-K-monotone multifunctions with values in ordered Banach space with increasing norm*

by

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Abstract: Let (X, d) be a metric space. Let Y be an ordered Banach space with increasing norm. Let Φ be a separable linear family (a class) of Lipschitz functions defined on X and with values in Y. Let $\alpha(\cdot)$ be a nondecreasing function mapping the interval $[0, +\infty)$ into itself such that $\lim_{t\downarrow 0} \frac{\alpha(t)}{t} = 0$. We say that a multifunction Γ mapping X into Φ is Φ-$\alpha(\cdot)$-K-*monotone* if for all k in the interior of K, $k \in \text{Int } K$, there is a constant $C_k > 0$ such that for all $\phi_x \in \Gamma(x)$,$\phi_y \in \Gamma(y)$ we have

$$\phi_x(x) + \phi_y(y) - \phi_x(y) - \phi_y(x) \geq_K -C_k \alpha(d(x, y))k.$$

It is shown in the paper that under certain conditions on Φ each Φ-$\alpha(\cdot)$-K-monotone multifunction is single-valued and continuous on a dense G_δ-set.

Keywords: vector valued functions, normal cone, cone with bounded basis, Φ-$\alpha(\cdot)$-K-subgradi-ents, increasing norm, Φ-$\alpha(\cdot)$-k-subdifferential Fréchet Φ-differentiability

1. Φ-$\alpha(\cdot)$-K-subgradients and Φ-$\alpha(\cdot)$-K-supergradients of vector valued functions

Let (X, d) be a metric space. Let $f(x)$ and $\phi(x)$ be two functions defined on X with values in a Banach space $(Y, \|\cdot\|)$ partially ordered by a pointed closed convex cone K with non-empty interior, Int $K \neq \emptyset$. Recall that the cone K introduces the order in the following way. We write $x \leq_K y$ if $y \in x + K$ ($x \geq_K y$ if $x \in y + K$) and $x <_K y$ if $y \in x + \text{Int } K$ ($x >_K y$ if $x \in y + \text{Int } K$).

Let $\alpha(\cdot)$ be a nondecreasing function mapping the interval $[0, +\infty)$ into itself such that

$$\lim_{t\downarrow 0} \frac{\alpha(t)}{t} = 0. \tag{1.1}$$

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Let $k \in \text{Int } K$. The function $\phi(x)$ will be called a Φ-$\alpha(\cdot)$-k-*subgradient* (Φ-$\alpha(\cdot)$-k-*supergradient*) of the function $f(x)$ at a point x_0 if there is a constant $C_k > 0$ such that

$$f(x) - f(x_0) \geq_K \phi(x) - \phi(x_0) - C_k \alpha(d(x,y))k \tag{1.2}$$

$$(\text{resp.,}\ \ f(x) - f(x_0) \leq_K \phi(x) - \phi(x_0)\ ) + C_k \alpha(d(x,y))k) \tag{1.2$'$}$$

for all $x \in X$.

If a function ϕ is Φ-$\alpha(\cdot)$-k_0-subgradient (respectively Φ-$\alpha(\cdot)$-k_0- supergradient) of the function $f(x)$ at a point x_0 for a certain $k_0 \in \text{Int } K$, then by (1.1) it is Φ-$\alpha(\cdot)$-k-subgradient (respectively Φ-$\alpha(\cdot)$-k-supergradient) of the function $f(x)$ at a point x_0 for all $k \in \text{Int } K$. Therefore the natural definition is that a function ϕ is Φ-$\alpha(\cdot)$-K-subgradient (respectively Φ-$\alpha(\cdot)$-K-supergradient) of the function $f(x)$ at a point x_0 if it is Φ-$\alpha(\cdot)$-k_0-*subgradient* (respectively Φ-$\alpha(\cdot)$-k_0-*supergradient*) of the function $f(x)$ at a point x_0 for a certain $k_0 \in \text{Int } K$.

