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Sequential optimization for semilinear divergent hyperbolic equation with a boundary control and state inequality constraint

by

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Abstract: An optimal control problem with a state constraint of inequality type and with dynamics described by a semilinear hyperbolic equation in divergence form with the non-homogeneous boundary condition of the third kind is considered. The state constraint contains a functional parameter that belongs to the class of continuous functions and occurs as an additive term. We study the properties of solutions of linear hyperbolic equations in divergence form with measures in the original data and compute the first variations of functionals on the basis of a so-called two-parameter needle variation of controls. We consider the necessary conditions for minimizing sequences in an optimal control problem with a pointwise in time state constraint of inequality type and with dynamics described by a semilinear hyperbolic equation in divergence form with the non-homogeneous boundary condition of the third kind. For the parametric optimization problem, we also consider regularity and normality conditions stipulated by the differential properties of its value function.

Keywords: sequential optimization, maximum principle for minimizing sequences, semilinear hyperbolic partial differential equations, pointwise in time state constraints, boundary control, value function, sensitivity, normality, regularity, Radon measures, two-parameter needle variation of controls

1. Introduction

The present paper deals with an extention to the theory of Pontryagin maximum principle to parametric (i.e., parameter-dependent) problems of sequential optimization for semilinear divergent hyperbolic equations with boundary controls and with state constraints. The words "sequential optimization" mean here that we use the concept of a sequence of admissible elements as a main concept of an optimization theory, instead of a classical concept of an optimal element. In other words, we use a sequential language of minimizing sequences instead of a classical language of optimal elements.

The continuing interest in optimal control problems for distributed systems with pointwise state constraints (PSC), which has lasted for more than four decades: see, e.g., Novozhenov and Plotnikov (1982), Mackenroth (1982, 1986), Bergounioux (1992), Casas (1993, 1997), Li and Yong (1995), Bonnans and Casas (1995), Raymond and Zidani (1998), Casas, Raymond, and Zidani (2000), Mordukhovich and Raymond (2004, 2005). But the majority of publications on optimal control problems with PSC are devoted to finding necessary optimality conditions, in particular, the Pontryagin's maximum principle. Other classical optimization problems related to the specified class of systems have received little attention in the literature. Among these are issues related to sequential optimization problems (suboptimality conditions), regularity, normality, differential properties of the value function, stability of values of problems (sensitivity), etc. Similar issues for parametric optimization problems with PSC were earlier considered by Sumin (2000a, 2001) in the case of elliptic equations, by Gavrilov and Sumin (2004, 2005) in the case of nonlinear hyperbolic Goursat-Darboux systems, and by Gavrilov and Sumin (2011a, b, c) in the case of divergent hyperbolic equations.

Foremost, it should be said, that, as in Gavrilov and Sumin (2011a, b, c) and Mordukhovich and Raymond (2004, 2005), in the present paper we consider an optimal control problem with pointwise in time state constraints. This is because the regularity properties of solutions of hyperbolic divergent equations are much weaker than in the case of elliptic and parabolic equations, for details, see Sumin (2009). In contrast to Gavrilov and Sumin (2011a, b), here we consider an optimal control problem where a controlled hyperbolic equation is semilinear, and where the corresponding initial-boundary value problems contain a boundary control. In distinction to Gavrilov and Sumin (2011c), in the present paper we find solutions to the initial-boundary value problems in the class of functions z which are such that: 1) for any fixed t a function z belongs to a Sobolev space, with respect to spatial variables; 2) for almost every t a function z_t is summable with square, with respect to spatial variables; 3) a function $[0,T] \ni t \mapsto z(\cdot,t)$ is continuous with respect to t, in the sense of the weak topology of the Sobolev space; 4) the inclusion $z_t \in L_\infty([0,T],L_2(\Omega))$ holds. In this paper, we denote this functional class as $\mathfrak{C}_2^1(Q_T)$. For linear hyperbolic equation, such solutions were first considered in Chapter 3, section 8.4 of Lions and Magenes (1968). In contrast to solutions belonging to the Sobolev space $W_2^1(Q_T)$, a consideration of solutions belonging to $\mathfrak{C}_2^1(Q_T)$ allowed for imposing much weaker conditions than in Gavrilov and Sumin (2011c). More precisely, in Gavrilov and Sumin (2011c) we addressed an optimization problem with linear order of growth of all source data with respect to a state variable z. In the present paper, we use an essentially narrower class of solutions (namely, the class $\mathfrak{C}_2^1(Q_T)$). The use of this class allows for considering the source data growing super linearly, with exponents that are near the bounding exponents. The bounding exponents are obtained from embedding theorems for Sobolev spaces.

Let us note that we do not know any works of other authors, where these questions of theory are considered for controlled nonlinear (semilinear) divergent hyperbolic equations. Difficulties in the study of optimization problems for controlled divergent hyperbolic equations are inherent to the problem class and caused by the lack of regularity properties of hyperbolic equations solutions in comparison to solutions of parabolic and elliptic equations. These regularity properties are necessary for the earlier methods of research of such problems.

Let us emphasize that we use the sequential language in studying the optimization problem of the present paper. This use is an essential feature of the paper and is associated with the following important circumstances: 1) obtaining of classical optimality conditions is connected with very hard assumptions on the source data of problems[1], and in case of existence of an optimal control under general assumptions, classical optimality conditions are the limit case of "optimality" conditions in the sequential form, see Sumin (2000a, 2001), Gavrilov and Sumin (2004, 2005, 2011a, b, c); 2) minimizing sequences (more precisely, minimizing approximate solutions in the sense of Warga, 1972) that we use in the paper, have regularizing properties (see, e.g., Sumin and Trushina, 2008), in contrast to classical optimal elements for constrained optimization problems (these elements are instable with respect to perturbations of input data, see Sumin, 2011, 2012).

Here, as in Gavrilov and Sumin (2011a, b, c), first of all, we study issues related to the theory of linear hyperbolic divergent equations with a Radon measure in the right–hand side part. Such equations appear (in the form of adjoint equations of a maximum principle) in the proof of the Pontryagin maximum principle (or the generalizations of it) for optimal control problems with pointwise state constraints. We study the following questions: existence, uniqueness and stability of solutions to such equations with boundary condition of the third kind; special integral representations of solutions to such equations; and the stability of solutions of linear hyperbolic divergent equations with respect to initial conditions on a hyperplane $t = \tau$ (but not top or bottom of the cylinder $Q_T \equiv \Omega \times (0, T)$), and with respect to its position $t = \tau$. As mentioned above, we consider an essentially narrower class of solutions than in Gavrilov and Sumin (2011c). Namely, we do not consider solutions from just a Sobolev space, but we consider solutions from class $\mathfrak{E}_2^1(Q_T)$. Hence, we investigate in the sense of the class the stability of solutions with respect to the position of a section of cylinder Q_T. Let us note that we do not know any analogous results concerning third boundary–value problems for linear hyperbolic divergent equations involving the Radon measures in the right–hand side part.

The problem of calculating first variations of functionals has received much attention in literature. In the present paper, like in Gavrilov and Sumin (2011a), to solve this problem efficiently under natural conditions on the input data of the optimization problem with PSC, we use the so-called two-parameter (many-point) needle variation of controls, Sumin (1983, 1991, 2009, 2000b). Such

[1] First of all this refers to optimization problems for partial differential equations.

a modification for optimization problems related to hyperbolic systems with generalized solutions in Sobolev spaces is justified for the following reasons.

Firstly, it is motivated by the "instability" of the classical Lebesgue points (see, e.g., Stane, 1970) of functions in the topology of the Sobolev classes they belong to.

This "instability" means the following. Let $S^n_\varepsilon(\bar{x}) \equiv \{x \in R^n : |x-\bar{x}| < \varepsilon\}$, and let $\eta(x)$, $x \in D \subset R^n$, be a function that is summable over a domain D. If $\bar{x} \in D$ is a Lebesgue point of η, then, as is well known,

$$\lim_{\varepsilon\to 0} \frac{1}{\operatorname{meas} S^n_\varepsilon(\bar{x})} \int\limits_{S^n_\varepsilon(\bar{x})} \eta(x)dx = \eta(\bar{x}).$$

And if $\|\eta_\varepsilon - \eta\|_{\infty,D} \to 0$, $\varepsilon \to 0$, where η_ε, $\varepsilon > 0$, is a family of functions that are summable over D, then the limit relation

$$\lim_{\varepsilon\to 0} \frac{1}{\operatorname{meas} S^n_\varepsilon(\bar{x})} \int_{S^n_\varepsilon(\bar{x})} \eta_\varepsilon(x)dx = \eta(\bar{x}) \tag{1}$$

holds. But the relation (1) may not hold if the convergence $\eta_\varepsilon \to \eta$ as $\varepsilon \to 0$ is not uniform, and, for example, is a convergence in some Sobolev space and if η_ε, η belong the space. Namely, let $\eta(x) \equiv 0$ and $\bar{x} \in R^n$ be an arbitrary point. Obviously, this $\bar{x}$ is a Lebesgue point of η. Let us put

$$\eta_\varepsilon(x;\bar{x}) \equiv \begin{cases} \exp\left[\frac{|x-\bar{x}|^2}{|x-\bar{x}|^2-\varepsilon^2}\right], & |x-\bar{x}|^2 < \varepsilon^2, \\ 0, \ |x-\bar{x}|^2 \geqslant \varepsilon^2. \end{cases}$$

It is not difficult to prove that

$$\lim_{\varepsilon\to 0} \|\eta_\varepsilon(\cdot;\bar{x})\|_{p,\Omega} = 0, \ \text{ if } p \in [1,\infty),\ n \geqslant 1; \ \lim_{\varepsilon\to 0} \|\eta_\varepsilon(\cdot;\bar{x})\|^{(1)}_{p,\Omega} = 0, \ \text{ if } p < n;$$

$$\lim_{\varepsilon\to 0} \frac{1}{\operatorname{meas} S^n_\varepsilon(\bar{x})} \int\limits_{S^n_\varepsilon(\bar{x})} \eta_\varepsilon(x;\bar{x})dx = n \int\limits_0^1 r^{n-1} e^{\frac{r^2}{r^2-1}} dr \neq 0.$$

The last limit equality means that for the point $\bar{x}$ and for the selected family $\eta_\varepsilon(\cdot;\bar{x})$ we have $\eta_\varepsilon(\cdot;\bar{x}) \to \eta$, $\varepsilon \to 0$, in the norm of W^1_p, but the limit relation (1) is not true.

Secondly, the classical needle variation approach may fail for divergent hyperbolic equations due to the fact that the regularity properties of solutions are much weaker than in the case of elliptic and parabolic equations. For details, see Sumin (2009).

The analysis of sequential optimization problems with PSC is based on a method of Sumin (1986) (as in Gavrilov and Sumin, 2011a, b, c) and can be divided into three main stages.

1. Approximating the original problem with PSC by problems each of which is "equivalent" to a problem with finitely many function constraints. The original problem with PSC is treated as a problem with infinitely many inequality-type functional constraints. This approximation permits one to use the advantages of nonsmooth finite-dimensional analysis over its infinite-dimensional counterpart. In particular, one essential advantage from the viewpoint of obtaining results on sensitivity in the present paper is that if the subdifferential (in the sense of Clarke, 1983, or Mordukhovich, 2006a) of a lower semicontinuous function of n variables is bounded at some point, then the function has the Lipschitz property in a neighborhood of that point (see Proposition 2.9.7 of Clarke, 1983, Mordukhovich, 2006a, Corollary 8.5 of Mordukhovich and Shao, 1996)[2].

2. Obtaining an "approximate" maximum principle in each approximating optimization problem on the basis of a two-parameter needle variation, Sumin (2009), in an "ordinary" way, i.e. in analogy to the method applied in optimal control problems with finitely many inequality-type functional constraints. This approximate maximum principle is stated in terms of the adjoint functions corresponding to each constraint and satisfying the usual adjoint linear hyperbolic equations in divergence form.

3. Passing to the limit in the family of approximating maximum principles as the number of constraints tends to infinity and deriving the resulting maximum principle in the original problem with PSC. Here, the families of adjoint equations corresponding to each inequality-type constraint in the approximating problem are glued together to form a single resulting adjoint equation corresponding to the original state constraint and containing the corresponding Radon measure in the right-hand side.

2. Problem statement

We begin with some notation. Suppose $U \subset R^m$ is a compact set, $V \subset R$ is a segment, $T > 0$ is a constant, $\Omega \subset R^n$ $(n > 1)$ is a bounded domain having a sectionally smooth boundary S, $S_T \equiv S \times (0,T)$, $Q_T \equiv \Omega \times (0,T)$, $\mathcal{D} \equiv \{\pi \equiv (u,v,w) \,:\, \pi \in \mathcal{D}_1 \times \mathcal{D}_2 \times \mathcal{D}_3\}$, $\mathcal{D}_1 \equiv \{u \in L^m_\infty(Q_T) \,:\, u(x,t) \in U$ for a.e. $(x,t) \in Q_T\}$, $\mathcal{D}_2 \equiv \{v \in L_\infty(\Omega) : v(x) \in V$ for a.e. $x \in \Omega\}$, $\mathcal{D}_3 \equiv \{w \in W^{0,1}_{2,1}(S_T) \,:\, w \in \mathbf{W}\}$, where $\mathbf{W}$ is a convex closed bounded subset of $W^{0,1}_{2,1}(S_T)$.

Here and in what follows the following notation is used: $\|\varphi\|_{p,\Omega}$ is a norm in the space $L_p(\Omega)$ of functions $\varphi\colon \Omega \to R$ summable to p–th power (essentially bounded for $p = \infty$); $\|\cdot\|^{(1)}_{2,\Omega}$ is a norm in the space $W^1_2(\Omega)$; $|\cdot|^{(0)}_X$ is the standard norm in the space $C(X)$ of continuous functions $\varphi\colon X \to R$ on a compact set X; $M(X)$ is the set of all Radon measures on a compact

[2]This is not necessarily true in the infinite-dimensional case. But for the reasonably wide class of spaces (so-called Asplund spaces) this is true (see Theorem 3.52 in Mordukhovich, 2006a, Lemma 8.5 in Mordukhovich and Shao, 1996).

set X, $\|\mu\|$ is the total variation of a measure $\mu \in M(X)$; $L_{2,1}(Q_T)$ is a Banach space of all Lebesgue measurable functions $\varphi\colon Q_T \to R$ such that the norm $\|\varphi\|_{2,1,Q_T} \equiv \int\limits_0^T (\int\limits_\Omega |\varphi(x,t)|^2 dx)^{1/2} dt$ is finite; $L_{2,1}(S_T)$ is a Banach space of all Lebesgue measurable functions $\varphi\colon S_T \to R$ such that the norm $\|\varphi\|_{2,1,S_T} \equiv \int\limits_0^T (\int\limits_S |\varphi(s,t)|^2 ds)^{1/2} dt$ is finite. By $W_{2,1}^{0,1}(S_T)$ we denote the set of all functions $\varphi \in L_{2,1}(S_T)$ such that $\varphi_t \in L_{2,1}(S_T)$. The norm in the space $W_{2,1}^{0,1}(S_T)$ is defined by $\|\varphi\|_{2,1,S_T}^{(0,1)} \equiv \|\varphi\|_{2,1,S_T} + \|\varphi_t\|_{2,1,S_T}$. By $C^r([0,T],Y)$, where Y is a infinite–dimensional Banach space, we denote the space of r times strongly continuously differentiable functions $\varphi : [0,T] \to Y$ for $r > 0$, and the space of strongly continuous functions $\varphi : [0,T] \to Y$ for $r = 0$. A norm in the space $C^r([0,T],Y)$ is defined by $|z|_Y^{(r)} \equiv \sum\limits_{i=0}^{r} \max\limits_{t\in[0,\,T]} \|z_{t^{(i)}}(t)\|_Y$. Let us put $C([0,T],Y) \equiv C^0([0,T],Y)$. By $C_s([0,T],Y)$, where Y is a infinite–dimensional Banach space, we denote the space of weak continuous functions $\varphi : [0,T] \to Y$, i.e. $\lim\limits_{t\to\tau} \langle \varphi(t), y^* \rangle = \langle \varphi(\tau), y^* \rangle$ for all $\tau \in [0,T]$, $y^* \in Y^*$. The norm in the space $C_s([0,T],Y)$ is defined by $\|\varphi\|_{C_s([0,T],Y)} \equiv \sup\limits_{t\in[0,\,T]} \|\varphi(t)\|_Y$. Finally, by $\mathfrak{E}_2^1(Q_T)$ denote the space of functions $z : Q_T \to R$ such that $z \in C_s([0,T], W_2^1(\Omega))$, $z_t \in L_\infty([0,T], L_2(\Omega))$. A norm in the space $\mathfrak{E}_2^1(Q_T)$ is defined by $\|z\|_{\mathfrak{E}_2^1(Q_T)} \equiv \sup\limits_{t\in[0,\,T]} \|z(\cdot,t)\|_{2,\Omega}^{(1)} + \operatorname{vraisup}\limits_{t\in[0,\,T]} \|z_t(\cdot,t)\|_{2,\Omega}$.

Consider the following parametric optimization problem:

$$I_0(\pi) \to \inf,\ \ \pi \in \mathcal{D},\ \ I_1(\pi) \in \mathcal{M} + q,\ \ q \in C(X) \text{ is a parameter}, \qquad (P_q)$$

where $\mathcal{M}$ is the set of all continuous nonpositive functions on the compact set $X \subseteq [0,\,T]$, the functional $I_0\colon \mathcal{D} \to R$ and the operator $I_1\colon \mathcal{D} \to C(X)$ are defined by

$$I_0(\pi) \equiv \int\limits_\Omega G(x, z[\pi](x,T), v(x)) dx,\ \ I_1(\pi)(\tau) \equiv \int\limits_\Omega \Phi(x, \tau, z[\pi](x,\tau), v(x)) dx,$$

$$\tau \in [0,\,T],$$

$z[\pi] \in \mathfrak{E}_2^1(Q_T)$ is a unique generalized solution (see Gavrilov, 2012) to the initial–boundary value problem

$$z_{tt} - \frac{\partial}{\partial x_i}(a_{ij} z_{x_j} + a_i z) + a(x,t,z,u) + b_i z_{x_i} = 0,\ \ (x,t) \in Q_T, \qquad (2)$$

$$z|_{t=0} = \varphi(x),\ \ z_t|_{t=0} = v(x),\ \ x \in \Omega;\ \ \frac{\partial z}{\partial \mathcal{N}} + \sigma(s,t) z = w(s,t),\ \ (s,t) \in S_T;$$

corresponding to a triple $\pi \equiv (u,v,w) \in \mathcal{D}$. Here $\frac{\partial z}{\partial \mathcal{N}} \equiv (a_{ij} z_{x_j} + a_i z) \cos\alpha_i(x,t)$, and $\alpha_i(x,t)$ is an angle between the outward normals to S_T and Ox_i–axis.

