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On existence of shape optimization for a p-Laplacian equation over a class of open domains[*][†]

by

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Abstract: In this paper, we introduce four new classes of open sets in general Euclidean space $\mathbb{R}^N$. It is shown that every such class of open sets is compact under the Hausdorff distance. The result is applied to a shape optimization problem of p-Laplacian equation. The existence of the optimal solution is presented.

Keywords: Laplacian, shape optimization, existence

1. Introduction

The existence of optimal solution is one of the major concerns in most of the shape optimization problems. Many approaches aiming to achieve the existence are available in literature. Under a regularity assumption on the boundary of unknown domain, the existence of various shape optimizations can be found in Chenais (1975), Pironneau (1984), Tiba (2003), Wang, Wang, and Yang (2006), Wang and Yang (2008), Yang (2009). In Tiba (2003), the existence of shape optimization for an elliptic equation over a class of special interior domains is considered. Wang, Wang, and Yang (2006) generalizes the work of Tiba (2003) to the stationary Navier-Stokes equations over a class of exterior domains. In

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Wang and Yang (2008), the solution space of stationary Navier-Stokes equations over a class of domains by using some geometric methods is considered. Similar interesting studies have also been presented in Tiba and Halanay (2009) and Tiba (2013). Some generalizations based on Wang, Wang, and Yang (2006) have been developed in Delay (2012). The generalized perimeter and constraints, or the penalty terms constructed from generalized perimeter and constraints are used in dealing with existence in Guo and Yang (2013), He and Guo (2012), where, for the second case, the conditions on the dimension of underlying Euclidean spaces are imposed to obtain the compactness of certain families of open sets with respect to the Hausdorff distance.

Let $U_R = U(0,R) \subset \mathbb{R}^N$ be an open ball centered at the origin with the radius R in a general Euclidean space $\mathbb{R}^N$ and let $\mathcal{C}$ be a class of open sets inside of U_R, this class to be specified later. Consider the following p-Laplacian equation:

$$\begin{cases} -\Delta_p u_\Omega = f \text{ in } \Omega, \\ u_\Omega \in W_0^{1,p}(\Omega), \end{cases} \tag{1.1}$$

where Δ_p denotes the p-Laplace operator: $\Delta_p u_\Omega = \operatorname{div}(|\nabla u_\Omega|^{p-2}\nabla u_\Omega)$ with $2 \le p < +\infty$ and $f \in L^{p'}(U_R)$ is a given function, $p' = \frac{p}{p-1}$.

We say that u_Ω is a (weak) solution of equation (1.1) if $u_\Omega \in W_0^{1,p}(\Omega)$ satisfies

$$\int_\Omega |\nabla u_\Omega|^{p-2}\nabla u_\Omega \cdot \nabla\varphi dx = \int_\Omega f \cdot \varphi dx,\ \forall\, \varphi \in W_0^{1,p}(\Omega). \tag{1.2}$$

In this paper, we are concerned with the existence of the following shape optimization problem:

$$\inf_{\Omega\in\mathcal{C}} E(\Omega) = \inf_{\Omega\in\mathcal{C}} \int_\Omega |u_\Omega - u_0|^p dx, \tag{1.3}$$

where u_Ω is the solution of equation (1.2) corresponding to $\Omega \in \mathcal{C}$ with zero extension outside of Ω, and $u_0 \in L^p(U_R)$ is a given function.

It is natural that in order to study problem (1.3), we need to define the topology for the open sets class $\mathcal{C}$. This is realized by the Hausdorff distance between their complementary sets for any two given open sets. That is, for any $\Omega_1, \Omega_2 \in \mathcal{C}$, the Hausdorff distance $\rho(\Omega_1,\Omega_2)$ is defined as

$$\rho(\Omega_1,\Omega_2) = \max\left\{ \sup_{x\in B_R\setminus\Omega_1} \operatorname{dist}(x, B_R\setminus\Omega_2),\ \sup_{y\in B_R\setminus\Omega_2} \operatorname{dist}(B_R\setminus\Omega_1, y) \right\}, \tag{1.4}$$

where $\operatorname{dist}(\cdot,\cdot)$ denotes the Euclidean metric of $\mathbb{R}^N$ and $B_R = \overline{U_R}$ is the closure of U_R in $\mathbb{R}^N$. In this way, $(\mathcal{C}, \rho)$ becomes a metric space (see Pironneau, 1984). A sequence $\{\Omega_n\} \subset \mathcal{C}$ is said to be convergent to $\Omega \in \mathcal{C}$, which is denoted by $\Omega_n \xrightarrow{\rho} \Omega$, if $\rho(\Omega_n,\Omega) \to 0$ as $n \to \infty$.

In this work, we introduce four new classes of open sets $\mathcal{C}_i, i = 1, 2, 3, 4$, in $\mathbb{R}^N (N \geq 1)$ and $\mathcal{C} = \mathcal{C}_3 \cap \mathcal{C}_4$. We show that each class is compact under the Hausdorff distance (1.4). The existence of the optimal solution (1.3) over the class $\mathcal{C}$ is demonstrated.

We proceed as follows. In Section 2, we first introduce some preliminary notation and define the classes of open sets $\mathcal{C}_i, i = 1, 2, 3, 4$, respectively. The main results are stated. Section 3 is devoted to the proof of the main results.

2. Main results

Throughout the paper, we denote by $U(x, r) \subset \mathbb{R}^N$ the open ball centered at $x \in \mathbb{R}^N$ with radius r and by $B(x, r) \subset \mathbb{R}^N$ the closure of $U(x, r)$. Define

$$\delta(K_1, K_2) = \max \left\{ \sup_{x \in K_1} \text{dist}(x, K_2) \sup_{y \in K_2} \text{dist}(y, K_1) \right\},$$

which is also called the Hausdorff distance between two compact subsets K_1 and K_2 of $\mathbb{R}^N$. It is seen from equation (1.4) that $\rho(\Omega_1, \Omega_2) = \delta(B_R \setminus \Omega_1, B_R \setminus \Omega_2)$ for any open sets $\Omega_1, \Omega_2 \subset U_R$. Hence

$$\Omega_n \xrightarrow{\rho} \Omega \Longleftrightarrow B_R \setminus \Omega_n \xrightarrow{\delta} B_R \setminus \Omega.$$

Lemmas 2.1-2.5 below are brought from Guo and Yang (2012), Pironneau (1984), and Schneider (1993).

LEMMA 2.1 *Let $K, K_n, n \in \mathbb{N}$, be compact subsets of $\mathbb{R}^N$ such that $K_n \xrightarrow{\delta} K$. Then K is the set of all accumulation points of the sequences $\{x_n\}_{n \in \mathbb{N}}$ such that $x_n \in K_n$ for every $n \in \mathbb{N}$.*

REMARK 2.1 *It follows from Lemma 2.1 and the definition of δ that for any given $\varepsilon > 0$, there exists an integer $M(\varepsilon) > 0$ such that $K \subset \bigcup_{x \in K_m} U(x, \varepsilon)$ for all $m \geq M(\varepsilon)$ and $K_m \subset \bigcup_{x \in K} U(x, \varepsilon)$.*

LEMMA 2.2 *Let $K, \tilde{K}, K_n, \tilde{K}_n, n \in \mathbb{N}$, be compact subsets of $\mathbb{R}^N$ such that $K_n \xrightarrow{\delta} K$ and $\tilde{K}_n \xrightarrow{\delta} \tilde{K}$. If $K_n \subset \tilde{K}_n$ for every n, then $K \subset \tilde{K}$.*

