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Dynamic programming approach to shape optimization*

by

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Abstract: We provide a dynamic programming approach through the level set setting to structural optimization problems. By constructing a dual dynamic programming method we provide the verification theorem for optimal and $\varepsilon-$optimal solutions of shape optimization problem.

Keywords: dynamic programming, shape optimization

1. Introduction

1.1. The background

One of the most typical problems we meet in shape optimization of deformable structures can be formulated as follows: determine the shape of the structure of a prescribed volume exhibiting the highest stiffness. The area of optimization of the shape of continuum structures is now dominated by methods that employ the material distribution concept. The typical ones are the homogenization approach and the variable density approach. In the variable density approach, a density function is introduced into the problem formulation to represent the material distribution in the design domain. In order to achieve the goal of topology design, the density function is related to the stiffness of the material by a power law. This choice has the effect of penalizing the intermediate densities, since in this case volume is proportional, while stiffness is less than proportional to density. In this way, it is hoped that the optimal structure may almost entirely consist of elements, which only have minimal or maximal densities. Numerical algorithms based on these two approaches are element-based. One of the disadvantage of the element-based optimization model is that in the representation of the geometric information, such as the location and shape of the boundary, the normal vector or curvature of the boundary are not straightforward (see the detailed description in Sethian, 1996). To overcome that disadvantage, some more geometry-oriented topology optimization algorithms have been proposed recently by Sethian and Wiegmann (Sethian, 2000). The essential feature of that method is the introduction of a

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function, so called level set function, that describes the shape and topology of the structure implicitly. The level set approach, instead of the interface itself, takes the original interface and adds an extra dimension to the problem. Level set methods, basing on the dynamic programming approach (Hamilton-Jacobi equation) provide mathematical and computational tools for tracking evolving interfaces with sharp corners and cusps, topological changes, and multidimensional complications. They efficiently compute optimal robot path around obstacles, extract clinically useful features from the noisy output of medical images, and model the manufacturing steps that transfer a street map of circuitry onto a tiny piece of silicon. The Hamilton-Jacobi equation has been used for tracing the motion of the structural boundary based on ad hoc constructed speed function. In the present paper, the framework of dual dynamic programming, together with sufficient optimality conditions (the so-called verification theorem for relative minimum) is proposed for a solution of the optimum design problem. Different approaches are given in Bellman (1957) or Sokolowski and Zochowski (1999). The shape problem is formulated in terms of the level set functions which satisfy the Hamilton-Jacobi PDE. There is a lack of convexity in our optimization problem ($\tilde{\mathrm{P}}$), since the set of admissible level set functions (deformation) given by the Hamilton-Jacobi PDE is nonconvex. Our goal is not the standard analysis of the problem as, e.g., in Delfour and Zolesio (2001), but the approximate solution by application of the sufficient optimality conditions given by dual dynamic programming. This approach seems to be new and the result obtained is original, to our best knowledge. We construct a dual dynamic programming approach to our shape control problems. This approach allows us to obtain the sufficient conditions for optimality in the problem considered, as well as sufficient optimality conditions for the approximate solutions. We believe that the conditions for problems of type ($\tilde{\mathrm{P}}$) in terms of dual dynamic programming, that we formulate here, have not been provided earlier.

There are two main difficulties that must be overcome in problems such as ($\tilde{\mathrm{P}}$). The first one consists in the following observation. We have no possibility to perform perturbations of the problem - as it is considered in the fixed set with boundary condition - which can be compared to the one-dimensional case given in Bellman (1957) or Fleming and Rishel (1975). The second one is that we deal with elliptic type equation for state. The technique we apply is similar to the methods from Galewska and Nowakowski (2006) or Nowakowski (1992 and 2013). The main idea of the methods from Galewska and Nowakowski (2006) and Nowakowski (2013) is that they carry over all objects used in dynamic programming to dual space: space of multipliers (similar to those which appear in the Pontryagin maximum principle). Next, instead of the classical value function (which for problem ($\tilde{\mathrm{P}}$) makes no sense), we define an auxiliary function $V(\tau, x, p)$, satisfying the first order partial differential equation of dual dynamic programming, which is very similar to level set PDE (compare Galewska and Nowakowski, 2006). Investigations of properties of this function lead to an appropriate verification theorem. Our dual dynamic equation (sufficient optimality conditions and approximate sufficient optimality conditions) is just of the type of Hamilton-Jacobi PDE. The main advantage and difference of our approach is:

after solving our "dual Hamilton-Jacobi PDE" we substitute this solution to inequality for approximate solution and check whether it is really the solution we are looking for, or at least we know the error indicating how wrong is the solution obtained numerically.

The shape optimization problems are considered, for instance, in Haslinger and Mäkinen (2003) or Sokolowski and Zolesio (1992), where necessary optimality conditions, results concerning convergence of finite-dimensional approximation and numerical results are provided. In the monograph of Sokolowski and Zolesio (1992), the material derivative method is employed to calculate both the sensitivity of solutions to shape problems and the derivatives of domain depending functionals with respect to variations of the boundary of the domain occupied by the body. In Myśliński (2004), Myśliński (2005), or Myśliński (2010), the level set based method is applied to find numerically the optimal topology and shape in elastic contact problems. To formulate the necessary optimality condition for simultaneous shape and topology optimization, the shape and topological derivatives are employed. The notion of topological derivative and the results concerning its application in optimization of elastic structures are reported, in particular, in the papers by Burger, Hackl and Ring (2004), or Fulmański, Laurin, Scheid, and Sokołowski (2007), or Garreau, Guillaume and Masmoudi (2001), or Myśliński (2005), or Sokolowski and Zochowski (2003), or Sokolowski and Zochowski (2004).

The approach presented in this article is different from the one given in the mentioned papers. It stays close to the classical dynamic programming approach to optimization problems and gives sufficient $\varepsilon-$optimality conditions (see, e.g., Nowakowski, 2008, or Nowakowski, 2013, or Nowakowski and Sokołowski 2012), while in Sokołowski and Zolesio (1992), or Myśliński (2010) only necessary optimality conditions are stated. We provide a dynamic programming approach through the level set setting to structural optimization problems. This approach allows us to obtain conditions for $\varepsilon-$optimality in the problem considered, with the application of known numerical tools, such as the fast marching method (FMM) (see Sethian, 1987, or Sethian, 1996, compare also Hüber, Stadler and Wohlmuth, 2008).

Another advantage of the dynamic programming approach is that we derive the sufficient optimality ($\varepsilon-$optimality) conditions for the given shape optimization problem - in such a case we do not need any existence of minimum with respect to some family of admissible sets (compact family) followed by necessary optimality conditions. The existence of minimum is in itself a challenging issue, which requires usually the sufficiently rich (or very poor) family of admissible sets and semicontinuity of the functional with respect to a topology introduced in the family of admissible sets (the idea of Hilbert). We follow the ideas originated by Weierstrass, Carathēodory and Bellman, constructing sufficient optimality conditions (then existence and necessary optimality conditions are not needed), which can be done, (in most cases) for a given (analytically) family of admissible sets - more convenient in practice.

1.2. Model problem formulation: optimization of deformable structures

Let a body occupying a bounded domain $\Omega \subset \mathbb{R}^3$ be loaded by body forces of density $f = (f_1, f_2, f_2)$ and *surface tractions* of density $P = (P_1, P_2, P_3)$ on a portion Γ_P of the boundary of Ω. On the remaining part $\Gamma_u = \partial\Omega \setminus \bar{\Gamma}_P$, the body is fixed. We want to find an equilibrium state of Ω. This state is characterized by a symmetric stress tensor σ, defined in Ω, with values in $\mathbb{R}^3$, in equilibrium with f and P, i.e., satisfying

$$\begin{aligned} -\frac{\partial \sigma}{\partial x} &= f \text{ in } \Omega, \\ \sigma\nu &= P \text{ on } \Gamma_P, \end{aligned} \tag{1}$$