Assume that

a) the functions a_{ij}, a_{ijt}, a_i, a_{it}, b_i, b_{it}, $i,\, j = \overline{1,\, n}$, are Lebesgue measurable on Q_T;
b) the functions σ and σ_t are Lebesgue measurable on S_T;
c) the function $a\colon Q_T \times R \times U \to R$, together with $\nabla_z a$, is measurable in the Lebesgue sense with respect to (x, t, z, u) and continuous with respect to (z, u) for a.e. $(x, t) \in Q_T$;
d) the function $G\colon \Omega \times R \times V \to R$, together with $\nabla_z G$, $\nabla_v G$, is measurable in the Lebesgue sense with respect to (x, z, v) and continuous with respect to (z, v) for a.e. $x \in \Omega$;
e) the function $\Phi\colon \Omega \times [0, T] \times R \times R \to R$, together with gradients $\nabla_z \Phi$, $\nabla_v \Phi$, is measurable in the Lebesgue sense with respect to $(x, t, z, v) \in \Omega \times [0,\, T] \times R \times V$ and continuous with respect to $(t, z, v) \in [0,\, T] \times R \times V$ for a.e. $x \in \Omega$;
f) the functions $a\colon Q_T \times R \times U \to R$ and $\nabla_z a\colon Q_T \times R \times U \to R$ are Lebesgue measurable with respect to $(x, t, z, u) \in Q_T \times R \times U$ and continuous with respect to $(z, u) \in R \times U$ for a.e. $(x, t) \in Q_T$; there exists a function $K_0 \in L_1[0, T]$ such that

$$|\nabla_z a(x, t, z, u)| \le K_0(t) \;\; \forall\, (x, t, z, u) \in Q_T \times R \times U;$$

moreover, there exists $K_1 \in L_{2,1}(Q_T)$ such that

$$|a(x, t, 0, u)| \le K_1(x, t) \;\; \forall\, u \in U \text{ for a.e. } (x, t) \in Q_T;$$

g) the following conditions and estimates are fulfilled:

$$\begin{aligned}
&a_{ij} = a_{ji},\;\; \varphi \in W_2^1(\Omega),\;\; \nu_1|\xi|^2 \le a_{ij}(x, t)\xi_j\xi_i \le \nu_2|\xi|^2\\
&\forall\, (x, t) \in Q_T,\;\; \xi \in R^n \;\; (\nu_1,\;\; \nu_2 > 0);\\
&\|a_{ij}\|_{\infty,Q_T} + \|a_{ijt}\|_{\infty,Q_T} + \|a_i\|_{\infty,Q_T} + \|a_{it}\|_{\infty,Q_T} +\\
&+\|b_i\|_{\infty,Q_T} + \|b_{it}\|_{\infty,Q_T} + \|\sigma\|_{\infty,S_T} + \|\sigma_t\|_{\infty,S_T} \le \nu_3,\;\; i,\, j = \overline{1,\, n};
\end{aligned}$$

h) the following condition is fulfilled:

$$\begin{aligned}
&|G(x, z, v)| + |\nabla_v G(x, z, v)| \le K_2[1 + |z|^{\gamma_1}],\\
&|\nabla_z G(x, z, v)| \le K_2[1 + |z|^{\gamma_2}] \;\; \forall\, (x, z, v) \in \Omega \times R \times V;\\
&|\Phi(x, \tau, z, v)| + |\nabla_v \Phi(x, \tau, z, v)| \le K_2[1 + |z|^{\gamma_1}],\\
&|\nabla_z \Phi(x, \tau, z, v)| \le K_2[1 + |z|^{\gamma_2}] \;\; \forall\, (x, \tau, z, v) \in \Omega \times [0, T] \times R \times V;\\
&|\nabla_z G(x, z_1, v_1) - \nabla_z G(x, z_2, v_2)| \le K_2|z_1 - z_2|^{\gamma_2} + K_3(|v_1 - v_2|),\\
&|\nabla_v G(x, z_1, v_1) - \nabla_v G(x, z_2, v_2)| \le K_2|z_1 - z_2|^{\gamma_1} + K_3(|v_1 - v_2|)\\
&\forall\, (x, z_i, v_i) \in \Omega \times R \times V,\;\; i = 1,\, 2;\\
&|\nabla_z \Phi(x, \tau, z', v') - \nabla_z \Phi(x, \tau, z'', v'')| \le K_2|z' - z''|^{\gamma_2} + K_3(|v' - v''|),\\
&\\
&|\nabla_v \Phi(x, \tau, z', v') - \nabla_v \Phi(x, \tau, z'', v'')| \le K_2|z' - z''|^{\gamma_1} + K_3(|v' - v''|)\\
&\forall\, (x, \tau, z', v'),\;\; (x, \tau, z'', v'') \in \Omega \times [0, T] \times R \times V;
\end{aligned}$$

where $\gamma_1,\ \gamma_2 \in [1,+\infty)$, if $n=2$; $\gamma_1 \in \left[1, \frac{2n}{n-2}\right)$, $\gamma_2 \in \left[1, \frac{n}{n-2}\right)$, if $n>2$; and $K_3\colon [0, \operatorname{meas} V] \to [0,+\infty)$ is a nonnegative nondecreasing function, such that $\lim\limits_{\xi\to+0} K_3(\xi) = K_3(0) = 0$;

i) the following estimates are fulfilled:

$$|\Phi(x,t_1,z,v) - \Phi(x,t_2,z,v)| + |\nabla_z\Phi(x,t_1,z,v) - \nabla_z\Phi(x,t_2,z,v)| +$$
$$+|\nabla_v\Phi(x,t_1,z,v) - \nabla_v\Phi(x,t_2,z,v)| \le K_4(|t_1-t_2|)$$
$$\forall\,(x,t_i,z,v) \in \Omega\times[0,T]\times R\times V,\ \ i=1,2;$$

where a function $K_4\colon [0,T]\to[0,+\infty)$ is such that $\lim\limits_{\tau\to+0} K_4(\tau) = K_4(0) = 0$;

j) there exists a function $K_5 \in L_1[0,T]$ such that

$$|\nabla_z a(x,t,z_1,u) - \nabla_z a(x,t,z_2,u)| \le K_5(t)|z_1-z_2|^{\gamma_2}$$
$$\forall\,(x,t,z_i,u)\in Q_T\times R\times U,\ \ i=1,\,2.$$

By definition, put $\mathcal{D}_q^\varepsilon \equiv \{\pi\in\mathcal{D} : I_1(\pi)(\tau) - q(\tau) \le \varepsilon, \tau\in X\}, \varepsilon\geqslant 0$, $\beta(q) \equiv \beta_{+0}(q) \equiv \lim\limits_{\varepsilon\to+0}\beta_\varepsilon(q)$, where $\beta_\varepsilon(q) \equiv \{\inf\limits_{\pi\in\mathcal{D}_q^\varepsilon} I_0(\pi)$, if $\mathcal{D}_q^\varepsilon \neq \emptyset; +\infty$, if $\mathcal{D}_q^\varepsilon = \emptyset\}, \varepsilon\geqslant 0$. The function $\beta\colon C(X)\to R\cup\{+\infty\}$ is called the value function of the problem (P_q). It is obvious that $\beta(q)\le\beta_0(q)\ \forall\, q\in C(X)$, where $\beta_0\colon C(X)\to R$ is a classic value function. Suppose that $\beta(q) < +\infty$. According to Warga (1972), a minimizing approximate solution (m.a.s.) in the problem (P_q) is a sequence of triples $\pi^i\in\mathcal{D}$, $i=1,2,\dots$, such that

$$I_0(\pi^i) \le \beta(q) + \delta^i,\ \ \pi^i\in\mathcal{D}_q^{\varepsilon^i},\ \ i=1,2,\dots, \tag{3}$$

where δ^i, ε^i, $i=1,2,\dots$, δ^i, $\varepsilon^i\to 0$, $i\to\infty$, are sequences of nonnegative numbers.

3. Preliminary results

3.1. Main equation

3.1.1. Uniqueness and existence of a solution to a divergent hyperbolic equation

We need results concerning the third initial–boundary value problem for a semilinear hyperbolic partial differential equation.

Consider the following third initial–boundary value problem:

$$z_{tt} - \frac{\partial}{\partial x_i}(a_{ij}z_{x_j} + a_i z) + a(x,t,z) + b_i z_{x_i} = 0,\ \ (x,t)\in Q_T, \tag{4}$$

$$z|_{t=0} = \varphi(x),\ \ z_t|_{t=0} = \psi(x),\ \ x\in\Omega;\ \ \frac{\partial z}{\partial\mathcal{N}} + \sigma(s,t)z = f(s,t),\ \ (s,t)\in S_T;$$

where coefficients a_{ij}, a_i, b_i, φ, ψ, σ, w are such that

$$\begin{aligned}
&a_{ij} = a_{ji},\ \ \varphi \in W_2^1(\Omega),\ \ \psi \in L_2(\Omega),\ \ f \in W_{2,1}^{0,1}(S_T); \\
&\nu_1|\xi|^2 \le a_{ij}(x,t)\xi_j\xi_i \le \nu_2|\xi|^2\ \forall\,(x,t) \in Q_T,\ \ \xi \in R^n\ \ (\nu_1,\ \ \nu_2 > 0); \\
&\|a_{ij}\|_{\infty,Q_T} + \|a_{ijt}\|_{\infty,Q_T} + \|a_i\|_{\infty,Q_T} + \|a_{it}\|_{\infty,Q_T} + \\
&+\|b_i\|_{\infty,Q_T} + \|b_{it}\|_{\infty,Q_T} + \|\sigma\|_{\infty,S_T} + \|\sigma_t\|_{\infty,S_T} \le \nu_3,\ \ i,\,j = \overline{1,\,n};
\end{aligned} \tag{5}$$

and the function $a\colon Q_T \times R \to R$ is measurable in the Lebesgue sense with respect to (x,t,z), and there exist functions $\mathbf{K}_0 \in L_1[0,T]$ and $\mathbf{K}_1 \in L_{2,1}(Q_T)$ such that

$$\begin{aligned}
&|a(x,t,z_1) - a(x,t,z_2)| \le \mathbf{K}_0(t)|z_1 - z_2|\ \ \forall\,(x,t,z_i) \in Q_T \times R,\ \ i = 1,2; \\
&|a(x,t,0)| \le \mathbf{K}_1(x,t)\ \ \forall\,(x,t) \in Q_T.
\end{aligned} \tag{6}$$

Definition 1 *(Gavrilov, 2012) A function $z \in \mathfrak{E}_2^1(Q_T)$ is said to be a solution to the initial–boundary value problem (4), if z satisfies the following integral identity:*

$$\begin{aligned}
&\int\limits_{Q_T} [-z_t\eta_t + a_{ij}z_{x_j}\eta_{x_i} + a_i z\eta_{x_i} + a(x,t,z)\eta + b_i z_{x_i}\eta]dxdt + \int\limits_{S_T} \sigma z\eta dsdt = \\
&= \int\limits_{S_T} f\eta dsdt + \int\limits_{\Omega} \psi(x)\eta(x,0)dx\ \ \forall\,\eta \in \hat{\mathfrak{E}}_2^1(Q_T);\ \ z(x,0) = \varphi(x),\ \ x \in \Omega.
\end{aligned} \tag{7}$$

Here $\hat{\mathfrak{E}}_2^1(Q_T) \equiv \{\eta \in \mathfrak{E}_2^1(Q_T) : \eta(\cdot,T) = 0\}$.

Theorem 1 *(Gavrilov, 2012) The problem (4) has a unique solution $z \in \mathfrak{E}_2^1(Q_T)$, and there exists a constant $B > 0$ such that*

$$\|z\|_{\mathfrak{E}_2^1(Q_T)} \le B[\|\varphi\|_{2,\Omega}^{(1)} + \|\psi\|_{2,\Omega} + \|f\|_{2,1,S_T}^{(1,0)} + \|a(\cdot,\cdot,0)\|_{2,1,Q_T}]. \tag{8}$$

The constant B depends on the dimension n, numbers T, ν_1, ν_2, $\nu_3 > 0$, a function $\mathbf{K}_0 \in L_1[0,T]$, and a domain Ω.

Proof. The proof consists of three steps.

Step 1. Let us prove that a solution is unique. Indeed, let z_1, $z_2 \in \mathfrak{E}_2^1(Q_T)$, and let $w \equiv z_1 - z_2$. Then $w \in \mathfrak{E}_2^1(Q_T)$ and w satisfies the identity

$$\begin{aligned}
&\int\limits_{Q_T} [-w_t\eta_t + a_{ij}w_{x_j}\eta_{x_i} + a_i w\eta_{x_i} + [a(x,t,z_2+w) - a(x,t,z_2)]\eta + \\
&+b_i w_{x_i}\eta]dxdt + \int\limits_{S_T} \sigma w\eta dsdt = 0\ \ \forall\,\eta \in \hat{\mathfrak{E}}_2^1(Q_T);\ \ w(x,0) = 0,\ \ x \in \Omega.
\end{aligned} \tag{9}$$

Let us introduce functions $\eta^\alpha : Q_T \to R$, $\beta_i : Q_T \to R$, $i = \overline{1,n}$ (here $\alpha \in [0,T]$ is a parameter), by relations

$$\eta^\alpha(x,t) = -\chi_{[0,\alpha]}(t)\int\limits_t^\alpha w(x,\xi)d\xi, \;\; \beta_i(x,t) = -\int\limits_0^t w_{x_i}(x,\xi)d\xi, \;\; i = \overline{1,n},$$

where χ_E is a characteristic function of the set E.

By substituting $\eta = \eta^\alpha$ into (9), integrating by parts, and using conditions (5)–(6) (for details see Gavrilov, 2012), we obtain that for any $\varepsilon > 0$

$$\int\limits_\Omega [|\eta_t^\alpha(x,\alpha)|^2 + (\nu_1 - \gamma_1\varepsilon)|\nabla_x\eta^\alpha(x,0)|^2]dx \le$$
$$\le \int\limits_0^\alpha dt \int\limits_\Omega [\gamma_2|\nabla_x\eta^\alpha|^2 + \gamma_3(t,\varepsilon)[|\eta^\alpha|^2 + |\eta_t^\alpha|^2]]dx,$$

where γ_1, $\gamma_2 >$ are some constants, $\gamma_3(t,\varepsilon) > 0$, $t \in [0,T]$, is some function. Upon setting $\varepsilon = \frac{\nu_1}{2\gamma_1}$ in the last inequality, we get

$$\int\limits_\Omega [|\eta_t^\alpha(x,\alpha)|^2 + \frac{\nu_1}{2}|\nabla_x\eta^\alpha(x,0)|^2]dx$$
$$\le \int\limits_0^\alpha dt \int\limits_\Omega [\gamma_2|\nabla_x\eta^\alpha|^2 + \gamma_4(t)[|\eta^\alpha|^2 + |\eta_t^\alpha|^2]]dx \forall\, \alpha \in [0,T],$$

where $\gamma_4(t) \equiv \gamma_3(t, \frac{\nu_1}{2\gamma_1})$, $t \in [0,T]$. It follows from this inequality that

$$\int\limits_\Omega \left[w^2(x,\alpha) + \frac{\nu_1}{2}\sum_{i=1}^n \beta_i^2(x,\alpha)\right]dx \le \int\limits_0^\alpha dt \int\limits_\Omega \left[\gamma_2 \sum_{i=1}^n (\beta_i(x,\alpha) - \beta_i(x,t))^2 + \right.$$
$$\left. + \gamma_4(t)\left[\left|\int\limits_t^\alpha w(x,\xi)d\xi\right|^2 + w^2(x,t)\right]\right]dx \le 2\gamma_2\alpha \int\limits_\Omega \sum_{i=1}^n \beta_i^2(x,\alpha)dx +$$

$$+2\gamma_2\int_0^\alpha dt\int_\Omega \sum_{i=1}^n \beta_i^2 dx + \int_0^\alpha dt\int_\Omega \gamma_4(t)\Big[w^2(x,t) + (\alpha - t)\int_0^\alpha w^2(x,\xi)d\xi\Big]dx \le$$

$$\le 2\gamma_2\alpha\int_\Omega \sum_{i=1}^n \beta_i^2(x,\alpha)dx + 2\gamma_2\int_0^\alpha dt\int_\Omega \sum_{i=1}^n \beta_i^2 dx +$$

$$+\int_0^\alpha d\xi\int_\Omega \Big[T\int_0^\alpha \gamma_4(t)dt + \gamma_4(\xi)\Big]w^2(x,\xi)dx \le 2\gamma_2\alpha\int_\Omega \sum_{i=1}^n \beta_i^2(x,\alpha)dx +$$

$$+\int_0^\alpha d\xi\int_\Omega \Big[T\int_0^\alpha \gamma_4(t)dt + \gamma_4(\xi) + 2\gamma_2\Big]\Big[w^2(x,\xi) + \sum_{i=1}^n \beta_i^2(x,\xi)\Big]dx.$$

Hence,

$$\int_\Omega [w^2(x,\alpha) + \Big(\frac{\nu_1}{2} - 2\gamma_2\alpha\Big)\sum_{i=1}^n \beta_i^2(x,\alpha)]dx \le \int_0^\alpha dt\int_\Omega \gamma_5(t)[w^2 + \sum_{i=1}^n \beta_i^2]dx, \tag{10}$$

where $\gamma_5(t) \equiv T\int_0^T \gamma_4(\xi)d\xi + \gamma_4(t) + 2\gamma_2$.

Let $\omega_m \equiv m\theta$, $m = \overline{0,\lambda}$, where $\theta = \frac{\nu_1}{8\gamma_2}$, $\lambda = \lceil \frac{T}{\theta} \rceil$. Let us put $J_m \equiv [\omega_m, \omega_{m+1}] \cap [0,T]$, $m = \overline{0, \lambda - 1}$.

Suppose that $\alpha \in J_0$ in the inequality (10). Then $\frac{\nu_1}{2} - 2\alpha\gamma_2 \geqslant \frac{\nu_1}{4}$, whence

$$\int_\Omega [w^2(x,\alpha) + \sum_{i=1}^n \beta_i^2(x,\alpha)]dx \le \int_0^\alpha dt\int_\Omega \gamma_6(t)[w^2(x,t) + \sum_{i=1}^n \beta_i^2(x,t)]dx,$$

where $\gamma_6(t) = \gamma_5(t)/\min\{1, \frac{\nu_1}{4}\}$. By applying the Gronwall lemma, we get that

$$w(x,t) \equiv 0,\ \ \beta_i(x,t) \equiv 0,\ \ (x,t) \in \Omega \times J_0,\ \ i = \overline{1,n}.$$

Arguing in a similar fashion, we obtain in the finite number of steps that

$$w(x,t) \equiv 0,\ \ \beta_i(x,t) \equiv 0,\ \ (x,t) \in \Omega \times J_m,\ \ i = \overline{1,n},\ \ m = \overline{0, \lambda - 1}.$$

Thus, a difference of any two solutions of the initial–boundary value problem (4) is equal to zero almost everywhere in Q_T. Hence, the problem (4) can have no more than one solution.

Step 2. Let us prove the existence of the solution. Let a sequence $g_k \in W_2^1(\Omega)$, $k = 1, 2, \dots$, be orthonormal in $L_2(\Omega)$, orthogonal in $W_2^1(\Omega)$ and be such that for any functions $\bar\varphi \in W_2^1(\Omega)$, $\bar\psi \in L_2(\Omega)$

$$\lim_{m\to\infty} \|\bar\varphi^N - \bar\varphi\|_{2,\Omega}^{(1)} = 0,\ \ \lim_{m\to\infty} \|\bar\psi^N - \bar\psi\|_{2,\Omega} = 0,$$

where

$$\bar{\varphi}^N(x) \equiv \sum_{m=1}^{N} \bar{\varphi}_m g_m(x),\ \ \bar{\psi}^N(x) \equiv \sum_{m=1}^{N} \bar{\psi}_m g_m(x),$$
$$\bar{\varphi}_j \equiv \int\limits_{\Omega} \bar{\varphi}(y) g_j(y) dy,\ \ \bar{\psi}_j \equiv \int\limits_{\Omega} \bar{\psi}(y) g_j(y) dy,\ \ j,\ N = 1, 2, \ldots$$

We will find an approximate solution z^N to the problem (4) in the form

$$z^N(x,t) \equiv \sum_{m=1}^{N} h_m^N(t) g_m(x),$$

where a collection of functions $h_m^N \in W_1^2[0,T]$, $m = \overline{1,N}$, is a solution to the following Cauchy problem:

$$\ddot{h}_l^N + \sum_{m=1}^{N} [\mathfrak{p}_{lm}(t)\dot{h}_m^N + \mathfrak{q}_{lm}(t) h_m^N] + \mathfrak{r}_l^N(t, h_1^N(t), \ldots, h_N^N(t)) = 0, \tag{11}$$
$$h_l^N(0) = \varphi_l,\ \ \dot{h}_l^N(0) = \psi_l,\ \ l = \overline{1,N},$$

where

$$\mathfrak{p}_{lm}(t) \equiv \int\limits_{\Omega} c(x,t) g_l g_m dx,\ \ \mathfrak{r}_l^N(t, h_1, \ldots, h_N) \equiv \int\limits_{\Omega} a\left(x, t, \sum_{m'=1}^{N} h_{m'} g_{m'}\right) g_l dx +$$
$$+ \int\limits_{S} [\sigma(s,t) \sum_{m'=1}^{N} h_{m'} g_{m'} - f(s,t)] g_l ds,$$
$$\mathfrak{q}_{lm}(t) \equiv \int\limits_{\Omega} [a_{ij}(x,t) g_{mx_j} g_{lx_i} + a_i(x,t) g_m g_{lx_i} + b_i(x,t) g_{mx_i} g_l] dx,$$

Obviously, such collection exists and will be unique.