The set of all Φ-$\alpha(\cdot)$-K-subgradients (respectively, Φ-$\alpha(\cdot)$-K-supergradients) of the function f at a point x_0 we shall call Φ-$\alpha(\cdot)$-K-*subdifferential* (respectively, Φ-$\alpha(\cdot)$-K-*superdifferential*) of the function f at a point x_0 and we shall denote it by $\partial^{\Phi}_{\alpha,k} f\big|_{x_0}$ (respectively, $\partial^{\alpha,k}_{\Phi} f\big|_{x_0}$).

Proposition 1 *Φ-$\alpha(\cdot)$-K-subdifferential (respectively, Φ-$\alpha(\cdot)$-K-superdifferential) of the function f at a point x_0 is a convex set.*

Proof Let $\phi(x)$ and $\psi(x)$ be two Φ-$\alpha(\cdot)$-K-subgradients (Φ-$\alpha(\cdot)$-K-supergradients) of a function $f(x)$ at a point x_0.

By the definition of Φ-$\alpha(\cdot)$-K-subgradient there are $k \in \text{Int } K$ and a constant C_k, such that

$$f(x) - f(x_0) \geq_K \phi(x) - \phi(x_0) - C_k \alpha(d(x,y))k, \tag{1.2$_\phi$}$$

$$f(x) - f(x_0) \geq_K \psi(x) - \psi(x_0) - C_k \alpha(d(x,y))k. \tag{1.2$_\psi$}$$

Multiplying $(1.2)_\phi$ by t and $(1.2)_\psi$ by $(1-t)$ and adding them, by the convexity of K we obtain that

$$f(x) - f(x_0) \geq_K [t\phi(x) + (1-t)\psi(x)] - [t\phi(x_0) + (1-t)\psi(x_0)] - C_k \alpha(d(x,y))k., \tag{1.3}$$

which shows that $t\phi(x) + (1-t)\psi(x)$ is a Φ-$\alpha(\cdot)$-k-subgradient of a function $f(x)$ at a point x_0. The proof for Φ-$\alpha(\cdot)$-k-supergradients is similar. □

Observe that the introduced notions of Φ-$\alpha(\cdot)$-k-subgradients, Φ-$\alpha(\cdot)$-k-supergradients, Φ-$\alpha(\cdot)$-K- subdifferentials, Φ-$\alpha(\cdot)$-K-superdifferentials do not depend on the norm in the space Y, and as a consequence we get that if $\|\cdot\|_1$ is a norm in Y equivalent to the norm $\|\cdot\|$, then Φ-$\alpha(\cdot)$-K-subgradients, Φ-$\alpha(\cdot)$-K-supergradients, Φ-$\alpha(\cdot)$-K- subdifferentials, Φ-$\alpha(\cdot)$-K-superdifferentials are the same with respect to both these norms.

2. Φ-$\alpha(\cdot)$-k-monotone vector-valued multifunctions

Let X, Z be two sets. Let $\Gamma : X \to 2^Z$ be a multifunction, i.e., the mapping of the set X into subsets of Z. We shall call the *domain* of Γ, $\mathrm{dom}(\Gamma)$, the set of such x, that $\Gamma(x) \neq \emptyset$,

$$\mathrm{dom}(\Gamma) = \{x \in X : \Gamma(x) \neq \emptyset\}.$$

By the *graph* of Γ, $G(\Gamma)$, we shall call the set $G(\Gamma) = \{(x, z) \in X \times Z : z \in \Gamma(x)\}$.

Let, as before, (X, d) be a metric space. Let Φ be a linear family of functions defined on (X, d) with values in a Banach space $(Y, \|\cdot\|)$ partially ordered by a pointed closed convex cone K with non-empty interior.