LEMMA 2.3 [**Γ-property for open sets class**] *For any given class of open sets $\mathcal{C}$, if $\{\Omega_n\}_{n=1}^{\infty} \subset \mathcal{C}, \Omega \in \mathcal{C}$, and $\Omega_n \xrightarrow{\rho} \Omega$, then for each open subset Λ with $\overline{\Lambda} \subset \Omega$, there exists a positive integer n_Λ depending on Λ such that $\overline{\Lambda} \subset \Omega_n$ for all $n \geq n_\Lambda$.*

LEMMA 2.4 *Suppose that $\Omega_n \subset U_R$, $n \in \mathbb{N}$, are bounded open sets of $\mathbb{R}^N$. Then there exist an open set $\Omega \subset U_R$ and a subsequence $\{\Omega_{n_k}\}_{k=1}^{\infty}$ of $\{\Omega_n\}_{n=1}^{\infty}$ such that $\Omega_{n_k} \xrightarrow{\rho} \Omega$. In particular, $(\mathcal{O}, \delta)$ is a compact metric space, where $\mathcal{O} = \{K \subset U_R |\ K \text{ is compact}\}$.*

LEMMA 2.5 [**Blaschke selection theorem**] *Any bounded sequence of convex sets contains a convergent subsequence under the Hausdorff distance.*

Let $C \subset \mathbb{R}^N$ be a given nonempty convex domain. C' is said to be congruent with C, if C' is equal to C upon rotation and translation in $\mathbb{R}^N$. Therefore, C' is congruent with C if and only if C is congruent with C'.

DEFINITION 2.1 *Let Ω be a bounded set in $\mathbb{R}^N$, $x_0 \in \partial\Omega$. We say that Ω satisfies the* interior convex domain condition *at x_0 if there exists a C' that is congruent with C such that $C' \subset \Omega$ and $x_0 \in \partial C'$.*

We say that Ω satisfies the uniformly interior convex domain condition *if Ω satisfies the* interior convex domain condition *at every $x_0 \in \partial\Omega$.*

DEFINITION 2.2 *Let Ω be a bounded set in $\mathbb{R}^N$, $x_0 \in \partial\Omega$. Ω is said to satisfy the* exterior convex domain condition *at x_0 if there exists a C' that is congruent with C such that $C' \subset U_R \setminus \Omega$ and $x_0 \in \partial C'$.*

We say that Ω satisfies the uniformly interior convex domain condition *if Ω satisfies the* exterior convex domain condition *at every $x_0 \in \partial\Omega$.*

The following Definition 2.3 and Lemma 2.6 appeared first in Chenais (1975), and can also be found in Pironneau (1984) as well as in Delfour and Zolésio (2001). Definition 2.3 is available in Adams and Fournier (2003) on page 81.

DEFINITION 2.3 *Let $C(\varepsilon, \xi, x)$ be the half-cone with angle ε, direction ξ, and vertex x, intersecting with the ball $U(x, \varepsilon)$.*

*Ω is said to have the ε-*cone property *if for all $x \in \partial\Omega$, there exists a direction $\xi(x)$ such that*

$$C(\varepsilon, \xi(x), y) \subset \Omega, \ \forall\, y \in U(x, \varepsilon) \cap \Omega.$$

*$C(\varepsilon, \xi(x), y)$ is then called an ε-*cone *at y. Set*

$$\mathcal{O}_\varepsilon = \{\Omega \subset \mathbb{R}^N \mid \Omega \text{ is open and } \Omega \text{ has the } \varepsilon\text{-cone property}\}. \tag{2.1}$$

LEMMA 2.6 *Let $\frac{\pi}{2} > \varepsilon > 0$ and let $\mathcal{O}_\varepsilon$ be defined by equation (2.1). Then $(\mathcal{O}_\varepsilon, \rho)$ is a compact metric space; and Ω has the ε-cone property if and only if $\partial\Omega$ is Lipschitz continuous with constant $k(\varepsilon) > 0$.*

DEFINITION 2.4 [**The cone condition**] *Ω is said to satisfy the* cone condition *if there exists a finite cone C_0 such that for any $x \in \Omega$, there exists a finite cone $C_x \subset \Omega$ that is congruent with C_0 and x is the vertex of C_x. Note that C_x is not necessarily obtained from C_0 by the parallel translation, but it is simply obtained by the rigid motion.*

DEFINITION 2.5 *Let C_0 be a given cone. We say that Ω satisfies the C_0-*cone condition *if for every $x \in \Omega$, there exists a cone $C_x \subset \Omega$ that is congruent with C_0 and x is the vertex of C_x.*

We say that Ω satisfies the exterior C_0-cone condition *if for every $x \in U_R \setminus \Omega$, there exists a cone C_x that is congruent with C_0, where x is the vertex of C_x, such that $C_x \subset U_R \setminus \Omega$.*

Let $R, r_0 > 0$, and let C_0 be a given cone. We introduce four classes of open sets $\mathcal{C}_i, i = 1, 2, 3, 4$:

$$\begin{cases} \mathcal{C}_1 = & \{\Omega \subset U(0, \frac{R}{2}) | \\ & \Omega \text{ satisfies the uniformly interior convex domain condition}\}, \\ \mathcal{C}_2 = & \{\Omega \subset U(0, \frac{R}{2}) | \ U(x_\Omega, r_\Omega) \subset \Omega, r_\Omega \geq r_0, \\ & \Omega \text{ satisfies the uniformly interior convex domain condition}\}, \\ \mathcal{C}_3 = & \{\Omega \subset U(0, \frac{R}{2}) | \ \Omega \text{ satisfies the } C_0\text{-cone condition}\}, \\ \mathcal{C}_4 = & \{\Omega \subset U(0, \frac{R}{2}) | \ U(x_\Omega, r_\Omega) \subset \Omega, r_\Omega \geq r_0, \\ & \Omega \text{ satisfies the exterior } C_0\text{-cone condition}\}, \\ \mathcal{C} = & \mathcal{C}_3 \cap \mathcal{C}_4. \end{cases} \tag{2.2}$$

We are now in a position to state the main results of this paper.

THEOREM 2.1 *For every given $i \in \{1, 2, 3, 4\}$, if $\{\Omega_m\}_{m=1}^{\infty} \subset \mathcal{C}_i$, then there exist a subsequence $\{\Omega_{m_k}\}_{k=1}^{\infty}$ of $\{\Omega_m\}_{m=1}^{\infty}$ and $\Omega \in \mathcal{C}_i$ such that*

$$\Omega_{m_k} \xrightarrow{\rho} \Omega \text{ as } k \to \infty.$$

In other words, each $(\mathcal{C}_i, \rho)$ is a compact metric space. In particular, $(\mathcal{C}, \rho)$ is also a compact metric space.

The following Remark 2.2 establishes the relationship between $\mathcal{C}_1 \cap \mathcal{C}_2$ and $\mathcal{O}_\varepsilon$ defined by equation (2.1).

REMARK 2.2 *Let $\mathcal{O}_\varepsilon$ be defined by equation (2.1). For any $\varepsilon > 0$, all ε-cones in $\mathcal{O}_\varepsilon$ are congruent with each other. If we take some ε-cone $C(\varepsilon, \xi, y)$ in $\mathcal{O}_\varepsilon$ as the convex domain C in $\mathcal{C}_1 \cap \mathcal{C}_2$, then $\mathcal{O}_\varepsilon \subset \mathcal{C}_1 \cap \mathcal{C}_2$. If $0 < \varepsilon < \frac{\pi}{4}$, then there may be $\mathcal{O}_\varepsilon \neq \mathcal{C}_1 \cap \mathcal{C}_2$. Obviously, if one takes ε-cone as the cone C_0, then $\mathcal{O}_\varepsilon \subset \mathcal{C}_3 \cap \mathcal{C}_4$.*

In general, the open set Ω in $\mathcal{C}_1$ or $\mathcal{C}_2$ may not satisfy the ε-cone property. The following Example 2.1 is an appropriate counterexample.