By multiplying l–th equation (11) by $\dot{h}_l^N(t)$, $l = \overline{1,N}$, adding all the obtained equations, and integrating the result over $t \in [0,\tau]$, we conclude that

$$\frac{1}{2} \int\limits_{\Omega} [|z_t^N(x,\tau)|^2 + a_{ij}(x,\tau) z_{x_j}^N(x,\tau) z_{x_i}^N(x,\tau)] dx -$$
$$\frac{1}{2} \int\limits_{\Omega} [|\psi^N|^2 + a_{ij}(x,0) \varphi_{x_j}^N \varphi_{x_i}^N] dx -$$

$$-\int_0^\tau dt \int_\Omega \Big[\frac{1}{2} a_{ijt} z^N_{x_j} z^N_{x_i} + (a_i - b_i) z^N_t z^N_{x_i} + a_{it} z^N z^N_{x_i} - c|z^N_t|^2 - [a(x,t,z^N) -$$

$$-a(x,t,0)] z^N_t \Big] dx + \Big[\int_\Omega a_i z^N z^N_{x_i} dx + \frac{1}{2}\int_S \sigma |z^N|^2 ds - \int_S f z^N ds\Big]\Big|_{t=0}^{t=\tau} +$$

$$+\int_0^\tau dt \int_\Omega a(x,t,0) z^N_t dx - \frac{1}{2}\int_0^\alpha dt \int_S \sigma_t |z^N|^2 ds + \int_0^\alpha dt \int_S f_t z^N ds = 0.$$

Thus,

$$\frac{1}{2}\int_\Omega [|z^N_t(x,\tau)|^2 + \nu_1 |\nabla_x z^N(x,\tau)|] dx \le \frac{1}{2}\int_\Omega [|\psi^N|^2 + \nu_2 |\nabla \varphi^N|^2] dx +$$

$$+\int_0^\tau dt \int_\Omega |a(x,t,0)||z^N_t| dx - \int_\Omega a_i z^N z^N_{x_i} dx \Big|_{t=0}^{t=\tau} + \int_0^\tau dt \int_\Omega \Big[\frac{1}{2} a_{ijt} z^N_{x_j} z^N_{x_i} +$$

$$+(a_i - b_i) z^N_t z^N_{x_i} + a_{it} z^N z^N_{x_i} - c|z^N_t|^2 + K_0(t)|z^N||z^N_t|\Big] dx + \int_0^\alpha dt \int_S |f_t||z^N| ds +$$

$$+\Big[\int_S f z^N ds - \frac{1}{2}\int_S \sigma |z^N|^2 ds\Big]\Big|_{t=0}^{t=\tau} + \frac{1}{2}\int_0^\alpha dt \int_S |\sigma_t||z^N|^2 ds.$$

By estimating the left–hand side part of this inequality we obtain that, for any $\varepsilon > 0$

$$\frac{1}{2}\int_\Omega [|z^N_t(x,\tau)|^2 + (\nu_1 - \rho_1\varepsilon)|\nabla_x z^N(x,\tau)|] dx \le \rho_2(\varepsilon)\Big[\int_\Omega [|\psi^N|^2 + |\nabla\varphi^N|^2 +$$

$$+|\varphi^N|^2] dx + [\|a(\cdot,\cdot,0)\|_{2,1,Q_T} + \|f\|^{(0,1)}_{2,1,S_T}] \max_{t\in[0,\tau]} \Big[\int_\Omega [|z^N(x,t)|^2 + |z^N_t(x,t)|^2 +$$

$$+|\nabla_x z^N(x,t)|^2] dx\Big]^{\frac{1}{2}}\Big] + \int_0^\tau dt \int_\Omega \rho_3(t,\varepsilon)[|z^N|^2 + |z^N_t|^2 + |\nabla_x z^N|^2] dx,$$

where ρ_1 and $\rho_2(\varepsilon)$ are some positive constants, and $\rho_3(t,\varepsilon)$, $t \in [0,T]$, is a nonnegative summable function.

Adding an expression $\frac{1}{2}\int_\Omega |z^N(x,\tau)|^2 dx$ to both parts of the last inequality and putting $\varepsilon \equiv \frac{\nu_1}{2\rho_3}$, yields that for any $\tau \in [0,T]$

$$y^N(\tau) \le \rho_4[y^N(0) + [\|a(\cdot,\cdot,0)\|_{2,1,Q_T} + \|f\|^{(0,1)}_{2,1,S_T}] \max_{t\in[0,\tau]} \sqrt{y^N(t)}] + \int_0^\tau \rho_5 y^N dt, \tag{12}$$

where $\rho_4 \equiv 4\rho_2(\frac{\nu_1}{2\rho_1})/\min\{2,\nu_1\}$, $\rho_5(t) \equiv 4\rho_3(t, \frac{\nu_1}{2\rho_3})/\min\{2,\nu_1\}$,

$$y^N(t) \equiv \int\limits_\Omega [|z^N(x,t)|^2 + |z_t^N(x,t)|^2 + |\nabla_x z^N(x,t)|^2]dx.$$

By applying the Gronwall lemma to (12), we obtain

$$\max_{t\in[0,T]} \sqrt{y^N(t)} \le \rho_6[\sqrt{y^N(0)} + \|a(\cdot,\cdot,0)\|_{2,1,Q_T} + \|f\|_{2,1,S_T}^{(0,1)}], \tag{13}$$

where $\rho_6 \equiv \rho_4 \exp\left[\int\limits_0^T \rho_5(t)dt\right]$.

It follows from (13) that

$$\|z^N\|_{\mathfrak{E}_2^1(Q_T)} \le 2\rho_6[\|\varphi^N\|_{2,\Omega}^{(1)} + \|\psi^N\|_{2,\Omega} + \|a(\cdot,\cdot,0)\|_{2,1,Q_T} + \|f\|_{2,1,S_T}^{(0,1)}]. \tag{14}$$

It is easy to see, that

$$\lim_{N\to\infty} \|\varphi^N - \varphi\|_{2,\Omega}^{(1)} = 0, \quad \lim_{N\to\infty} \|\psi^N - \psi\|_{2,\Omega} = 0. \tag{15}$$

Hence,

$$\lim_{N\to\infty} [\|\varphi^N\|_{2,\Omega}^{(1)} + \|\psi^N\|_{2,\Omega}] = \|\varphi\|_{2,\Omega}^{(1)} + \|\psi\|_{2,\Omega}. \tag{16}$$

It follows from (16) that there exists a constant $\rho_7 > 0$ such that

$$\|z^N\|_{\mathfrak{E}_2^1(Q_T)} \le \rho_7 \ \ \forall N = 1,2,\dots. \tag{17}$$

It is not difficult to see that there exist a subsequence z^{N_k}, $k = 1,2,\dots$, of a sequence z^N, $N = 1,2,\dots$, and a function $z \in \mathfrak{E}_2^1(Q_T)$ such that

$$\lim_{k\to\infty} \max_{t\in[0,T]} \|z^{N_k}(\cdot,t) - z(\cdot,t)\|_{2,\Omega} = 0, \tag{18}$$

$$z^{N_k} \to z, \ \ k\to\infty, \ \text{weakly in } W_2^1(Q_T),$$

$$z^{N_k} \to z, \ \ *\text{–weakly in } L_\infty([0,T], W_2^1(\Omega)), \ \ k\to\infty; \tag{19}$$

$$\lim_{k\to\infty} \max_{t\in[0,T]} \|z^{N_k}(\cdot,t) - z(\cdot,t)\|_{2,S} = 0. \tag{20}$$

Similarly to the argumentation provided for linear equations in Chapter 3, section 8.3 of Lions and Magenes (1968), we obtain that z is a solution to the problem (4).

Step 3. Let us obtain an a priori estimate. Indeed, in view of (16), for any $\varepsilon > 0$ there exists a number $m_0(\varepsilon) \geqslant 1$ such that for all $m \geqslant m_0(\varepsilon)$

$$\|\varphi^{N_m}\|_{2,\Omega}^{(1)} + \|\psi^{N_m}\|_{2,\Omega} \le \|\varphi\|_{2,\Omega}^{(1)} + \|\psi\|_{2,\Omega} + \varepsilon. \tag{21}$$

Further, it follows from (14) that

$$\|z^{N_m}\|_{\mathfrak{E}_2^1(Q_T)} \le 2\rho_6[\|\varphi^{N_m}\|_{2,\Omega}^{(1)} + \|\psi^{N_m}\|_{2,\Omega} + \|a(\cdot,\cdot,0)\|_{2,1,Q_T} + \|f\|_{2,1,S_T}^{(0,1)}].$$

Limiting in this inequality with the numbers $m \geqslant m_0(\varepsilon)$, in view of (21) we will have

$$\|z^{N_m}\|_{\mathfrak{E}_2^1(Q_T)} \le 2\rho_6[\|\varphi\|_{2,\Omega}^{(1)} + \|\psi\|_{2,\Omega} + \|a(\cdot,\cdot,0)\|_{2,1,Q_T} + \|f\|_{2,1,S_T}^{(0,1)} + \varepsilon]$$
$$\forall\, m \geqslant m_0(\varepsilon).$$

It follows from relations (18), (19) that

$$\|z\|_{\mathfrak{E}_2^1(Q_T)} \le 2\rho_6[\|\varphi\|_{2,\Omega}^{(1)} + \|\psi\|_{2,\Omega} + \|a(\cdot,\cdot,0)\|_{2,1,Q_T} + \|f\|_{2,1,S_T}^{(0,1)} + \varepsilon].$$

Passing then with ε to zero, we get estimate (8) with $B = 2\rho_6$. The theorem is proved. □

3.1.2. Properties of solutions to the main equation

First of all, we need results concerning the main initial–boundary value problem (2). In order to formulate these results, let us introduce some notation. For any triples $\pi^i \equiv (u^i, v^i, w^i) \in \mathcal{D}$, $i = 1, 2$, by definition, put $\mathcal{R}(u^1, u^2) \equiv \{(x,t) \in Q_T : u^1(x,t) \neq u^2(x,t)\}$, $\mathcal{R}_t(u^1, u^2) \equiv \{x \in \Omega : (x,t) \in \mathcal{R}(u^1, u^2)\}$, $d(\pi^1, \pi^2) \equiv \|v^1 - v^2\|_{\infty,\Omega} + \operatorname{meas} \mathcal{R}(u^1, u^2) + \|w^1 - w^2\|_{2,1,S_T}^{(0,1)}$. Equip the set $\mathcal{D}$ with the metric $d(\cdot,\cdot)$. Then $\mathcal{D}$ is a complete metric space (see, e.g., Ekeland, 1974).

The following result follows from the assumptions on input data of problem (P_q) and the results of Gavrilov (2012).

Lemma 1 *For any triple $\pi \equiv (u, v, w) \in \mathcal{D}$ there exists a unique solution $z[\pi] \in \mathfrak{E}_2^1(Q_T)$ to the initial–boundary value problem (2), and there exists a constant $c_0 > 0$ such that*

$$\|z[\pi]\|_{\mathfrak{E}_2^1(Q_T)} \le c_0 \;\; \forall \pi \in \mathcal{D}. \tag{22}$$

This constant is independent of the triple $\pi \in \mathcal{D}$. Moreover, for any two triples $\pi^i \equiv (u^i, v^i, w^i) \in \mathcal{D}$, $i = 1, 2$, the following inequality is fulfilled

$$\|z[\pi^1] - z[\pi^2]\|_{\mathfrak{E}_2^1(Q_T)} \le c_1 \Big[\|v^1 - v^2\|_{\infty,\Omega} + \|w^1 - w^2\|_{2,1,S_T}^{(0,1)} + \tag{23}$$
$$+ \int_0^T K_0(t)\sqrt{\operatorname{meas} \mathcal{R}_t(u^1, u^2)}\,dt + \int_0^T \|K_1(\cdot,t)\|_{2,\mathcal{R}_t(u^1,u^2)}dt\Big],$$

where $c_1 > 0$ is a constant independent of the triples π^1, $\pi^2 \in \mathcal{D}$.

The following result follows from Lemma 1 and from the embedding theorem for the space $W_2^1(\Omega)$.

Lemma 2 *Suppose the sequences of triples $\pi^{i,k} \equiv (u^{i,k}, v^{i,k}, w^{i,k}) \in \mathcal{D}$, $k = 1, 2$, $i = 1, 2, \ldots$, are such that*

$$\lim_{i\to\infty} d(\pi^{i,1}, \pi^{i,2}) = 0;$$

then

$$\lim_{i\to\infty} \|z[\pi^{i,1}] - z[\pi^{i,2}]\|_{\mathfrak{E}_2^1(Q_T)} = 0,$$
$$\lim_{i\to\infty} \max_{t\in[0,T]} \|z[\pi^{i,1}](\cdot,t) - z[\pi^{i,2}](\cdot,t)\|_{2,S} = 0.$$

Finally, from Lemmas 1–2 the following result is concluded.

Lemma 3 *The functional $I_0 : \mathcal{D} \to R$ and the operator $I_1 : \mathcal{D} \to C(X)$ are uniformly continuous and uniformly bounded on the complete metric space $\mathcal{D}$. Moreover, the set $\{I_1(\pi) : \pi \in \mathcal{D}\}$ is a precompact set in the space $C(X)$.*

3.2. Adjoint equations

3.2.1. Equations with Radon measure in the right–hand side part: a unique existence of solutions

Consider the following initial–boundary value problem:

$$\eta_{tt} - \frac{\partial}{\partial x_i}(a_{ij}\eta_{x_j} + b_i\eta) + a_i\eta_{x_i} + a\eta = f(x,t) + g(x,t)\mu(dt), \ \ (x,t) \in Q_T, \tag{24}$$
$$\eta(x,T) = \varphi(x), \ \ \eta_t(x,T) = \psi(x), \ \ x \in \Omega, \ \ \left[\frac{\partial\eta}{\partial\mathcal{N}'} + \sigma\eta\right]\Big|_{S_T} = \omega(s,t),$$

where $\frac{\partial\eta}{\partial\mathcal{N}'} \equiv (a_{ij}z_{x_j} + b_i z)\cos\alpha_i(x,t)$, $f \in L_{2,1}(Q_T)$, $g \in C([0,T], L_2(\Omega))$, $\psi \in L_2(\Omega)$, $\varphi \in W_2^1(\Omega)$, $\omega \in W_{2,1}^{0,1}(S_T)$, $\mu \in M[0,T]$, and coefficients a_{ij}, a_i, b_i, a, σ, i, $j = \overline{1,n}$, are such that

$$a_{ij} = a_{ji}, \ \ \nu_1|\xi|^2 \le a_{ij}(x,t)\xi^i\xi^j \le \nu_2|\xi|^2 \tag{25}$$
$$\forall\,(x,t) \in Q_T, \ \ \xi \in R^n \ \ (\nu_1, \ \ \nu_2 > 0); \tag{26}$$
$$\|a_{ij}\|_{\infty,Q_T} + \|a_i\|_{\infty,Q_T} + \|b_i\|_{\infty,Q_T} + \|a\|_{\infty,1,Q_T} + \|a_{ijt}\|_{\infty,Q_T} + \|a_{it}\|_{\infty,Q_T} +$$
$$+ \|b_{it}\|_{\infty,Q_T} + \|\sigma\|_{\infty,S_T} + \|\sigma_t\|_{\infty,S_T} \le \nu_3 \ \ i,\, j = \overline{1,n}.$$

Let us give the following

Definition 2 *(Gavrilov, 2012) A function $\eta \in \mathfrak{E}_2^1(Q_T)$ is said to be a solution to the initial–boundary value problem (24), if the following integral identity is* *fulfilled:*

$$\int\limits_{Q_T} [-\eta_t z_t + a_{ij}\eta_{x_j} z_{x_i} + b_i \eta z_{x_i} + a_i \eta_{x_i} z + a\eta z]\, dxdt + \int\limits_{S_T} \sigma\eta z dsdt +$$

$$+ \int\limits_{\Omega} \psi(x) z(x,T) dx + \int\limits_{Q_T} f z dxdt + \int\limits_{[0,\,T]} \left[\int\limits_{\Omega} g(x,t) z(x,t) dx \right] \mu(dt)$$

$$\forall\, z \in \mathfrak{C}_2^1(Q_T),\ z(\cdot,0) = 0;\ \ \eta(x,T) = \varphi(x),\ \ x \in \Omega.$$

The following result holds.

Lemma 4 *(Gavrilov, 2012) Under the above–mentioned conditions, there exist a unique solution $\eta \in \mathfrak{C}_2^1(Q_T)$ to the initial–boundary value problem (24), and a constant $\bar{c}_0 > 0$ such that*

$$\|\eta\|_{\mathfrak{C}_2^1(Q_T)} \le \bar{c}_0[\|\varphi\|_{2,\Omega}^{(1)} + \|\psi\|_{2,\Omega} + \|f\|_{2,1,Q_T} + \max_{t\in[0,\,T]} \|g(\cdot,t)\|_{2,\Omega}\|\mu\| + \|\omega\|_{2,1,S_T}^{(0,1)}].$$

The constant $\bar{c}_0 > 0$ depends only on T, ν_1, ν_2, $\nu_3 > 0$, on the domain Ω and on the dimension n.

Proof. We shall give here just the scheme of the proof. Firstly, we approximate (in the $*$–weak sense) the Radon measure μ by a sequence of Radon measures μ^k, $k = 1, 2, \dots$ Each of the approximating measures is absolutely continuous with respect to the Lebesgue measure. Then, for each $k = 1, 2, \dots,$ we write out the initial–boundary value problem (24), where μ is replaced by μ^k. Using the result of Theorem 1 and taking then the limit for $k \to \infty$, we obtain the required results. For details, see Gavrilov (2012).

3.2.2. Equations with Radon measure in the right–hand side part: integral representation of the solution

In this section, we obtain a special integral representation of the solution to a homogeneous third initial–boundary value problem for a linear divergent hyperbolic partial differential equation with Radon measure in the right–hand side part. To formulate the results of this section, let us introduce the following notation.

Suppose $t_1 < t_2$, t_1, $t_2 \in [0,\,T]$. By definition, put $Q_{(t_1,t_2)} \equiv \Omega \times (t_1,t_2)$, $Q_{[t_1,t_2]} \equiv \Omega \times [t_1,t_2]$, $S_{(t_1,t_2)} \equiv S \times (t_1,t_2)$. Suppose functions $a_{i\,j}$, a_i, b_i, $i,\,j = \overline{1,\,n}$, a, σ satisfy (25). Suppose $g \in C([0,\,T], L_2(\Omega))$, $\mu \in M[0,\,T]$. By definition, put $\chi(t,\tau) \equiv \{1,\, 0 \le t \le \tau \le T;\ \ 0,\, 0 \le \tau < t \le T\}$. Consider the following initial–boundary value problem:

$$\mathfrak{f}_{tt} - \frac{\partial}{\partial x_i}(a_{i\,j}\mathfrak{f}_{x_j} + b_i\mathfrak{f}) + a_i\mathfrak{f}_{x_i} + a\mathfrak{f} = g(x,t)\mu(dt),\ \ (x,t) \in Q_T, \tag{27}$$

$$\mathfrak{f}(x,T)=\mathfrak{f}_t(x,T)=0,\;\; x\in\Omega,\;\; \frac{\partial\mathfrak{f}}{\partial\mathcal{N}'}+\sigma(s,t)\mathfrak{f}=0,\;\; (s,t)\in S_T.$$

By $\mathfrak{f}[a,g,\mu]$ denote the solution of this problem.

Let us put $\mathfrak{p}[a,g](x,t,\tau)\equiv\mathfrak{f}[a,g,\delta_\tau](x,t)$, where δ_τ is a Radon δ–measure, concentrated at the point $t=\tau$.

Let us define a function $\mathfrak{x}[a,g](x,t,\tau)$, $(x,t)\in Q_T$, $\tau\in[0,T]$, for $(x,t)\in Q_{[0,\tau]}$ as a solution to the initial–boundary value problem

$$\mathfrak{x}_{tt}-\frac{\partial}{\partial x_i}(a_{ij}\mathfrak{x}_{x_j}+b_i\mathfrak{x})+a_i\mathfrak{x}_{x_i}+a\mathfrak{x}=0,\;\; (x,t)\in Q_{[0,\tau]}, \tag{28}$$

$$\mathfrak{x}|_{t=\tau}=0,\;\; \mathfrak{x}_t|_{t=\tau}=-g(x,\tau),\;\; x\in\Omega,\;\; \frac{\partial\mathfrak{x}}{\partial\mathcal{N}'}+\sigma(s,t)\mathfrak{x}=0,\;\; (s,t)\in S_{(0,\tau)},$$

and as a solution to the problem

$$\mathfrak{x}_{tt}-\frac{\partial}{\partial x_i}(a_{ij}\mathfrak{x}_{x_j}+b_i\mathfrak{x})+a_i\mathfrak{x}_{x_i}+a\mathfrak{x}=0,\;\; (x,t)\in Q_{[\tau,T]}, \tag{29}$$

$$\mathfrak{x}|_{t=\tau}=0,\;\; \mathfrak{x}_t|_{t=\tau}=-g(x,\tau),\;\; x\in\Omega,\;\; \frac{\partial\mathfrak{x}}{\partial\mathcal{N}'}+\sigma(s,t)\mathfrak{x}=0,\;\; (s,t)\in S_{(\tau,T)},$$

for $(x,t)\in Q_{[\tau,T]}$

The function $\mathfrak{x}[a,g]$ can be interpreted as a solution to the initial–boundary value problem

$$\mathfrak{x}_{tt}-\frac{\partial}{\partial x_i}(a_{ij}\mathfrak{x}_{x_j}+b_i\mathfrak{x})+a_i\mathfrak{x}_{x_i}+a\mathfrak{x}=0,\;\; (x,t)\in Q_T, \tag{30}$$

$$\mathfrak{x}|_{t=\tau}=0,\;\; \mathfrak{x}_t|_{t=\tau}=-g(x,\tau),\;\; x\in\Omega,\;\; \frac{\partial\mathfrak{x}}{\partial\mathcal{N}'}+\sigma(s,t)\mathfrak{x}=0,\;\; (s,t)\in S_T,$$

where initial conditions are given on a section of the cylinder Q_T by a hyperplane $t=\tau$.