We say that a multifunction Γ mapping (X, d) into Φ is *Φ-$\alpha(\cdot)$-k-monotone* if there is $C_k > 0$ such that for $\phi_x \in \Gamma(x)$, $\phi_y \in \Gamma(y)$ we have

$$\phi_x(x) + \phi_y(y) - \phi_x(y) - \phi_y(x) \geq_K -C_k \alpha(d(x, y))k. \tag{2.1}$$

In particular, when (X, d) is a metric linear space, and Φ is a linear space consisting of linear operators $\phi(x) = \langle \phi, x \rangle$, we can rewrite (2.1) in the more classical form

$$\langle \phi_x - \phi_y, x - y \rangle \geq_K -C_k \alpha(d(x, y))k. \tag*{$(2.1)_\ell$}$$

A multifunction Γ mapping (X, d) into Φ is called *n-cyclic Φ-$\alpha(\cdot)$-k-monotone* if there is $C_k > 0$ such that for arbitrary $x_0, x_1, ..., x_n = x_0 \in X$ and $\phi_{x_i} \in \Gamma(x_i)$, $(i = 0, 1, 2, ..., n)$, we have

$$\sum_{i=1}^{n} [\phi_{x_{i-1}}(x_{i-1}) - \phi_{x_{i-1}}(x_i)] \geq_K -C_k \sum_{i=1}^{n} \alpha(d(x_i, x_{i-1}))k. \tag*{$(2.1)_n$}$$

A multifunction Γ mapping (X, d) into Φ is called *cyclic Φ-$\alpha(\cdot)$-k-monotone* if it is n-cyclic Φ-$\alpha(\cdot)$-k-monotone for $n = 2, 3, ...$. Of course, just from the definition a multifunction Γ is Φ-$\alpha(\cdot)$-k-monotone if and only if it is 2-cyclic Φ-$\alpha(\cdot)$-k-monotone.

Observe that the introduced notions of Φ-$\alpha(\cdot)$-k-monotone multifunctions, n-cyclic Φ-$\alpha(\cdot)$-k-monotone multifunctions, cyclic Φ-$\alpha(\cdot)$-k-monotone multifunctions do not depend on the norm in the space Y, and so, as a consequence we get that if $\|\cdot\|_1$ is a norm in Y equivalent to the norm $\|\cdot\|$, then Φ-$\alpha(\cdot)$-k-monotone multifunctions, n-cyclic Φ-$\alpha(\cdot)$-k-monotone multifunctions, cyclic Φ-$\alpha(\cdot)$-k-monotone multifunctions are the same with respect to both norms.

Proposition 2 *For a given function f the subdifferential $\partial^{\Phi}_{\alpha,k} f\big|_x$, considered as a multifunction of x, is cyclic Φ-$\alpha(\cdot)$-k-monotone.*

Proof Take arbitrary $x_0, x_1, ..., x_n = x_0 \in X$ and $\phi_{x_i} \in \partial^{\Phi}_{\alpha,k} f\big|_{x_i}$, $i = 0, 1, 2, ..., n$. Since $\phi_{x_i} \in \partial^{\Phi}_{\alpha,k} f\big|_{x_i}$ we have that for $i = 1, 2, , , .n$ there are C^i_k such that

$$f(x_i) - f(x_{i-1}) \geq_K \phi_{x_{i-1}}(x_i) - \phi_{x_{i-1}}(x_{i-1}) - C^i_k \alpha(d(x_i, x_{i-1}))k. \tag*{$(1.3)^i$}$$

Adding all equations $(1.3)^i$ for $i = 1, 2, ..., n$ and changing the sign we obtain

$$\sum_{i=1}^{n}[\phi_{x_{i-1}}(x_{i-1}) - \phi_{x_i}(x_{i-1})] \geq_K -C_k^i \sum_{i=1}^{n} \alpha(d(x_i, x_{i-1}))k \geq_K$$

$$-C_k \sum_{i=1}^{n} \alpha(d(x_i, x_{i-1}))k, \qquad (2.1)_n$$

where $C_k = \max C_k^i$. □

Let $\mathcal{L}$ be the space of all Lipschitzian functions defined on (X, d) with values in $(Y, \|\cdot\|)$. We define on $\mathcal{L}$ a quasinorm

$$\|\phi\|_L = \sup_{\substack{x_1, x_2 \in X, \\ x_1 \neq x_2}} \frac{\|\phi(x_1) - \phi(x_2)\|}{d(x_1, x_2)}. \qquad (2.2)$$