EXAMPLE 2.1 *(a). Set $\Omega = ((-1, 0) \cup (0, 1)) \times (-1, 1)$ and $C(\varepsilon, \boldsymbol{e}_1, 0)$ with $\varepsilon < \frac{1}{4}$ and $\boldsymbol{e}_1 = (1, 0)$. Then $\Omega \in \mathcal{C}_1$, but $\Omega \notin \mathcal{O}_\varepsilon$.*

(b). Let $C(\varepsilon, \xi, 0)$ be any ε-cone. Set $\Omega = (-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}) \times (-1, 1)$. Then $\Omega \in \mathcal{C}_2$ but $\Omega \notin \mathcal{O}_\varepsilon$.

The open set Ω in $\mathcal{C}$ may not satisfy uniform segment property (see e.g., Neittaanmaki, Sprekels and Tiba, 2006; Tiba, 2003). The following Example 2.2 is an appropriate counterexample.

EXAMPLE 2.2 *Let $\Omega = \{(-1, 2) \times (-2, 2)\} \setminus \{(x, y) \in [-1, 1]^2 | \ |y| \leq |x|\}$ and cone $C_0 = C(\frac{\pi}{8}, \frac{1}{8}, 0)$. Then $\Omega \in \mathcal{C}$ is a connected domain. But Ω does not satisfy the uniform segment property.*

The following property is called the exterior $\Gamma-$property.

THEOREM 2.2 [**Exterior Γ-property for $\mathcal{C}$**] *If $\{\Omega_n\}_{n=1}^{\infty} \subset \mathcal{C}$ and $\Omega_n \xrightarrow{\rho} \Omega$, then for each open subset Λ satisfying $\overline{\Lambda} \subset U_R \setminus \overline{\Omega}$, there exists a positive integer n_Λ depending on Λ such that $\overline{\Lambda} \subset U_R \setminus \overline{\Omega}_n$ for all $n \geq n_\Lambda$.*

It is noted that the exterior Γ-property cannot be deduced from the Hausdorff convergence as the (interior) Γ-property stated in Lemma 2.3.

EXAMPLE 2.3 *Set $\Omega_n = \{(0,1)\times(-1,0)\cup\{(0,1/n)\times[0,1)\}$, $\Omega = (0,1)\times(-1,0)$, $\Lambda_\eta = \{(x,y)|(x^2+(y-1/2)^2 < \eta\}$. Obviously, $\Omega_n \xrightarrow{\rho} \Omega$ and $\Lambda_\eta \cap \Omega = \emptyset$, but $\Lambda_\eta \cap \Omega_n$ cannot be empty for any n no matter how small η is.*

The existence of the optimal solution for the shape optimization (1.3) in class $\mathcal{C}$ is stated as the following Theorem 2.3.

THEOREM 2.3 *Suppose that $p \neq N$. Then the shape optimization problem (1.3) admits at least one solution for the class of open sets $\mathcal{C}$.*

REMARK 2.3 *The p-Laplacian problem (1.3) has been studied in Bucur and Buttazzo (2005) for $p = 2$ under a different class of open sets.*

3. Proof of main results

The following Lemmas 3.1 and 3.2 are brought from Guo and Yang (2012):

LEMMA 3.1 *Let $\{x_n\} \subset \mathbb{R}^N$ and $\{r_n\} \subset \mathbb{R}$ be such that $r_n \geq r^* > 0$ and $U(x_n, r_n) \xrightarrow{\rho} D \subset \mathbb{R}^N$. Then there exist an $x \in \mathbb{R}^N$ and a subsequence $\{x_{nk}\}$ of $\{x_n\}$ such that $U(x, r^*) \subset D$ and $x_{nk} \to x$ as $k \to \infty$. Furthermore, if $r_n \to r^*$ as $n \to \infty$, then $D = U(x, r^*)$.*

LEMMA 3.2 *Suppose that $\Omega_n \xrightarrow{\rho} \Omega$ and $x \in \partial\Omega$. Then there exist $x_{n_l} \in \partial\Omega_{n_l}$ for all $l \in \mathbb{N}$ such that $x_{n_l} \to x$ as $l \to \infty$.*

Assume being given a sequence $\{C_n\}_{n=1}^{\infty}$ where C_n is congruent with C for every $n \in \mathbb{N}$. By Lemma 2.4, there exists a subsequence of $\{C_n\}_{n=1}^{\infty}$, still denoted by itself, and $\tilde{C} \subset \mathbb{R}^N$, such that $C_n \xrightarrow{\rho} \tilde{C}$.

LEMMA 3.3 *Let $C_n \xrightarrow{\rho} \tilde{C}$, where C_n is congruent with C for every $n \in \mathbb{N}$. Then $\tilde{C}$ is also congruent with C. In addition, if $\overline{C}_n \xrightarrow{\delta} F$, then $\mathrm{Int}(F) = \tilde{C}$ and $F = \overline{\mathrm{Int}(F)}$, where $\overline{C}_n$ is the closure of C_n and $\mathrm{Int}(F)$ is the interior of F.*

PROOF Since C_n is congruent with C, there exist an orthogonal matrix $A_n \in \mathbb{R}^{N\times N}$ and an x_n such that $C_n = A_nC + x_n$; here and throughout the paper we denote by $AC + x_0 = \{Ay + x_0;\ y \in C\}$, where $A \in \mathbb{R}^{N\times N}$ is an orthogonal matrix and $x_0 \in \mathbb{R}^N$.

Since $\{A_n\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}^{N\times N}$, there exists a subsequence of $\{A_n\}_{n=1}^{\infty}$, still denoted by itself, and $A_0 \in \mathbb{R}^{N\times N}$, such that $A_n \to A_0$. By orthogonality of $\{A_n\}_{n=1}^{\infty}$, A_0 is orthogonal as well. By the same reasoning, there exists a subsequence of $\{x_n\}_{n=1}^{\infty}$, still denoted by itself, and $x_0 \in \mathbb{R}^N$ such that $x_n \to x_0$.

Next, we show that $\tilde{C} = A_0C + x_0$.

(i). For any $y \in C$, there exists an $r_y > 0$ such that $U(y, r_y) \subset C$ and hence $U(A_ny + x_n, r_y) = A_nU(y, r_y) + x_n \subset C_n$. On the other hand, since $A_ny + x_n \to A_0y + x_0$, by Lemma 3.1, $U(A_ny + x_n, r_y) \xrightarrow{\rho} U(A_0y + x_0, r_y)$. By Lemma 2.2, $A_0U(y, r_y) + x_0 = U(A_0y + x_0, r_y) \subset \tilde{C}$, which implies that $A_0C + x_0 \subset \tilde{C}$.

(ii). If $\tilde{C} \not\subset A_0C+x_0$, there exists a $z \in \tilde{C}\setminus(A_0C + x_0)$. Obviously, $\overline{A_0C+x_0}$ is a convex domain. There are two cases. a) If $z \in \tilde{C} \setminus \overline{(A_0C + x_0)}$, then there exists an $r_z > 0$ such that $U(z, r_z) \subset \tilde{C}$ and $U(z, r_z) \cap (A_0C + x_0) = \emptyset$; b) If $z \in \tilde{C} \cap \partial (A_0C + x_0)$, since $A_0C + x_0$ is convex, there also exist $z' \in \tilde{C}$ and $r_{z'} > 0$ such that $U(z', r_{z'}) \cap (A_0C + x_0) = \emptyset$. By combining two cases, we get that there exist $z' \in \mathbb{R}^N$ and $r_{z'} > 0$ such that $U(z', r_{z'}) \subset \tilde{C}$ and $U(z', r_{z'}) \cap (A_0C + x_0) = \emptyset$. In other words,

$$\operatorname{dist}(z', A_0C + x_0) \geq r_{z'}. \tag{3.1}$$

Since $A_n \to A_0$ and $x_n \to x_0$, there exists an $M > 0$ such that $\|A_n - A_0\| < \min\{\frac{r_{z'}}{4}, \frac{r_{z'}}{4R}\}$ and $|x_n - x_0| < \min\{\frac{r_{z'}}{4}, \frac{r_{z'}}{4R}\}$ for all $n \geq M$. Hence

$$\begin{aligned} |(A_ny + x_n) - (A_0y + x_0)| &= |(A_n - A_0)y + (x_n - x_0)| \\ &\leq \|A_n - A_0\||y| + |x_n - x_0| \leq \frac{r_{z'}}{2}, \forall y \in C. \end{aligned} \tag{3.2}$$