In connection with the selected method of investigation of the considered optimal control problem, the question arises about an integral representation of the solution $\mathfrak{f}[a,g,\mu]$ to the initial–boundary value problem (27). Let us remind here the scheme of this method. Firstly, we approximate the source problem (P_q) with pointwise state constraints by problems with finite number of functional constraints. Secondly, by applying the "standard" methods, we obtain "approximating" maximum principle in each "approximating" problem. Thirdly, we go to the limit in the family of approximating maximum principles as the number of functional inequality constraints tends to infinity. To this aim, we must "glue" the family of adjoint equations (each of the adjoint equations corresponds to some functional inequality constraint) into one adjoint equation. The "glued" adjoint equation corresponds to the source state constraint, and involves a Radon measure in the right–hand side part. To provide this "gluing", we prove a result on the representation of a solution to the problem (27) in the form of an integral of Green $\mathfrak{p}[a,g]$ function by measure μ with respect to τ.

To prove this representation result, we need to prove that the solution $\mathfrak{x}[a,g]$ of the problem (30) depends continuously on the position of the initial conditions hyperplane. Additionally, we need the connection of the Green function $\mathfrak{p}[a,g]$ of problem (27) with $\mathfrak{x}[a,g]$.

THEOREM 2 *The inclusion $\mathfrak{x}[a,g] \in C([0,\,T], \mathfrak{E}_2^1(Q_T))$ holds, i.e., for any $\tau \in [0,\,T]$ there exists a trace $\mathfrak{x}[a,g](\cdot,\cdot,\tau) \in \mathfrak{E}_2^1(Q_T)$. This trace depends continuously on $\tau \in [0,\,T]$ in the norm of $\mathfrak{E}_2^1(Q_T)$. In addition,*

$$\mathfrak{p}[a,g](x,t,\tau) \equiv \mathfrak{x}[a,g](x,t,\tau)\chi(t,\tau), \;\; (x,t) \in Q_T, \;\; \tau \in [0,\,T]; \tag{31}$$

$$\mathfrak{f}[a,g,\mu](x,t) = \int\limits_{[0,T]} \mathfrak{p}[a,g](x,t,\tau)\mu(d\tau), \;\; \textit{for a.e. } (x,t) \in Q_T, \tag{32}$$

and the following a priori estimate holds:

$$\max_{\tau\in[0,\,T]} \|\mathfrak{x}[a,g](\cdot,\cdot,\tau)\|_{\mathfrak{E}_2^1(Q_T)} \le \tilde{c} \max_{t\in[0,\,T]} \|g(\cdot,t)\|_{2,\Omega}. \tag{33}$$

Here the constant $\tilde{c} > 0$ depends only on numbers T, ν_1, ν_2, $\nu_3 > 0$, on the domain Ω, and on the dimension n.

To prove this theorem, we need two lemmas. First of them is the consequence of Theorem 6.1 from Chapter 1 of Ladyzhenskaya (1973) and Theorem 2.2 in page 157 of Osipov, Vasil'ev, and Potapov (1999).

LEMMA 5 *Suppose $\Omega \subset R^n$ is a bounded domain with the sectionally smooth boundary; then there exists an orthonormal (in the space $L_2(\Omega)$) sequence $h_k \in W_2^1(\Omega)$, $k = 1,2,\ldots$, such that for any $\varphi \in W_2^1(\Omega)$, $\psi \in L_2(\Omega)$, the following equalities are fulfilled:*

$$\lim_{N\to\infty} \|\varphi^N - \varphi\|_{2,\Omega}^{(1)} = 0, \;\; \lim_{N\to\infty} \|\psi^N - \psi\|_{2,\Omega} = 0.$$

Here $\varphi^N(x) \equiv \sum\limits_{k=1}^{N} \varphi_k h_k(x)$, $\psi^N(x) \equiv \sum\limits_{k=1}^{N} \psi_k h_k(x)$, $\varphi_k \equiv \int\limits_\Omega \varphi h_k\,dx$, $\psi_k \equiv \int\limits_\Omega \psi h_k\,dx$, k, $N = 1,2,\ldots$.

LEMMA 6 *Suppose $h_k \in W_2^1(\Omega)$, $k = 1,2,\ldots$, to be an orthonormal (in the space $L_2(\Omega)$) sequence from the previous lemma; then*

$$\lim_{N\to\infty} |g_0^N - g_0|_{L_2(\Omega)}^{(0)} = 0, \;\; \lim_{N\to\infty} |g_1^N - g_1|_{W_2^1(\Omega)}^{(r)} = 0,$$

if $g_0 \in C^r([0,\,T], L_2(\Omega))$, $g_1 \in C^r([0,\,T], W_2^1(\Omega))$ ($r \geqslant 0$ is a fixed integer), where

$$g_0^N(x,t) \equiv \sum_{k=1}^{N} g_{0k}(t)h_k(x), \;\; {g_1}^N(x,t) \equiv \sum_{k=1}^{N} g_{1k}(t)h_k(x),$$

$$g_{0k}(t) \equiv \int\limits_\Omega g_0(x,t)h_k(x)dx, \;\; g_{1k}(t) \equiv \int\limits_\Omega g_1(x,t)h_k(x)dx, \;\; k,\, N \geqslant 1, \;\; t \in [0,\,T].$$

Moreover, the set $C^2([0,T], W_2^1(\Omega))$ is dense in the space $C([0,T], L_2(\Omega))$.

Proof. The proof of the lemma's first assertion is analogous to the proof of Lemma 4.1 from Chapter IV, §4 of Osipov, Vasil'ev, and Potapov (1999). By this reason, the proof of the first assertion is omitted. The second assertion follows immediately from the first assertion and the classic Weierstrass theorem about uniform approximation of continuous functions by algebraic polynoms. Thus, Lemma 6 is proved. □

The proof of Theorem 2. The proof is done in three steps.

Step 1. Let us show that the equality (31) holds. First of all, according to results of Gavrilov (2012), the function $\mathfrak{x}[a,g]$ is uniquely determined in the cylinder Q_T, for any $\tau \in [0,\, T]$. Besides, in view of Definition 2, the following identities hold

$$\int\limits_{Q_T} [-\mathfrak{p}_t[a,g](x,t,\tau)z_t + a_{ij}\mathfrak{p}_{x_j}[a,g](x,t,\tau)z_{x_i} + b_i\mathfrak{p}[a,g](x,t,\tau)z_{x_i} + \tag{34}$$

$$+a_i\mathfrak{p}_{x_j}[g](x,t,\tau)z + a\mathfrak{p}[g](x,t,\tau)z]dxdt + \int\limits_{S_T} \sigma\mathfrak{p}[a,g](x,t,\tau)zdsdt =$$

$$\int\limits_{\Omega} g(x,\tau)z(x,\tau)dx$$

$$\forall\ z \in \mathfrak{C}_2^1(Q_T),\ z(\cdot,0) \equiv 0;\ \mathfrak{p}[a,g](x,T,\tau) \equiv 0,\ x \in \Omega,\ \tau \in [0,T].$$

In these identities, let us consider three cases.

Case A. Suppose that $\tau = 0$; then the function $\mathfrak{p}[a,g](x,t,0)$, $(x,t) \in Q_T$, is the solution to the homogeneous initial–boundary value problem

$$\mathfrak{p}_{tt} - \frac{\partial}{\partial x_i}(a_{i\,j}\mathfrak{p}_{x_j} + b_i\mathfrak{p}) + a_i\mathfrak{p}_{x_i} + a\mathfrak{p} = 0,\ \ (x,t) \in Q_T;$$

$$\mathfrak{p}(x,T) = \mathfrak{p}_t(x,T) = 0,\ \ x \in \Omega,\ \ \frac{\partial \mathfrak{p}}{\partial \mathcal{N}'} + \sigma(s,t)\mathfrak{p} = 0,\ \ (x,t) \in S_T.$$

According to Gavrilov (2012), this problem has only trivial solution. Hence, in the case of $\tau = 0$, the equality (31) holds.

Case B. Suppose that $\tau = T$; then the function $\mathfrak{p}[a,g](\cdot,\cdot,T)$ coincides identically with the function $\mathfrak{x}[a,g](\cdot,\cdot,T)$ whence the equality (31) holds, if $\tau = T$.

Case C. Suppose that $\tau \in (0,\, T)$. In the identities (34), let z be a linear combination of the form $\sum\limits_{k=1}^{N} c_k(t)h_k(x)$, where N is some natural number, and functions c_k, $k = \overline{1,\, N}$, are piecewise differentiable on $[0,\, T]$, $c_k|_{[0,\,\tau]} \equiv 0$, $k = \overline{1,\, N}$. Here $h_k \in W_2^1(\Omega)$, $k = 1, 2, \ldots$, is the sequence from Lemma 5. Because the set of restrictions of all such z is everywhere dense in the set of all functions belonging to $W_2^1(Q_{(\tau,T)})$ and such that $z|_{t=\tau} = 0$, we conclude that

$$\int\limits_{Q_{(\tau,\,T)}} [-\mathfrak{p}_t[a,g](x,t,\tau)z_t + a_{ij}\mathfrak{p}_{x_j}[a,g](x,t,\tau)z_{x_i} + b_i\mathfrak{p}[a,g](x,t,\tau)z_{x_i} +$$

$$+a_i\mathfrak{p}_{x_j}[a,g](x,t,\tau)z + a\mathfrak{p}[a,g](x,t,\tau)z]\,dxdt + \int\limits_{S_{(\tau,\,T)}} \sigma\mathfrak{p}[a,g](s,t,\tau)zdsdt = 0,$$

$$\forall\, z \in \mathfrak{C}_2^1(Q_{(\tau,T)}),\;\; z(\cdot,\tau) \equiv 0;\;\; \mathfrak{p}[a,g](x,T,\tau) \equiv 0,\;\; x \in \Omega.$$

Therefore, if $(x,t) \in Q_{(\tau,T)}$, then the function $\mathfrak{p}[a,g](\cdot,\cdot,\tau)$ is the solution to the homogeneous initial–boundary value problem

$$\mathfrak{p}_{tt} - \frac{\partial}{\partial x_i}(a_{i\,j}\mathfrak{p}_{x_j} + b_i\mathfrak{p}) + a_i\mathfrak{p}_{x_i} + a\mathfrak{p} = 0, (x,t) \in Q_{(\tau,T)},$$

$$\mathfrak{p}(x,T) = \mathfrak{p}_t(x,T) = 0,\;\; x \in \Omega,\;\; \frac{\partial\mathfrak{p}}{\partial\mathcal{N}'} + \sigma(s,t)\mathfrak{p} = 0,\;\; (s,t) \in S_{(\tau,T)}.$$

According to Gavrilov (2012), this problem has only trivial solution, whence

$$\mathfrak{p}[a,g](x,t,\tau) \equiv 0,\;\; (x,t) \in Q_{(\tau,T)}.$$

Hence, the relations (34) can be rewritten in the form

$$\int\limits_{Q_{(0,\,\tau)}} [-\mathfrak{p}_t[a,g](x,t,\tau)z_t + a_{ij}\mathfrak{p}_{x_j}[a,g](x,t,\tau)z_{x_i} + b_i\mathfrak{p}[a,g](x,t,\tau)z_{x_i} +$$

$$+a_i\mathfrak{p}_{x_j}[g](x,t,\tau)z + a\mathfrak{p}[g](x,t,\tau)z]dxdt + \int\limits_{S_{(0,\,\tau)}} \sigma\mathfrak{p}[a,g](s,t,\tau)zdsdt =$$

$$= \int\limits_{\Omega} g(x,\tau)z(x,\tau)\,dx\;\; \forall\, z \in \mathfrak{C}_2^1(Q_T),\;\; z(\cdot,0) \equiv 0\;\; \tau \in [0,\,T];$$

$$\mathfrak{p}[a,g](x,\tau,\tau) \equiv 0,\;\; x \in \Omega,\;\; \tau \in [0,\,T].$$

In view of the last equalities, if $(x,t) \in Q_{(0,\tau)}$, then the function $\mathfrak{p}[a,g](\cdot,\cdot,\tau)$ is the solution to the initial–boundary value problem

$$\mathfrak{p}_{tt} - \frac{\partial}{\partial x_i}(a_{i\,j}\mathfrak{p}_{x_j} + b_i\mathfrak{p}) + a_i\mathfrak{p}_{x_i} + a\mathfrak{p} = 0,\;\; (x,t) \in Q_{(0,\tau)},$$

$$\mathfrak{p}(x,\tau) = 0,\;\; \mathfrak{p}_t(x,\tau) = -g(x,\tau),\;\; x \in \Omega,\;\; \left[\frac{\partial\mathfrak{p}}{\partial\mathcal{N}'} + \sigma(s,t)\mathfrak{p}\right]\Bigg|_{S_{(0,\,\tau)}} = 0.$$

Thus, in the case of $\tau \in (0,\,T)$, the equality (31) holds. Hence, (31) is completely proved.

Step 2. Using the Galerkin method, let us show the inclusion

$$\mathfrak{x}[a,g] \in C([0,T],\mathfrak{C}_2^1(Q_T)).$$

Suppose that $g \in C^2([0,T], W_2^1(\Omega))$. Let $h_k \in W_2^1(\Omega)$, $k = 1, 2, \dots$, be the sequence from Lemma 5, and let $g^N \equiv \sum\limits_{m=1}^{N} g_m(\tau) h_m(x)$, $g_m(\tau) \equiv \int_\Omega g(x,\tau) h_m(x) dx$, $m = \overline{1,\, N}$, $N = 1, 2, \dots$, $\tau \in [0,\, T]$. Then, by virtue of Lemma 5,

$$\lim_{N\to\infty} |g^N - g|^{(2)}_{W_2^1(\Omega)} = 0. \tag{35}$$

We will find an approximation $\mathfrak{x}^N[a,g]$ to the function $\mathfrak{x}[a,g]$ in the form

$$\mathfrak{x}^N[a,g](x,t,\tau) \equiv \sum_{k=1}^{N} e_k^N(t,\tau) h_k(x),$$

where a collection of functions $e_k^N \in C^1(\Delta)$, $\Delta \equiv [0,\, T]^2$, $k = \overline{1,\, N}$, is a unique continuous solution to the integral equation

$$e^N(t,\tau) + \int_t^\tau (y-t)\mathcal{A}^N(y) e^N(y,\tau)\, d\tau = \bar{g}^N(\tau)(\tau - t), \;\; (t,\tau) \in \Delta. \tag{36}$$

Here the matrix–valued function $\mathcal{A}^N(t) = [\alpha_{km}(t)]_{k,\, m=\overline{1,\, N}}$, $t \in [0,T]$, and vector–valued functions $e^N(t,\tau) \equiv [e_1^N(t,\tau), \dots, e_N^N(t,\tau)]^*$, $\bar{g}^N(\tau) \equiv [g_1(\tau), \dots, g_N(\tau)]^* \in R^N$ are such that

$$\alpha_{km}(t) \equiv$$
$$\int\limits_\Omega [a_{ij}(x,t) h_{kx_j} h_{mx_i} + b_i(x,t) h_k h_{mx_i} + a_i(x,t) h_{kx_i} h_m + a(x,t) h_k h_m] dx +$$
$$+ \int\limits_S \sigma(s,t) h_k(s) h_m(s) ds, \;\; k,\, m = 1, 2, \dots, \;\; t \in [0,\, T].$$

By differentiating (36) with respect to t twice, we obtain that (36) is equivalent to the Cauchy problem

$$e_{tt}^N(t,\tau) + \mathcal{A}^N(t) e^N(t,\tau) = 0, \;\; e^N(t,\tau)|_{t=\tau} = 0, \;\; e_t^N(t,\tau)|_{t=\tau} = -\bar{g}^N(\tau).$$

Then, by differentiating (36) with respect to τ once, and, then, with respect to t twice, we have

$$(e_\tau^N)_{tt} + \mathcal{A}^N(t) e_\tau^N = 0, \;\; e_\tau^N|_{t=\tau} = \bar{g}^N(\tau), \;\; (e_\tau^N)_t|_{t=\tau} = -\bar{g}^{N\prime}(\tau).$$

Applying reasoning similar to that in the proof of an energetic inequality from Chapter IV, §3 of Ladyzhenskaya (1973), and in the proof of an a priori estimate from Gavrilov (2012), we conclude that

$$\|\mathfrak{x}^N[a,g](\cdot,\cdot,\tau_1) - \mathfrak{x}^N[a,g]](\cdot,\cdot,\tau_2)\|_{\mathfrak{E}_2^1(Q_T)} \le T\sqrt{C}|\tau_1 - \tau_2|^{1/2} |g^N|^{(1)}_{W_2^1(\Omega)},$$
$$\|\mathfrak{x}^N[a,g](\cdot,\cdot,\tau)\|_{\mathfrak{E}_2^1(Q_T)} \le \sqrt{CT}|g^N|^{(0)}_{L_2(\Omega)} \;\; \forall \tau, \;\; \tau_1, \;\; \tau_2 \in [0,\, T]. \tag{37}$$

Let τ_1, $\tau_2 \in [0, T]$ be fixed.

In analogy to the existence proof for solutions to an initial–boundary value problem in Gavrilov (2012), we obtain that there exist a subsequence N_m, $m = 1, 2, \ldots,$ of the sequence $N = 1, 2, \ldots,$ and the functions $\mathfrak{x}[a, g](\cdot, \cdot, \tau_i)$, $i = 1, 2$, such that

$$\begin{aligned}
&\mathfrak{x}^{N_m}[a, g](\cdot, \cdot, \tau_i) \to \mathfrak{x}[a, g](\cdot, \cdot, \tau_i) \text{ weak in } W_2^1(Q_T), \qquad (38)\\
&\max_{t\in[0,\,T]} \|\mathfrak{x}^{N_m}[a, g](\cdot, t, \tau_i) - \mathfrak{x}[a, g](\cdot, t, \tau_i)\|_{2,\Omega} \to 0,\\
&\|\mathfrak{x}^{N_m}[a, g](\cdot, \cdot, \tau_i) - \mathfrak{x}[a, g](\cdot, \cdot, \tau_i)\|_{2,S_T} \to 0, \ \ i = 1, 2, \ \ m \to \infty.
\end{aligned}$$

Moreover, each of functions $\mathfrak{x}[a, g](\cdot, \cdot, \tau_i)$, $i = 1, 2$, is a solution to initial–boundary value problem (28) for $(x, t) \in Q_{[0,\,\tau_i]}$, and is a solution to initial–boundary value problem (29) for $(x, t) \in Q_{[\tau_i,\,T]}$, where $\tau = \tau_i$, $i = 1, 2$. Using the limit relations (35) and (38) together with the weak compactness of a closed ball of Hilbert space, we get

$$\|\mathfrak{x}[a, g](\cdot, \cdot, \tau_1) - \mathfrak{x}[a, g](\cdot, \cdot, \tau_2)\|_{\mathfrak{E}_2^1(Q_T)} \le T\sqrt{C}|\tau_1 - \tau_2|^{1/2}|g|^{(2)}_{W_2^1(\Omega)}; \qquad (39)$$

$$\|\mathfrak{x}[a, g](\cdot, \cdot, \tau)\|_{\mathfrak{E}_2^1(Q_T)} \le \sqrt{CT}|g|^{(0)}_{L_2(\Omega)}. \qquad (40)$$

Thus, if $g \in C^2([0, T], W_2^1(\Omega))$, then $\mathfrak{x}[a, g] \in C([0, T], \mathfrak{E}_2^1(Q_T))$, and, additionally, $\mathfrak{x}[a, g]$ depends linearly on $g \in C^2([0, T], W_2^1(\Omega))$. Using the density of the space $C^2([0, T], W_2^1(\Omega))$ in the space $C([0, T], L_2(\Omega))$, and linearly depending $\mathfrak{x}[a, g]$ on $g \in C^2([0, T], W_2^1(\Omega))$, we obtain that $\mathfrak{x}[a, g] \in C([0, T], \mathfrak{E}_2^1(Q_T))$ for all functions $g \in C([0, T], L_2(\Omega))$.

