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Calculus and applications of Studniarski's derivatives to sensitivity and implicit function theorems*

by

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Abstract: We first discuss basic calculus rules for Studniarski's derivatives. Then, we apply these derivatives to sensitivity analysis of solutions to inclusions and to computing the derivative of implicit multifunctions.

Keywords: Studniarski's derivatives, sum rule, chain rule, product rule, quotient rule, sensitivity analysis, implicit multifunction theorems

1. Introduction

In set-valued analysis, one of the most popular and useful higher-order derivatives is the following contingent derivative introduced by Aubin (1981). Let X and Y be normed spaces, $F : X \to 2^Y$, $y_0 \in F(x_0)$, and $(u_1, v_1), ..., (u_{m-1}, v_{m-1}) \in X \times Y$. The value at $u \in X$ of the contingent derivative of order m of F at (x_0, y_0) relative to $(u_1, v_1), ..., (u_{m-1}, v_{m-1})$ is

$$D^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(u) := \{v \in Y : \exists t_n \to 0^+, \exists (u_n, v_n) \to (u, v),$$

$$\forall n, y_0 + t_n v_1 + ... + t_n^{m-1} v_{m-1} + t_n^m v_n \in F(x_0 + t_n u_1 + ... + t_n^{m-1} u_{m-1} + t_n^m u_n)\}.$$

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Observe that $D^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(u)$ is nonempty only if $v_1 \in DF(x_0, y_0)(u_1)$, ..., $v_{m-1} \in D^{m-1}F(x_0, y_0, u_1, v_1, ..., u_{m-2}, v_{m-2})(u_{m-1})$. In Studniarski (1986), another higher-order derivative was proposed, but only for an extended-real-valued function. As a direct extension to the case of a set-valued map, we have: the value at $u \in X$ of the Studniarski derivative of order m of F at (x_0, y_0) is

$$D^m F(x_0, y_0)(u) := \{v \in Y : \exists t_n \to 0^+, \exists (u_n, v_n) \to (u, v), \ \forall n,$$

$$y_0 + t_n^m v_n \in F(x_0 + t_n u_n)\}.$$

We can write the following two equivalent formulations for this derivative, where Limsup is the Painlevé-Kuratowski upper set-limit,

$$D^m F(x_0, y_0)(u) = \operatorname*{Limsup}_{(t,u') \to (0^+, u)} \frac{F(x_0 + tu') - y_0}{t^m},$$

and, by setting $(x_n, y_n) := (x_0 + t_n u_n, y_0 + t_n^m v_n)$, $\gamma_n = t_n^{-1}$, and gr$F$ as the graph of F,

$$D^m F(x_0, y_0)(u) = \{v \in Y : \ \exists \gamma_n > 0, \ \exists (x_n, y_n) \in \mathrm{gr}F : \ (x_n, y_n) \to (x_0, y_0),$$

$$(\gamma_n(x_n - x_0), \gamma_n^m(y_n - y_0)) \to (u, v)\}.$$

In nonsmooth optimization, this object was applied in obtaining optimality conditions, e.g., in Studniarski (1986), Jiménez (2003), Jimenez and Novo (2008), Luu (2008), Sun and Li (2012), and Li et al. (2012), and in discussing sensitivity analysis in Sun and Li (2011).

In Anh et al. (2011) and Diem et al. (2013), several notions of higher-order derivatives were developed, combining the Studniarski derivative and the extension of the radial derivative proposed in Taa (1998) (for the first-order) to higher orders. In that way, global (not local as with the above two derivatives) higher-order optimality conditions were established for nonconvex optimization. (The main technical change in the above definitions is replacing $\exists t_n \to 0^+$ by $\exists t_n > 0$.) But for some other topics like sensitivity analysis or implicit function theorems, this may be inconvenient. In Diem et al. (2013), further modifications of the derivatives of Anh et al. (2011) were introduced in order to obtain other objects suitable for higher-order sensitivity analysis.

In this paper we return to the Studniarski derivative proposed in Studniarski (1986), since it is simpler than the derivatives in Anh et al. (2011) and Diem et al. (2013). Namely, we are concerned with two topics. First, we develop calculus rules for this derivative, observing that these rules have not been studied, but a kind of derivatives is significant only if it is endowed with sufficiently developed calculus rules. Next, we use the Studniarski derivative to sensitivity analysis and implicit function theorems to ensure that we can investigate the issues that are difficult for the derivatives considered in Anh et al. (2011).

Throughout the paper, if not otherwise specified, let X, Y, Z be normed spaces, and $C \subset Y$ a closed convex cone. For a subset A of a normed space, cl$A$ denotes its closure. B_Y stands for the closed unit ball in Y. $\mathcal{U}(x_0)$ and $\mathcal{U}(y_0)$ are used for the collections of the neighborhoods of x_0 in X and of y_0 in Y, respectively. $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}^n_+$ are the set of the natural numbers, the set of the real numbers, and the nonnegative orthant of the n-dimensional space. For a set-valued map $F : X \to 2^Y$, its profile map F_+ is defined by $F_+(x) := F(x) + C$. The domain, graph, and epigraph of F are defined as

$$\text{dom}F = \{x \in X : F(x) \neq \emptyset\}, \text{gr}F = \{(x, y) \in X \times Y : y \in F(x)\}, \text{epi}F = \text{gr}\, F_+.$$

The closure map of F, denoted by $\text{cl}\, F$, is defined by $\text{gr}(\text{cl}\, F) := \text{cl}(\text{gr}F)$. If $(\text{cl}\, F)(x) = F(x)$, one says that F is closed at x.

DEFINITION 1.1 Let $F : X \to 2^Y$, $(x_0, y_0) \in \text{gr}F$.

(i) F is a convex map on a convex set $S \subset X$ if, for all $\lambda \in [0, 1]$ and $x_1, x_2 \in S$,

$$(1 - \lambda)F(x_1) + \lambda F(x_2) \subset F((1 - \lambda)x_1 + \lambda x_2).$$

(ii) F is lower semicontinuous at (x_0, y_0) if, for each $V \in \mathcal{U}(y_0)$, there is some neighborhood $U \in \mathcal{U}(x_0)$ such that for each $x \in U$, $V \cap F(x) \neq \emptyset$.

(iii) F is locally pseudo-Hölder calm of order m at $(x_0, y_0) \in \text{gr}\, F$ if $\exists \lambda > 0$, $\exists U \in \mathcal{U}(x_0)$, $\exists V \in \mathcal{U}(y_0)$, $\forall x \in U$,

$$F(x) \cap V \subset \{y_0\} + \lambda ||x - x_0||^m B_Y.$$

When $m = 1$, the word "Hölder" is replaced by "Lipschitz". If $V = Y$, then "locally pseudo-Hölder calm" is replaced by "locally Hölder calm".

EXAMPLE 1.1 (i) The set-valued map $F : \mathbb{R} \to 2^{\mathbb{R}}$ defined by $F(x) = \{y : -x^2 \leq y \leq x^2\}$ is locally pseudo-Hölder calm of order 2 at $(0, 0)$.

(ii) Let $F : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by

$$F(x) = \begin{cases} \{0, 1/x\}, & \text{if } x \neq 0, \\ \{0, (1/n)_{n \in \mathbb{N}}\}, & \text{if } x = 0. \end{cases}$$

For any $m \geq 1$, F is not locally pseudo-Hölder calm of order m at $(0, 0)$.

Observe that if F is locally pseudo-Hölder calm (or locally Hölder calm) of order m at (x_0, y_0), it is also locally pseudo-Hölder calm (locally Hölder calm,

respectively) of order n at (x_0, y_0) for all $m > n$. However, the converse may not hold. The following example shows the case.
EXAMPLE 1.2 Let $F : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by

$$F(x) = \begin{cases} \{x^2 \sin(1/x)\}, & \text{if } x \neq 0, \\ \{0\}, & \text{if } x = 0. \end{cases}$$

Obviously, F is locally Hölder calm of order 2 at $(0, 0)$, but F is not locally Hölder calm of order 3 at $(0, 0)$.

2. Studniarski's derivatives

Let $F : X \to 2^Y$, $(x_0, y_0) \in \text{gr}F$, $u \in X$, and $m \geq 1$.
DEFINITION 2.1 Let $F : X \to 2^Y$ and $(x_0, y_0) \in \text{gr}F$.

(i) (Li et al., 2012) The mth-order Studniarski derivative of F at (x_0, y_0) is defined by, for $u \in X$,

$$D^m F(x_0, y_0)(u) = \operatorname*{Limsup}_{(t,u') \to (0^+, u)} \frac{F(x_0 + tu') - y_0}{t^m},$$

or, equivalently,

$$D^m F(x_0, y_0)(u) = \{v \in Y : \exists t_n \to 0^+, \exists (u_n, v_n) \to (u, v), \forall n,$$

$$y_0 + t_n^m v_n \in F(x_0 + t_n u_n)\}.$$

(ii) (Sun and Li, 2012) The lower mth-order Studniarski derivative of F at (x_0, y_0) is defined by, for $u \in X$,

$$D_l^m F(x_0, y_0)(u) = \{v \in Y : \forall t_n \to 0^+, \forall u_n \to u, \exists v_n \to v, \forall n,$$

$$y_0 + t_n^m v_n \in F(x_0 + t_n u_n)\}.$$

(iii) If $D^m F(x_0, y_0)(u) = D_l^m F(x_0, y_0)(u)$ for all $u \in X$, then $D^m F(x_0, y_0)$ is called the mth-order proto Studniarski derivative of F at (x_0, y_0).

(iv) (Sun and Li, 2012) If

$$D^m F(x_0, y_0)(u) = \{v \in Y : \forall t_n \to 0^+, \exists (u_n, v_n) \to (u, v), \ \forall n,$$

$$y_0 + t_n^m v_n \in F(x_0 + t_n u_n)\},$$

then $D^m F(x_0, y_0)$ is called the mth-order strict Studniarski derivative of F at (x_0, y_0).