ν being the normal vector to Γ_P. The deformation of Ω is characterized by a *displacement vector* $u = (u_1, u_2, u_3)$ and the respective linearized strain tensor $\varepsilon(u) = (\varepsilon_{ij}(u))_{i,j=1}^3$ with $\varepsilon_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. We seek a displacement vector u such that (1) is satisfied with $\sigma = \sigma(u)$, where $\sigma(u) = \mathbf{K}\varepsilon(u)$, ($\mathbf{K}$-matrix), i.e.

$$\begin{aligned} &-\text{div } \sigma = f \text{ in } \Omega, \\ &u = 0 \text{ on } \Gamma_u = \partial\Omega \setminus \bar{\Gamma}_P, \\ &\sigma\nu = P \text{ on } \Gamma_P. \end{aligned} \tag{2}$$

The shape optimization of deformable structures P_m can be formulated as follows (see Haslinger and Mäkinen, 2003): determine the shape of the structure of a prescribed volume exhibiting the highest stiffness. The mathematical formulation of this problem is as follows:

$$\begin{aligned} \text{Find } \Omega^* &\in \Theta \text{ such that} \\ J(\Omega^*, u(\Omega^*)) &\leq J(\Omega, u(\Omega)), \, \Omega \in \Theta, \end{aligned}$$

where Θ is a certain family of subdomains of $D \subset \mathbb{R}^3$ bounded with Lipschitz boundary, and

$$J(\Omega, u) = \int_\Omega f u dx + \int_{\Gamma_P} P u dx \tag{3}$$

is the compliance functional, where $u(\Omega)$ solves (2) in Ω.

2. Formulation of a general problem

In this section, the general approach, not limited only to the problem presented in the previous section, is described. Consider the following shape optimization problem:

$$\text{minimize } J(\Omega) = \int_\Omega L(x, u(x), \nabla u(x)) dx \tag{4}$$

subject to

$$\Omega \in \Theta,$$

$$\begin{aligned} Au(x) &= f(x, u(x)) \text{ a.e. on } \Omega, && (5)\\ u(x) &= \varphi(x) \text{ on } \partial\Omega \backslash \tilde{\Gamma}, && (6)\\ Bu(x) &= \Upsilon(x) \text{ on } \tilde{\Gamma}, && (7) \end{aligned}$$

where $\tilde{\Gamma} \subset \partial\Omega$, Θ is a certain family of subdomains of $D \subset \mathbb{R}^n$ bounded with Lipschitz boundary, which will be defined precisely in the subsection below, A is a differential operator, e.g., div σ from the former section and B is the differential operator acting on $\partial\Omega$. We assume that $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$ is Lipschitz continuous with respect to all variables, $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous with respect to all variables and Lipschitz continuous with respect to the last variable, $\varphi(\cdot)$ is continuous on D and $\Upsilon(\cdot)$ is L^2 on D. As it is usually done in optimization problems (control theory) we assume that the problem (5)-(7) has at least one weak solution for some Ω's from Θ.

3. The level set method - Hamilton-Jacobi equation

3.1. Definition of the level set and contour function

We recall some facts from the level set approach to shape optimization. The level set (contour) of a function $f : \mathbb{R}^n \to \mathbb{R}$, corresponding to a real value c, is a set of points $x \in \mathbb{R}^n$, for which

$$f(x) < c,$$

$$(f(x) = c).$$

Let Ω be an open and connected subset $\mathbb{R}^n$ with a Lipschitz boundary, for which there exists Lipschitz continuous function $\Psi(x) : \mathbb{R}^n \to \mathbb{R}$ such that

$$\Omega = \{x \in \mathbb{R}^n : \Psi(x) < 0\}. \tag{8}$$

In consequence, boundary Γ of Ω is a set of all points $x \in \mathbb{R}^n$ such that $\Psi(x) = 0$. Let $\phi : (t, x) \in [0, 1] \times \mathbb{R}^n \to \mathbb{R}$ be any Lipschitz function, such that

$$\phi(0, x) = \Psi(x), \quad x \in \mathbb{R}^n.$$

According to the definition of function $\Psi(\cdot)$ and the level set function, the level set of the function $\phi(t, \cdot)$ at time $t = 0$ is identical to Ω, i.e.