Step 3. Let us prove other assertions of the Lemma. From the inequality (40) and density $C^2([0, T], W_2^1(\Omega))$ in $C([0, T], L_2(\Omega))$ it follows that the estimation (33) holds. Since, by the second step of the proof, $\mathfrak{x}[a, g] \in C([0, T], \mathfrak{E}_2^1(Q_T))$ for all $g \in C([0, T], L_2(\Omega))$, the functions

$$\int\limits_{[0,T]} \mathfrak{p}[a, g](x, t, \tau)\mu(d\tau), \int\limits_{[0,T]} \mathfrak{p}_t[a, g](x, t, \tau)\mu(d\tau),$$
$$\int\limits_{[0,T]} \mathfrak{p}_{x_i}[a, g](x, t, \tau)\mu(d\tau), \ i = \overline{1, n}$$

are square summable over Q_T for any Radon measure $\mu \in M[0, T]$. Besides, using a definition of generalized derivatives in the sense of Sobolev, we conclude that

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{[0,T]} \mathfrak{p}[a, g](x, t, \tau)\mu(d\tau) = \int_{[0,T]} \mathfrak{p}_t[a, g](x, t, \tau)\mu(d\tau), \qquad (41)\\
&\frac{\partial}{\partial x_i}\int_{[0,T]} \mathfrak{p}[a, g](x, t, \tau)\mu(d\tau) = \int_{[0,T]} \mathfrak{p}_{x_i}[a, g](x, t, \tau)\mu(d\tau),\\
&i = \overline{1,\, n}, \ \ (x, t) \in Q_T.
\end{aligned}$$

Let us show that (32) takes place. Indeed, because $\mathfrak{p}[a,g] \equiv \mathfrak{f}[a,g,\delta_\tau]$, then, writing out the corresponding integral identity, integrating this identity in $\tau \in [0,T]$ by a measure μ, and taking into account (41), we get that the function $\zeta(x,t) \equiv \int\limits_0^T \mathfrak{p}[a,g](x,t,\tau)\mu(d\tau)$, $(x,t) \in Q_T$, is a solution to the problem (27). Because the problem (27) has a unique solution, then the equality (32) holds. Theorem 2 is completely proved. □

Finally, let us prove the following result.

Lemma 7 *Let us consider the initial–boundary value problem*

$$z_{tt} - \frac{\partial}{\partial x_i}(a_{ij}z_{x_j} + a_i z) + b_i z_{x_i} + az = f(x,t), \ \ (x,t) \in Q_T, \tag{42}$$
$$z(x,0) = 0, \ \ z_t(x,0) = \psi(x), \ \ x \in \Omega, \ \ \frac{\partial z}{\partial \mathcal{N}} + \sigma z = \omega(s,t), \ \ (s,t) \in S_T,$$

where coefficients a_{ij}, a_i, b_i, a, σ satisfy the conditions (25), and $f \in L_{2,1}(Q_T)$, $\psi \in L_2(\Omega)$, $\omega \in W_{2,1}^{0,1}(S_T)$.

Suppose that function $z \in \mathfrak{E}_2^1(Q_T)$ is a solution to the problem (42), $\tau \in [0,T]$. Suppose also that a function $g : \Omega \times [0,T] \to R$ has the trace $g(\cdot,\tau) \in L_2(\Omega)$ for any $\tau \in [0,T]$, and this trace depends continuously on $\tau \in [0,T]$ in the norm of the space $L_2(\Omega)$. Then

$$\int\limits_\Omega g(x,\tau)z(x,\tau)dx = \int\limits_{Q_T} f\mathfrak{p}[a,g](x,t,\tau)dxdt + \tag{43}$$
$$+ \int\limits_\Omega \psi\mathfrak{p}[a,g](x,0,\tau)dx + \int\limits_{S_T} \mathfrak{p}[a,g](s,t,\tau)\omega dsdt.$$

Proof. Because $z \in \mathfrak{E}_2^1(Q_T)$ is a unique solution to the problem (42), then the restriction $z|_{Q_\tau}$ is a unique solution to the problem

$$z_{tt} - \frac{\partial}{\partial x_i}(a_{ij}z_{x_j} + a_i z) + b_i z_{x_i} + az = f(x,t), \ \ (x,t) \in Q_\tau, \tag{44}$$
$$z(x,0) = 0, \ \ z_t(x,0) = \psi(x), \ \ x \in \Omega, \ \ \frac{\partial z}{\partial \mathcal{N}} + \sigma z = \omega(s,t), \ \ (s,t) \in S_\tau.$$

In view of (44) we can write out the following identity, which holds for all $\eta \in \hat{\mathfrak{E}}_2^1(Q_T)$:

$$\int\limits_{Q_\tau}[-z_t\eta_t + a_{ij}z_{x_j}\eta_{x_i} + a_i z\eta_{x_i} + b_i z_{x_i}\eta + az\eta]dxdt + \int\limits_{S_\tau}\sigma z\eta dsdt = \tag{45}$$
$$= \int\limits_{S_\tau}\omega\eta dsdt + \int_{Q_\tau} f\eta dxdt + \int\limits_\Omega \psi(x)\eta(x,0)dx; \ \ z(x,0) = 0, \ \ x \in \Omega.$$

However, in view of the lemma's conditions, all conditions of the unique existence theorem from Gavrilov (2012) of a solution $\mathfrak{x} \in \mathfrak{C}_2^1(Q_T)$ to the adjoint problem (28) are fulfilled, whence the following identity holds for all $z \in \mathfrak{C}_2^1(Q_T)$, $z(\cdot, 0) = 0$:

$$\int\limits_{Q_\tau} [-z_t \mathfrak{x}_t + a_{ij} z_{x_j} \mathfrak{x}_{x_i} + a_i z \mathfrak{x}_{x_i} + b_i z_{x_i} \mathfrak{x} + az\mathfrak{x}] dxdt + \int\limits_{S_\tau} \sigma z \mathfrak{x} dsdt = \tag{46}$$
$$= \int\limits_{\Omega} g(x, \tau) z(x, \tau) dx; \;\; \mathfrak{x}(x, \tau) = 0, \;\; x \in \Omega.$$

Substituting $\mathfrak{x}[a, g]$ for η in (45), and substituting the solution to the problem (44) for z in (46), and taking into account the fact that the left–hand side parts of (45) and (46) coincide, we conclude that

$$\int\limits_{\Omega} g(x, \tau) z(x, \tau) dx = \int\limits_{Q_\tau} f(x, t) \mathfrak{x}[a, g](x, t, \tau) dxdt +$$
$$+ \int\limits_{\Omega} \psi(x) \mathfrak{x}[a, g](x, 0, \tau) dx + \int\limits_{S_\tau} \mathfrak{x}[a, g](s, t, \tau) \omega(s, t) dsdt.$$

The last equality can be rewritten in the form

$$\int\limits_{\Omega} g(x, \tau) z(x, \tau) dx = \int\limits_{Q_T} f(x, t) \mathfrak{x}[a, g](x, t, \tau) \chi(t, \tau) dxdt + \tag{47}$$
$$+ \int\limits_{\Omega} \psi(x) \mathfrak{x}[a, g](x, 0, \tau) \chi(0, \tau) dx + \int\limits_{S_T} \mathfrak{x}[a, g](s, t, \tau) \chi(t, \tau) \omega(s, t) dsdt.$$

According to equality (31) from Lemma 2, we have that $\mathfrak{x}[a, g](x, t, \tau)\chi(t, \tau) = \mathfrak{p}[a, g](x, t, \tau)$. Therefore, (47) can be rewritten in the form (43). The lemma is completely proved. □

3.2.3. Adjoint equations of the maximum principle

In this section, we formulate facts concerning the stability of solutions of the maximum principle's adjoint equations under perturbations of controls. These facts will be used to prove the main results. First of all, after having investigated the special questions of the theory of linear hyperbolic equations with Radon measures in right–hand side parts, let us define the Pontryagin maximum principle's adjoint functions of $\mathfrak{p}_0[\pi^1, \pi^2]$ and $\mathfrak{p}_1[\pi^1, \pi^2]$, $\pi^i \equiv (u^i, v^i, w^i) \in \mathcal{D}$, $i = 1, 2$, by

$$\mathfrak{p}_0[\pi^1, \pi^2](x, t) \equiv \mathfrak{p}[\mathfrak{a}[\pi^1, \pi^2], \mathfrak{g}_0[\pi^1, \pi^2]](x, t, T),$$
$$\mathfrak{p}_1[\pi^1, \pi^2](x, t, \tau) \equiv \mathfrak{p}[\mathfrak{a}[\pi^1, \pi^2], \mathfrak{g}_1[\pi^1, \pi^2]](x, t, \tau), \;\; (x, t) \in Q_T, \;\; \tau \in [0, T],$$

where

$$\mathfrak{a}[\pi^1,\pi^2](x,t) \equiv \int_0^1 \nabla_z a(x,t,z_2(x,t)+\gamma\Delta z(x,t),u^1(x,t))d\gamma,$$
$$\mathfrak{g}_0[\pi^1,\pi^2](x) \equiv -\int_0^1 \nabla_z G(x,z_2(x,T)+\gamma\Delta z(x,T))d\gamma,$$
$$\mathfrak{g}_1[\pi^1,\pi^2](x,\tau) \equiv -\int_0^1 \nabla_z \Phi(x,z_2(x,\tau)+\gamma\Delta z(x,\tau),v^2(x)+\gamma\Delta v(x))d\gamma.$$

Here $z_i \equiv z[\pi^i]$, $i=1,2$, $\Delta z \equiv z_1 - z_2$, $\Delta v \equiv v^1 - v^2$. For brevity, set $\mathfrak{g}_0[\pi,\pi] \equiv \mathfrak{g}_0[\pi]$, $\mathfrak{g}_1[\pi,\pi] \equiv \mathfrak{g}_1[\pi]$, $\mathfrak{p}_0[\pi,\pi] \equiv \mathfrak{p}_0[\pi]$, $\mathfrak{p}_1[\pi,\pi] \equiv \mathfrak{p}_1[\pi]$, $\mathfrak{a}[\pi,\pi] \equiv \mathfrak{a}[\pi]$. Let us remember that $\mathfrak{p}[a,g](x,t,\tau) \equiv \mathfrak{f}[a,g,\delta_\tau](x,t)$, $(x,t) \in Q_T$, $\tau \in [0,T]$, where $\mathfrak{f}[a,g,\delta_\tau]$ is a solution to the problem (27) for $\mu = \delta_\tau$.

From the assumptions on input data of problem (P_q), and from Theorem 2, it follows that the following lemma holds.

Lemma 8 *If sequences $\pi^{i,k} \in \mathcal{D}$, $k=\overline{1,4}$, $i=1,2,\dots,$ are such that*

$$\lim_{i\to\infty} d(\pi^{i,1},\pi^{i,3}) = 0, \quad \lim_{i\to\infty} d(\pi^{i,2},\pi^{i,4}) = 0,$$

then

$$\lim_{i\to\infty} \|\mathfrak{p}_0[\pi^{i,1},\pi^{i,2}] - \mathfrak{p}_0[\pi^{i,3},\pi^{i,4}]\|_{\mathfrak{C}_2^1(Q_T)} = 0,$$
$$\lim_{i\to\infty} \sup_{\tau\in[0,T]} \|\mathfrak{p}_1[\pi^{i,1},\pi^{i,2}](\cdot,\cdot,\tau) - \mathfrak{p}_1[\pi^{i,3},\pi^{i,4}](\cdot,\cdot,\tau)\|_{\mathfrak{C}_2^1(Q_T)} = 0.$$

3.3. A calculation of first variations

In the sequel, in the definition and calculation of first variations of functionals, the notion of an iterative limit will play the main role. Besides, the corresponding generalizations of the classic notion of Lebesgue point will also play an important role.

Definition 3 *(Sumin, 1983, 1991, 2000b) Let $G \subset R^n$ be an open set. A point $x \in G$ is said to be an (l,m)–Lebesgue point of a summable function $f : G \to R^1$, $1 \le l \le m \le n$, if $f(x) \ne \infty$ and*

$$\lim_{h\to 0} \frac{1}{(2h)^{m-l+1}} \int\limits_{x_l-h}^{x_l+h} \cdots \int\limits_{x_m-h}^{x_m+h} |f(x_1,\dots,x_{l-1},y_1,\dots,y_{m-l+1},x_{m+1},\dots,x_n) -$$
$$- f(x)|dy_1 \dots dy_{m-l+1} = 0.$$

It is easy to see that an $(1,n)$–Lebesgue point is a Lebesgue point in usual sense, Stane (1970).

Lemma 9 *(Sumin, 1983, 1991, 2000b, 2009) For any fixed* l, m*,* $1 \le l \le m \le n$*, almost all points of an open set* G *are* (l, m)*–Lebesgue points of a summable function* $f\colon G \to R^1$*.*

On the basis of these notions and results, we calculate the first variations of functionals $I_0(\cdot)$ and $I_1(\cdot)(\tau)$ $\forall \tau \in [0, T]$. Let $\pi \equiv (u, v, w) \in \mathcal{D}$ be arbitrary. Let us construct a collection of variation parameters $\mathfrak{m} \equiv (\{(x^i, t^i),\ \gamma^{i,r},\ u^{i,r}, i = \overline{1,\, i_1},\ r = \overline{1,\, r_0(i)}\}, \tilde v, \tilde w)$, where $(x^i, t^i) \in Q_T$, $\gamma^{i,r} \ge 0$, $i = \overline{1,\, i_1}$, $r = \overline{1,\, r_0(i)}$, $\sum\limits_{i=1}^{i_1} \sum\limits_{r=1}^{r_0(i)} \gamma^{i,r} \le 1$, $\tilde v \in \mathcal{D}_2$, $\tilde w \in \mathcal{D}_3$, $u^{i,r} \in U^*$, $i = \overline{1,\, i_1}$, $r = \overline{1,\, r_0(i)}$; $U^* \subseteq U$ is a countable set, U^* is everywhere dense in U . Let us denote the set of all such collections $\mathfrak{m}$ by $\mathfrak{M}$.

A triple $\pi_\varepsilon \equiv (u_\varepsilon, v_\varepsilon, w_\varepsilon) \in \mathcal{D}$ is called a variation of triple $\pi \equiv (u, v, w) \in \mathcal{D}$, where $\varepsilon \equiv (\varepsilon_1, \varepsilon_2)$, $\varepsilon_1, \varepsilon_2 \geqslant 0$, $0 \le \varepsilon_1, \varepsilon_2 \le \varepsilon_0 < 1$, if

$$u_\varepsilon(x,t) \equiv \begin{cases} u^{i,r}, & (x,t) \in Q^\varepsilon_{i,r},\ i = \overline{1,\, i_1},\ r = \overline{1,\, r_0(i)}; \\ u(x,t), & (x,t) \in Q_T \setminus \bigcup\limits_{i=1}^{i_1} \bigcup\limits_{r=1}^{r_0(i)} Q^\varepsilon_{i,r}; \end{cases}$$

$$v_\varepsilon(x) \equiv v(x) + \varepsilon_1^n \varepsilon_2 (\tilde v(x) - v(x)) \equiv v(x) + \varepsilon_1^n \varepsilon_2 \delta v(x), \quad x \in \Omega;$$
$$w_\varepsilon(s,t) \equiv w(s,t) + \varepsilon_1^n \varepsilon_2 (\tilde w(s,t) - w(s,t)) \equiv w(s,t) + \varepsilon_1^n \varepsilon_2 \delta w(s,t),\ (s,t) \in S_T;$$

where $Q^\varepsilon_{i,r} \equiv Q^{\varepsilon_1,\varepsilon_2}_{i,r} \equiv Q^{\varepsilon_1}_{i,r,1} \times Q^{\varepsilon_2}_{i,r,2}$, $Q^{\varepsilon_1}_{i,r,1} \equiv \prod\limits_{\alpha=1}^{n} (x^i_\alpha - \varepsilon_1, x^i_\alpha - \varepsilon_1(r-1)]$, $Q^{\varepsilon_2}_{i,r,2} \equiv (t^i - \varepsilon_2 \sum\limits_{\alpha=1}^{r} \gamma^{i,\alpha}, t^i - \varepsilon_2 \sum\limits_{\alpha=1}^{r-1} \gamma^{i,\alpha}]$. Here, $\varepsilon_0 > 0$ is a small enough number depending on $\gamma^{i,r}$ and (x^i, t^i), $i = \overline{1,\, i_1}$, $r = \overline{1,\, r_0(i)}$, such that sets $Q^{\varepsilon_0}_i \equiv Q^{\varepsilon_0}_{i,1} \times Q^{\varepsilon_0}_{i,2} \equiv \prod\limits_{\alpha=1}^{n} [x^i_\alpha - \varepsilon_0 r_0(i), x^i_\alpha] \times [t^i - \varepsilon_0 \sum\limits_{\alpha=1}^{r_0(i)} \gamma^{i,\alpha}, t^i])$, $i = \overline{1,\, i_0(i)}$, do not pairwise intersect.

Let us formulate the following obvious result.

Lemma 10 *There exists a constant* $L \equiv 1 + \operatorname{meas} V + \operatorname{diam} \mathbf{W}$ *such that for any* $\pi \equiv (u, v, w) \in \mathcal{D}$

$$d(\pi, \pi_\varepsilon) \le L \varepsilon_1^n \varepsilon_2.$$

Now, let us prove that the following lemma holds.

Lemma 11 *1) For any triple* $\pi \equiv (u, v, w) \in \mathcal{D}$ *there exists a subset* $Q_0[\pi] \subseteq Q_T$ *such that* $\operatorname{meas} Q_0[\pi] = \operatorname{meas} Q_T$ *and for all* $\mathfrak{m} \equiv (\{(x^i, t^i),\ \gamma^{i,r},\ u^{i,r},\ i = \overline{1,\, i_1},\ r = \overline{1,\, r_0(i)}\}, \tilde v, \tilde w) \in \mathfrak{M}$*,* $(x^i, t^i) \in Q_0[\pi]$*,* $i = \overline{1,\, i_1}$*, there exists a variation*

$$\delta I_0(\pi; \mathfrak{m}) \equiv \lim_{\varepsilon_1 \to +0} \frac{1}{\varepsilon_1^n} \lim_{\varepsilon_2 \to +0} \frac{I_0(\pi_{\varepsilon_1,\varepsilon_2}) - I_0(\pi)}{\varepsilon_2}.$$

In addition, the following representation holds

$$\delta I_0(\pi;\mathfrak{m}) \equiv \int\limits_{\Omega} [\mathfrak{p}_0[\pi](x,0) + \nabla_v G(x, z[\pi](x,T), v)]\delta v dx + \int\limits_{S_T} \mathfrak{p}_0[\pi](s,t)\delta w ds dt-$$
$$-\sum_{i=1}^{i_1}\sum_{r=1}^{r_0(i)} \gamma^{i,r}\mathfrak{p}_0[\pi](x^i,t^i)\Delta_u a(x^i,t^i,z[\pi](x^i,t^i);u^{i,r},u(x^i,t^i)).$$

2) For any point $\tau \in [0,T]$ and any triple $\pi \equiv (u,v,w) \in \mathcal{D}$ there exists a subset $Q_1[\pi,\tau] \subseteq Q_T$ such that $\operatorname{meas} Q_1[\pi,\tau] = \operatorname{meas} Q_T$ and for all collections $\mathfrak{m} \equiv (\{(x^i,t^i),\ \gamma^{i,r},\ u^{i,r},\ i=\overline{1,\, i_1},\ r=\overline{1,\, r_0(i)}\}, \tilde v, \tilde w) \in \mathfrak{M}$, $(x^i,t^i) \in Q_1[\pi,\tau]$, $i=\overline{1,\, i_1}$, there exists a variation

$$\delta I_1(\pi,\tau;\mathfrak{m}) \equiv \lim_{\varepsilon_1\to+0}\frac{1}{\varepsilon_1^n}\lim_{\varepsilon_2\to+0}\frac{I_1(\pi_{\varepsilon_1,\varepsilon_2})(\tau) - I_1(\pi)(\tau)}{\varepsilon_2}.$$

This variation can be represented in the form

$$\delta I_1(\pi,\tau;\mathfrak{m}) \equiv$$
$$\int\limits_{\Omega} [\mathfrak{p}_1[\pi](x,0,\tau) + \nabla_v \Phi(x,\tau, z[\pi](x,\tau), v)]\delta v dx + \int\limits_{S_T} \mathfrak{p}_1[\pi](s,t,\tau)\delta w ds dt$$
$$-\sum_{i=1}^{i_1}\sum_{r=1}^{r_0(i)} \gamma^{i,r}\mathfrak{p}_1[\pi](x^i,t^i,\tau)\Delta_u a(x^i,t^i,z[\pi](x^i,t^i);u^{i,r},u(x^i,t^i)).$$

Proof. Let us prove only the second assertion of the lemma, because the calculation of the iterative limit $\delta I_0(\pi;\mathfrak{m})$ is completely analogous.