On the other hand, one can find a unique $y'_n \in \partial C$ such that

$$\begin{aligned} \operatorname{dist}(z', A_nC + x_n) &= |z' - (A_ny'_n + x_n)| \\ &\geq |z' - (A_0y'_n + x_0)| - |(A_0y'_n + x_0) - (A_ny'_n + x_n)| \\ &\geq \operatorname{dist}(z', A_0C + x_0) - |(A_0y'_n + x_0) - (A_ny'_n + x_n)|. \end{aligned} \tag{3.3}$$

By combining (3.1), (3.2), and (3.3), we obtain

$$\operatorname{dist}(z', A_nC + x_n) \geq \frac{r_{z'}}{2}, \ \forall\, n \geq M.$$

In other words, there exists an $M > 0$ such that $U(z', \frac{r_{z'}}{2}) \cap [A_nC + x_n] = \emptyset$ for all $n \geq M$. Hence, $z' \notin A_nC + x_n$ for all $n \geq M$. Upon letting $K_n = B_R \setminus (A_nC + x_n)$ and $K = B_R \setminus \tilde{C}$, it follows from Lemma 2.1 that $z' \in B_R \setminus \tilde{C}$, that is, $z' \notin \tilde{C}$. This is a contradiction, and hence the first part follows.

Now, we show the second part.

Let $\overline{C}_n \xrightarrow{\delta} F$. Since $C_n = A_nC + x_n$ and C is a convex domain, it has $\overline{C}_n = A_n\overline{C} + x_n$ for every $n \in \mathbb{N}$. Using the same notation as above, we may assume $A_n \to A_0$ and $x_n \to x_0$ as $n \to \infty$. For any $y \in \overline{C}$, since $A_ny + x_n \to A_0y + x_0$, it follows from Lemma 2.1 and $A_ny + x_n \in \overline{C}_n$ for each $n \in \mathbb{N}$ that $\overline{A_0C + x_0} = A_0\overline{C} + x_0 \subset F$. On the other hand, for any $y \in F$, there exists $y_{nk} \in \overline{C}_{nk}$ where $\{\overline{C}_{nk}\}_{k=1}^{\infty}$ is a subsequence of $\{\overline{C}_n\}_{n=1}^{\infty}$, such that $y_{nk} \to y$ as $k \to \infty$. Since there exists $z_{nk} \in \overline{C}$ such that $y_{nk} = A_{nk}z_{nk} + x_{nk}$ for each

$k \in \mathbb{Z}$, and $\overline{C}_{nk} = A_{nk}\overline{C} + x_{nk}$, there is $z_{nk} = A_{nk}^{-1}(y_{nk} - x_{nk}) \to A_0^{-1}(y - x_0)$. Furthermore, $A_0^{-1}(y-x_0) \in \overline{C}$. That is, $y \in A_0\overline{C}+x_0$, and hence $F \subset \overline{A_0C + x_0}$, so $F = \overline{A_0C + x_0} = A_0\overline{C} + x_0$. Moreover, since C is an open convex domain, one arrives at $\mathrm{Int}(F) = \mathrm{Int}\left(A_0\overline{C} + x_0\right) = A_0C + x_0 = \tilde{C}$ and $F = \overline{A_0C + x_0} = \overline{\mathrm{Int}(F)}$.

Proof of Theorem 2.1. Let $i \in \{1, 2, 3, 4\}$. For any given $\{\Omega_n\}_{n=1}^{\infty} \subset \mathcal{C}_i$, by Lemma 2.4, there exists a subsequence of $\{\Omega_n\}_{n=1}^{\infty}$, still denoted by itself, and an open set Ω_0 such that $\Omega_n \xrightarrow{\rho} \Omega_0$. Hence, we only need to show that $\Omega_0 \in \mathcal{C}_i$.

We first show that $(\mathcal{C}_1, \rho)$ is a compact metric space.

For every $z_0 \in \partial\Omega_0$, by Lemma 3.2 and $\Omega_n \xrightarrow{\rho} \Omega_0$, there exists a subsequence of $\{\Omega_n\}_{n=1}^{\infty}$, still denoted by itself, and $z_n \in \partial\Omega_n$, such that $z_n \to z_0$ as $n \to \infty$. For every $z_n \in \partial\Omega_n$, there exist an orthogonal matrix A_n and $x_n \in \mathbb{R}^N$, such that $A_nC + x_n \subset \Omega_n$ and $z_n \in \partial(A_nC + x_n)$. Since $\{A_n\}_{n=1}^{\infty}$ is bounded, there exists a subsequence of $\{A_n\}_{n=1}^{\infty}$, still denoted by itself, and an orthogonal matrix A_0 such that $A_n \to A_0$. In the same way, one can find a subsequence of $\{x_n\}_{n=1}^{\infty}$, still denoted by itself, such that $x_n \to x_0$. With the same arguments as in the proof of Lemma 3.3, one has $A_nC + x_n \xrightarrow{\rho} A_0C + x_0$. By Lemma 2.2, $A_0C + x_0 \subset \Omega_0$. On the other hand, since $z_n \in \partial(A_nC + x_n)$ for every $n \in \mathbb{N}$, it follows that $\mathrm{dist}(z_n, A_nC + x_n) = 0$. Since $A_n \to A_0$ and $x_n \to x_0$, $\mathrm{dist}(z_0, A_0C + x_0) = 0$. So, $z_0 \in \partial(A_0C + x_0)$. This is the required result.

Next, we show that $(\mathcal{C}_2, \rho)$ is also a compact metric space.

By the definition of $\mathcal{C}_2$, there exists an $U(x_n, r_n) \subset \Omega_n$ with $r_n \geq r_0 > 0$ for every $n \in \mathbb{N}$. Hence, one can extract a subsequence of $\{U(x_n, r_n)\}_{n=1}^{\infty}$, still denoted by itself, and $x^* \in \mathbb{R}^N$, $r^* > r_0$ such that $U(x_n, r_n) \xrightarrow{\rho} U(x^*, r^*)$. Since $U(x_n, r_n) \subset \Omega_n$, by Lemma 2.2, $U(x^*, r^*) \subset \Omega$. So, $\Omega_0 \neq \emptyset$.

Since $B_R \setminus \Omega_n \xrightarrow{\delta} B_R \setminus \Omega_0$, for every $x \in \partial\Omega_0 \subset B_R \setminus \Omega_0$, there exists a sequence $\{x_{nk}\}_{k=1}^{\infty}$ with $x_{nk} \in B_R \setminus \Omega_{nk}$ for every $k \in \mathbb{N}$, where $\{\Omega_{nk}\}_{k=1}^{\infty}$ is a subsequence of $\{\Omega_n\}_{n=1}^{\infty}$, such that $x_{nk} \to x$ as $k \to \infty$. By the definition of $\mathcal{C}_2$, there exists a C_k that is congruent with C, such that $C_k \subset B_R \setminus \Omega_{nk}$ and $x_{nk} \in \partial C_k$. Obviously, $\overline{C}_k \subset B_R \setminus \Omega_{nk}$. Since $\{\overline{C}_k\}_{k=1}^{\infty}$ is bounded in B_R, by Lemma 2.4 there exists a subsequence of $\{\overline{C}_k\}_{k=1}^{\infty}$, still denoted by itself, and $F \subset B_R$ such that $\overline{C}_k \xrightarrow{\rho} F$. By Lemma 2.1 $x \in F$, and by Lemma 3.3, $\mathrm{Int}(F)$ is congruent with C. On the other hand, by Lemma 2.2, $F \subset B_R \setminus \Omega_0$, which implies that for every $x \in \partial\Omega_0$, there exists a C' that is congruent with C such that $C' \subset B_R \setminus \Omega_0$.