$$\Omega = \{x \in \mathbb{R}^n : \phi(0, x) < 0\}$$

while the boundary of Ω is equal to Γ, i.e.

$$\Gamma = \{x \in \mathbb{R}^n : \phi(0, x) = 0\}.$$

Thus, if Ω is subject to change in time, we can describe a deformation of Ω and its boundary Γ at time t (denoted as Ω_t and Γ_t) as

$$\Omega_t(\phi) = \{x \in \mathbb{R}^n : \phi(t,x) < 0\}$$

and

$$\Gamma_t(\phi) = \{x \in \mathbb{R}^n : \phi(t,x) = 0\}.$$

Let $x : [0,1] \times \Gamma_0(\phi) \to \mathbb{R}^n$ be a continuous function, which for every point $x_0 \in \Gamma_0(\phi)$ is absolutely continuous and assigns its location at time t, $t \in [0,1]$, i.e.

$$x(t,x_0) = x \in \Gamma_t(\phi).$$

In particular, we obtain

$$x(0,x_0) = x_0.$$

The function $x(\cdot,x_0)$ represents the location of point x_0 at successive time steps t, and determines in this way a trajectory, starting from the point $x_0 \in \Gamma_0(\phi)$.

To derive the level set formula (Hamilton-Jacobi equation) according to which the changes of the function $\phi(t,\cdot)$ affect boundary Γ_t (represented by the zero contour of $\phi(t,\cdot)$) we require for every $x_0 \in \Gamma_0$ that

$$\phi(t, x(t,x_0)) = 0 \tag{9}$$

along the deformation $x(\cdot,\cdot)$ of the point x_0. By differentiating (9) with respect to t we get

$$\frac{\partial \phi}{\partial t}(t, x(t,x_0)) + \nabla\phi(t, x(t,x_0))x'(t,x_0) = 0. \tag{10}$$

Let $Q(x(t,x_0))$, $t \in [0,1]$, $x_0 \in \Gamma_0(\phi)$, to be a Lipschitz mapping assigning to every point $x(t,x_0)$ its speed of moving in normal (outer) direction to the boundary $\Gamma_t(\phi)$. We have

$$Q(x(t,x_0)) = x'(t,x_0) \cdot n, \tag{11}$$

where

$$n = \frac{\nabla\phi}{|\nabla\phi|}(t,x_0) \tag{12}$$

is a normal vector to the boundary Γ_t at the point $x(t,x_0)$. It is clear that having Q and ϕ we can find $x(t,x_0)$. Applying (11) and (12) to (10), finally we have

$$\frac{\partial \phi}{\partial t}(t, x(t,x_0)) + |\nabla\phi(t, x(t,x_0))|\, Q(x(t,x_0)) = 0,$$

i.e. ϕ has to satisfy the following equation of Hamilton-Jacobi type:

$$\frac{\partial \phi}{\partial t}(t,x) + |\nabla \phi(t,x)|\, Q(x) = 0,\ (t,x) \in (0,1) \times \mathbb{R}^n \tag{13}$$

with initial condition

$$\phi(0,x) = \Psi(x), \quad x \in \mathbb{R}^n. \tag{14}$$

It transforms the pure geometry problems into the language of partial differential equations, where theoretical results about existence of solutions may be used to analyze the solutions for different speed functions Q (see Sethian, 1996). One of the novelties of the paper is just to consider a family of the level set functions ϕ generated by a certain set of functions Q (speed functions) with fixed initial function Ψ according to (13) and (14).