Let a triple $\pi \equiv (u,v,w) \in \mathcal{D}$ be fixed. By linearizing the equation (2), and setting $z_\varepsilon \equiv z[\pi_\varepsilon]$, $z \equiv z[\pi]$, $\Delta_\varepsilon z \equiv z_\varepsilon - z$, $\Delta_\varepsilon v \equiv v_\varepsilon - v$, $\Delta_\varepsilon w \equiv w_\varepsilon - w$, we obtain that

$$\Delta_\varepsilon z_{tt} - \frac{\partial}{\partial x_i}(a_{ij}\Delta_\varepsilon z_{x_j} + a_i\Delta_\varepsilon z) + b_i\Delta_\varepsilon z_{x_j} + \mathfrak{a}[\pi_\varepsilon,\pi]\Delta_\varepsilon z = -\Delta_u a(x,t,z;u_\varepsilon,u);$$
$$\Delta_\varepsilon z|_{t=0} = 0,\ \ \Delta_\varepsilon z_t|_{t=0} = \Delta_\varepsilon v, x\in\Omega;\ \ \frac{\partial(\Delta_\varepsilon z)}{\partial\mathcal{N}} + \sigma\Delta_\varepsilon z = \Delta_\varepsilon w,\ \ (s,t)\in S_T.$$

As we linearize the increment $\Delta_\varepsilon I_1(\tau) \equiv I_1(\pi_\varepsilon)(\tau) - I_1(\pi)(\tau)$ of the functional $I_1(\cdot)(\tau)$, we obtain

$$\Delta_\varepsilon I_1(\tau) = \int\limits_{\Omega}\left[\int\limits_0^1 \nabla_v\Phi(x,\tau,z(x,\tau)+y\Delta_\varepsilon z(x,\tau), v(x)+y\Delta_\varepsilon v(x))dy\right]\Delta_\varepsilon v(x)dx-$$
$$-\int\limits_{\Omega}\mathfrak{g}_1[\pi_\varepsilon,\pi](x,\tau)\Delta_\varepsilon z(x,\tau)dx.$$

By applying Lemma 7 to the last equality, we obtain

$$\Delta_\varepsilon I_1(\tau) =$$

$$= \left\{ \int\limits_\Omega \left[\mathfrak{p}_1[\pi_\varepsilon, \pi](x,0,\tau) + \int\limits_0^1 \nabla_v \Phi(x,\tau,z(x,\tau) + y\Delta_\varepsilon z(x,\tau), v + y\Delta_\varepsilon v) dy \right] \Delta_\varepsilon v dx \right.$$

$$\left. + \int\limits_{S_T} \mathfrak{p}_1[\pi_\varepsilon,\pi](s,t,\tau)\Delta_\varepsilon w ds dt \right\} + \left\{ - \int\limits_{Q_T} \Delta_\varepsilon \mathfrak{p}_1(x,t,\tau)\Delta_u a(x,t,z;u_\varepsilon,u) dx dt \right\} +$$

$$+ \left\{ - \int\limits_{Q_T} \mathfrak{p}_1[\pi](x,t,\tau)\Delta_u a(x,t,z;u_\varepsilon,u) dx dt \right\}$$

$$\equiv \{\Delta_\varepsilon I_1^{(1)}(\tau)\} + \{\Delta_\varepsilon I_1^{(2)}(\tau)\} + \{\Delta_\varepsilon I_1^{(3)}(\tau)\}.$$

Here $\Delta_\varepsilon \mathfrak{p}_1 \equiv \mathfrak{p}_1[\pi_\varepsilon,\pi] - \mathfrak{p}_1[\pi]$.

Using assumptions on an integrand Φ, Lemma 2, Lemma 8, and the definition of variation π_ε of controls, we conclude that

$$\lim_{\varepsilon_1 \to +0} \frac{1}{\varepsilon_1^n} \lim_{\varepsilon_2 \to +0} \frac{\Delta_\varepsilon I_1^{(1)}(\tau)}{\varepsilon_2} = \int\limits_\Omega [\nabla_v \Phi(x,\tau,z(x,\tau),v) + \mathfrak{p}_1[\pi](x,0,\tau)]\delta v dx + \tag{48}$$

$$+ \int\limits_{S_T} \mathfrak{p}_1[\pi](s,t,\tau)\delta w ds dt.$$

Let

$$\mathbf{b}_{i,r}(x,t) \equiv \begin{cases} -\Delta_u a(x,t,z(x,t);u^{i,r},u(x,t)),\ (x,t) \in Q_{i,r}^{\varepsilon_0}; \\ 0, \ \ (x,t) \in \mathbb{R}^{n+1} \setminus Q_{i,r}^{\varepsilon_0}; \end{cases}$$

$$\mathbf{B}_{i,r}(t) \equiv \|\mathbf{b}_{i,r}(\cdot,t)\|_{2,\mathbb{R}^n}.$$

It is not difficult to see that $\mathbf{B}_{i,r} \in L_1(\mathbb{R})$. Hence,

$$\mathbf{b}_{i,r}(\cdot,t) \in L_2(\mathbb{R}^n) \text{ for a.e. } t \in [0,T].$$

Let us introduce a maximal function

$$(M\mathbf{B}_{i,r})(t) \equiv \sup_{\delta>0} \frac{1}{2\delta} \int\limits_{t-\delta}^{t+\delta} |\mathbf{B}_{i,r}(\kappa)| d\kappa.$$

By virtue of the classical maximal function theorem (see, e.g., proposition a) of Theorem 1 on p.15 in Stane, 1970),

$$(M\mathbf{B}_{i,r})(t) \quad \text{is finite for a.e. } t \in \mathbb{R}^1, \ \ i = \overline{1,\, i_1}, \ \ r = \overline{1,\, r_0(i)}. \tag{49}$$

Let us require that the following assumptions on points (x^i,t^i) be fulfilled:

1) points (x^i,t^i), $i=\overline{1,\,i_1}$, are $(1,n+1)$, $(1,n)$–Lebesgue points $(t\equiv x_{n+1})$ (see Definition 3) of all functions

$$-\mathfrak{p}_1[\pi](\cdot,\cdot,\tau)\Delta_u a(\cdot,\cdot,z(\cdot,\cdot);u',u(\cdot,\cdot)),\;\; u'\in U^*; \tag{50}$$

2) almost all points of sections $Q_T^{t^i}\equiv\{(x,t)\in Q_T:\,(x,t)=(x,t^i)\}$, $i=\overline{1,\,i_1}$, are $(n+1,n+1)$–Lebesgue points of all functions (50);

3) the following conditions hold:

$$(M\mathbf{B}_{i,r})(t^i)\ \text{ is finite},\;\; i=\overline{1,\,i_1},\;\; r=\overline{1,\,r_0(i)}.$$

Due to Lemma 9 and conditions (49), such selection of points (x^i,t^i) is possible, and, additionally, the Lebesgue measure of the set $Q_1[\pi,\tau]$ of all such points coincides with the Lebesgue measure of the cylinder Q_T.

Let us calculate a similar limit for the summand $\Delta_\varepsilon I_1^{(2)}(\tau)$:

$$\left|\frac{\Delta_\varepsilon^{(2)}I(\tau)}{\varepsilon_2}\right|\le\sum_{i=1}^{i_1}\sum_{r=1}^{r_0(i)}\int\limits_{Q_{i,r,1}^{\varepsilon_1}}\left[\frac{1}{\varepsilon_2}\int\limits_{Q_{i,r,2}^{\varepsilon_2}}|\Delta_\varepsilon\mathfrak{p}_1(x,t,\tau)||\mathbf{b}_{i,r}(x,t)|dt\right]dx=$$

$$=\frac{1}{\varepsilon_2}\sum_{i=1}^{i_1}\sum_{r=1}^{r_0(i)}\int\limits_{Q_{i,r,2}^{\varepsilon_2}}dt\int\limits_{Q_{i,r,1}^{\varepsilon_1}}|\Delta_\varepsilon\mathfrak{p}_1(x,t,\tau)||\mathbf{b}_{i,r}(x,t)|dx\le$$

$$\le\frac{1}{\varepsilon_2}\sum_{i=1}^{i_1}\sum_{r=1}^{r_0(i)}\int\limits_{Q_{i,r,2}^{\varepsilon_2}}\|\Delta_\varepsilon\mathfrak{p}_1(\cdot,t,\tau)\|_{2,Q_{i,1}^{\varepsilon_0}}\|\mathbf{b}_{i,r}(\cdot,t)\|_{2,Q_{i,1}^{\varepsilon_0}}dt\le$$

$$\le\sup_{\xi\in[0,T]}\|\Delta_\varepsilon\mathfrak{p}_1(\cdot,\cdot,\xi)\|_{\mathfrak{E}_2^1(Q_T)}\sum_{i=1}^{i_1}\sum_{r=1}^{r_0(i)}\frac{1}{\varepsilon_2}\int\limits_{Q_{i,r,2}^{\varepsilon_2}}\|\mathbf{b}_{i,r}(\cdot,t)\|_{2,Q_{i,1}^{\varepsilon_0}}dt\le$$

$$\le\sup_{\xi\in[0,T]}\|\Delta_\varepsilon\mathfrak{p}_1(\cdot,\cdot,\xi)\|_{\mathfrak{E}_2^1(Q_T)}\sum_{i=1}^{i_1}\sum_{r=1}^{r_0(i)}2\delta^{i,r}\frac{1}{2\varepsilon_2\delta^{i,r}}\int\limits_{t^i-\varepsilon_2\delta^{i,r}}^{t^i+\varepsilon_2\delta^{i,r}}\|\mathbf{b}_{i,r}(\cdot,t)\|_{2,Q_{i,1}^{\varepsilon_0}}dt\le$$

$$\le\sup_{\xi\in[0,T]}\|\Delta_\varepsilon\mathfrak{p}_1(\cdot,\cdot,\xi)\|_{\mathfrak{E}_2^1(Q_T)}\sum_{i=1}^{i_1}\sum_{r=1}^{r_0(i)}2\delta^{i,r}(M\mathbf{B}_{i,r})(t^i),$$

where $\delta^{i,r}=\sum\limits_{l=1}^{r}\gamma^{i,l}$.

Hence, in view of Lemma 8, the definition of a variation π_ε, and conditions on points (x^i,t^i),

$$\lim_{\varepsilon_1\to+0}\frac{1}{\varepsilon_1^n}\lim_{\varepsilon_2\to+0}\frac{\Delta_\varepsilon I_1^{(2)}(\tau)}{\varepsilon_2}=0. \tag{51}$$

Using the notation introduced before, let us rewrite the expression for $\Delta_\varepsilon I_1^{(3)}(\tau)$:

$$\frac{1}{\varepsilon_2}\Delta_\varepsilon I_2^{(3)}(\tau) \equiv \sum_{i=1}^{i_1}\sum_{r=1}^{r_0(i)} \int\limits_{Q_{i,r,1}^{\varepsilon_1}} \left[\frac{1}{\varepsilon_2} \int\limits_{Q_{i,r,2}^{\varepsilon_2}} \mathbf{b}_{i,r}(x,t)\mathfrak{p}_1[\pi](x,t,\tau)dt \right] dx.$$

Using a maximal function definition, the classical Lebesgue dominated convergence theorem, and the (l,m)–Lebesgue points definition, we obtain that

$$\lim_{\varepsilon_1\to+0}\frac{1}{\varepsilon_1^n}\lim_{\varepsilon_2\to+0}\frac{1}{\varepsilon_2}\Delta_\varepsilon I_1^{(3)}(\tau) = -\sum_{i=1}^{i_1}\sum_{r=1}^{r_0(i)} \gamma^{i,r}\mathfrak{p}_1[\pi](x^i,t^i,\tau)\Delta_u a(x^i,t^i,z[\pi](x^i,t^i);$$
$$u^{i,r}, u(x^i,t^i)).$$

Combining this equality, (48) and (51), we get second assertion of the lemma. This completes the proof of the lemma. □

4. Main results

The main results of the present paper deal with the necessary conditions for elements of minimizing sequences, that is, the Pontryagin maximum principle for m.a.s. and the properties of normality, regularity, and sensitivity. In the sequel, we use the following standard notation: $H(x,t,z,u,\eta) \equiv -\eta a(x,t,z,u)$.

The following theorem gives us the necessary conditions for a m.a.s. which this being referred to as the maximum principle for the m.a.s.

THEOREM 3 *Let $\pi^i \equiv (u^i,v^i,w^i) \in \mathcal{D}$, $i=1,2,\dots,$ be an m.a.s. to the problem (P_q); then there exist a sequence of numbers $\gamma^i \geqslant 0$, $i=1,2,\dots,$ $\gamma^i \to 0$, $i\to\infty$, a sequence of nonnegative numbers λ^i, $i=1,2,\dots,$ and a sequence of nonnegative Radon measures $\mu^i \in M(X)$, $i=1,2,\dots,$ where μ^i is concentrated on the set*

$$X_i \equiv \{\tau \in X \,:\, |I_1(\pi^i)(\tau) - q(\tau)| \le \gamma^i\}, \tag{52}$$

such that the following conditions are fulfilled:

a) the nontriviality condition of the Lagrange multipliers:

$$\pi^i \in \mathcal{D}_q^{\gamma^i},\ \ \lambda^i + \|\mu^i\| = 1; \tag{53}$$

b) the maximum condition with respect to u:

$$\int\limits_{Q_T} [\max_{u'\in U} H(x,t,z[\pi],u',\eta[\pi,\lambda,\mu](x,t)) - \tag{54}$$
$$-H(x,t,z[\pi],u(x,t),\eta[\pi,\lambda,\mu](x,t))]dxdt \le \gamma,$$

c) the transversality condition with respect to v:

$$\max_{v'\in\mathcal{D}_2}\left\{\int_\Omega [\eta[\pi,\lambda,\mu](x,0)+\lambda\nabla_v\Phi(x,z[\pi](x,T),v(x))](v(x)-v'(x))dx+\right. \tag{55}$$
$$\left.+\int_X \mu(d\tau)\int_\Omega \nabla_v\Phi(x,\tau,z[\pi](x,\tau),v(x))(v(x)-v'(x))dx\right\}\le\gamma,$$

d) the transversality condition with respect to w:

$$\max_{w'\in\mathcal{D}_3}\int_{S_T}\eta[\pi,\lambda,\mu](s,t)(w(s,t)-w'(s,t))dsdt\le\gamma, \tag{56}$$

for $\pi=\pi^i$, $\lambda=\lambda^i$, $\mu=\mu^i$, $\gamma=\gamma^i$, $i=1,2,\dots$.

Here $\eta[\pi,\lambda,\mu]$ is a solution to the adjoint initial–boundary value problem

$$\eta_{tt}-\frac{\partial}{\partial x_i}(a_{ij}\eta_{x_j}+b_i\eta)+a_i\eta_{x_i}+\nabla_z a(x,t,z[\pi],u)\eta=\nabla_z\Phi(x,t,z[\pi],v)\mu(dt), \tag{57}$$
$$\eta|_{t=T}=0,\ \ \eta_t|_{t=T}=-\lambda\nabla_z G(x,z[\pi](x,T),v(x)),\ x\in\Omega,\ \left[\frac{\partial\eta}{\partial\mathcal{N}'}+\sigma\eta\right]\Bigg|_{S_T}=0,$$

where $\frac{\partial\eta}{\partial\mathcal{N}'}\equiv(a_{ij}(x,t)\eta_{x_j}+b_i(x,t)\eta)\cos\alpha_i(x,t)$.

It follows from Theorem 3 that, under the condition $I_0(\pi^0)=\beta(q)$, control $\pi^0\in\mathcal{D}_q^0$ satisfies the ordinary maximum principle for $\pi^i\equiv\pi^0$, $\gamma^i\equiv 0$, $(\lambda^i,\mu^i)\equiv(\lambda,\mu)$, $i=1,2,\dots$

Next, following, say, Sumin (1997, 2000c, d), we introduce some natural definitions.

Definition 4 *A sequence $\pi^i\in\mathcal{D}$, $i=1,2,\dots$, is said to be stationary in problem (P_q) if there exists a sequence of nonnegative numbers $\gamma^i\to 0$, $i\to\infty$, and a bounded sequence of pairs (λ^i,μ^i), where $\lambda^i\geqslant 0$, and $\mu^i\in M[0,T]$ is a nonnegative Radon measure concentrated on the set X_i, such that $\pi^i\in\mathcal{D}_q^{\gamma^i}$, $i=1,2,\dots$, inequalities (54)–(56) are satisfied, and all limit points of the sequence of pairs (λ^i,μ^i), $i=1,2,\dots$, (in the $*$-weak sense for the second component), are nonzero.*

Definition 5 *A stationary sequence $\pi^i\in\mathcal{D}$, $i=1,2,\dots$, in problem (P_q) is said to be normal if the first components of all limit points of each corresponding sequence (λ^i,μ^i), $i=1,2,\dots$, are nonzero. Problem (P_q) is said to be normal if all of its stationary sequences are normal. A stationary sequence $\pi^i\in\mathcal{D}$, $i=1,2,\dots$, in problem (P_q) is said to be regular if the first components of all limit points of some corresponding sequence (λ^i,μ^i), $i=1,2,\dots$, are nonzero. Problem (P_q) is said to be regular if there exist regular stationary sequences in it.*

Next, let us state that regularity and normality conditions for the problem (P_q) (e.g., see Sumin, 2000c, d). Consideration of such conditions is possible owing to the presence of a parameter in the original optimization problem. First, let us state a theorem on the stability of the optimal value of problem (P_q) in the case of its normality.

THEOREM 4 *If problem (P_q) is normal, then its value function β satisfies the Lipschitz condition in a neighborhood of the point $q \in C(X)$.*

It turns out that the converse assertion holds in a sense as well.

THEOREM 5 *Let the value function β of problem (P_q) satisfy the Lipschitz condition in a neighborhood of the point q. Then problem $(P_{q'})$ has regular m.a.s. for all q' in that neighborhood.*

In the general case, where the value function $\beta(q)$ does not necessarily possess the Lipchitz property, we have the following general result.

THEOREM 6 *For any point $q \in \mathrm{dom}\beta \equiv \{q' \in C(X) : \beta(q') < +\infty\}$ and any continuous positive function $\xi \in C(X)$ all m.a.s. are regular in problem $(P_{q'})$ for almost all points q' on the ray $\{q + t\xi : t \geqslant 0\}$, i.e., the property that any m.a.s. in problem $(P_{q+t\xi})$ with these q and ξ is regular is a property of general position for $t \geqslant 0$.*

5. Proof of the main results

In this section, we prove the main results, essentially related to the presence of a parameter in problem (P_q).