Observe that, if $\|\phi_1 - \phi_2\|_L = 0$, then the difference of ϕ_1 and ϕ_2 is a constant function, i.e., there is $c \in Y$ such that $\phi_1(x) = \phi_2(x) + c$. Thus, we consider the quotient space $\tilde{\mathcal{L}} = {}^{\mathcal{L}}\!/_{Y}$. The quasinorm $\|\phi\|_L$ induces the norm in the space $\tilde{\mathcal{L}}$. Since this will not lead to any misunderstanding, we shall also denote this norm by $\|\phi\|_L$.

Let Φ be a linear family of Lipschitz functions. If there is an element h, belonging to the interior of K, $h \in \operatorname{Int} K$, $\|h\| < 1$, such that for all $x \in X$ and all $\phi \in \Phi$ and all $t > 0$, there is a $y \in X$ such that $0 < d(x, y) < t$ and

$$\phi(y) - \phi(x) \geq_K \|\phi\|_L d(y, x) h, \qquad (2.3)$$

we say that the family Φ has the *monotonicity property with respect to the element h* (briefly: the family Φ has the *h-monotonicity property*).

It is easy to see, that if $a \in Int\ K$, $0 \leq_K a \leq_K h$, then each family Φ having the h-monotonicity property also has a-monotonicity property.

Write for any $\phi \in \Phi$, $a \in \operatorname{Int} K$, $x \in X, \varrho \in \mathbf{R}_+$ (see Preiss and Zajiček, 1984; Rolewicz, 1994; Pallaschke and Rolewicz, 1997; Rolewicz, 1999)

$$K(\phi, a, x, \varrho) = \{y \in X : \phi(y) - \phi(x) \geq_K \|\phi\|_L d(y, x) a,\ d(x, y) < \varrho\}. \qquad (2.4)$$

The set $K(\phi, a, x, \varrho)$ will be called a (a, ϱ)*-cone with vertex at x and direction* ϕ. Of course, it may happen that $K(\phi, a, x, \varrho) = \{x\}$. However, if $h \in a + \operatorname{Int} K$, it is obvious that the set $K(\phi, a, x, \varrho)$ has a non-empty interior and, even more,

$$x \in \overline{\operatorname{Int} K(\phi, a, x, \varrho)}. \qquad (2.5)$$

Observe that just from the definition it follows that if $a_1 <_K a_2$, then $K(\phi, a_1, x, \varrho) \supset K(\phi, a_2, x\varrho)$.

A set $M \subset X$ is said to be (a, ϱ)*-cone meagre* if for arbitrary ε, $0 < \varepsilon < \varrho$ there are $z \in X$, $d(x, z) < \varepsilon$ and $\phi \in \Phi$ such that

$$M \cap \text{Int } K(\phi, a, z, \varrho) = \emptyset. \tag{2.6}$$

The arbitrariness of ε and (2.5) imply that an (a, ϱ)-cone meagre set M is nowhere dense. A set $M \subset X$ is called (a, ϱ)-*small-angle* if it can be represented as a union of a countable number of (a, ϱ)-cone meagre sets M_n,

$$M = \bigcup_{n=1}^{\infty} M_n. \tag{2.7}$$

Of course, every (a, ϱ)-small-angle set M is of the first category.