EXAMPLE 2.1 Let $X = Y = \mathbb{R}$ and $F_n : X \to 2^Y$, $n \in \mathbb{N}$, be defined by $F_n(x) = \{y \in Y : y \geq x^n\}$ for $x \in X$. By direct calculations, we can find the mth-order Studniarski derivative of F_n at $(x_0, y_0) = (0, 0)$ as follows:

If $m = n$, then $D^m F_n(x_0, y_0)(u) = \{y \in Y : y \geq u^n\}$ for $u \in X$.

If $m < n$, then $D^m F_n(x_0, y_0)(u) = \mathbb{R}_+$ for $u \in X$.

If $m > n$, then

$$D^m F_n(x_0, y_0)(u) = \begin{cases} \mathbb{R}, & \text{if } n = 2k-1\ (k = 1, 2, ..) \text{ and } u \leq 0, \\ \mathbb{R}_+, & \text{if } n = 2k\ (k = 1, 2, ..) \text{ and } u = 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

In the following example, we compute the Studniarski derivative of a map into an infinite dimensional space.

EXAMPLE 2.2 Let $X = \mathbb{R}$ and $Y = l^2$, the Hilbert space of the numerical sequences $x = (x_i)_{i\in\mathbb{N}}$ with $\sum_{i=1}^{\infty} x_i^2$ being convergent. By $(e_i)_{i\in\mathbb{N}}$ we denote the standard unit basis of l^2. Let $F : X \to 2^Y$ be defined by

$$F(x) = \begin{cases} \{\frac{1}{n}(-e_1 + 2e_n)\}, & \text{if } x = \frac{1}{n}, \\ \{0\}, & \text{otherwise,} \end{cases}$$

and $(x_0, y_0) = (0, 0)$. We see that $v \in D^m F(x_0, y_0)(u)$ means the existence of $t_k \to 0^+$, $u_k \to u$, and $v_k \to v$ such that

$$y_0 + t_k^m v_k \in F(x_0 + t_k u_k). \tag{2.1}$$

For all $u \in X$, we can choose $t_k \to 0^+$, $u_k \to u$ such that $t_k u_k \neq 1/k$. So, for all $u \in X$, $\{0\} \subset D^m F(x_0, y_0)(u)$. We now prove that, for each $v \in Y \setminus \{0\}$, $v \notin D^m F(x_0, y_0)(u)$ for any $u \in X$. Suppose, to the contrary, that there exist $u \in U$ and $v \in Y \setminus \{0\}$ such that $v \in D^m F(x_0, y_0)(u)$, i.e., there are $t_k \to 0^+$, $u_k \to u$, $v_k \to v$ such that (2.1) holds. If $t_k u_k \neq 1/k$ for infinitely many $k \in \mathbb{N}$, we get a contradiction easily. Hence, assume that $t_k u_k = 1/k$. Then, (2.1) becomes $v_k = \dfrac{1}{kt_k^m}(-e_1 + 2e_k)$. If $1/kt_k^m \to +\infty$, we get a contradiction with the convergence of the sequence $(-e_1 + 2e_k)/kt_k^m$. Suppose $1/kt_k^m \to a \geq 0$. As $e_1/kt_k^m \to ae_1$, the sequence e_k/kt_k^m converges to some c, i.e.,

$$||\frac{2}{kt_k^m} e_k - c||^2 \to 0,$$

that is,

$$||\frac{2}{kt_k^m} e_k - c||^2 = (\frac{2}{kt_k^m})^2 + ||c||^2 + 2\left\langle \frac{2}{kt_k^m} e_k, -c \right\rangle \to 0. \tag{2.2}$$

Since (e_k) converges to 0 with respect to the weak topology, then $\langle e_k, -c\rangle \to 0$. From (2.2), we get $4a^2 + ||c||^2 = 0$. If $a = 0$, then $c = v$ ($\neq \emptyset$) since $(-e_1 + 2e_k)/kt_k^m \to v$. If $a > 0$, then $4a^2 + ||c||^2 \neq 0$. Therefore, we always have a contradiction. Thus, for all $u \in X$, $D^m f(x_0, y_0)(u) = \{0\}$.

We now present a condition for mth-order Studniarski's derivatives to be nonempty.

PROPOSITION 2.1 *Let* $\dim Y < +\infty$, $(x_0, y_0) \in \mathrm{gr}F$, *and* $x_0 \in \mathrm{int}(\mathrm{dom}\, F)$. *Suppose that*

(i) F *is lower semicontinuous at* (x_0, y_0)*;*

(ii) F *is locally pseudo-Hölder calm of order* m *at* (x_0, y_0).

Then, $D^m F(x_0, y_0)(x) \neq \emptyset$ *for all* $x \in X$.

Proof. For $x = 0$, this is trivial because we always have $0 \in D^m F(x_0, y_0)(0)$. By assumption (ii), there exist $\lambda > 0$, $U_1 \in \mathcal{U}(x_0)$ and $V \in \mathcal{U}(y_0)$ such that, for all $x' \in U_1$,

$$F(x') \cap V \subset \{y_0\} + \lambda ||x' - x_0||^m B_Y.$$

By assumption (i), with V above, there exists $U_2 \in \mathcal{U}(x_0)$ such that $\forall \hat{x} \in U_2$, $V \cap F(\hat{x}) \neq \emptyset$. By setting $\hat{U} = U_1 \cap U_2$, we get $\hat{U} \in \mathcal{U}(x_0)$. Let an arbitrary $x \in X$ ($x \neq 0$) and $t_n \to 0^+$. Because $x_0 + t_n x \to x_0$, we get $x_0 + t_n x \in \hat{U}$ for large n. Hence, there eixsts $y_n \in F(x_0 + t_n x) \cap V$ such that $t_n^{-m} ||y_n - y_0|| \leq \lambda ||x||^m$. So, $t_n^{-m}(y_n - y_0)$ is a bounded sequence and hence has a convergent subsequence. By definition, the limit of this subsequence is an element of $D^m F(x_0, y_0)(x)$. ■

EXAMPLE 2.3 (assumption (ii) is essential) Let $F : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by

$$F(x) = \begin{cases} \{x^{1/3}\}, & \text{if } 0 \leq x \leq 1, \\ \{x\}, & \text{if } x > 1, \\ \{-x\}, & \text{if } -1 \leq x < 0, \\ \{-x^{1/3}\}, & \text{if } x < -1. \end{cases}$$

Direct computations yield that $D^m F(0,0)(1) = \emptyset$ for all $m \geq 1$. Here, F is lower semicontinuous at $(0,0)$, but the locally pseudo-Hölder calmness of order m at $(0,0)$ fails.

EXAMPLE 2.4 (assumption (i) cannot be dropped) Let $F : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by

$$F(x) = \begin{cases} \{1\}, & \text{if } x = 0, \\ \{y : y \leq x\}, & \text{if } x \neq 0. \end{cases}$$

Then, assumption (ii) is satisfied at $(0,1)$. Direct calculations give that
$D^m F(0,1)\,(1) = \emptyset$ for all $m \geq 1$. The cause is that F is not lower semicontinuous

at $(0,1)$, since F is locally pseudo-Holder calm of order m at $(0,1)$. Indeed, pick $\lambda = 1, U = \{x \in \mathbb{R} : -1/2 < x < 1/2\}$ and $V = \{y \in \mathbb{R} : 1/2 < y < 3/2\}$. Then, $F(x) = \{y \in \mathbb{R} : y \leq x\} \subset (-\infty, 1/2)$ for all $x \in U \setminus \{0\}$. Therefore, $F(x) \cap V = \emptyset$ for all $x \in U \setminus \{0\}$, and

$$F(0) \cap V = \{1\} \subset \{y_0\} + ||x||^m B_Y$$

for all $m \geq 1$.

PROPOSITION 2.2 *Let $F : X \to 2^Y$, $(x_0, y_0) \in \mathrm{gr}F$, and F be a convex map and have a strict Studniarski derivative at (x_0, y_0). Then, $D^m F(x_0, y_0)$ is convex.*

Proof. Let $x^1, x^2 \in X$ and $y^i \in D^m F(x_0, y_0)(x^i)$, $i = 1, 2$, i.e., for any $t_n \to 0^+$, there exists $(x_n^i, y_n^i) \to (x^i, y^i)$ such that, for all n, $y_n^i \in t_n^{-m}(F(x_0 + t_n x_n^i) - y_0)$. Since F is convex, for all $\lambda \in [0, 1]$,

$$\lambda(\frac{F(x_0 + t_n x_n^1) - y_0}{t_n^m}) + (1 - \lambda)(\frac{F(x_0 + t_n x_n^2) - y_0}{t_n^m}) \subset$$

$$\frac{F(\lambda(x_0 + t_n x_n^1) + (1 - \lambda)(x_0 + t_n x_n^2)) - y_0}{t_n^m}.$$

Therefore,

$$\lambda y_n^1 + (1 - \lambda) y_n^2 \in \frac{F(x_0 + t_n(\lambda x_n^1 + (1 - \lambda) x_n^2)) - y_0}{t_n^m}.$$

Hence, $\lambda y^1 + (1 - \lambda) y^2 \in D^m F(x_0, y_0)(\lambda x^1 + (1 - \lambda) x^2)$. ■

The next statement is a relation between the Studniarski derivative of F and that of the profile map.

PROPOSITION 2.3 *Let $F : X \to 2^Y$, and $(x_0, y_0) \in \mathrm{gr}F$. Then, for all $x \in X$,*

$$D^m F(x_0, y_0)(x) + C \subset D^m F_+(x_0, y_0)(x). \tag{2.3}$$

If $\dim Y < +\infty$ and F is locally Hölder calm of order m at (x_0, y_0), then (2.3) becomes an equality.