Proof of Theorem 3. Suppose $\pi^k \equiv (u^k, v^k, w^k),\ k = 1, 2, \ldots$, is a m.a.s. in problem (P_q). Consider the problem

$$J(\pi) \to \inf,\ \ \pi \in \mathcal{D}, \tag{58}$$

where $J(\pi) \equiv \max\{I_0(\pi) - \beta(q),\ I_1(\pi)(\tau) - q(\tau),\ \tau \in X\}$. Obviously, the sequence $\pi^k,\ k = 1, 2, \ldots$, is also a minimizing sequence in problem (58), and the value of problem (58) is equal to zero. Let $\hat{X}^k \equiv \{\tau^{k,j} : j = 1, \ldots, l_k\} \subset X$ be a $1/k$-net in X, $\hat{X}^k \subseteq \hat{X}^{k+1}$, $k = 1, 2, \ldots$. Consider the family of auxiliary problems $J_k(\pi) \to \inf,\ \pi \in \mathcal{D}$, where $J_k(\pi) \equiv \max\{I_0(\pi) - \beta(q);\ I_1(\pi)(\tau) - q(\tau),\ \tau \in \hat{X}^k\}$. By virtue of Lemma 3, a functional $J_k(\cdot)$ is continuous and bounded on $\mathcal{D}$. Using a precompactness of the family $\{I_1(\pi) : \pi \in \mathcal{D}\} \subset C(X)$ in the space $C(X)$ (see Lemma 3), it is not difficult to show that $\lim\limits_{k\to\infty} \inf\limits_{\pi\in\mathcal{D}} J_k(\pi) = \inf\limits_{\pi\in\mathcal{D}} J(\pi) = \lim\limits_{k\to\infty} J(\pi^k) = 0$. It follows that there exists a sequence $\varkappa^k \geq 0,\ k = 1, 2, \ldots,\ \varkappa^k \to 0,\ k \to \infty$, such that $J_k(\pi^k) \leq \inf\limits_{\pi\in\mathcal{D}} J_k(\pi) + \varkappa^k$. Hence, according to the Ekeland variational principle (Ekeland, 1974), there exists a sequence $\bar{\pi}^k \equiv (\bar{u}^k, \bar{v}^k, \bar{w}^k) \in \mathcal{D},\ k = 1, 2, \ldots$, such that for any $k = 1, 2, \ldots$, $\bar{\pi}^k$ is a solution to the problem

$$J_k(\pi) + \sqrt{\varkappa^k} d(\bar{\pi}^k, \pi) \to \min,\ \ \pi \in \mathcal{D}, \tag{59}$$

and

$$d(\pi^k,\, \bar{\pi}^k) \leq \sqrt{\varkappa^k}, \;\; J_k(\bar{\pi}^k) \leq J_k(\pi^k). \tag{60}$$

Suppose $\pi \in \mathcal{D}$, $I \in R$. By definition, put

$$\hat{I}_j(\pi) = \begin{cases} I_0(\pi), \;\; j = 0; \\ I_1(\pi)(\tau^{k,j}), \;\; j = \overline{1, l_k}; \end{cases} \qquad F_j(I) = \begin{cases} I - \beta(q), \;\; j = 0; \\ I - q(\tau^{k,j}), \;\; j = \overline{1, l_k}. \end{cases}$$

Then, obviously, $J_k(\pi) \equiv \max\limits_{s=\overline{0,l_k}} F_j(\hat{I}_j(\pi))$. Let $\Gamma_k \equiv \{j = \overline{0, l_k} : J_k(\bar{\pi}^k) = F_j(\hat{I}_j(\bar{\pi}^k))\}$. Then

$$J_k(\bar{\pi}^k) = F_j(\hat{I}_j(\bar{\pi}^k)), \;\; j \in \Gamma_k; \;\; J_k(\bar{\pi}^k) > F_j(\hat{I}_j(\bar{\pi}^k)), \;\; j \notin \Gamma_k.$$

Let $\bar{k}$ be the number of elements in Γ_k. We introduce the first variations vector

$$\delta\hat{I}(\bar{\pi}^k; \mathsf{m}) \equiv (\delta\hat{I}_0(\bar{\pi}^k; \mathsf{m}), \delta\hat{I}_1(\bar{\pi}^k; \mathsf{m}), \ldots, \delta\hat{I}_{l_k}(\bar{\pi}^k; \mathsf{m})) \in R^{l_k+1}.$$

By $\mathcal{K}(\bar{\pi}^k)$ we denote the set of all first variations vectors. Using standard methods of optimal control, it can be shown that $\mathcal{K}(\bar{\pi}^k)$ is convex. Let us project $\mathcal{K}(\bar{\pi}^k)$ on the subspace of R^{l_k+1} spanned vectors e_j, $j \in \Gamma_k$ (e_j, $j = \overline{0, l_k}$, is the standard basis of R^{l_k+1}), and let us denote the projection by $\mathcal{K}_{\bar{k}}(\bar{\pi}^k)$. Consider the set $\mathcal{K}^-_{\bar{k}} \equiv \{\sum\limits_{j \in \Gamma_k} x_j e_j : x_j \leq -2L\sqrt{\varkappa^k}, \;\; j \in \Gamma_k\}$, where the constant $L \equiv 1 + \operatorname{meas} V + 2A$ is defined in Lemma 10. Let us show that the following lemma holds.

Lemma 12 $\mathcal{K}^-_{\bar{k}} \cap \mathcal{K}_{\bar{k}}(\bar{\pi}^k) = \emptyset$.

Proof Assume the converse. Then, there exists $\mathfrak{m} \in \mathfrak{M}$ such that $\delta\hat{I}_j(\bar{\pi}^k; \mathsf{m}) \leq -2L\sqrt{\varkappa^k}$, $j \in \Gamma_k$. In view of first variations' form of functionals in Lemma 11, we get $F_j(\hat{I}_j(\bar{\pi}^k_\varepsilon)) - F_j(\hat{I}_j(\bar{\pi}^k)) = \hat{I}_j(\bar{\pi}^k_\varepsilon) - \hat{I}_j(\bar{\pi}^k) = \varepsilon_1^n \varepsilon_2 \delta\hat{I}_j(\bar{\pi}^k; \mathsf{m}) + \varepsilon_1^n \varepsilon_2 \omega_1(\varepsilon_1) + \varepsilon_2 \omega_2(\varepsilon_1, \varepsilon_2)$, where ω_1 and ω_2 are such that $\lim\limits_{\varepsilon_2 \to 0} \omega_2(\varepsilon_1, \varepsilon_2) = \lim\limits_{\varepsilon_1 \to 0} \omega_1(\varepsilon_1) = 0$. Hence,

$$F_j(\hat{I}_j(\bar{\pi}^k_\varepsilon)) - F_j(\hat{I}_j(\bar{\pi}^k)) = \hat{I}_j(\bar{\pi}^k_\varepsilon) - \hat{I}_j(\bar{\pi}^k) = \varepsilon_1^n \varepsilon_2 \delta\hat{I}_j(\bar{\pi}^k; \mathsf{m}) + \varepsilon_1^n \varepsilon_2 \omega_1(\varepsilon_1) +$$
$$+ \varepsilon_2 \omega_2(\varepsilon_1, \varepsilon_2) \leq -\frac{19}{10} L \varepsilon_1^n \varepsilon_2 \sqrt{\varkappa^k} + \varepsilon_1^n \varepsilon_2 \omega_1(\varepsilon_1) + \left[-\frac{19}{10} L \varepsilon_1^n \sqrt{\varkappa^k} + \omega_2(\varepsilon_1, \varepsilon_2)\right] \varepsilon_2 <$$
$$< -\frac{19}{10} L \varepsilon_1^n \varepsilon_2 \sqrt{\varkappa^k} + \varepsilon_1 \varepsilon_2^n \omega_1(\varepsilon_1) = -\frac{9}{5} L \varepsilon_1^n \varepsilon_2 \sqrt{\varkappa^k} + \left[-\frac{1}{10} L \sqrt{\varkappa^k} + \omega_1(\varepsilon_1)\right] \varepsilon_1^n \varepsilon_2 <$$
$$< -\frac{9}{5} L \varepsilon_1^n \varepsilon_2 \sqrt{\varkappa^k}$$

for all small enough ε. Therefore, $F_j(\hat{I}_j(\bar{\pi}^k_\varepsilon)) \le F_j(\hat{I}_j(\bar{\pi}^k)) - \frac{9}{5}L\varepsilon_1^n\varepsilon_2\sqrt{\varkappa^k}$, $j \in \Gamma_k$. According to Lemma 10, it follows that

$$F_j(\hat{I}_j(\bar{\pi}^k_\varepsilon)) + \sqrt{\varkappa^k}d(\bar{\pi}^k_\varepsilon, \bar{\pi}^k) \le F_j(\hat{I}_j(\bar{\pi}^k)) - \frac{9}{5}L\varepsilon_1^n\varepsilon_2\sqrt{\varkappa^k} + L\varepsilon_1^n\varepsilon_2\sqrt{\varkappa^k}+$$
$$+\sqrt{\varkappa^k}d(\bar{\pi}^k, \bar{\pi}^k) = J_k(\bar{\pi}^k) + \sqrt{\varkappa^k}d(\bar{\pi}^k, \bar{\pi}^k) - \frac{4}{5}L\varepsilon_1^n\varepsilon_2\sqrt{\varkappa^k}.$$

Thus, $J_k(\bar{\pi}^k_\varepsilon) + \sqrt{\varkappa^k}d(\bar{\pi}^k_\varepsilon, \bar{\pi}^k) < J_k(\bar{\pi}^k) + \sqrt{\varkappa^k}d(\bar{\pi}^k, \bar{\pi}^k)$. But this contradicts the optimality of $\bar{\pi}^k$ in problem (59). The lemma is proved. □

Since $\mathcal{K}^-_{\bar{k}} \cap \mathcal{K}_{\bar{k}}(\bar{\pi}^k) = \emptyset$, these sets are separable; i.e., there exists a vector $\lambda^{\bar{k}} \in R^{\bar{k}}$, $\lambda^{\bar{k}}_j \geqslant 0$, $j \in \Gamma_k$, $\sum_{j\in\Gamma_k} \lambda^{\bar{k}}_j = 1$, such that

$$\sum_{j\in\Gamma_k} \lambda^{\bar{k}}_j \delta\hat{I}_j(\bar{\pi}^k;\mathfrak{m}) \geqslant \sum_{j\in\Gamma_k} \lambda^{\bar{k}}_j x_j \ \ \forall\, \mathfrak{m} \in \mathfrak{M}\ \forall\, x = \sum_{j\in\Gamma_k} x_j e_j \in \mathcal{K}^-_{\bar{k}}.$$

Putting $x_j = -2L\sqrt{\varkappa^k}$, $j \in \Gamma_k$, in the last inequality, and completing vectors $\lambda^{\bar{k}} \in R^{\bar{k}}$ to vectors $\lambda^k \in R^{l_k+1}$ by zeros, we conclude that

$$\lambda^k_j \geqslant 0,\ \ j = 0, \ldots, l_k;\ \ \lambda^k_0 + \sum_{j=1}^{l_k} \lambda^k_j = 1; \tag{61}$$

$$\lambda^k_j(J_k(\bar{\pi}^k) - (I_1(\bar{\pi}^k)(\tau^{k,j}) - q(\tau^{k,j}))) = 0,\ \ j = 1, \ldots, l_k; \tag{62}$$

$$\lambda^k_0 \delta I_0(\bar{\pi}^k;\mathfrak{m}) + \sum_{j=1}^{l_k} \delta I_1(\bar{\pi}^k, \tau^{k,j};\mathfrak{m}) \geqslant -2L\sqrt{\varkappa^k},\ \ \forall\, \mathfrak{m} \in \mathfrak{M}. \tag{63}$$

In view of the first variations expressions (see Lemma 11), it follows from the last inequality that

$$H(x,t,z[\bar{\pi}^k](x,t),u,\lambda_0^k\mathfrak{p}_0[\bar{\pi}^k](x,t)+\sum_{j=1}^{l_k}\lambda_j^k\mathfrak{p}_1[\bar{\pi}^k](x,t,\tau^{k,\,j}))- \tag{64}$$

$$-H(x,t,z[\bar{\pi}^k](x,t),\bar{u}^k(x,t),\lambda_0^k\mathfrak{p}_0[\bar{\pi}^k](x,t)+\sum_{j=1}^{l_k}\lambda_j^k\mathfrak{p}_1[\bar{\pi}^k](x,t,\tau^{k,\,j}))\le 2L\sqrt{\varkappa^k}$$

$\forall\, u\in U$ for a.e. $(x,\,t)\in Q_T$;

$$\int_\Omega\left[\lambda_0^k\mathfrak{p}_0[\bar{\pi}^k](x,0)+\sum_{j=1}^{l_k}\lambda_j^k\mathfrak{p}_1[\bar{\pi}^k](x,0,\tau^{k,j})\right](\bar{v}^k(x)-\tilde{v}(x))\,dx+ \tag{65}$$

$$+\sum_{j=1}^{l_k}\lambda_j^k\int_\Omega\nabla_v\Phi(x,\tau^{k,j},z[\bar{\pi}^k](x,\tau^{k,j}),\bar{v}^k)(\bar{v}^k(x)-\tilde{v}(x))dx\le 2\sqrt{\varkappa^k}\ \forall\,\tilde{v}\in\mathcal{D}_2;$$

$$\int_{S_T}[\lambda_0^k\mathfrak{p}_0[\bar{\pi}^k](s,t)+\sum_{j=1}^{l_k}\lambda_j^k\mathfrak{p}_1[\bar{\pi}^k](s,t,\tau^{k,j})](\bar{w}^k-\tilde{w})\,dsdt\le 2\sqrt{\varkappa^k}\ \ \forall\,\tilde{w}\in\mathcal{D}_3. \tag{66}$$

By definition, we put $\mu^k\equiv\sum\limits_{j=1}^{l_k}\lambda_j^k\delta_{\tau^{k,j}}$, where δ_τ is a Radon δ–measure concentrated in the point τ. Then, $\sum\limits_{j=1}^{l_k}\lambda_j^k\mathfrak{p}_1[\bar{\pi}^k](x,t,\tau^{k,j})=\int\limits_X\mathfrak{p}_1[\bar{\pi}^k](x,t,\tau)\mu^k(d\tau)$. According to Theorem 2,

$$\int_X\mathfrak{p}_1[\bar{\pi}^k](x,t,\tau)\mu^k(d\tau)\equiv\mathfrak{f}[\mathfrak{a}[\bar{\pi}^k],\mathfrak{g}_1[\bar{\pi}^k],\mu^k](x,t).$$

Hence, relations (61), (62), (64)–(66) can be rewritten in the form

$$\lambda_0^k\geqslant 0,\ \ \lambda_0^k+\|\mu^k\|=1; \tag{67}$$

$$\|\mu^k\|J_k(\bar{\pi}^k)-\int_X[I_1(\bar{\pi}^k)(\tau)-q(\tau)]\mu^k(d\tau)=0; \tag{68}$$

$$H(x,t,z[\bar{\pi}^k](x,\,t),u,\eta[\bar{\pi}^k,\lambda_0^k,\mu^k](x,t))-H(x,t,z[\bar{\pi}^k](x,\,t),\bar{u}^k(x,t), \tag{69}$$

$\eta[\bar{\pi}^k,\lambda_0^k,\mu^k](x,t))\le 2L\sqrt{\varkappa^k}\ \ \forall\ u\in U$ for a.e. $(x,\,t)\in Q_T$;

$$\int_X \mu^k(d\tau) \int_\Omega \nabla_v \Phi(x,\tau,z[\bar\pi^k](x,\tau),\bar v^k(x))(\bar v^k(x)-\tilde v(x))dx+ \tag{70}$$

$$+\int_\Omega \eta[\bar\pi^k,\lambda_0^k,\mu^k](x,0)(\bar v^k(x)-\tilde v(x))dx \le 2L\sqrt{\varkappa^k} \ \ \forall \tilde v \in \mathcal{D}_2;$$

$$\int_{S_T} \eta[\bar\pi^k,\lambda_0^k,\mu^k](s,t)(\bar w^k-\tilde w)\,dsdt \le 2L\sqrt{\varkappa^k} \ \ \forall \tilde w \in \mathcal{D}_3. \tag{71}$$

Using assumptions on source data of problem (P_q), Lemma 2, Lemma 8, the first inequality in (60), from relations (69), (70), and (71) we obtain relations (54), (55), and (56) respectively. From the second inequality in (60), relation (68), and the uniform continuity of I_1 on $\mathcal{D}$ (see Lemma 3), it follows that the nondegenerate condition holds, and the measure μ^k is concentrated on the set X_k. This completes the proof of Theorem 3. □

Let us approximate the original problem with PSC by problems with finitely many functional constraints. We denote a $1/k$-net in X by $\hat X^k \equiv \{\tau^{k,j} : j = 1,\dots,l_k\} \subset X$, $\hat X^k \subseteq \hat X^{k+1}$, $k = 1,2,\dots$ Consider a sequence of families of optimization problems depending on the vector parameter $q^k \equiv (q_1^k,\dots,q_{l_k}^k) \in R^{l_k}$ and approximating the original family (P_q):

$(P^k_{q^k})$

$I_0(\pi) \to \inf,\ \ I^k(\pi) \in \mathcal{M}^k + q^k,\ \ \pi \in \mathcal{D},$ $\qquad q^k \in R^{l_k}$ being a parameter,

where $\mathcal{M}^k \equiv \{y \in R^{l_k} :\ y_i \le 0,\ i = 1,\dots,l_k\}$, $I^k(\pi) \equiv (I_1^k(\pi),\dots,I^k_{l_k}(\pi))$, $I_i^k(\pi) \equiv I_1(\pi)(\tau^{k,\,i})$, and $\tau^{k,\,i} \in \hat X^k$. Just as in problem (P_q), in problem $(P^k_{q^k})$ the value function $\beta_k : R^{l_k} \to R \cup \{+\infty\}$ is defined by

$$\beta_k(q^k) \equiv \lim_{\varepsilon\to+0} \beta_{k,\,\varepsilon}(q^k),\ \ \beta_{k,\,\varepsilon}(q^k) \equiv \{\inf_{\pi\in\mathcal{D}^{k,\,\varepsilon}_{q^k}} I_0(\pi),\ \mathcal{D}^{k,\,\varepsilon}_{q^k} \ne \emptyset;\ +\infty,\ \mathcal{D}^{k,\,\varepsilon}_{q^k} = \emptyset\},$$

where $\mathcal{D}^{k,\,\varepsilon}_{q^k} \equiv \{\pi \in \mathcal{D} : I_j^k(\pi) \le q_j^k + \varepsilon,\ \ j = 1,\dots,l_k\}$. We have the following approximation lemma, which is similar to the corresponding lemmas in Gavrilov and Sumin (2004, 2005), Sumin (2000d), and follows from the precompactness of the image of the operator I_1 in $C(X)$ (see Lemma 1.4 in Gavrilov and Sumin, 2011a).

Lemma 13 *Let $\beta(q) < +\infty$ and $q \in C(X)$. Then the sequence of vectors $\bar q^k \in R^{l_k}$, $\bar q^k \equiv (\bar q_1^k,\dots,\bar q^k_{l_k})$, $\bar q_i^k = q(\tau^{k,\,i})$, $i = 1,\dots,l_k$, $k = 1,2,\dots$, satisfies the limit relation $\beta_k(\bar q^k) \to \beta(q)$, $k \to \infty$.*

In the sequel, we need the following definition and results from Mordukhovich (1976, 1988, 2006a, b) (see also Mordukhovich and Shao, 1996). Suppose that $A \subset R^s$ is a nonempty closed set, $\varepsilon \geqslant 0$, and $x \in A$. The nonempty set

$$\hat N(x;A) \equiv \left\{ x^* \in R^s \ :\ \limsup_{u \xrightarrow{A} x} \frac{\langle x^*, u-x\rangle}{|u-x|} \le 0 \right\}$$

is called the cone of Fréchet normals to the set A at the point x. We define the Mordukhovich basic/limiting normal cone at a point $\hat{x} \in A$ by the formula $N(\hat{x};\, A) \equiv \limsup_{x \xrightarrow{A} \hat{x}} \hat{N}(x;\, A)$ (see Mordukhovich, 1976, 1988, 2006a, b).

For a lower semicontinuous function $f\,:\, R^s \to R \cup \{+\infty\}$ and for $\bar{x} \in \mathrm{dom} f$, the Fréchet subdifferential $\hat{\partial} f(\bar{x})$ of the function f at the point $\bar{x} \in \mathrm{dom}\, f$ is given by the relation

$$\hat{\partial} f(\bar{x}) \equiv \left\{ x^* \in R^s \,:\, \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*,\, x - \bar{x} \rangle}{|x - \bar{x}|} \geqslant 0 \right\},$$

or, which is equivalent, by the relation

$$\hat{\partial} f(\bar{x}) \equiv \left\{ x^* \in R^s \,:\, (x^*, -1) \in \hat{N}((\bar{x},\, f(\bar{x})); \mathrm{epi} f) \right\}.$$

For any $\bar{x} \in \mathrm{dom} f$, the sets

$$\partial f(\bar{x}) \equiv \{x^* \in R^s \,:\, (x^*, -1) \in N((\bar{x},\, f(\bar{x})); \mathrm{epi} f)\},$$
$$\partial^\infty f(\bar{x}) \equiv \{x^* \in R^s \,:\, (x^*, 0) \in N((\bar{x},\, f(\bar{x})); \mathrm{epi} f)\}$$

are referred to as the subdifferential and the singular subdifferential, respectively, of the function f at the point $\bar{x}$ in the sense of Mordukhovich (1976, 1980). If f is a lower semicontinuous function, then

$$\partial f(\bar{x}) = \limsup_{x \xrightarrow{f} \bar{x}} \hat{\partial} f(x), \quad \partial^\infty f(\bar{x}) = \limsup_{x \xrightarrow{f} \bar{x};\, \varepsilon \downarrow 0} \varepsilon \hat{\partial} f(x), \tag{72}$$

where $x \xrightarrow{f} \bar{x}$ means that $x \to \bar{x}$, $f(x) \to f(\bar{x})$. We have $\partial^\infty f(\bar{x}) = \{0\}$ if the function f satisfies the Lipschitz condition in a neighborhood of the point x. The following assertion holds (Mordukhovich 2006a, b, see also Mordukhovich and Shao, 1996).

Lemma 14 *Let $A \subset R^s$ be a nonempty closed set. Then the set $\{x \in A \,:\, \hat{N}(x; A) \neq \{0\}\}$ that is, the set of all boundary points of A at which there exists a nonzero Fréchet normal, is everywhere dense in the set of all boundary points of A. In addition, for any lower semicontinuous function $f\,:\, R^s \to R \cup \{\pm\infty\}$ the set $\{x \in \mathrm{dom} f \,:\, \hat{\partial} f(x) \neq \emptyset\}$ is everywhere dense in* $\mathrm{dom} f$.

The definition of a Fréchet subdifferential of a function f at a point x directly implies the following assertion.