In further considerations we shall assume that the cone K is *normal*, i.e. for all $k \in Int\ K$, the set $(-k + K) \cap (k - K)$ is a bounded neighbourhood of 0. This is equivalent to the fact that K has a bounded basis, i.e. there exists in K a closed bounded convex subset, $B \subset K$, such that $0 \notin B$ and each $k \in K$ can be represented in the form $k = tb$, where t is a non-negative real number and $b \in B$. Let K be a convex cone having non-empty interior and a bounded basis. It can be shown (Peressini, 1967; Jahn, 1986, 2004) that in this case there is in Y an equivalent norm $\|\cdot\|_i$ such that if $k \in K$, $k \leq_K h$, i.e. $h \in k + K$, then

$$\|k\|_i \leq \|h\|_i. \tag{2.8}$$

Any norm satisfying (2.8) shall be called *increasing*. By adapting the method of Preiss and Zajiček (1984) to metric spaces we obtain

Theorem 1 *(compare Rolewicz 1994, 1999; Pallaschke and Rolewicz, 1997). Let (X, d) be a metric space. Let $(Y, \|\cdot\|)$ be a Banach space, ordered by a closed pointed convex cone K, such that the norm is increasing. Let Φ be a linear family of Lipschitz functions mapping (X, d) into $(Y, \|\cdot\|)$ having the monotonicity property with respect to the element $h \in$ Int K with $\|h\| < 1$. Assume that Φ is separable in the metric d_L. Let a multifunction Γ mapping (X, d) into 2^{Φ} be Φ-$\alpha(\cdot)$-k-monotone and such that* dom $\Gamma = X$ *(i.e., $\Gamma(x) \neq \emptyset$ for all $x \in X$). Then there are $\varrho > 0$ and a (k, ϱ)-small-angle set A such that Γ is single-valued and continuous on the set $X \setminus A$.*

Proof It is sufficient to show that the set

$$A = \{x \in X : \lim_{\delta \to 0} \text{diam } \Gamma(B(x, \delta)) > 0\},$$

where by diam we denote the diameter of the set measured in the Lipschitz metric d_L, is (k, ϱ)-small-angle. Of course, we can represent A as a union of sets

$$A_n = \{x \in X : \lim_{\delta \to 0} \text{diam } \Gamma(B(x, \delta)) > \frac{1}{n}\}. \tag{2.9}$$

Let $\{\phi_m\}$ be a dense sequence in the space Φ in the metric d_L. Suppose that $0 <_K a <_K h$ and $\|a\| < 1$. Let

$$A_{n,m} = \{x \in A_n : \text{dist}(\phi_m, \Gamma(x)) < \frac{\|a\|}{4n}\}, \tag{2.10}$$

where, as usual, we denote $\operatorname{dist}(\phi_m, \Gamma(x)) = \inf\{\|\phi_m - \phi\|_L : \phi \in \Gamma(x)\}$.

By the density of the sequence $\{\phi_m\}$ in Φ,

$$\bigcup_{m=1}^{\infty} A_{n,m} = A_n.$$

We will show that there is $\varrho > 0$ such that the sets $A_{n,m}$ are (a, ϱ)-cone meagre. Suppose that $x \in A_{n,m}$. Let ε be an arbitrary positive number. Since $x \in A_n$, there are $0 < \delta < \varepsilon$ and $z_1, z_2 \in X$, $\phi_1 \in \Gamma(z_1)$, $\phi_2 \in \Gamma(z_2)$ such that $d(z_1, x) < \delta$, $d(z_2, x) < \delta$ and

$$\|\phi_1 - \phi_2\|_L > \frac{1}{n}. \tag{2.11}$$

Thus, by the triangle inequality, for every $\phi \in \Gamma(x)$ either $\|\phi_1 - \phi\| > \frac{1}{2n}$ or $\|\phi_2 - \phi\| > \frac{1}{2n}$. By the definition of $A_{n,m}$, we can find $\phi_x \in \Gamma(x)$ such that $\|\phi_x - \phi_m\| < \frac{\|a\|}{4n}$. Therefore, choosing as z either z_1 or z_2, we can say that there are $z \in X$ and $\phi_z \in \Gamma(z)$ such that $d(z, x) < \delta$ and

$$\|\phi_z - \phi_m\|_L \geq \|\phi_z - \phi_x\|_L - \|\phi_x - \phi_m\|_L > \frac{1}{2n} - \frac{\|a\|}{4n}. \tag{2.12}$$