Proof. Let $w \in D^m F(x_0, y_0)(x) + C$, i.e., there exist $v \in D^m F(x_0, y_0)(x)$ and $c \in C$ such that $w = v + c$. We then have sequences $t_n \to 0^+$, $x_n \to x$, and $v_n \to v$ such that, for all n,

$$y_0 + t_n^m (v_n + c) \in F(x_0 + t_n x_n) + t_n^m c \subset F(x_0 + t_n x_n) + C.$$

So, $v + c \in D^m F_+(x_0, y_0)(x)$.

Let $w \in D^m F_+(x_0, y_0)(x)$, i.e., there exist $t_n \to 0^+$, $x_n \to x$, $w_n \to w$ such that $y_0 + t_n^m w_n \in F(x_0 + t_n x_n) + C$. Then, there exist $y_n \in F(x_0 + t_n x_n)$ and $c_n \in C$ satisfying

$$w_n = t_n^{-m}(y_n - y_0) + t_n^{-m} c_n. \tag{2.4}$$

Because F is locally Hölder calm of order m at (x_0, y_0), there exists $\lambda > 0$ such that, for large n,

$$y_n \in F(x_0 + t_n x_n) \subset \{y_0\} + \lambda ||t_n x_n||^m B_Y.$$

So,

$$t_n^{-m} ||y_n - y_0|| \leq \lambda ||x_n||^m.$$

Since $\dim Y < +\infty$, $t_n^{-m}(y_n - y_0)$ (using a subsequence, if necessary) converges to some v and $v \in D^m F(x_0, y_0)(x)$. From (2.4), the sequence c_n / t_n^m converges to some $c \in C$ and $w = v + c$. Thus, $w \in D^m F(x_0, y_0)(x) + C$. ■

Observe that, for the special case of $m = 1$, (2.3) collapses to the result of Proposition 2.1 of Tanino (1988) and also of Theorem 3 of Jahn and Rauh (1997). Moreover, the equality

$$D^m F(x_0, y_0)(x) + C = D^m F_+(x_0, y_0)(x)$$

asserted in Proposition 2.3 was also asserted in Proposition 2 of Bednarczuk and Song (1998) for C being a pointed closed convex cone (under assumptions different from those imposed in Proposition 2.3) since, for such a pointed C, the above equality implies that

$$\operatorname{Min} D^m F_+(x_0, y_0)(x) = \operatorname{Min} (D^m F(x_0, y_0)(x)),$$

where $a_0 \in \operatorname{Min} A$ means $(A - a_0) \cap (-C) = \{0\}$, i.e., a_0 is a Pareto minimum of the set A.

3. Calculus rules

PROPOSITION 3.1 (sum rule) *Let $F_1, F_2 : X \to 2^Y$, $x_0 \in \operatorname{dom} F_1 \cap \operatorname{dom} F_2$, $y_i \in F(x_i)$ (i=1,2) and $u \in X$. Suppose either F_1 or F_2 has an mth-order proto Studniarski's derivative at (x_0, y_1) or (x_0, y_2), respectively. Then,*

$$D^m F_1(x_0, y_1)(u) + D^m F_2(x_0, y_2)(u) \subset D^m (F_1 + F_2)(x_0, y_1 + y_2)(u). \tag{3.1}$$

If, additionally, $\dim Y < +\infty$ and either F_1 or F_2 is locally Hölder calm of order m at (x_0, y_1) or at (x_0, y_2), respectively, then (3.1) becomes an equality.

Proof. Suppose F_2 has an mth-order proto Studniarski's derivative at (x_0, y_2) and $v^i \in D^m F_i(x_0, y_i)(u)$, $i = 1, 2$. For v^1, there exist $t_n \to 0^+$, $u_n \to u$, and $v_n^1 \to v^1$ such that $y_1 + t_n^m v_n^1 \in F_1(x_0 + t_n u_n)$ for all n. For these t_n and u_n, there exists $v_n^2 \to v^2$ such that $y_2 + t_n^m v_n^2 \in F_2(x_0 + t_n u_n)$. Hence, $y_1 + y_2 + t_n^m(v_n^1 + v_n^2) \in (F_1 + F_2)(x_0 + t_n u_n)$ and $v^1 + v^2 \in D^m(F_1 + F_2)(x_0, y_1 + y_2)(u)$.

To consider the equality case, suppose F_1 is locally Hölder calm of order m at (x_0, y_1). Let $v \in D^m(F_1 + F_2)(x_0, y_1 + y_2)(u)$, i.e., there exist $t_n \to 0^+$, $u_n \to u$, and $v_n \to v$ such that, for all n,

$$y_1 + y_2 + t_n^m v_n \in (F_1 + F_2)(x_0 + t_n u_n) = F_1(x_0 + t_n u_n) + F_2(x_0 + t_n u_n).$$

This means that there exist $y_n^i \in F_i(x_0 + t_n u_n)$, i = 1, 2, such that

$$v_n = t_n^{-m}(y_n^1 - y_1) + t_n^{-m}(y_n^2 - y_2). \tag{3.2}$$

Applying a Hölder calmness argument similarly as for Propositions 2.1 and 2.3, we obtain $v^1 \in D^m F_1(x_0, y_1)(u)$ and $v^2 \in D^m F_2(x_0, y_2)(u)$ such that $v^2 = v - v^1$. Thus, $v \in D^m F_1(x_0, y_1)(u) + D^m F_2(x_0, y_2)(u)$. ■

PROPOSITION 3.2 (chain rule) *Let* $F : X \to 2^Y$, $G : Y \to 2^Z$, $(x_0, y_0) \in \mathrm{gr}F$, $(y_0, z_0) \in \mathrm{gr}G$, *and* $\mathrm{Im}F \subset \mathrm{dom}G$.

(i) *Suppose G has an mth-order proto Studniarski's derivative at* (y_0, z_0). *Then, for all* $u \in X$,

$$D^m G(y_0, z_0)(D^1 F(x_0, y_0)(u)) \subset D^m(G \circ F)(x_0, z_0)(u). \tag{3.3}$$

If, additionally, $\dim Y < +\infty$ *and F is locally Lipschitz calm at* (x_0, y_0), *then (3.3) becomes an equality.*

(ii) *Suppose G has a first order proto Studniarski derivative at* (y_0, z_0). *Then, for all* $u \in X$,

$$D^1 G(y_0, z_0)(D^m F(x_0, y_0)(u)) \subset D^m(G \circ F)(x_0, z_0)(u). \tag{3.4}$$

If, additionally, $\dim Y < +\infty$ *and F is locally Hölder calm of order m at* (x_0, y_0), *then (3.4) becomes an equality.*

Proof. By the similarity, we prove only (i). Let $w \in D^m G(y_0, z_0)(D^1 F(x_0, y_0)(u))$, i.e., there exists $v \in D^1 F(x_0, y_0)(u)$ such that $w \in D^m G(y_0, z_0)(v)$. There exist $t_n \to 0^+$, $u_n \to u$, and $v_n \to v$ such that, for all n, $y_0 + t_n v_n \in F(x_0 + t_n u_n)$. With t_n, v_n above, we have $w_n \to w$ such that, for all n, $z_0 + t_n^m w_n \in G(y_0 + t_n v_n)$. So, $z_0 + t_n^m w_n \in G(F(x_0 + t_n u_n))$. Thus, $w \in D^m(G \circ F)(x_0, z_0)(u)$.

Let $w \in D^m(G \circ F)(x_0, z_0)(u)$, i.e., there exist $t_n \to 0^+$, $u_n \to u$, and $w_n \to w$ such that $z_0 + t_n^m w_n \in G(F(x_0 + t_n u_n))$ for all n. Then, there exists $y_n \in$

$F(x_0+t_nu_n)$ such that $z_0+t_n^m w_n \in G(y_n)$. Due to the local Lipschitz calmness of F and the finiteness of dim$Y$, the sequence $v_n := t_n^{-1}(y_n - y_0)$, or a subsequence, converges to some v and $v \in D^1F(x_0, y_0)(u)$. This implies that $z_0 + t_n^m w_n \in G(y_0 + t_n v_n)$ and hence $w \in D^m G(y_0, z_0)(v)$. ■

We next discuss calculus rules for the following operations.

DEFINITION 3.1 (i) For $F_1, F_2 : X \to 2^{\mathbb{R}^k}$, $\mathbb{R}^k$ being an Euclidean space, the product of F_1 and F_2 is the set-valued map $\langle F_1, F_2\rangle : X \to 2^{\mathbb{R}}$ defined by $\langle F_1, F_2\rangle (x) := \{\langle y_1, y_2\rangle : y_1 \in F_1(x), y_2 \in F_2(x)\}$.

(ii) For $F_1, F_2 : X \to 2^{\mathbb{R}}$, the quotient of F_1 and F_2 is the set-valued map $F_1/F_2 : X \to 2^{\mathbb{R}}$ defined by $(F_1/F_2)(x) := \{y_1/y_2 : y_1 \in F_1(x), y_2 \in F_2(x), y_2 \neq 0\}$.