Thirdly, we show that $(\mathcal{C}_3, \rho)$ is a compact metric space.

Indeed, for every $x \in \Omega_0$, there exists an $r_* > 0$ such that $B(x, r_*) \subset \Omega_0$. This, together with Lemma 2.3, shows that there exists an $n_x > 0$ such that $B(x, r_*) \subset \Omega_n$ for all $n \geq n_x$. Since Ω_n satisfies the C_0-cone condition for every $n \geq n_x$, then, there exists a cone $C_{x,n}$ for which x is the vertex of $C_{x,n}$ and $C_{x,n}$ is congruent with C_0 such that $C_{x,n} \subset \Omega_n$. By Lemma 3.3, there exists a subsequence of $\{C_{x,n}\}_{n=n_x}^{\infty}$, still denoted by itself, such that $C_{x,n} \xrightarrow{\rho} \tilde{C}$ and $\tilde{C}$ is congruent with C_0. Obviously, x is the vertex of $\tilde{C}$. Moreover, by Lemma

2.2, $\tilde{C} \subset \Omega_0$. Hence, Ω_0 satisfies the C_0-cone condition. In other words, $(\mathcal{C}_3, \rho)$ is a compact metric space.

Finally, by Lemma 3.1, $\Omega_0 \neq \emptyset$. Along the same line as that in the proof of $(\mathcal{C}_2, \rho)$, we can obtain that $(\mathcal{C}_4, \rho)$ is also a compact metric space. The proof is complete. □

In order to prove Remark 2.2, we need the following Lemma 3.4 that has been proven in Lemma III.1 of Chenais (1975).

Lemma 3.4 *Let $\varepsilon > 0$ and let $\mathcal{O}_\varepsilon$ be defined by equation (2.1). Then, for every $x_0 \in \partial\Omega$, there exists a unit vector $\xi(x_0) \in \mathbb{R}^N$ such that $C(\varepsilon, \xi(x_0), x_0) \subset \Omega$ and $C(\varepsilon, -\xi(x_0), x_0) \subset U_R \setminus \overline{\Omega}$.*

Proof of Remark 2.2. For any $x_1, x_2 \in \partial\Omega, \Omega \in \mathcal{O}_\varepsilon$, by the ε-cone property, there exist unit vectors $\xi(x_1)$ and $\xi(x_2)$ such that $C(\varepsilon, \xi(x_1), y_1) \subset \Omega$, $C(\varepsilon, \xi(x_2), y_2) \subset \Omega$, respectively, for all $y_1 \in B(x_1, \varepsilon) \cap \Omega$, $y_2 \in B(x_2, \varepsilon) \cap \Omega$. Take A_{x_1,x_2} as an orthogonal matrix satisfying $A_{x_1,x_2}\xi(x_1) = \xi(x_2)$. Then, $C(\varepsilon, \xi(x_2), y_2) = A_{x_1,x_2}C(\varepsilon, \xi(x_1), y_1) + (y_2 - y_1)$, that is, $C(\varepsilon, \xi(x_2), y_2)$ and $C(\varepsilon, \xi(x_1), y_1)$ are congruent.

If we take the convex domain C in $\mathcal{C}_1 \cap \mathcal{C}_2$ as some ε-cone $C(\varepsilon, \xi, y)$ in $\mathcal{O}_\varepsilon$, then it follows from Lemma 3.4 that $\mathcal{O}_\varepsilon \subset \mathcal{C}_1 \cap \mathcal{C}_2$.

The following is a counterexample showing that $\mathcal{O}_\varepsilon \neq \mathcal{C}_1 \cap \mathcal{C}_2$ for $0 < \varepsilon < \frac{\pi}{4}$.

Example 3.1 *Let $\Omega_1 \in \mathbb{R}^2$ be the interior domain surrounded by the following three curves $\Gamma_i, i = 1, 2, 3$*

$$\Gamma_1 : \left\{(x, y) \in \mathbb{R}^2 |\; x \in [-1, 1], y = |x| \cos\varepsilon\right\};$$

$$\Gamma_2 : \left\{(x, y) \in \mathbb{R}^2 |\; x = \sqrt{\sec^2\varepsilon - (y - \sec\varepsilon\csc\varepsilon)^2}, y \in [\cot\varepsilon, \sec\varepsilon(1 + \csc\varepsilon)]\right\};$$

$$\Gamma_3 : \left\{(x, y) \in \mathbb{R}^2 |\; x = -\sqrt{\sec^2\varepsilon - (y - \sec\varepsilon\csc\varepsilon)^2}, y \in [\cot\varepsilon, \sec\varepsilon(1 + \csc\varepsilon)]\right\}.$$

It is easy to verify that $\Omega \equiv \Omega_1 \cup (-\Omega_1) \in \mathcal{C}_1 \cap \mathcal{C}_2$ with $-\Omega_1 = \{x \in \mathbb{R}^N |\; -x \in \Omega_1\}$. But $\Omega \notin \mathcal{O}_\varepsilon$ in terms of Theorem 2.6.

The following Lemma 3.5, which is a direct consequence of the definition of δ, and Lemma 3.6, are used to prove Theorem 2.2.

Lemma 3.5 *Let $K, K_n, n \in \mathbb{N}$ be compact sets in B_R, and $K_n \xrightarrow{\delta} K^*$.Then $K \cup K_n \xrightarrow{\delta} K \cup K^*$.*

Lemma 3.6 *Let $\Omega \in \mathcal{C}_3$. For every $x_0 \in \partial\Omega$, there exists a C_{x_0} that is congruent with C_0, where x_0 is the vertex of C_{x_0}, such that $C_{x_0} \subset \Omega$.*

Proof Since $x_0 \in \partial\Omega$, there exists a sequence $\{x_n\}_{n=1}^\infty \subset \Omega$ such that $x_n \to x_0$ as $n \to \infty$. For every $x_n \in \Omega$, one can find a C_{x_n} such that $C_{x_n} \subset \Omega$

and $x_n \in \partial C_{x_n}$ in terms of $\Omega \in \mathcal{C}_3$. By Lemmas 2.4 and 3.3, it follows that there exists a subsequence of $\{C_{x_n}\}_{n=1}^{\infty}$, still denoted by itself, and C^* such that $C_{x_n} \xrightarrow{\rho} C^*$ where C^* is congruent with C_0. Since $C_{x_n} \subset \Omega$, by Lemma 2.1 there also holds $C^* \subset \Omega$. Obviously, $x_0 \in \partial C^*$. The lemma is then proved by taking $C^* = C_{x_0}$. □

REMARK 3.1 *Following from the definition of $\mathcal{C}_3$ and Lemma 3.6, one can obtain that for every $x_0 \in \overline{\Omega}$ with $\Omega \in \mathcal{C}_3$, there exists a C_{x_0} that is congruent with C_0, such that $C_{x_0} \subset \Omega$, where x_0 is the vertex of C_{x_0}.*

Proof of Theorem 2.2. By Lemma 2.3, we need only to show, by the definitions of ρ and δ, that $U_R \setminus \overline{\Omega}_n \xrightarrow{\rho} U_R \setminus \overline{\Omega}$. This is equivalent to showing that

$$\partial B_R \cup \overline{\Omega}_n = B_R \setminus \left(U_R \setminus \overline{\Omega}_n\right) \xrightarrow{\delta} B_R \setminus \left(U_R \setminus \overline{\Omega}\right) = \partial B_R \cup \overline{\Omega}. \tag{3.4}$$

By Lemma 3.5 and $\overline{\Omega}_n \subset B\left(0, \frac{R}{2}\right)$ for all $n \in \mathbb{N}$, in order to prove (3.4), it suffices to prove $\overline{\Omega}_n \xrightarrow{\delta} \overline{\Omega}$. Since $\{\overline{\Omega}_n\}_{n=1}^{\infty}$ is a sequence of compact sets, by Lemma 2.4, there exists a subsequence of $\{\overline{\Omega}_n\}_{n=1}^{\infty}$, still denoted by itself, and a compact set F such that $\overline{\Omega}_n \xrightarrow{\delta} F$. Now, we show that $F = \overline{\Omega}$.