We shall show that there is $\varrho > 0$ such that

$$A_{n,m} \cap K(\phi_z - \phi_m, a, z, \varrho) =$$
$$\{y \in A_{n,m} : d(y, z) < \varrho,\ \phi_z(y) + \phi_m(z) - \phi_m(y) - \phi_z(z) \geq_K \|\phi_z - \phi_m\|_L d(y, z) a\} =$$
$$= \emptyset. \tag{2.13}$$

Indeed, let $\varrho > 0$ be chosen in such a way that

$$\sup_{0<t<\varrho} C_k \|k\| \frac{\alpha(t)}{t} < r = \frac{1 - \|a\|}{4n}. \tag{2.14}$$

Since $r > 0$, by (1.1) such ϱ exists. Now we shall show (2.13). Suppose that $y \in K(\phi_z - \phi_m, a, z, \varrho)$. This means that

$$d(y, z) < \varrho \tag{2.15}$$

and

$$[\phi_z(y) - \phi_m(y)] - [\phi_z(z) - \phi_m(z)] =$$
$$\phi_y(y) + \phi_m(z) - \phi_m(y) - \phi_y(z) \geq_K \|\phi_z - \phi_m\|_L d(y, z) a. \tag{2.16}$$

Suppose that $\phi_y \in \Gamma(y)$. Then, by the Φ-$\alpha(\cdot)$-k-monotonicity of Γ,

$$\phi_y(y) - \phi_y(z) \geq_K \phi_z(y) - \phi_z(z) - C_k \alpha(d(z, y)) k \tag{2.17}$$

and by (2.16)

$$\begin{aligned}&\phi_y(y) + \phi_m(z) - \phi_m(y) - \phi_y(z)\\ &\geq_K \phi_z(y) + \phi_m(z) - \phi_m(y) - \phi_z(z) - C_k\alpha(d(z,y))k\\ &\geq_K \|\phi_z - \phi_m\|_L d(y,z)a - C_k\alpha(d(z,y))k.\end{aligned}$$

Using the fact that the norm is increasing and (2.12) we get

$$\|\phi_y(y) + \phi_m(z) - \phi_m(y) - \phi_y(z)\| \geq \left([\frac{1}{2n} - \frac{\|a\|}{4n}]\right) d(y,z) - rd(y,z) \geq \frac{1}{4n} d(y,z)$$

$$> \frac{\|a\|}{4n} d(y,z).$$

This implies that

$$\|\phi_y - \phi_m\|_L > \frac{\|a\|}{4n}$$

and by the definition of $A_{n,m}$, $y \notin A_{n,m}$. □

Let K be a convex cone K having non-empty interior and a bounded basis. We have the following corollary

Corollary 1 *(compare Rolewicz, 1994, 1999; Pallaschke and Rolewicz, 1997). Let (X,d) be a metric space. Let $(Y,\|\cdot\|)$ be a Banach space, ordered by a closed pointed convex cone K with bounded basis. Let Φ be a linear family of Lipschitz functions mapping X into Y having the monotonicity property with respect to the element $h \in \operatorname{Int} K$. Assume that Φ is separable in the metric d_L. Let a multifunction Γ mapping X into 2^Φ be Φ-$\alpha(\cdot)$-k-monotone and such that* $\operatorname{dom}\Gamma = X$ *(i.e., $\Gamma(x) \neq \emptyset$ for all $x \in X$). Then there are $\varrho > 0$ and a (k,ϱ)-small-angle set A such that Γ is single-valued and continuous on the set $X \setminus A$.*