PROPOSITION 3.3 (product rule) *Let* $F_1, F_2 : X \to 2^{\mathbb{R}^k}$, $x_0 \in \text{dom}F_1 \cap \text{dom}F_2$, $y_i \in F_i(x_0)$, *i=1,2. Suppose either* F_1 *or* F_2 *has an mth-order proto Studniarski's derivative at* (x_0, y_1) *or* (x_0, y_2), *respectively. Then, for all* $u \in X$,

$$\langle y_2, D^mF_1(x_0, y_1)(u)\rangle + \langle y_1, D^mF_2(x_0, y_2)(u)\rangle \subset D^m(\langle F_1, F_2\rangle)(x_0, \langle y_1, y_2\rangle)(u). \tag{3.5}$$

If, additionally, both F_i *are locally Hölder calm of order* m *at* (x_0, y_i), *then (3.5) becomes an equality.*

Proof. Suppose F_2 has an mth-order proto Studniarski's derivative at (x_0, y_2) and $v^i \in D^mF_i(x_0, y_i)(u)$, $i = 1, 2$. Then, there exist $t_n \to 0^+$, $u_n \to u$, $v_n^1 \to v^1$, and $v_n^2 \to v^2$ such that, for all n, $y_1 + t_n^m v_n^1 \in F_1(x_0 + t_nu_n)$ and $y_2 + t_n^m v_n^2 \in F_2(x_0 + t_nu_n)$. We have

$$\left\langle y_1 + t_n^m v_n^1, y_2 + t_n^m v_n^2\right\rangle = \langle y_1, y_2\rangle + t_n^m(\left\langle y_1, v_n^2\right\rangle + \left\langle y_2, v_n^1\right\rangle + t_n^m \left\langle v_n^1, v_n^2\right\rangle),$$

and

$$\left\langle y_1 + t_n^m v_n^1, y_2 + t_n^m v_n^2\right\rangle \in \langle F_1, F_2\rangle (x_0 + t_nu_n).$$

This implies that $\left\langle y_1, v^2\right\rangle + \left\langle y_2, v^1\right\rangle \in D^m(\langle F_1, F_2\rangle)(x_0, \langle y_1, y_2\rangle)(u)$.

Let $v \in D^m(\langle F_1, F_2\rangle)(x_0, \langle y_1, y_2\rangle)(u)$, i.e., there exist $t_n \to 0^+$, $u_n \to u$, $v_n \to v$, and $y_n^i \in F_i(x_0 + t_nu_n)$ such that $\langle y_1, y_2\rangle + t_n^m v_n = \left\langle y_n^1, y_n^2\right\rangle$ for all n. We have

$$\left\langle y_n^1, y_n^2\right\rangle = \left\langle y_n^1 - y_1 + y_1, y_n^2 - y_2 + y_2\right\rangle =$$

$$= \left\langle y_n^1 - y_1, y_n^2 - y_2\right\rangle + \left\langle y_n^1 - y_1, y_2\right\rangle + \left\langle y_n^2 - y_2, y_1\right\rangle + \langle y_1, y_2\rangle .$$

This implies that

$$v_n = \left\langle \frac{y_n^1 - y_1}{t_n^m}, y_2\right\rangle + \left\langle \frac{y_n^2 - y_2}{t_n^m}, y_1\right\rangle + t_n^m \left\langle \frac{y_n^1 - y_1}{t_n^m}, \frac{y_n^2 - y_2}{t_n^m}\right\rangle . \tag{3.6}$$

Because F_i are locally Hölder calm of order m at (x_0, y_i), there exist $L_i > 0$ such that, for $i = 1, 2$ and large n,

$$y_n^i \in F_i(x_0 + t_n^m u_n) \subset \{y_i\} + L_i ||t_n u_n||^m B_Y.$$

This implies that there exists a subsequence $\{n_k\}$ such that $t_{n_k}^{-m}(y_{n_k}^i - y_i)$ converges to some $v^i \in \mathbb{R}^k$ and $v^i \in D^m F_i(x_0, y_i)(u)$, $i = 1, 2$. Thus, from (3.6), $v \in \langle D^m F_1(x_0, y_1)(u), y_2\rangle + \langle D^m F_2(x_0, y_2)(u), y_1\rangle$. ■

PROPOSITION 3.4 (quotient rule) *Let $F_1, F_2 : X \to 2^{\mathbb{R}}$, $x_0 \in \mathrm{dom}F_1 \cap \mathrm{dom}F_2$, and $y_i \in F_i(x_0)$ (i=1,2) with $y_2 \neq 0$. Suppose either F_1 or F_2 has an mth-order proto Studniarski's derivative at (x_0, y_1) or (x_0, y_2), respectively. Then, for all $u \in X$,*

$$\frac{1}{y_2^2}(y_2 D^m F_1(x_0, y_1)(u) - y_1 D^m F_2(x_0, y_2)(u)) \subset D^m((F_1/F_2)(x_0, y_1/y_2)(u). \quad (3.7)$$

If, in addition, F_2 is locally Hölder calm of order m at (x_0, y_2), then (3.7) becomes an equality.

Proof. Assume that F_2 has an mth-order proto Studniarski's derivative at (x_0, y_2) and $v^i \in D^m F_i(x_0, y_i)(u)$, $i = 1, 2$. There exist $t_n \to 0^+$, $u_n \to u$, and $v_n^1 \to v^1$ such that $y_1 + t_n^m v_n^1 \in F_1(x_0 + t_n u_n)$ for all n. With these t_n, u_n, there exists $v_n^2 \to v^2$ such that $y_2 + t_n^m v_n^2 \in F_2(x_0 + t_n u_n)$. We have

$$\frac{y_1 + t_n^m v_n^1}{y_2 + t_n^m v_n^2} = \frac{y_1}{y_2} + t_n^m\left(\frac{y_2 v_n^1 - y_1 v_n^2}{y_2^2 + t_n^m v_n^2 y_2}\right) \in (F_1/F_2)(x_0 + t_n u_n).$$

This implies that $y_2^{-2}(y_2 v^1 - y_1 v^2) \in D^m((F_1/F_2)(x_0, y_1/y_2)(u)$.

Let $v \in D^m(F_1/F_2)(x_0, (y_1/y_2))(u)$, i.e., there exist $t_n \to 0^+$, $u_n \to u$, and $v_n \to v$ such that $(y_1/y_2) + t_n^m v_n \in (F_1/F_2)(x_0 + t_n u_n)$ for all n. So, there exist $y_n^i \in F_i(x_0 + t_n u_n)$ such that $(y_1/y_2) + t_n^m v_n = y_n^1/y_n^2$. Therefore,

$$\frac{y_n^1}{y_n^2} = \frac{y_1}{y_2} + \frac{y_2(y_n^1 - y_1) - y_1(y_n^2 - y_2)}{y_2^2 + y_2(y_n^2 - y_2)},$$

and hence

$$v_n = \frac{y_2(y_n^1 - y_1)/t_n^m - y_1(y_n^2 - y_2)/t_n^m}{y_2^2 + t_n^m y_2(y_n^2 - y_2)/t_n^m}. \quad (3.8)$$

Applying a Hölder calmness argument as above, we obtain $v^2 \in D^m F_2(x_0, y_2)(u)$ and $v^1 \in D^m F_1(x_0, y_1)(u)$ such that $v = y_2^{-2}(y_2 v^1 - y_1 v^2)$. Thus,

$$v \in y_2^{-2}(y_2 D^m F_1(x_0, y_1)(u) - y_1 D^m F_2(x_0, y_2)(u).$$

■

COROLLARY 3.5 (reciprocal rule) *Let* $F : X \to 2^{\mathbb{R}}$, $y_0 \in F(x_0)$ *with* $y_0 \neq 0$, *and* $u \in X$. *Then,*

$$-y_0^{-2} D^m F(x_0, y_0)(u) \subset D^m(1/F)(x_0, 1/y_0)(u). \tag{3.9}$$

If, in addition, F *is locally Hölder calm of order* m *at* (x_0, y_0)*, then (3.9) becomes an equality.*

In the rest of this section, we discuss other sum and chain rules, which may be more useful in some cases (see, e.g., Section 4). To investigate the sum $M+N$ of multifunctions $M, N : X \to 2^Y$, we express $M + N$ as a composition as follows: Define $F : X \to 2^{X\times Y}$ and $G : X \times Y \to 2^Y$ by, for I being the identity map on X and $(x, y) \in X \times Y$,

$$F = I \times M \qquad \text{and} \qquad G(x, y) = y + N(x). \tag{3.10}$$

Then, clearly $M + N = G \circ F$.

First, we develop a chain rule. Let general multimaps $F : X \to 2^Y$ and $G : Y \to 2^Z$ be considered. The so-called resultant set-valued map $C : X \times Z \to 2^Y$ is defined by

$$C(x, z) := F(x) \cap G^{-1}(z).$$

Then, dom$C$ = gr$(G \circ F)$. We need the following compactness property:

DEFINITION 3.2 (Penot, 1983) A set-valued map $H : X \to 2^Y$ is said to be compact at $x \in \mathrm{cl}(\mathrm{dom}H)$ if any sequence $y_n \in H(x_n)$ satisfying $x_n \to x$ has a convergent subsequence.

Note that when H is compact at x, the image $H(x)$ still may be not closed. Simply think of $H : \mathbb{R} \to 2^{\mathbb{R}}$ equal to $(0, 1)$ if $x = 0$, and to $\{0\}$ if $x \neq 0$. Then, H is compact at 0, but $H(0) = (0, 1)$ is not closed.

We define other kinds of mth-order Studniarski's derivatives of $G \circ F$ with respect to variable y as follows.

DEFINITION 3.3 Let $((x, z), y) \in \mathrm{gr}\, C$.

(i) The mth-order y-Studniarski derivative of $G \circ F$ at $((x, z), y)$ is defined as, for $u \in X$,

$$\begin{aligned} D^m(G \circ_y F)(x, z)(u) \quad = \quad & \{w \in Z : \exists t_n \to 0^+, \exists (u_n, y_n, w_n) \to (u, y, w), \forall n, \\ & y_n \in C(x + t_n u_n, z + t_n^m w_n)\}. \end{aligned}$$

(ii) For an integer k, the mth-order pseudo-Studniarski derivative of the map C at (x, z) with respect to k is defined as, for $(u, w) \in X \times Z$,

$$\begin{aligned} D_{ps}^{m(k)} C((x, z), y)(u, w) \quad = \quad & \{\overline{y} \in Y : \exists t_n \to 0^+, \exists (u_n, \overline{y}_n, w_n) \to (u, \overline{y}, w), \forall n, \\ & y + t_n^k \overline{y}_n \in C(x + t_n u_n, z + t_n^m w_n)\}. \end{aligned}$$

If $k = m$, the set in Definition 3.3(ii) is denoted shortly by $D^m_{ps}C((x,z),y)(u,w)$. One has a relationship between $D^m(G \circ_y F)(x,z)(u)$ and $D^m(G \circ F)(x,z)(u)$ in the following statement:

PROPOSITION 3.6 *Let* $(x,z) \in \mathrm{gr}(G \circ F)$ *and* $u \in X$.