For every $x \in \Omega$, there exists an $r^* > 0$ such that $B(x, r^*) \subset \Omega$. By Lemma 2.3, one can find an $n_x > 0$ such that $B(x, r^*) \subset \Omega_n$ for all $n \geq n_x$. By Lemma 2.1, $x \in F$. This shows that $\Omega \subset F$, and hence $\overline{\Omega} \subset F$.

Next, we show that $F \subset \overline{\Omega}$.

Indeed, for every $x \in F$, there exists a subsequence $\{x_{nk}\}_{k=1}^{\infty}$ of $\{\Omega_n\}_{n=1}^{\infty}$, where $x_{nk} \in \overline{\Omega}_{nk}$ for every $k \in \mathbb{N}$, such that $x_{nk} \to x$ as $k \to \infty$. By Remark 3.1, there exists a $C_{nk} \subset \Omega_{nk}$ such that x_{nk} is the vertex of C_{nk} for every $k \in \mathbb{N}$. Assume that $C_{nk} \xrightarrow{\rho} \tilde{C}$ and $\overline{C}_{nk} \xrightarrow{\delta} D$. Then $x \in D$, $D = \overline{(\tilde{C})}$, $\mathrm{Int}(D) = \tilde{C}$, and $\tilde{C}$ is congruent with C_0 owing to Lemma 3.3. These facts show that $x \in \overline{\Omega}$; and hence $F \subset \overline{\Omega}$. □

The next Lemma 3.7 is crucial for the proof of Theorem 2.3.

LEMMA 3.7 *Assume that $p \neq N$ and $\Omega \in \mathcal{C}$. Then $W_0^{1,p}(\overline{\Omega}) = W_0^{1,p}(\Omega)$, that is,*

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(U_R) |\ u = 0 \text{ a.e. on } U_R \setminus \overline{\Omega}\}, \tag{3.5}$$

where $W_0^{1,p}(\overline{\Omega}) = \cap_{G \supset \overline{\Omega}} W_0^{1,p}(G)$ with open sets $G \subset \mathbb{R}^N$.

PROOF To prove Lemma 3.7, we need some definitions of capacity in potential theory, for which we refer to Adams and Hedberg (1999), Hedberg (1980), Heinonen, Kilpelainen, and Martio (2006), as well as Landkof (1972).

First, by the arguments from Section 2 of Landkof (1972), (3.5) holds naturally for $p > N$, so we need only to show that (3.5) for $p < N$. Following Theorem 2.17 of Hedberg (1980), it suffices to show that $U_R \setminus \overline{\Omega}$ is $(1,p)$-thick for every $x \in \partial\Omega$. By Theorem 2.16 of Hedberg (1980), this is equivalent to showing that $\sum_{n=1}^{\infty} a_n(x, U_R \setminus \overline{\Omega})^{q-1} = \infty$, where $a_n(x, U_R \setminus \overline{\Omega}) =$

$2^{n(N-p)}C_{1,p}([U_R \setminus \overline{\Omega}] \cap U(x, 2^{-n}))$ with $\frac{1}{p} + \frac{1}{q} = 1$. However, this is an obvious fact since $\Omega \in \mathcal{C}$ means that Ω satisfies the exterior C_0-cone condition, and hence for $1 < p < N$, $\lim_{n\to\infty} a_n(x, U_R \setminus \overline{\Omega})$ is finite and positive (see, e.g., Hedberg, 1980, p.10). This concludes the result for $1 < p < N$. □

Open question: Our method here is not applied to the case of $p = N$. We leave this case as an open problem.

Proof of Theorem 2.3. Let $d = \min_{\Omega\in\mathcal{C}} \int_\Omega |u_\Omega - u_0|^p\, dx \geq 0$. Then, there exists a minimizing sequence $\{\Omega_n\}_{n=1}^\infty \subset \mathcal{C}$ such that

$$d = \lim_{n\to\infty} \int_{\Omega_n} |u_n - u_0|^p\, dx, \tag{3.6}$$

where $u_n \equiv u_{\Omega_n}$ is the (weak) solution of equation (1.1) in Ω_n. By Theorem 2.1, there exists a subsequence of $\{\Omega_n\}_{n=1}^\infty$, still denoted by itself, and $\Omega_0 \in \mathcal{C}$ such that $\Omega_n \xrightarrow{\rho} \Omega_0$.

By equation (1.2), it follows that

$$\int_{\Omega_n} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla u_n\, dx = \int_{\Omega_n} f \cdot u_n\, dx,$$

and hence

$$\int_{\Omega_n} |\nabla u_n|^p\, dx = \int_{\Omega_n} f \cdot u_n\, dx. \tag{3.7}$$

By virtue of the Poincaré-type inequality, we have

$$\|u_n\|_{L^p(\Omega_n)} \leq L|U_R|^{\frac{1}{N}} \|\nabla u_n\|_{L^p(\Omega_n)}$$

where and in what follows we use $L = L(N, R)$ to denote a positive constant independent of n although its value may vary in different contexts. Therefore,

$$\int_{\Omega_n} |\nabla u_n|^p\, dx \leq L. \tag{3.8}$$

Let

$$\hat{u}_n(x) = \begin{cases} u_n(x) & \text{in } \Omega_n, \\ 0 & \text{in } U_R \setminus \Omega_n. \end{cases} \tag{3.9}$$

Then $\{\hat{u}_n\}_{n=1}^\infty$ is bounded in $W_0^{1,p}(U_R)$. By the Sobolev embedding theorem, there exists a subsequence of $\{\hat{u}_n\}_{n=1}^\infty$, still denoted by itself, such that

$$\hat{u}_n \to \hat{u} \text{ in } W_0^{1,p}(U_R) \text{ weakly and in } L^p(U_R) \text{ strongly} \tag{3.10}$$

for some $\hat{u} \in W_0^{1,p}(U_R)$. We claim that

$$\hat{u}(x) \in W_0^{1,p}(\Omega_0). \tag{3.11}$$

To this end, by Lemma 3.7, we need only to show that

$$\hat{u}(x) = 0 \text{ a.e. in } U_R \setminus \overline{\Omega}_0. \tag{3.12}$$

Indeed, for any open subset K with $\overline{K} \subset U_R \setminus \overline{\Omega}_0$, it follows from Theorem 2.2 that there exists an $n_K > 0$ such that $\overline{K} \subset U_R \setminus \overline{\Omega}_n$ for all $n \geq n_K$. Thus

$$\int_K |\hat{u}(x)|^p \, dx = \lim_{n\to\infty} \int_K |\hat{u}_n(x)|^p \, dx \leq \overline{\lim_{n\to\infty}} \int_{U_R\setminus\overline{\Omega}_n} |\hat{u}_n(x)|^p \, dx = 0,$$

which implies that $\hat{u}(x) = 0$ almost everywhere in K. Since $K \subset \overline{K} \subset U_R \setminus \overline{\Omega}_0$ is arbitrary, we obtain (3.12), and so is for (3.11).