We recall that a set B of the second category is called *residual* if its complement is of the first category. Since the (a,ϱ)-small-angle sets are always of the first category we immediately obtain the following extension of the result from Kenderov (1974) on metric spaces and vector valued functions

Theorem 2 *Let (X,d) be a metric space of the second category on itself (in particular, let X be a complete metric space). Let $(Y,\|\cdot\|)$ be a Banach space. We assume that $(Y,\|\cdot\|)$ is an ordered Banach space and that the order is given by a closed convex cone K with bounded basis. Let Φ be a linear family of Lipschitz functions mapping X into Y. We assume that Φ has the monotonicity property with respect to an element $h \in \operatorname{Int} K, \|h\| < 1$. Assume that Φ is separable in the metric d_L. Let Γ be a Φ-$\alpha(\cdot)$-k- monotone multifunction mapping X into 2^Φ such that $\Gamma(x) \neq \emptyset$ for all $x \in X$. Then there is a residual set B such that the multifunction Γ is single-valued and continuous on B.*

Corollary 2 *Let (X, d) be a metric space of the second category on itself (in particular, let X be a complete metric space). Let $(Y, \|\cdot\|)$ be a Banach space. We assume that Y is an ordered Banach space and that the order is given by a closed convex cone K with bounded basis. Let Φ be a linear family of Lipschitz functions mapping X into Y. We assume that Φ has monotonicity property with respect to element $k \in \operatorname{Int} K$, $\|k\| < 1$. Assume that Φ is separable in the metric d_L. Let $f(x)$ be a function having at each point a Φ-subgradient. Then there is a residual set B such that on B the subdifferential $\partial_\Phi^{\alpha,k} f\big|_x$ is single-valued and it is continuous in the metric d_L.*

We shall say that a function $f(x)$ mapping a metric space (X, d) into a normed space $(Y, \|\cdot\|_Y)$ is *Fréchet Φ-differentiable* at a point x_0 if there are a function $\gamma(t)$ mapping the interval $[0, +\infty)$ into the interval $[0, +\infty]$ such that

$$\lim \frac{\gamma(t)}{t} = 0$$

and a function $\phi_{x_0} \in \Phi$ such that

$$\|[f(x) - f(x_0)] - [\phi(x) - \phi(x_0)]\|_Y \leq \gamma(d(x, x_0)).$$

The function ϕ will be called a *Fréchet Φ-gradient* of the function $f(x)$ at the point x_0. The function $\gamma(t)$ will be called the *modulus of smoothness*.

In the case of normed spaces the continuity of Gâteaux differentials in the norm operator topology implies that these differentials are the Fréchet differentials. Similarly, for metric spaces we obtain the following generalization of the Asplund Theorem (Asplund, 1968) (see also Mazur, 1933).

Proposition 3 *(compare Rolewicz, 1995a, 1995b). Let (X, d) be a metric space of the second category on itself (in particular, let X be a complete metric space). Let $(Y, \|\cdot\|)$ be a Banach space. We assume that Y is an ordered Banach space and that the order is given by a closed convex cone K with bounded basis. Let Φ be a linear family of Lipschitz functions mapping X into Y. We assume that Φ has monotonicity property with respect to an element $k \in \operatorname{Int}_r K$, $\|k\| < 1$. Assume that Φ is separable in the metric d_L. Let ϕ_{x_0} be a Φ-subgradient of the function $f(x)$ at a point x_0. Suppose that there is a neighbourhood U of x_0 such that for all $x \in U$ the subdifferential $\partial^{\alpha,k} f\big|_x$ is not empty and it is lower semi-continuous at x_0 in the Lipschitz norm, i.e., for every $\varepsilon > 0$ there is a neighbourhood $V_\varepsilon \subset U$ such that for $x \in V_\varepsilon$ there is $\phi_x \in \partial_\Phi^{\alpha,k} f\big|_X$ such that*

$$\|\phi_x - \phi_{x_0}\|_L \leq \varepsilon d(x, x_0). \tag{2.18}$$

Then ϕ_{x_0} is the Fréchet Φ-gradient of the function $f(x)$ at the point x_0.