(i) *For* $y \in C(x,z)$, *one has*

$$D^m(G \circ_y F)(x,z)(u) \subset D^m(G \circ F)(x,z)(u).$$

(ii) *If* C *is compact and closed at* (x,z), *then*

$$\bigcup_{y \in C(x,z)} D^m(G \circ_y F)(x,z)(u) = D^m(G \circ F)(x,z)(u).$$

Proof. (i) This follows immediately from the definitions.

(ii) "$\subset$" follows from (i). For "$\supset$", let $w \in D^m(G \circ F)(x,z)(u)$, i.e., there exist sequences $t_n \to 0^+$ and $(u_n, w_n) \to (u,w)$ such that $z + t_n^m w_n \in (G \circ F)(x + t_n u_n)$. So, there exists $y_n \in Y$ with $y_n \in C(x + t_n u_n, z + t_n^m w_n)$. Since C is compact at (x,z), y_n (or a subsequence) has a limit y. Since $(x + t_n u_n, z + t_n^m w_n, y_n) \to (x,z,y)$, $(x,z,y) \in \mathrm{cl}(\mathrm{gr}\, C) = \mathrm{gr}(\mathrm{cl}\, C)$. It follows from the closedness of C at (x,z) that $y \in C(x,z)$, and $w \in D^m(G \circ_y F)(x,z)(u)$ with this y. ■

The first chain rule for $G \circ F$ using these new Studniarski derivatives is

PROPOSITION 3.7 *Let* $(x,z) \in \mathrm{gr}(G \circ F)$ *and* $y \in C(x,z)$. *Suppose, for all* $(u,w) \in X \times Z$,

$$D^m F(x,y)(u) \cap (D^1 G(y,z))^{-1}(w) \subset D^m_{ps}C((x,z),y)(u,w). \tag{3.11}$$

Then,

$$D^1 G(y,z)[D^m F(x,y)(u)] \subset D^m(G \circ_y F)(x,z)(u).$$

Proof. Let $v \in D^1 G(y,z)[D^m F(x,y)(u)]$, i.e., there exists $\overline{y} \in D^m F(x,y)(u)$ such that $\overline{y} \in (D^1 G(y,z))^{-1}(v)$. Then, (3.11) ensures that $\overline{y} \in D^m_{ps}C((x,z),y)(u,v)$. This means the existence of $t_n \to 0^+$ and $(u_n, \overline{y}_n, v_n) \to (u, \overline{y}, v)$ such that $y + t_n^m \overline{y}_n \in C(x + t_n u_n, z + t_n^m v_n)$ for all n. We have $y_n := y + t_n^m \overline{y}_n \in C(x + t_n u_n, z + t_n^m v_n)$. So, $v \in D^m(G \circ_y F)(x,z)(u)$ and we are done. ■

PROPOSITION 3.8 *Let* $(x,z) \in \mathrm{gr}(G \circ F)$ *and* $y \in C(x,z)$. *Suppose, for all* $(u,w) \in X \times Z$,

$$D^1 F(x,y)(u) \cap (D^m G(y,z))^{-1}(w) \subset D^{m(1)}_{ps}C((x,z),y)(u,w). \tag{3.12}$$

Then,

$$D^m G(y,z)[D^1 F(x,y)(u)] \subset D^m(G \circ_y F)(x,z)(u).$$

Proof. The proof is similar to that of Proposition 3.7. ■

Note that, when $m = 1$, we have $(D^1 G(y,z))^{-1} = D^1 G^{-1}(z,y)$. However, this is not true for $m \geq 2$ as shown in the following example.
EXAMPLE 3.1 Let $F : \mathbb{R} \to \mathbb{R}$ be defined by $F(x) = x^2$. Then,

$$F^{-1}(y) = \begin{cases} \{-\sqrt{y}, \sqrt{y}\}, & \text{if } y \geq 0, \\ \emptyset, & \text{if } y < 0. \end{cases}$$

Direct computations yield that $D^1 F(0,0)(u) = \{0\}$ for all $u \in \mathbb{R}$, which implies that $(D^1 F(0,0))^{-1}(0) = \mathbb{R}$ and $(D^1 F(0,0))^{-1}(v) = \emptyset$ for $v \neq 0$. It is easy to check that $D^1 F^{-1}(0,0)$ coincides with $(D^1 F(0,0))^{-1}$.

For $m = 2$, $D^2 F(0,0)(u) = \{u^2\}$ for all $u \in \mathbb{R}$, which implies

$$(D^2 F(0,0))^{-1}(y) = \begin{cases} \{-\sqrt{y}, \sqrt{y}\}, & \text{if } y \geq 0, \\ \emptyset, & \text{if } y < 0. \end{cases}$$

However,

$$D^2 F^{-1}(0,0)(v) = \begin{cases} \mathbb{R}, & \text{if } v = 0, \\ \emptyset, & \text{if } v \neq 0. \end{cases}$$

To get a chain rule for Studniarski's derivatives in the form of equalities, we first prove the inclusions reverse to those in Propositions 3.7 and 3.8 under additional assumptions as follows:
PROPOSITION 3.9 *Let $y \in C(x,z)$ and Y be finite dimensional.*

(i) *If*

$$D^m_{ps} C((x,z),y)(0,0) = \{0\}, \tag{3.13}$$

then

$$D^m(G \circ_y F)(x,z)(u) \subset D^1 G(y,z)[D^m F(x,y)(u)].$$

(ii) *If*

$$D^{m(1)}_{ps} C((x,z),y)(0,0) = \{0\}, \tag{3.14}$$

then

$$D^m(G \circ_y F)(x,z)(u) \subset D^m G(y,z)[D^1 F(x,y)(u)].$$

Proof. By the similarity, we prove only (i). Let $w \in D^m(G \circ_y F)(x,z)(u)$, i.e., there exist $t_n \to 0^+$ and $(u_n, y_n, w_n) \to (u, y, w)$ such that $y_n \in C(x + t_n u_n, z + t_n^m w_n)$ for all n. If $y_k = y$ for infinitely many $k \in \mathbb{N}$, one has $0 \in D^m F(x,y)(u)$, $w \in D^1 G(y,z)(0)$ and we are done. Thus, suppose $y_n \neq y$ for all n and, for $s_n := ||y_n - y||^{1/m}$, the sequence $v_n := s_n^{-m}(y_n - y)$ or some subsequence has a limit v of norm one. If $t_n/s_n \to 0$, since

$$y + s_n^m v_n = y_n \in C(x + s_n(\frac{t_n u_n}{s_n}), z + s_n^m(\frac{t_n^m w_n}{s_n^m})),$$

one sees that $v \in D^m_{ps}C((x,z),y)(0,0)$, contradicting (3.13). Consequently, $t_n^{-1}s_n$ has a bounded subsequence and one may assume that $t_n^{-1}s_n$ tends to $q \in R_+$. So,

$$y + t_n^m(s_n^m v_n/t_n^m) = y_n \in C(x + t_n u_n, z + t_n^m w_n)$$

and then one gets $q^m v \in D^m_{ps}C((x,z),y)(u,w)$. It follows from the definition of $D^m_{ps}C((x,z),y)(u,w)$ that $q^m v \in D^m F(x,y)(u)$ and $w \in D^1 G(y,z)(q^m v)$. ■

Combining Propositions 3.6-3.9, we arrive at the following chain rule:
PROPOSITION 3.10 *Suppose Y is finite dimensional and $(x,z) \in \mathrm{gr}(G \circ F)$ is such that C is compact and closed at (x,z).*

(i) *Assume that (3.13) holds for every $y \in C(x,z)$. Then,*

$$D^m(G \circ F)(x,z)(u) \subset \bigcup_{y \in C(x,z)} D^1 G(y,z)[D^m F(x,y)(u)]. \tag{3.15}$$

If, additionally, (3.11) holds for every $y \in C(x,z)$, then (3.15) is an equality.

(ii) *Assume that (3.14) holds for every $y \in C(x,z)$. Then,*

$$D^m(G \circ F)(x,z)(u) \subset \bigcup_{y \in C(x,z)} D^m G(y,z)[D^1 F(x,y)(u)]. \tag{3.16}$$

If, additionally, (3.12) holds for every $y \in C(x,z)$, then (3.16) is an equality.