Now, we show that

$$\int_{\Omega_0} |\nabla \hat{u}|^{p-2} \nabla \hat{u} \cdot \nabla \varphi \, dx = \int_{\Omega_0} f \cdot \varphi dx, \forall \ \varphi \in C_0^\infty(\Omega_0), \tag{3.13}$$

that is,

$$\int_{\text{supp}(\varphi)} |\nabla \hat{u}|^{p-2} \nabla \hat{u} \cdot \nabla \varphi \, dx = \int_{\text{supp}(\varphi)} f \cdot \varphi dx, \forall \ \varphi \in C_0^\infty(\Omega_0). \tag{3.14}$$

To this end, let

$$\hat{\varphi} = \begin{cases} \varphi & \text{in } \ \Omega_0, \\ 0 & \text{in } \ U_R \setminus \overline{\Omega}_0. \end{cases} \tag{3.15}$$

By Lemma 2.4, there exists a positive integer $n_1(\varphi)$ such that

$$\text{sppp}(\hat{\varphi}) = \text{supp}(\varphi) \subset \Omega_n \text{ for all } n \geq n_1(\varphi).$$

So, $\hat{\varphi} \in C_0^\infty(\Omega_n)$ for all $n \geq n_1(\varphi)$. By equation (1.2),

$$\int_{\Omega_n} |\nabla \hat{u}_n|^{p-2} \nabla \hat{u}_n \cdot \nabla \hat{\varphi} \, dx = \int_{\Omega_n} f \cdot \hat{\varphi} dx.$$

This, together with equation (3.15), gives

$$\int_{\text{supp}(\varphi)} |\nabla \hat{u}_n|^{p-2} \nabla \hat{u}_n \cdot \nabla \varphi dx = \int_{\text{supp}(\varphi)} f \cdot \varphi dx. \tag{3.16}$$

We claim

$$\int_{\text{supp}(\varphi)} |\nabla \hat{u}_n|^{p-2} \nabla \hat{u}_n \cdot \nabla \varphi dx \to \int_{\text{supp}(\varphi)} |\nabla \hat{u}|^{p-2} \nabla \hat{u} \cdot \nabla \varphi dx \text{ as } n \to \infty. \tag{3.17}$$

The proof of (3.17) is similar to the proof of Theorem 2.3.12 of Neittaanmaki, Sprekels, and Tiba (2006) from pages 61-62. Indeed, set $A_i(x, \nabla v) =$

$|\nabla v(x)|^{p-2}\frac{\partial \hat{v}}{\partial x_i}$ for $x=(x_1,x_2,\cdots,x_n)$. Then $\Delta_p \hat{u}_n=\sum_{i=1}^N A_i(x,\nabla\hat{u}_n)$. For any $n\in\mathbb{N}$, it follows from (3.8) that

$$\begin{aligned}\int_{U_R}|A_i(x,\nabla\hat{u}_n(x))|^{p'}\,dx &=\int_{U_R}\left||\nabla\hat{u}_n(x)|^{p-2}\frac{\partial\hat{u}_n}{\partial x_i}\right|^{p'}dx\\ &\le\int_{U_R}\left||\nabla\hat{u}_n(x)|^{p-1}\right|^{p'}dx=\int_{U_R}|\nabla\hat{u}_n(x)|^{p}\,dx\le L.\end{aligned} \tag{3.18}$$

Hence, there exists $a_i\in L^{p'}(U_R)$ such that

$$A_i(x,\nabla\hat{u}_n)\to a_i \text{ in } L^{p'}(U_R) \text{ weakly, } i=1,\cdots,N. \tag{3.19}$$

Let $K\subset\Omega_0$ be any given compact set and pick any nonnegative test function $\psi\in C_0^\infty(\Omega_0)$ so that $\psi(x)=1$ for all $x\in K$. We estimate

$$I_n=\int_{\Omega_0}\psi\sum_{i=1}^n\Big(A_i(x,\nabla\hat{u}_n)-A_i(x,\nabla\hat{u})\Big)\left(\frac{\partial\hat{u}_n}{\partial x_i}-\frac{\partial\hat{u}}{\partial x_i}\right)dx. \tag{3.20}$$

I_n makes sense by simply setting the integrand function on the right-hand side of (3.20) to be zero in $\Omega_0\setminus\operatorname{supp}(\psi)$, because by Theorem 2.2, $\operatorname{supp}(\psi)\subset\Omega_n$ for all sufficiently large n. Therefore,

$$\begin{aligned}I_n=&\int_{\Omega_0}\sum_{i=1}^N A_i(x,\nabla\hat{u}_n)\frac{\partial}{\partial x_i}(\psi(\hat{u}_n-\hat{u}))dx-\int_{\Omega_0}\psi\sum_{i=1}^N A_i(x,\nabla\hat{u})\left(\frac{\partial\hat{u}_n}{\partial x_i}-\frac{\partial\hat{u}}{\partial x_i}\right)dx\\ &-\int_{\Omega_0}\sum_{i=1}^N A_i(x,\nabla\hat{u}_n)Z_n^i dx\triangleq I_{1n}+I_{2n}+I_{3n},\end{aligned}$$

where we write shorthand

$$Z_n^i=\frac{\partial}{\partial x_i}(\psi(\hat{u}_n-\hat{u}))-\psi\left(\frac{\partial\hat{u}_n}{\partial x_i}-\frac{\partial\hat{u}}{\partial x_i}\right)=(\hat{u}_n-\hat{u})\frac{\partial\psi}{\partial x_i}. \tag{3.21}$$

First, by (3.10), $Z_n^i\to 0$ in $L^p(\Omega_0)$ strongly as $n\to\infty$. This, together with the fact that $\|A_i(x,\nabla\hat{u}_n)\|_{L^{p'}(\Omega_0)}\le L$ in terms of (3.18), shows that $\lim_{n\to\infty}I_{3n}=0$. Second, by (3.10), $\hat{u}_n\to\hat{u}$ in $W^{1,p}(\operatorname{supp}(\psi))$ weakly, which leads immediately to $\lim_{n\to\infty}I_{2n}=0$. Third, by definition given in (1.2) with $\varphi=\psi(\hat{u}_n-\hat{u})$

$$I_{1n}=\int_{\Omega_0}\sum_{i=1}^n A_i(x,\nabla\hat{u}_n)\frac{\partial}{\partial x_i}(\psi(\hat{u}_n-\hat{u}))dx=\int_{\Omega_0}f\psi(\hat{u}_n-\hat{u})dx.$$

It then follows from (3.10) that

$$\lim_{n\to\infty}I_{1n}=\lim_{n\to\infty}\int_{\operatorname{supp}(\psi)}f\psi(\hat{u}_n-\hat{u})dx\to 0.$$

Combining the aforementioned facts, we have proved

$$\lim_{n\to\infty} I_n = 0. \tag{3.22}$$

By the uniform monotonicity of Δ_p (see Dinca, Jebelean, and Mawhin, 2001), there exists $\gamma > 0$ such that

$$\langle -\Delta_p \hat{u}_n - (-\Delta_p \hat{u}), \hat{u}_n - \hat{u}\rangle^p_{W_0^{1,p}} \geq \gamma \|\hat{u}_n - \hat{u}\|^p_{W_0^{1,p}},$$

and we finally obtain

$$\lim_{n\to\infty} \left(\gamma \int_{\Omega_0} \psi \sum_{i=1}^{N} \left| \frac{\partial \hat{u}_n}{\partial x_i} - \frac{\partial \hat{u}}{\partial x_i} \right|^2 dx \right) \leq \lim_{n\to\infty} I_n = 0. \tag{3.23}$$