3.2. Level set functions in shape optimization

Let two connected domains D, Ω, $\bar{\Omega} \subset D \subset \mathbb{R}^n$, Ω with Lipschitz boundary and initial Lipschitz continuous function $\Psi : D \to \mathbb{R}$, with the property

$$\Psi(x) < 0 \text{ on } \Omega,\ \Psi(x) = 0 \text{ on } \partial\Omega,\ \Psi(x) > 0 \text{ on } D\backslash\bar{\Omega}$$

be given. In order to apply the level set function in shape optimization let us introduce the Heaviside function $H(\phi(t,\cdot))$ to change the dependence of our functional (4) on sets Ω_t to dependence of it upon the level set functions $\phi(t,\cdot)$, $t \in [0,1]$. These functions are defined as

$$H(\phi(t,\cdot)) = 1 \text{ if } \phi(t,\cdot) \le 0,\ H(\phi(t,\cdot)) = 0 \text{ if } \phi(t,\cdot) > 0,\ t \in [0,1].$$

Denote, for fixed $K \in \mathbb{R}^+$, by

$$\digamma = \{Q :\ Q \in Lips(D),\ -K \le Q(x) \le K,\ x \in D\}$$

and for each given $Q \in \digamma$ denote by ϕ_t^Q, $t \in [0,1]$ any Lipschitz solution of (13) in $(0,1) \times D$ with initial condition (14). Thus, $\Omega_t(\phi_t^Q)$, $t \in [0,1]$, has Lipschitz boundary. Next, put for each fixed $t \in [0,1]$

$$J(\phi_t^Q) = \int_D L(x, u^t(x), \nabla u^t(x)) H(\phi_t^Q(x)) dx$$

where u^t satisfies

$$Au^t(x) = f(x, u^t(x)),\ x \in \Omega_t(\phi_t^Q), \tag{15}$$

$$\begin{aligned} u^t(x) &= \varphi(x) \quad \text{on} \quad \partial\Omega_t(\phi_t^Q)\backslash\tilde{\Gamma}_t \\ Bu^t(x) &= \Upsilon(x) \text{ on } \tilde{\Gamma}_t \end{aligned} \tag{16}$$

and $\tilde{\Gamma}_t \subset \partial\Omega_t(\phi_t^Q)$ is a part of $\partial\Omega_t(\phi_t^Q)$ corresponding to $\tilde{\Gamma} \subset \partial\Omega$ by ϕ_t^Q. We assume, similarly as in Section 2, that the problem (15)-(16) has a weak solution. Denote

$$J(\phi_{t_f}^Q) = \min_{t\in[0,1]} J(\phi_t^Q). \tag{17}$$

The family Θ of sets, over which the (4) is considered, we define as:

$$\Theta = \{\Omega_t(\phi_t^Q) : t \in [0,1],\ Q \in \digamma\}. \tag{18}$$

The sets from Θ are called *admissible sets*. Put

$$\Phi = \left\{\phi_t^Q : \Omega_t(\phi_t^Q) \in \Theta,\, t \in [0,1]\right\}.$$

Now we can reformulate our shape optimal problem (4) with (5) into the following optimization problem:

$$J(\phi_{t_f}^Q) = \min_{\phi_t^Q \in \Phi} J(\phi_t^Q). \tag{19}$$

Notice that problem (19) is the free time optimal control problem, so we minimize J also with respect to time. This is why we cannot directly apply the classical dynamic programming approach. We use a certain idea of a trick from Maurer (see Maurer and Oberle, 2002) to transform the free time optimal problem (with final time t and variable $s \in [0,t]$) (19) to a problem with fixed final time $t = 1$ and then, to this new problem with fixed final time, we apply the dual dynamic programming method. The transformation proceeds by augmenting the state dimension and by introducing the free final time as an additional state variable. To this effect, we define the *new time variable* $\tau \in [0,1]$ by