Now we apply the preceding chain rules to establish sum rules for $M, N : X \to 2^Y$. For this purpose we use $F : X \to 2^{X \times Y}$ and $G : X \times Y \to 2^Y$ defined in (3.10). For $(x,z) \in X \times Y$, set

$$S(x,z) := M(x) \cap (z - N(x)).$$

Then, the so-called resultant map $C : X \times Y \to 2^{X \times Y}$ associated to these F and G is

$$C(x,z) = \{x\} \times S(x,z).$$

Given $((x,z),y) \in \mathrm{gr}S$, the mth-order y-Studniarski derivative of $M+N$ at (x,z) is defined as, for $u \in X$,

$$D^m(M +_y N)(x,z)(u) := \{w \in Y : \exists t_n \to 0^+, \exists (u_n, y_n, w_n) \to (u, y, w), \forall n,$$

$$y_n \in S(x + t_n u_n, z + t_n^m w_n)\}.$$

Observe that

$$D^m(M +_y N)(x,z)(u) = D^m(G \circ_y F)(x,z)(u). \tag{3.17}$$

One has a relationship between $D^m(M +_y N)(x,z)(u)$ and $D^m(M + N)(x,z)(u)$ as noted in the next statement.
PROPOSITION 3.11 *Let* $(x,z) \in \mathrm{gr}(M+N)$ *and* $y \in S(x,z)$.
(i) $D^m(M +_y N)(x,z)(u) \subset D^m(M+N)(x,z)(u)$.
(ii) *If* S *is compact and closed at* (x,z), *then*

$$\bigcup_{y \in S(x,z)} D^m(M +_y N)(x,z)(u) = D^m(M+N)(x,z)(u).$$

Proof. (i) This is an immediate consequence of the definitions.
(ii) When S is compact and closed at (x,z), C is compact and closed at (x,z). Hence, the equality in Proposition 3.6(ii) holds. In view of (3.17), this relation implies the required equality. ■

For higher-order sum rules, we have
PROPOSITION 3.12 *Let* $(x,z) \in \mathrm{gr}(M+N)$ *and* $y \in S(x,z)$. *Suppose, for all* $(u,v) \in X \times Y$,

$$D^m M(x,y)(u) \cap [v - D^m N(x, z-y)(u)] \subset D^m_{ps} S((x,z),y)(u,v). \tag{3.18}$$

Then,

$$D^m M(x,y)(u) + D^m N(x,z-y)(u) \subset D^m(M +_y N)(x,z)(u).$$

Proof. Let $w \in D^m M(x,y)(u) + D^m N(x,z-y)(u)$, i.e., there exists $\overline{y} \in D^m M(x,y)(u)$ such that $\overline{y} \in w - D^m N(x,z-y)(u)$. Hence, (3.18) ensures that $\overline{y} \in D^m_{ps} S((x,z),y)(u,w)$. Therefore, there exist $t_n \to 0^+$ and $(u_n, \overline{y}_n, w_n) \to (u, \overline{y}, w)$ such that $y + t_n^m \overline{y}_n \in S(x + t_n u_n, z + t_n^m w_n)$. Setting $y_n = y + t_n^m \overline{y}_n$, we have $y_n \in S(x + t_n u_n, z + t_n^m w_n)$. Consequently, $w \in D^m(M +_y N)(x,z)(u)$. ■

We can impose an additional condition to get equalities in the above sum rules as follows:

PROPOSITION 3.13 *Let Y be finite dimensional and $(x,z) \in \mathrm{gr}(M+N)$.*

(i) *Suppose, for $y \in S(x,z)$,*

$$D^m_{ps}S((x,z),y))(0,0) = \{0\}. \tag{3.19}$$

Then,

$$D^m(M +_y N)(x,z)(u) \subset D^m M(x,y)(u) + D^m N(x,z-y)(u).$$

(ii) *If S is compact and closed at (x,z) and (3.19) holds for every $y \in S(x,z)$, then one has*

$$D^m(M+N)(x,z)(u) \subset \bigcup_{y \in S(x,z)} (D^m M(x,y)(u) + D^m N(x,z-y)(u)). \tag{3.20}$$

If, additionally, (3.18) holds for every $y \in S(x,z)$, then (3.20) becomes an equality.

Proof. (i) Let $w \in D^m(M +_y N)(x,z)(u)$, i.e., there exist $t_n \to 0^+$ and $(u_n, y_n, w_n) \to (u,y,w)$ such that, for all n, $y_n \in S(x+t_n u_n, z+t_n^m w_n)$. If $y_k = y$ for infinitely many $k \in \mathbb{N}$, one has $0 \in D^m M(x,y)(u)$ and $w \in D^m N(x,z-y)(u)$, and we are done. Thus, suppose $y_n \neq y$ for all n and, for $s_n := ||y_n - y||^{1/m}$, the sequence $v_n := s_n^{-m}(y_n - y)$ converges to some v of norm one. If $t_n/s_n \to 0$, since

$$y + s_n^m v_n = y_n \in S(x + s_n(\frac{t_n u_n}{s_n}), z + s_n^m(\frac{t_n^m w_n}{s_n^m})),$$

one sees that $v \in D^m_{ps}S((x,z),y)(0,0)$, contradicting (3.19). Consequently, s_n/t_n has a bounded subsequence and we may assume that s_n/t_n tends to $q \in R_+$. So,

$$y + t_n^m(\frac{s_n^m}{t_n^m} v_n) = y_n \in S(x + t_n u_n, z + t_n^m w_n)$$

and then $q^m v \in D^m_{ps}S((x,z),y)(u,w)$. It follows from the definition of $D^m_{ps}S((x,z),y)$ (u,w) that $q^m v \in D^m M(x,y)(u)$ and $w - q^m v \in D^m N(x,z-y)(u)$.

(ii) This follows from (i) and Propositions 3.11 and 3.12. ■

Next, we define two other mth-order Studniarski's derivatives, which are slight modifications of those in the above definitions and suitable for applications to variational inequalities in Section 4. Let P be also a normed space, $F : P \times X \to 2^Y$ and $N : P \times X \to 2^Y$. Let $\hat{S} : P \times X \times Y \to 2^Y$ be given by

$$\hat{S}(p,x,y) := F(p,x) \cap (y - N(p,x)).$$

DEFINITION 3.4 Given $y_0 \in \hat{S}(p,x,y)$ and $(u,v) \in P \times X$, we define

$$D^m(F+_{y_0}N)((p,x),y)(u,v) := \{w \in Y : \exists t_n \to 0^+, \exists (u_n,v_n,y_n,w_n) \to (u,v,y_0,w),$$
$$y_n \in \hat{S}(p+t_nu_n, x+t_n^m v_n, y+t_n^m w_n)\},$$

and

$$D^m_{ps}\hat{S}((p,x,y),y_0)(u,v,s) := \{w \in Y : \exists t_n \to 0^+, \exists (u_n,v_n,s_n,w_n) \to (u,v,s,w),$$
$$y_0 + t_n^m w_n \in \hat{S}(p+t_nu_n, x+t_n^m v_n, y+t_n^m s_n)\}.$$

PROPOSITION 3.14 *Let Y be finite dimensional and $((p,x),y) \in \mathrm{gr}(F+N)$.*
(i) *Suppose, for $y_0 \in \hat{S}(p,x,y)$,*

$$D^m_{ps}\hat{S}((p,x,y),y_0))(0,0,0) = \{0\}. \tag{3.21}$$

Then,

$$D^m(F+_{y_0}N)((p,x),y)(u,v) \subset D^m_{ps}F((p,x),y_0)(u,v) + D^m_{ps}N((p,x),y-y_0)(u,v).$$

(ii) *If $\hat{S}$ is compact and closed at (p,x,y) and (3.21) holds for every $y_0 \in \hat{S}(p,x,y)$, then one has*

$$D^m_{ps}(F+N)((p,x),y)(u,v) \subset$$
$$\bigcup_{y_0 \in \hat{S}(p,x,y)} (D^m_{ps}F((p,x),y_0)(u,v) + D^m_{ps}N((p,x),y-y_0)(u,v)).$$

Proof. (i) Let $w \in D^m(F+_{y_0}N)((p,x),y)(u,v)$, i.e., there exist $t_n \to 0^+$ and $(u_n,v_n,y_n,w_n) \to (u,v,y_0,w)$ such that $y_n \in \hat{S}(p+t_nu_n, x+t_n^m v_n, y+t_n^m w_n)$ for all n. If $y_k = y_0$ for infinitely many $k \in \mathbb{N}$, one has $0 \in D^m_{ps}F((p,x),y_0)(u,v)$ and $w \in D^m_{ps}N((p,x),y-y_0)(u,v)$, and we are done. Now suppose $y_n \neq y_0$ for all n and, for $s_n := ||y_n - y_0||^{1/m}$, the sequence $l_n := s_n^{-m}(y_n - y_0)$ converges to some l of norm one. If $t_n/s_n \to 0$, since

$$y_0 + s_n^m l_n = y_n \in \hat{S}(p + s_n\frac{t_nu_n}{s_n}, x + s_n(\frac{t_n^m v_n}{s_n}), y + s_n^m(\frac{t_n^m w_n}{s_n^m})),$$

one sees that $l \in D^m_{ps}\hat{S}((p,x,y),y_0)(0,0,0)$, contradicting (3.21). Consequently, one may assume that s_n/t_n tends to a number $q \in R_+$. So,

$$y_0 + t_n^m(\frac{s_n^m}{t_n^m} l_n) = y_n \in \hat{S}(p+t_nu_n, x+t_n^m v_n, y+t_n^m w_n)$$

and thus $q^m l \in D^m_{ps}\hat{S}((p,x,y),y_0)(u,v,w)$. By the definition of $D^m_{ps}\hat{S}((p,x,y),y_0)$ (u,v,w), one has $q^m l \in D^m_{ps}F((p,x),y_0)(u,v)$ and $w - q^m l \in D^m_{ps}N((p,x),y-y_0)(u,v)$.