Since the test function ψ is nonnegative in Ω_0 with $\psi|_K = 0$, (3.23), shows that we can select a subsequence of n so that

$$\frac{\partial \hat{u}_n}{\partial x_i} \to \frac{\partial \hat{u}}{\partial x_i} \quad \text{as} \quad n \to \infty, \forall\, x \in K \text{ a.e., } i = 1, \cdots, N.$$

The above, together with the continuity of $A_i(x, \nabla \hat{u}_n)$ shows that

$$A_i(x, \nabla \hat{u}_n) \to A_i(x, \nabla \hat{u}),\ \forall\, x \in K \text{ a.e., } i = 1, \cdots, N.$$

This, together with (3.19), gives

$$A_i(x, \nabla \hat{u}) = a_i,\ \forall\, x \in K \text{ a.e., } i = 1, \cdots, N.$$

Since $K \subset \Omega_0$ is arbitrary, it follows that

$$A_i(x, \nabla \hat{u}) = a_i,\ \forall\, x \in \Omega_0 \text{ a.e., } i = 1, \cdots, N.$$

In other words,

$$A_i(x, \nabla \hat{u}_n) \to A_i(x, \nabla \hat{u}) \text{ in } L^{p'}(U_R) \text{ weakly, } i = 1, \cdots, N. \tag{3.24}$$

Therefore,

$$\begin{aligned}
&\int_{\text{supp}(\varphi)} |\nabla \hat{u}_n|^{p-2} \nabla \hat{u}_n \cdot \nabla \varphi dx = \sum_{i=1}^{N} \int_{\text{supp}(\varphi)} A_i(x, \nabla \hat{u}_n) \frac{\partial \varphi}{\partial x_i} \\
&= \sum_{i=1}^{N} \int_{\text{supp}(\varphi)} [A_i(x, \nabla \hat{u}_n) - A_i(x, \nabla \hat{u})] \frac{\partial \varphi}{\partial x_i} \\
&+ \int_{\text{supp}(\varphi)} |\nabla \hat{u}|^{p-2} \nabla \hat{u} \cdot \nabla \varphi dx \\
&\to \int_{\text{supp}(\varphi)} |\nabla \hat{u}|^{p-2} \nabla \hat{u} \cdot \nabla \varphi dx \text{ as } n \to \infty.
\end{aligned} \tag{3.25}$$

This proves (3.17). Passing to the limit as $n \to \infty$ in the left of (3.16) one gets (3.14) and hence (3.13).

Finally, we show that

$$d \geq \int_{\Omega_0} |\hat{u} - u_0|^p \, dx. \tag{3.26}$$

Indeed, let $\Omega_0 = \bigcup_{j=1}^{\infty} G_j$, where $G_j, j = 1, 2, \cdots$, are open and bounded subsets in Ω_0 such that $\overline{G}_j \subset G_{j+1}$. By Theorem 2.2, for each j, there exists a positive integer n_j such that

$$\overline{G}_j \subset \Omega_n \text{ as } n \geq n_j.$$

Hence

$$\varliminf_{n\to\infty} \int_{B\setminus\overline{\Omega}_n} |\hat{u}_n - u_0|^p \, dx \geq \varliminf_{n\to\infty} \int_{G_j} |\hat{u}_n - u_0|^p \, dx. \tag{3.27}$$

By Equations (3.6), (3.10), (3.27), and Fatou's Lemma, we have, for each j, that

$$d \geq \int_{G_j} |\hat{u} - u_0|^p dx. \tag{3.28}$$

Since

$$\lim_{j\to\infty} \chi_{G_j}(x) = \chi_{\Omega_0}(x),$$

where χ_{G_j} and χ_{Ω_0} are the characteristic functions of G_j and Ω_0, respectively, it follows from equation (3.28) and Fatou's Lemma again that

$$d \geq \varliminf_{j\to\infty} \int_{G_j} |\hat{u} - u_0|^p \, dxdx = \varliminf_{j\to\infty} \int_{\Omega_0} \chi_{G_j} |\hat{u} - u_0|^p dx \geq \int_{\Omega_0} |\hat{u} - u_0|^p dx.$$

Equation (3.26) then follows.

By combining equations (3.11), (3.14), and (3.26), Ω_0 is shown to be a solution to problem (1.3). This ends the proof. □

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References

Adams, R.A. and Fournier, John J.F. (2003) *Sobolev Spaces.* Academic Press, Boston.

Adams, D.R. and Hedberg, L.I. (1999) *Function Spaces and Potential Theory.* Springer, Berlin.

BUCUR, D. AND BUTTAZZO, G. (2005) *Variational Methods in Shape Optimization Problems.* Birkhäuser, Boston.

CHENAIS, D. (1975) On the existence of a solution in a domain identification problem. *J. Math. Anal. Appl.* **52** (2), 189-219.

DINCA, G., JEBELEAN, P. AND MAWHIN, J. (2001) Variational and topological methods for Dirichlet problems with p-Laplacian. *Port. Math.* **58** (3), 339-378.

DELAY, E. (2012) Smooth Compactly Supported Solutions of Some Undetermined Elliptic PDE with Gluing Applications. *Comm. Partial Differential Equations.* **37** (10), 1689-1716.

DELFOUR, M.C. AND ZOLÉSIO, J.P. (2001) *Shapes and Geometries.* SIAM, New York.

GUO, B.Z. AND YANG, D.H. (2012) Some compact classes of open sets under Hausdorff distance and application to shape optimization. *SIAM J. Control Optim.* **50** (1), 222-242.

GUO, B.Z. AND YANG, D.H. (2013) On convergence of boundary Hausdorff measure and application to a boundary shape optimization problem. *SIAM J. Control Optim.* **51** (1), 253-272.

HEDBERG, L.I. (1980) Spectral synthesis and stability in Sobolev spaces. In: *Euclidean Harmonic Analysis. Lecture Notes in Math.* **779**, Springer-Verlag, Berlin, 73–103.

HEINONEN, J., KILPELAINEN, T. AND MARTIO, O. (2006) *Nonlinear Potential Theory of Degenerate Elliptic Equation.* Dover Publications, Mineola.

HE, Y. AND GUO, B.Z. (2012) The existence of optimal solution for a shape optimization problem on starlike domain. *J. Optim. Theory Appl.* **152** (1), 21-30.

LANDKOF, N.S. (1972) *Foundations of Modern Potential Theory.* Springer-Verlag, Berlin.

NEITTAANMAKI, P., SPREKELS, J. AND TIBA, D. (2006) *Optimization of Elliptic Systems: Theory and Applications.* Springer-Verlag, New York.

PIRONNEAU, O. (1984) *Optimal Shape Design for Elliptic Systems.* Springer-Verlag, Berlin.

SCHNEIDER, R. (1993) *Convex Bodies: The Brunn-Minkowski Theory.* Cambridge University Press, London.

TIBA, D (2003) A property of Sobolev spaces and existence in optimal design. *Appl. Math. Optim.* **47** (1), 45-58.

TIBA, D. (2013) Finite element discretization in shape optimization problems for the stationary Navier-Stokes equation. *System Modeling and Optimization. IFIP Advances in Information and Communication Technology.* **391**, 437-444.

TIBA, D. AND HALANAY, A. (2009) Shape optimization for stationary Navier-Stokes equations. *Control Cybernet.* **38** (4B), 1359-1374.

WANG, G., WANG, L. AND YANG, D.H. (2006) Shape Optimization of exterior domain stationary Navier-Stokes equations. *SIAM. J. Control Optim.* **45** (2), 532-547.

Wang, G. and Yang, D.H. (2008) Decomposition of vector-valued divergence free Sobolev functions and shape optimization for stationary Navier-Stokes equations. *Comm. Partial Differential Equations.* **33** (1-3), 429-449.

Yang, D.H. (2009) Shape optimization of stationary Navier-Stokes equation over classes of convex domains. *Nonlinear Anal.* **71** (12), 6202-6211.