Lemma 15 *Let $f\,:\, R^s \to R \cup \{\pm\infty\}$ be a lower semicontinuous function, and let $x \in \mathrm{dom} f$. If $(x^*, -\eta) \in \hat{N}((x, f(x)); epi\, f)$, $\eta > 0$, then for each $\varepsilon > 0$ there exists a neighborhood S_ε of x such that $\eta f(x') - \eta f(x) - \langle x^*, x' - x \rangle + \varepsilon |x' - x| \geqslant 0 \;\forall\, x' \in S_\varepsilon$.*

The following lemma on the relationship of Lagrange multipliers and Fréchet normals holds.

Lemma 16 *Let $\beta_k(q^k) < \infty$ and let $(\zeta^k, -\varkappa^k) \in N((q^k, \beta_k(q^k)); \text{epi}\beta_k)$ be an arbitrary vector. Then there exist a sequence of nonnegative numbers $\gamma^i \to 0$, $i \to \infty$, a sequence of controls $\pi^i \in \mathcal{D}^{k,\gamma^i}_{q^k}$, $i = 1, 2, \dots,$ and a bounded sequence of Lagrange multipliers $\lambda^i \equiv (\lambda^i_0, \lambda^i_1, \dots, \lambda^i_{l_k}) \in R^{l_k}$, $i = 1, 2, \dots,$ such that*

$$|\lambda^i| \neq 0, \ \ \lambda^i_j \geqslant 0, \ \ j = \overline{0, l_k}; \ \ \lambda^i_j(I^k_j(\pi^i) - q^k_j) \geqslant -\gamma^i, \ \ j = \overline{1, l_k}, \tag{73}$$

inequalities (54)–(56) *hold for $\pi = \pi^i$, $\alpha = \lambda^i_0$, $\mu = \mu^i \equiv \sum\limits_{j=1}^{l_k} \lambda^i_j \delta_{\tau^{k,j}}$, $\gamma = \gamma^i$, and the relation*

$$\zeta^k + \sum_{j=1}^{l_k} \lambda_j e^j = 0, \tag{74}$$

holds, where $\lambda \equiv (\varkappa^k, \lambda_1, \dots, \lambda_{l_k}) \neq 0$ is some limit point of the sequence of vectors λ^i, $i = 1, 2, \dots;$ δ_τ is the Radon δ-measure concentrated at a point $\tau \in X$; and

$$e^j \equiv (\underbrace{0, \dots, 0}_{j-1}, 1, 0, \dots, 0) \in R^{l_k}.$$

Proof. The proof is similar to that of Lemma 6.4 in Gavrilov and Sumin (2011b), and so it is omitted here.

Let us give the following natural definitions.

Definition 6 *A sequence of controls $\pi^i \in \mathcal{D}$, $i = 1, 2, \dots,$ is said to be **stationary** in problem $(P^k_{q^k})$ if there exists a bounded sequence of vectors $\lambda^i \equiv (\lambda^i_0, \lambda^i_1, \dots, \lambda^i_{l_k}) \in R^{l_k+1}$, $i = 1, 2, \dots,$ and a sequence of nonnegative numbers γ^i, $\gamma^i \to 0$, such that $\pi^i \in \mathcal{D}^{k,\gamma^i}_{q^k}$, relations* (73) *are satisfied, inequalities* (54)–(56) *hold for $\pi = \pi^i$, $\alpha = \lambda^i_0$, $\mu = \mu^i \equiv \sum\limits_{j=1}^{l_k} \lambda^i_j \delta_{\tau^{k,j}}$, and $\gamma = \gamma^i$, and all limit points of the sequence λ^i, $i = 1, 2, \dots,$ are nonzero.*

Definition 7 *A stationary sequence $\pi^i \in \mathcal{D}$, $i = 1, 2, \dots,$ in problem $(P^k_{q^k})$ is said to be **normal** if the first components of all limit points of each corresponding sequence of vectors λ^i, $i = 1, 2, \dots,$ are nonzero. Problem $(P^k_{q^k})$ is said to be **normal** if all of its stationary sequences are normal.*

*A stationary sequence $\pi^i \in \mathcal{D}$, $i = 1, 2, \dots,$ in problem $(P^k_{q^k})$ is said to be **regular** if the first components of all limit points of some corresponding sequence of vectors λ^i, $i = 1, 2, \dots,$ are nonzero. Problem $(P^k_{q^k})$ is said to be **regular** if there exist regular stationary sequences in it.*

We introduce the sets of multipliers

$L^{k,\nu}_{q^k} \equiv \{-\sum\limits_{j=1}^{l_k} \lambda_j e^j \in R^{l_k} \ : \ \lambda = (\lambda_0, \lambda_1, \dots, \lambda_{l_k}) \in R^{l_k+1}, \ \lambda \neq 0, \ \lambda_0 = \nu$, in problem $(P^k_{q^k})$, there exists a stationary sequence for which the corresponding, in accordance with the definition of a stationary sequence, sequence of vectors λ^i, $i = 1, 2, \dots$, has the limit point being the vector $\lambda\}$, $\nu = 0, 1$; $M^{k,\,0}_{q^k} \equiv L^{k,\,0}_{q^k} \cup \{0\}$, and $M^{k,\,1}_{q^k} \equiv L^{k,\,1}_{q^k}$.

The following assertion is a straightforward consequence of Lemma 16 and the definition of generalized subdifferentials Mordukhovich (2006a, b), Mordukhovich and Shao (1996).

Theorem 7 *Let $\beta_k(q^k) < +\infty$. Then $\partial\beta_k(q^k) \subseteq M^{k,\,1}_{q^k}$ and $\partial^\infty\beta_k(q^k) \subseteq M^{k,\,0}_{q^k}$, where $\partial\beta_k$ and $\partial^\infty\beta_k$ are the ordinary and singular generalized subdifferentials, respectively, in the sense of Mordukhovich (1976, 1980).*

From Theorem 3.52 in Mordukhovich (2006a) (see also Corollary 8.5 in Mordukhovich and Shao, 1996) and Theorem 7, we obtain the following important result, in which the finite dimension of the space R^{l_k} plays an important role.

Lemma 17 *If in some neighborhood O_{q^k} of the point q^k all problems $(P^k_{y^k})$, $y^k \in O_{q^k}$, are normal, i.e., $M^{k,\,0}_{y^k} = \{0\}$, $y^k \in O_{q^k}$, and moreover, the sets $M^{k,\,1}_{y^k}$ are uniformly bounded by a constant K in some norm $\|\cdot\|$ (for example, the Euclidean norm $|\cdot|$), then the value function β_k satisfies the Lipschitz condition in the norm dual to $\|\cdot\|$ on O_{q^k} with the same constant K.*

In addition, the lower semicontinuity of the value function of the approximating problem, its monotonicity with respect to each of l_k arguments, the results in Ward (1935), and Lemma 16 imply the following assertion.

Theorem 8 *If the condition $\hat{\partial}\beta_k(q^k) \neq \emptyset$ is satisfied in problem $(P^k_{q^k})$, then problem $(P^k_{q^k})$ is regular. The set of all such points $q^k \in R^{l_k}$ has full measure in* $\mathrm{dom}\beta_k$.

Proof of Theorem 4. The proof of this theorem is similar to the proof of Theorem 5.2 in Gavrilov and Sumin (2011b), and so it is omitted here.

Proof of Theorem 5. Let us state the main idea of the proof. We consider the family of problems $(\bar{P}_\rho) \equiv (P_{q+\rho\tilde{q}})$, depending on a real parameter ρ, with $\tilde{q} \equiv 1$. Since the function β satisfies the Lipschitz condition in a neighborhood of the point q, it follows that the function $\bar{\beta}(\rho) \equiv \beta(q + \rho\tilde{q})$ of one variable satisfies the Lipschitz condition in a neighborhood of zero. Therefore, by virtue of the first formula in (72), there exist sequences ρ^i, ζ^i, and $\varkappa^i$, $i = 1, 2, \dots$, of real numbers such that

$$\rho^i \to 0,\ \ \bar{\beta}(\rho^i) \to \bar{\beta}(0),\ \ (\zeta^i, -\varkappa^i) \to (\zeta, -\varkappa) \neq 0,\ \ i \to \infty,\ \ \varkappa > 0,$$
$$(\zeta^i, -\varkappa^i) \in N((\rho^i, \bar{\beta}(\rho^i)); \mathrm{epi}\bar{\beta}).$$

By using the same considerations as in the statement of the auxiliary problem (6.5) in Gavrilov and Sumin (2011b), we find that any m.a.s. $\pi^{i,k} \in \mathcal{D}$, $k =$

$1, 2, \ldots$, in the sense of (3) in problem $(\bar{P}_{\rho^i})$ is a m.a.s. (together with ρ^i) in the problem

$$\varkappa^i I_0(\pi) - \zeta^i \rho' + \omega|\rho' - \rho^i| \to \inf, \ \ I_1(\pi) \in \mathcal{M} + q + \rho'\tilde{q}, \qquad (75)$$
$$\rho' \in (-\omega, \omega), \ \ \pi \in \mathcal{D},$$

as well, where $\omega > 0$ is large enough to ensure that $\rho^i \in (-\omega, \omega)$, $i = 1, 2, \ldots$. Then, by writing out the necessary conditions for m.a.s. in problem (75) and by passing to the limit as $i \to \infty$, we obtain the m.a.s. $\pi^s \in \mathcal{D}$, $s = 1, 2, \ldots$, in (P_q) satisfying all relations of Theorem 3; moreover, $\lambda_0^s \to \varkappa > 0$, $s \to \infty$, which implies the regularity of problem (P_q). □

Proof of Theorem 6. Indeed, one can readily see that the function $\tilde{\beta}(t) \equiv \beta(q + t\xi)$ is monotone nonincreasing on the ray $t \geqslant 0$. Consequently, by the classical result of the theory of functions of a real variable, $\tilde{\beta}$ is differentiable in the classical sense almost everywhere for $t \geqslant 0$. Therefore, $\hat{\partial}\tilde{\beta}(t) \neq \emptyset$ for almost all $t \geqslant 0$, which implies that, for almost all $t \geqslant 0$, there exists a nonzero Fréchet normal $(\zeta, -\varkappa) \in N((t, \tilde{\beta}(t)); \mathrm{epi}\tilde{\beta})$, $\varkappa > 0$. By the definition of a basic normal cone, there exist sequences t^i, ζ^i, and $\varkappa^i$, $i = 1, 2, \ldots$, of real numbers such that

$$t^i \to t, \ \ \tilde{\beta}(t^i) \to \tilde{\beta}(t), \ \ (\zeta^i, -\varkappa^i) \to (\zeta, -\varkappa) \neq 0, \ \ i \to \infty, \ \ \varkappa > 0,$$
$$(\zeta^i, -\varkappa^i) \in N((t^i, \tilde{\beta}(t^i)); \mathrm{epi}\bar{\beta}).$$

Just as in the proof of Theorem 5, hence we find that any m.a.s. $\pi^{i,k} \in \mathcal{D}$, $k = 1, 2, \ldots$, in the sense of inequality (3) in problem $(P_{q+t^i\xi})$ is a m.a.s. (together with t^i) in the problem

$$\varkappa^i I_0(\pi) - \zeta^i t' + \omega|t' - t^i| \to \inf, \ \ I_1(\pi) \in \mathcal{M} + q + \rho'\tilde{q}, \qquad (76)$$
$$t' \in (-\omega + t, \omega + t), \ \ \pi \in \mathcal{D},$$

as well, where $\omega > 0$ is large enough to ensure that $t^i \in (-\omega + t, \omega + t)$, $i = 1, 2, \ldots$. Next, by writing out the necessary conditions for a m.a.s. in problem (76) and by passing to the limit as $i \to \infty$, we obtain the m.a.s. $\pi^s \in \mathcal{D}$, $s = 1, 2, \ldots$, in problem $(P_{q+t\xi})$ satisfying all relations of Theorem 3; moreover, $\lambda_0^s \to \varkappa > 0$, $s \to \infty$, which implies the regularity of problem $(P_{q+t\xi})$ for almost all $t \geqslant 0$. □

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References

BERGOUNIOUX, M. (1992) A Penalization Method for Optimal Control of Elliptic Problems with State Constraints. *SIAM J. Control Optim.*, **30**, 2, 305-323.

BONNANS, J.F. and CASAS, E. (1995) An Extension of Pontryagin's Principle for State-Constrained Optimal Control of Semilinear Elliptic Equations and Variational Inequalities. *SIAM J. Control Optim.*, **33**, 1, 274-298.

CASAS, E. (1993) Boundary Control of Semilinear Elliptic Equations with Point wise State Constraints. *SIAM J. Control Optim.*, **31**, 4, 993– 1006.

CASAS, E. (1997) Pontryagin's Principle for State-Constrained Boundary Control Problems of Semilinear Parabolic Equations. *SIAM J. Control Optim.*, **35**, 4, 1297-1327.

CASAS, E., RAYMOND, J.-P., and ZIDANI, H. (2000) Pontryagin's Principle for Local Solutions of Control Problems with Mixed Control-State Constraints. *SIAM J. Control Optim.*, **39**, 4, 1182-1208.

CLARKE, F.H. (1983) *Optimization and Nonsmooth Analysis.* Wiley, New York.

EKELAND, I. (1974) On the Variational Principle. *J. Math. Anal. Appl.*, **47**, 324–353.

GAVRILOV, V.S. (2012) Divergent hyperbolic differential equation with various boundary value conditions on various parts of boundary. *Vestnik Nizhegorodskogo universiteta im. N.I. Lobachevskogo*, **4**, 183-192 [in Russian].

GAVRILOV, V.S. and SUMIN, M.I. (2004) Parametric Optimization of Nonlinear
Goursat-Darboux Systems with State Constraints. *Computational Mathematics and Mathematical Physics*, **44**, 6, 949-968.

GAVRILOV, V.S. and SUMIN, M.I. (2005) A Parametric Problem of the Suboptimal Control of the Goursat–Darboux System with a Pointwise Phase Constraint. *Russian Math. (Izv. vuzov)*, **49**, 6, 37-48.

GAVRILOV, V.S. and SUMIN, M.I. (2011a) Parametric Optimization for a Hyperbolic Equation in Divergence Form with a Pointwise State Constraint: I. *Differential Equations*, **47**, 4, 547-559.

GAVRILOV, V.S. and SUMIN, M.I. (2011b) Parametric Optimization for a Hyperbolic Equation in Divergence Form with a Pointwise State Constraint: II. *Differential Equations*, **47**, 5, 726-737.

GAVRILOV, V.S. and SUMIN, M.I. (2011c) Perturbation Method in the Theory of Pontryagin Maximum Principle for Optimal Control of Divergent Semilinear Hyperbolic Equations with Pointwise State Constraints. In: *Control Theory and its Applications*, Nova Science Publishers Inc., New-York: Chapter 4, 83-144.

LADYZHENSKAYA, O.A. (1973) *Boundary–Value Problems of Mathematical Physics.* Nauka, Moscow [in Russian].

LI, X. and YONG, J (1995) *Optimal Control Theory for Infinite Dimensional Systems*. Birkhäuser Verlag, Basel.

LIONS, J.-L., MAGENES, E. (1968) *Problemes aux limites non homogènes et applications*, volume 1. Dunod, Paris.

MACKENROTH, U. (1982) Convex Parabolic Boundary Control Problems with Pointwise State Constraints. *J. Math. Anal. Appl.*, **87**, 256-277.

MACKENROTH, U. (1986) On Some Elliptic Optimal Control Problems with State Constraints. *Optimization*, **17**, 595-607.

MORDUKHOVICH, B.S. (1976) Maximum principle in problems of time optimal control with state constraints. *J. Applied Math. Mech.*, **40**, 960-969.

MORDUKHOVICH, B.S. (1980) Metric approximation and necessary optimality conditions for general classes of extremal problems. *Soviet. Math. Dokl.*, **22**, 526-530.

MORDUKHOVICH, B.S. (1988) *Approximation Methods in Problems of Optimization and Control*. Nauka, Moscow [in Russian].

MORDUKHOVICH, B.S. (2006a) *Variational Analysis and Generalized Differentiation, I: Basic Theory.* Springer, Berlin.

MORDUKHOVICH, B.S. (2006b) *Variational Analysis and Generalized Differentiation, II: Applications.* Springer, Berlin.

MORDUKHOVICH, B.S. and RAYMOND, J.-P. (2004) Dirichlet Boundary Control of Hyperbolic Equations in the Presence of State Constraints. *Appl. Math. Optim.*, **49**, 145-157.

MORDUKHOVICH, B.S. and RAYMOND, J.-P. (2005) Neumann Boundary Control of Hyperbolic Equations with Pointwise State Constraints. *SIAM J. Control Optim.*, **43**, 4, 1354-1372.

MORDUKHOVICH, B.S. and SHAO, Y. (1996) Nonsmooth sequential analysis in Asplund spaces. *Trans. Amer. Math. Soc.*, **346**, 4, 1235-1280.

NOVOZHENOV, M.M. and PLOTNIKOV, V.I. (1982) Generalized Lagrange Multipliers Rule for Distributed Systems with State Constraints. *Diff. Uravn.*, **18**, 4, 584-692. [in Russian]

OSIPOV, Yu.S., VASIL'EV, F.P., and POTAPOV, M.M. (1999) *Fundamentals of the Dynamic Regularization Method*. Mosk. Gos. Univ., Moscow [in Russian].

RAYMOND, J.-P. and ZIDANI, H. (1998) Pontryagin's Principle for State-Constrained Control Problems Governed by Parabolic Equations with Unbounded Controls. *SIAM J. Control Optim.*, **36**,6, 1853-1879.

STANE, E.M. (1970) *Singular Integrals and Differentiability Properties of Functions*. Princeton Univ. Press, Princeton.

SUMIN, M.I. (1983) Optimal Control Problems for Lumped and Distributed Systems with Differentiable and Nondifferentiable Functionals and Functions Defining Systems. Candidate's Dissertation in Mathematics and Physics. Gor'kov. Gos. Univ., Gor'kii [in Russian].

SUMIN, M.I. (1986) On Minimizing Sequences in Optimal Control Problems with Constrained State Coordinates. *Diff. Uravn.*, **22**, 10, 1719-1731 [in Russian].

SUMIN, M.I. (1989) Optimal Control of Objects Described by Quasilinear Elliptic Equations. *Differ. Uravn.*, **25**, 8, 1406-1416 [in Russian].

SUMIN, M.I. (1991) On First Variation in Optimal Control Theory for Distributed Parameters Systems. *Diff. Uravn.*, **27**, 12. 2179-2181 [in Russian].

SUMIN, M.I. (1997) Suboptimal Control of Distributed Parameters Systems: Minimizing Sequences and the Value Function. *Computational Mathematics and Mathematical Physics*, **37**, 1, 21-39.

SUMIN, M.I. (2000a) Optimal Control of Semilinear Elliptic Equation with State Constraint: Maximum Principle for Minimizing Sequence, Regularity, Normality, Sensitivity. *Control and Cybernetics*, **29**, 2, 449-472.

SUMIN, M.I. (2000b) Mathematical Theory of Suboptimal Control for Distributed Systems. Doctoral Dissertation in Mathematics and Physics, Nizhegorodskii Gos. Univ., Nizhnii Novgorod [in Russian].

SUMIN, M.I. (2000c) Suboptimal Control of Semilinear Elliptic Equations with Phase Constraints, I: The Maximum Principle for Minimizing Sequences and Normality. *Russian Math. (Izv. vuzov)*, **44**, 6, 31-42.

SUMIN, M.I. (2000d) Suboptimal Control of Semilinear Elliptic Equations with Phase Constraints, II: Sensitivity, Genericity of the Regular Maximum Principle. *Russian Math. (Izv. vuzov)*, **44**, 8, 50-60.

SUMIN, M.I. (2001) Suboptimal Control of a Semilinear Elliptic Equation with a Phase Constraint and a Boundary Control. *Differential Equations*, **37**, 2, 281-300.

SUMIN, M.I. (2009) The First Variation and Pontryagin's Maximum Principle in Optimal Control for Partial Differential Equations. *Computational Mathematics and Mathematical Physics*, **49**, 6, 958-978.

SUMIN, M.I. (2011) Regularized parametric Kuhn–Tucker theorem in a Hilbert space. *Computational Mathematics and Mathematical Physics*, **51**, 9, 1489-1509.

SUMIN, M.I. (2012) On the Stable Sequential Kuhn–Tucker Theorem and its Applications. *Applied Mathematics*, **3**, 10A (Special issue "Optimization"), 1334-1350.

SUMIN, M.I. and TRUSHINA, E.V. (2008) Minimizing sequences in optimal control with approximately given input data and the regularizing properties of the Pontryagin maximum principle. *Computational Mathematics and Mathematical Physics*, **48**, 2, 209-224.

WARD, A.L. (1935) Differentiability of Vector Monotone Functions. *Proc. London Math. Soc.*, **32**, 2, 339-362.

WARGA, J. (1972) *Optimal Control of Differential and Functional Equations.* Academic Press, New York.