Proof Let

$$F(x) = [f(x) - f(x_0)] - [\phi_{x_0}(x) - \phi_{x_0}(x_0)].$$

It is easy to see that $F(x_0) = 0$. Since ϕ_{x_0} is a Φ-subgradient of the function $f(x)$ at a point x_0, then $F(x) \geq_K 0$. Let ε be an arbitrary positive number and let V_ε be a neighbourhood of x_0 such that for $x \in V_\varepsilon$ (2.18) holds. Since ϕ_x is a Φ-subgradient of the function $f(x)$ at a point x, $\psi_x = \phi_x - \phi_{x_0}$ is a Φ-subgradient of the function $F(x)$ at the point x. Thus

$$F(y) - F(x) \geq_K \psi_x(y) - \psi_x(x).$$

In particular, if $y = x_0$, then

$$F(x_0) - F(x) \geq_K \psi_x(x_0) - \psi_x(x). \tag{2.19}$$

Taking into account (2.14), we obtain that for $x \in V_\varepsilon$

$$0 \leq F(x) \leq \psi_x(x) - \psi_x(x_0) \leq_K \phi_x(x) - \phi_x(x_0). \tag{2.20}$$

Since the cone K has bounded basis, without loss of generality we may assume that the norm is increasing $0 \leq_K a \leq_K b$, which implies that

$$\|a\| \leq \|b\|. \tag{2.21}$$

Thus from (2.18),(2.20) and (2.21) we obtain that

$$\|[f(x) - f(x_0)] - [\phi_{x_0}(x) - \phi_{x_0}(x_0)]\| \leq \varepsilon d(x, x_0). \tag{2.22}$$

So, the fact that ε is arbitrary implies that ϕ_{x_0} is the Fréchet gradient of the function $f(x)$ at a point x_0. □

If we assume that the function $f(x)$ is continuous, then we do not need to assume that there is a neighbourhood U of x_0 such that for all $x \in U$, the subdifferential $\partial^{\alpha,k} f\big|_x$ is not empty. It is sufficient to assume that the subdifferential $\partial^{\alpha,k} f\big|_x$ is not empty on a dense set.

Proposition 4 *(compare Rolewicz, 1995a, 1995b). Let (X, d) be a metric space. Let $(Y, \|\cdot\|)$ be a Banach space. We assume that Y is an ordered Banach space and that the order is given by a closed convex cone K with bounded basis. Let Φ be a linear family of Lipschitz functions mapping X into Y. We assume that Φ has monotonicity property with respect to element $k \in \mathrm{Int}_r K$. Assume that Φ is separable in the metric d_L. Let ϕ_{x_0} be a Φ-subgradient of the function $f(x)$ at a point x_0. Suppose that there is a dense set A in a neighbourhood U of x_0 such that for all $x \in A$ the Φ-subdifferential $\partial^{\alpha,k} f_\Phi\big|_x$ is not empty and lower semi-continuous at x_0 in the Lipschitz norm. Then, ϕ_{x_0} is the Fréchet Φ-gradient of the function $f(x)$ at the point x_0.*

Proof The proof goes along the same line as the proof of Proposition 3. We obtain that for $x \in A \cap V_\varepsilon$

$$\|[f(x) - f(x_0)] - [\phi_{x_0}(x) - \phi_{x_0}(x_0)]\| \leq \varepsilon d(x, x_0). \tag{2.23}$$

Thus, by the continuity of $f(x)$ and the density of A, we obtain that (2.23) holds for all $x \in U$. The remaining part of the proof is the same. □

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