(ii) We need to prove that, if $\hat{S}$ is compact and closed at (p, x, y), then

$$D^m_{ps}(F+N)((p,x),y)(u,v) = \bigcup_{y_0 \in \hat{S}(p,x,y)} D^m(F +_{y_0} N)((p,x),y)(u,v).$$

In fact, we only need to prove the inclusion "$\subset$". Let $w \in D^m_{ps}(F+N)((p,x),y)(u,v)$. There exist $t_n \to 0^+$ and $(u_n, v_n, w_n) \to (u, v, w)$ such that $y + t_n^m w_n \in F(p + t_n u_n, x + t_n^m v_n) + N(p + t_n u_n, x + t_n^m v_n)$ for all n. Then, one can find $y_n \in F(p + t_n u_n, x + t_n^m v_n)$ such that $y + t_n^m w_n - y_n \in N(p + t_n u_n, x + t_n^m v_n)$. Therefore, $y_n \in \hat{S}(p + t_n u_n, x + t_n^m v_n, y + t_n^m w_n)$ for all n. Since $\hat{S}$ is compact at (p, x, y), one may assume that y_n converges to a point y_0. As $(p + t_n u_n, x + t_n^m v_n, y + t_n^m w_n, y_n) \to (p, x, y, y_0)$, one has $y_0 \in (\mathrm{cl}\,\hat{S})(p, x, y)$. It follows from the closedness of $\hat{S}$ at (p, x, y) that $y_0 \in \hat{S}(p, x, y)$. ■

4. Applications

4.1. Studniarski's derivatives of solution maps to inclusions

Let $M : P \times X \to 2^Z$ be a set-valued map between normed spaces. Then, the map S defined by

$$S(p) := \{x \in X : 0 \in M(p, x)\}, \tag{4.1}$$

is said to be the solution map of the parametrized inclusion $0 \in M(p, x)$.

THEOREM 4.1 *For a solution map S defined by (4.1) and $\overline{x} \in S(\overline{p})$, we have, for $p \in P$,*

$$D^m S(\overline{p}, \overline{x})(p) \subset \{x \in X : 0 \in D^m_{ps} M((\overline{p}, \overline{x}), 0)(p, x)\}.$$

Proof. Let $(p, x) \in \mathrm{gr} D^m S(\overline{p}, \overline{x})$, i.e., there exist sequences $p_n \to p$, $x_n \to x$, and $t_n \to 0^+$ such that $\overline{x} + t_n^m x_n \in S(\overline{p} + t_n p_n)$ for all n. This implies that 0 is an element of the set $M(\overline{p} + t_n p_n, \overline{x} + t_n^m x_n)$. Hence, for $z_n = 0$, the inclusion $0 + t_n^m z_n \in M(\overline{p} + t_n p_n, \overline{x} + t_n^m x_n)$ holds, i.e., $0 \in D^m_{ps} M((\overline{p}, \overline{x}), 0)(p, x)$. ■

In parameterized optimization, we frequently meet M of the form

$$M(p, x) = F(p, x) + N(p, x), \tag{4.2}$$

where $F : P \times X \to 2^Z$ and $N : P \times X \to 2^Z$. Let $\hat{S} : P \times X \times Z \to 2^Z$ be defined by

$$\hat{S}(p, x, z) := F(p, x) \cap (z - N(p, x)).$$

The following theorem gives an approximation of the mth-order Studniarski derivative of S when M is defined by (4.2).

THEOREM 4.2 *For the solution map* $S(p) = \{x \in X : 0 \in F(p,x) + N(p,x)\}$ *and* $\overline{x} \in S(\overline{p})$ *with* Z *being finite dimensional, suppose either of the following conditions holds*

(i) $\hat{S}$ *is compact and closed at* $(\overline{p},\overline{x},0)$ *and* $D^m_{ps}\hat{S}((\overline{p},\overline{x},0),y)(0,0,0) = \{0\}$ *for all* $y \in \hat{S}(\overline{p},\overline{x},0)$*;*

(ii) *there exists* $y \in \hat{S}(\overline{p},\overline{x},0)$ *such that either* F *or* N *is locally Hölder calm of order* m *at* $(\overline{p},\overline{x},y)$ *or at* $(\overline{p},\overline{x},-y)$*, respectively.*

Then,

$$D^m S(\overline{p},\overline{x})(p) \subset$$

$$\{x \in X : 0 \in \bigcup_{y\in(\mathrm{cl}\,\hat{S})(\overline{p},\overline{x},0)} (D^m_{ps}F((\overline{p},\overline{x}),y)(p,x) + D^m_{ps}N((\overline{p},\overline{x}),0-y)(p,x))\}.$$

Proof. We first prove that

$$D^m_{ps}M((\overline{p},\overline{x}),0)(p,x) \subset \bigcup_{y\in(\mathrm{cl}\,\hat{S})(\overline{p},\overline{x},0)} (D^m_{ps}F((\overline{p},\overline{x}),y)(p,x)+D^m_{ps}N((\overline{p},\overline{x}),0-y)(p,x)).$$

If (i) holds, the above inclusion follows from Proposition 3.14. For the case (ii), with $y \in \hat{S}(\overline{p},\overline{x},0)$, we see that $y \in F(\overline{p},\overline{x})$ and $-y \in N(\overline{p},\overline{x})$. Let $v \in D^m_{ps}M((\overline{p},\overline{x}),0)(p,x)$, i.e., there exist $t_n \to 0^+$, $(p_n,x_n) \to (p,x)$, and $v_n \to v$ such that, for all n,

$$0+t_n^m v_n \in M(\overline{p}+t_np_n,\overline{x}+t_n^m x_n) = F(\overline{p}+t_np_n,\overline{x}+t_n^m x_n)+N(\overline{p}+t_np_n,\overline{x}+t_n^m x_n).$$

Then, there exist $y_n^1 \in F(\overline{p}+t_np_n,\overline{x}+t_n^m x_n)$ and $y_n^2 \in N(\overline{p}+t_np_n,\overline{x}+t_n^m x_n)$ such that

$$v_n = t_n^{-m}(y_n^1 - y) + t_n^{-m}(y_n^2 - (-y)). \tag{4.3}$$

For the case (ii), suppose F is locally Hölder calm of order m at at $(\overline{p},\overline{x},y)$. Then, there exists $L > 0$ such that, for large n,

$$y_n^1 \in F(\overline{p}+t_np_n,\overline{x}+t_n^m x_n) \subset \{y\} + L||(t_np_n,t_n^m x_n)||^m B_Z.$$

Because $\dim Z < +\infty$, $t_n^{-m}(y_n^1 - y)$, or a subsequence, converges to some $v^1 \in Z$ and so $v^1 \in D^m_{ps}F((\overline{p},\overline{x}),y)\ (p,x)$. From (4.3), the sequence $t_n^{-m}(y_n^2 - (-y))$ also converges to some v^2 such that $v^2 = v - v^1$, and $v^2 \in D^m_{ps}N((\overline{p},\overline{x}),-y)(p,x)$. Thus, $v \in D^m_{ps}F((\overline{p},\overline{x}),y)(p,x) + D^m_{ps}N((\overline{p},\overline{x}),-y)(p,x)$. Now, application of Theorem 4.1 completes the proof. ■

4.2. Implicit multifunction theorems

Let $M : P \times X \to Z$ and $S(p) := \{x \in X : M(p,x) = 0\}$, be the set of solutions to the parameterized equation $M(x,p) = 0$. We impose the condition

$$(*) \begin{cases} \text{there exists } \overline{x} \in X \text{ such that } M(0,\overline{x}) = 0 \text{ and} \\ M_p \text{ is continuous in a neighborhood } (U,V) \in \mathcal{U}(0) \times \mathcal{U}(\overline{x}), \end{cases}$$

where M_p denotes the partial Fréchet derivative with respect to p. Let $H = V \cap M(0,.)^{-1}$, i.e.,

$$H(z) = \{x \in V : M(0,x) = z\}.$$

Under the hypotheses of the usual implicit function theorems for $M \in C^1$, S and H are single-valued and smooth (with derivatives DS, DH), and there holds

$$DS(0) = -DH(0)M_p(0,\overline{x}) = -M_x(0,\overline{x})^{-1}M_p(0,\overline{x}).$$

Now we are interested in a similar formula of the mth-order Studniarski derivative $D^mS(0,\overline{x})(.)$ of the map S under assumption $(*)$. For (p,x) near $(0,\overline{x})$, we consider the map

$$r(p,x) := M(p,x) - M(0,x) - M_p(0,\overline{x})p.$$

By the mean-value theorem, one obtains

$$r(p,x) = \int_0^1 [M_p(\theta p,x) - M_p(0,\overline{x})]p d\theta,$$

and

$$\alpha(p,x,\theta) := ||M_p(\theta p,x) - M_p(0,\overline{x})||$$

can be estimated (uniformly for $0 < \theta < 1$) by

$$\alpha(p,x,\theta) \le 0(p,x) \text{ with } 0(p,x) \to 0^+ \text{ as } x \to \overline{x} \text{ and } ||p|| \to 0^+.$$

Due to $||r(p,x)|| \le 0(p,x)||p||$, one easily sees that $||p||^{-1}||r(p,x)|| \to 0^+$ as $x \to \overline{x}$ and $||p|| \to 0^+$, and also

$$r(p(t),x(t)) = o_2(t) \text{ if } x(t) \to \overline{x} \text{ and } p(t) = tq + o_1(t) \text{ with some } q \in P,$$

where $o_k(t)$ means that $||o_k(t)||/t \to 0^+$ as $t \to 0^+$.

For (p,x) near $(0,\overline{x})$, we have

$$M(p,x) = 0 \text{ if and only if } M(0,x) = -M_p(0,\overline{x})p - r(p,x),$$

i.e.,

$$x \in S(p) \text{ if and only if } x \in H(-M_p(0,\overline{x})p - r(p,x)).$$

Let $\hat{M} : P \times X \to Z$ be defined by $\hat{M}(p,x) := -M_p(0,\overline{x})(p) - r(p,x)$. Then,

$$x \in S(p) \text{ if and only if } x \in H(\hat{M}(p,x)). \tag{4.4}$$

Set $C(p,x) := \hat{M}(p,x) \cap H^{-1}(x)$. It is easy to see that $C(0,\overline{x}) = \{0\}$.

The following result is a modification of that in Proposition 3.9(ii).

LEMMA 4.1 *Let Z be finite dimensional and either of the following conditions hold*

(i) $\hat{M}$ *is locally Lipschitz calm at* $(0,\overline{x},0)$*;*

(ii) C *is compact and closed at* $(0,\overline{x})$ *and*

$$D_{ps}^{m(1)}C((0,\overline{x}),0)(0,0) = \{0\}. \tag{4.5}$$

Then, $x \in D^m S(0,\overline{x})(q)$ *implies that* $x \in D^m H(0,\overline{x})[D_{ps}^{m(1)}\hat{M}((0,\overline{x}),0)(q,x)]$.

Proof. Let (i) hold and $x \in D^m S(0,\overline{x})(q)$, i.e., there exist $t_n \to 0^+$, $q_n \to q$, and $x_n \to x$ such that $\overline{x} + t_n^m x_n \in S(0 + t_n q_n)$. It follows from (4.4) that

$$\overline{x} + t_n^m x_n \in H(\hat{M}(0 + t_n q_n, \overline{x} + t_n^m x_n)). \tag{4.6}$$

Then, there exists $y_n \in \hat{M}(0 + t_n q_n, \overline{x} + t_n^m x_n)$ such that $\overline{x} + t_n^m x_n \in H(y_n)$. Because $\hat{M}$ is locally Lipschitz calm at $(0,\overline{x},0)$, there exists $L > 0$ such that, for large n,

$$y_n \in \hat{M}(0 + t_n q_n, \overline{x} + t_n^m x_n) \subset \{0\} + L||(t_n q_n, t_n^m x_n)||B_Z.$$

Since $\dim Z < +\infty$, $v_n := t_n^{-1}(y_n - 0)$ (or a subsequence) converges to some $v \in Z$. So, $v \in D_{ps}^{m(1)}\hat{M}((0,\overline{x}),0)\ (q,x)$. This implies that $\overline{x} + t_n^m x_n \in H(0 + t_n v_n)$. Thus, $x \in D^m H(0,\overline{x})(v)$.

If (ii) holds, it follows from (4.6) that there exists

$$y_n \in \hat{M}(0 + t_n q_n, \overline{x} + t_n^m x_n) \cap H^{-1}(\overline{x} + t_n^m x_n) = C(0 + t_n q_n, \overline{x} + t_n^m x_n).$$

Since C is compact at $(0,\overline{x})$, y_n (or a subsequence) has a limit y. Since $(0 + t_n q_n, \overline{x} + t_n^m x_n, y_n) \to (0,\overline{x},y)$, one has $y \in (\mathrm{cl}\, C)(0,\overline{x})$. It follows from the closedness of S at $(0,\overline{x})$ that $y \in C(0,\overline{x}) = \{0\}$.

If $y_k = 0$ for infinitely many $k \in \mathbb{N}$, one has $0 \in D_p^{m(1)}\hat{M}((0,\overline{x}),0)(q,x)$ and $x \in D^m H(0,\overline{x})(0)$, and we are done. Thus, one may suppose, for $s_n := ||y_n||$, that the sequence $v_n := y_n/s_n$ has a limit v of norm one. If $t_n/s_n \to 0$, since

$$0 + s_n v_n = y_n \in C(0 + s_n(\frac{t_n q_n}{s_n}), \overline{x} + s_n^m(\frac{t_n^m x_n}{s_n^m})),$$

one sees that $v \in D_{ps}^{m(1)}C((0,\overline{x}),0)(0,0)$, contradicting (4.5). Consequently, one may assume that s_n/t_n converges to some $\xi \in R_+$. So,

$$0 + t_n(\frac{s_n}{t_n}v_n) = y_n \in C(0 + t_n q_n, \overline{x} + t_n^m x_n)$$

and thus $\xi v \in D_{ps}^{m(1)}C((0,\overline{x}),0)(q,x)$. It follows from the definition of $D_{ps}^{m(1)}C((0,\overline{x}),0)(q,x)$ that $\xi v \in D_{ps}^{m(1)}\hat{M}((0,\overline{x}),0)(q,x)$ and $x \in D^m H(y,z)(\xi v)$. ■

LEMMA 4.2 *Let Z be finite dimensional, the asumptions of Lemma 4.1 be satisfied. and*

$$D_{ps}^{m(1)}\hat{M}((0,\overline{x}),0)(q,x) \cap (D^m H(0,\overline{x}))^{-1}(x) \subset D_{ps}^{m(1)}C((0,\overline{x}),0)(q,x). \quad (4.7)$$

Then, $x \in D^m S(0,\overline{x})(q)$ if and only if $x \in D^m H(0,\overline{x})[D_{ps}^{m(1)}\hat{M}((0,\overline{x}),0)(q,x)]$.
Proof. By Lemma 4.1, we need to prove that $x \in D^m H(0,\overline{x})[D_{ps}^{m(1)}\hat{M}((0,\overline{x}),0)(q,x)]$ implies $x \in D^m S(0,\overline{x})(q)$. $x \in D^m H(0,\overline{x})[D_{ps}^{m(1)}\hat{M}((0,\overline{x}),0)(q,x)]$ means the existence of $v \in D_{ps}^{m(1)}\hat{M}((0,\overline{x}),0)\ (q,x) \cap (D^m H(0,\overline{x}))^{-1}(x)$. Then, (4.7) ensures that $v \in D_{ps}^{m(1)}C((0,\overline{x}),0)(q,x)$. This means the existence of $t_n \to 0^+$ and $(q_n,x_n,v_n) \to (q,x,v)$ such that, for all n,

$$0 + t_n v_n \in C(0 + t_n q_n, \overline{x} + t_n^m x_n).$$

From the definition of C, we get $0+t_n v_n \in \hat{M}(0+t_n q_n, \overline{x}+t_n^m x_n)$ and $\overline{x}+t_n^m x_n \in H(0+t_n v_n)$, which imply that $\overline{x}+t_n^m x_n \in H(\hat{M}(0+t_n q_n,\overline{x}+t_n^m x_n))$. Thus, we have $\overline{x}+t_n^m x_n \in S(0+t_n q_n)$ and $x \in D^m S(0,\overline{x})(q)$. ■

THEOREM 4.3 *Impose the assumptions of Lemma 4.1. Then,*

$$D^m S(0,\overline{x})(q) \subset D^m H(0,\overline{x})[-M_p(0,\overline{x})(q)]. \quad (4.8)$$

If, additionally, (4.7) holds, then (4.8) becomes an equality.
Proof. By Lemmas 4.1 and 4.2, we need to prove that $D_{ps}^{m(1)}\hat{M}((0,\overline{x}),0)(q,x) = -M_p(0,\overline{x})(q)$. Let $v \in D_{ps}^{m(1)}\hat{M}((0,\overline{x}),0)(q,x)$. There exist $t_n \to 0^+$ and $(q_n,x_n,v_n) \to (q,x,v)$ such that, for all n,

$$0+t_n v_n = \hat{M}(0+t_n q_n,\overline{x}+t_n^m x_n) = -M_p(0,\overline{x})(0+t_n q_n) - r(0+t_n q_n,\overline{x}+t_n^m x_n).$$

Therefore,

$$v_n = -M_p(0,\overline{x})(q_n) - t_n^{-1} r(t_n q_n,\overline{x}+t_n^m x_n) \to -M_p(0,\overline{x})(q).$$

Thus, $v = -M_p(0,\overline{x})(q)$ and we are done. ■

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References

ANH, N.L.H., KHANH, P.Q. and TUNG, L.T. (2011) Higher-order radial derivatives and optimality conditions in nonsmooth vector optimization. *Nonlinear Anal. TMA*, **74**(18), 7365-7379.

AUBIN, J.-P. (1981) Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions. In : Nachbin, L. (ed.) *Advances in Mathematics, Supplementary studies*. Acad. Press, **7A**, 160-232.

BEDNARCZUK, E.M. and SONG, W. (1998) Contingent epiderivative and its applications to set-valued optimization. *Control Cybern.*, **27** (3), 375-386.

DIEM, H.T.H., KHANH, P.Q. and TUNG, L.T. (2013) On higher-order sensitivity analysis in nonsmooth vector optimization. *J. Optim. Theory Appl.*, DOI 10.1007/s10957-013-0424-3, OnlineFirst.

JAHN, J. and RAUH, R (1997) Contingent Epiderivatives and Set-Valued Optimization. *Math. Meth. Oper. Res.*, **46** (2), 193-211.

JIMÉNEZ, B. (2003) Strict minimality conditions in nondifferentiable multiobjective programming. *J. Optim. Theory Appl.*, **116** (1), 99-116.

JIMÉNEZ, B. and NOVO, V. (2008) Higher-order optimality conditions for strict local minima. *Ann. Oper. Res.*, **157** (1), 183-192.

LI, S.J., SUN, X.K. and ZHU, S.K. (2012) Higher-order optimality conditions for strict minimality in set-valued optimization. *J. Nonlinear Convex Anal.*, **13** (2), 281-291.

LUU, D.V. (2008) Higher-order necessary and sufficient conditions for strict local Pareto minima in terms of Studniarski's derivatives. *Optimization*, **57** (4), 593-605.

PENOT, J.-P. (1983) Compact nets, filters and relation. *J. Math. Anal. Appl.*, **93** (2), 400-417.

STUDNIARSKI, M. (1986) Necessary and sufficient conditions for isolated local minima of nonsmooth functions. *SIAM J. Control Optim.*, **24** (5), 1044–1049.

SUN, X.K. and LI, S.J. (2011) Lower Studniarski derivative of the perturbation map in parametrized vector optimization. *Optim. Lett.*, **5** (4), 601-614.

SUN, X.K. and LI, S.J. (2012) Weak lower Studniarski derivative in set-valued optimization. *Pacific J. Optim.*, **8** (2), 307-320.

TAA, A. (1998) Set-valued derivatives of multifunctions and optimality conditions. *Num. Funct. Anal. Optim.*, **19** (1), 121-140.

TANINO, T. (1988) Sensitivity analysis in multiobjective optimization. *J. Optim. Theory Appl.*, **56** (3), 479-499.