$$s = \tau \cdot t,\ 0 \le \tau \le 1,\ t \in [0,1],\ s \in [0,t]. \tag{20}$$

We shall use the same notation $\phi(\tau, x) = \phi(\tau \cdot t, x)$ for the deformation of Ω with respect to the new variable τ. The *augmented* state

$$\tilde{\phi} = \begin{pmatrix} \phi \\ \phi^1 \end{pmatrix} \in \mathbb{R}^2,\ \phi^1 = t, \tag{21}$$

satisfies the differential equations

$$(\partial\phi/\partial\tau) = -t \cdot |\nabla\phi(\tau,x)|\, Q(x), \qquad d\phi^1/d\tau \equiv 0.$$

To underline the dependence of $\tilde{\phi}$ on Q we shall write $\tilde{\phi}^Q$.

As a result of this time transformation, we consider the following control problem $(\tilde{\mathrm{P}})$ on the fixed interval $[0,1]$: minimize the functional

$$\begin{aligned} J(\tilde{\phi}^Q) &= J(\phi^Q, t) \\ &= \int_D \tilde{L}(x, \tilde{u}^{\tilde{\phi}^Q}(x), \nabla\tilde{u}^{\tilde{\phi}^Q}(x))\tilde{H}(\tilde{\phi}^Q(x))dx \end{aligned} \tag{22}$$

subject to

$$\tilde{\phi}^Q \in \tilde{\Phi},$$

i.e. $\tilde{\phi}^Q$ satisfies

$$\left(\partial\tilde{\phi}^Q/\partial\tau\right) = \tilde{\varphi}(\tau, x),$$

with

$$\tilde{\varphi}(\tau, x) = \begin{pmatrix} -t \cdot |\nabla\phi(\tau, x)|\, Q(x) \\ 0 \end{pmatrix}$$

and $\tilde{u}^{\tilde{\phi}^Q}$ satisfies

$$A\tilde{u}^{\tilde{\phi}^Q}(x) = \tilde{f}(x, \tilde{u}^{\tilde{\phi}^Q}(x)),\ x \in \tilde{\Omega}(\tilde{\phi}^Q), \tag{23}$$

$$\begin{aligned} \tilde{u}^{\tilde{\phi}^Q}(x) &= \varphi(x) \quad \text{on} \quad \partial\tilde{\Omega}(\tilde{\phi}^Q)\backslash\tilde{\Gamma}_{\tilde{\phi}^Q}, \\ \tilde{u}^{\tilde{\phi}^Q}(x) &= \Upsilon(x) \text{ on } \tilde{\Gamma}_{\tilde{\phi}^Q}, \end{aligned}$$

where $\tilde{\Gamma}_{\tilde{\phi}^Q} \subset \partial\tilde{\Omega}(\tilde{\phi}^Q)$ is a part of $\partial\tilde{\Omega}(\tilde{\phi}^Q)$ corresponding to $\tilde{\Gamma} \subset \partial\Omega$ by $\tilde{\phi}^Q$. The functions herein are given by

$$\tilde{L}(x, \tilde{u}^{\tilde{\phi}^Q}(x), \nabla\tilde{u}^{\tilde{\phi}^Q}(x)) = L(x, u^t(x), \nabla u^t(x)),$$

$$\tilde{H}(\tilde{\phi}^Q(x)) = H(\phi_t^Q(x))$$

$$\tilde{f}(x, \tilde{u}^{\tilde{\phi}^Q}(x)) = f(x, u^t(x))$$

and the sets

$$\tilde{\Phi} = \Phi \times [0, 1],$$

$$\tilde{\Omega}(\tilde{\phi}^Q) = \Omega_t(\phi_t^Q).$$

The transformed problem ($\tilde{\mathrm{P}}$) on the fixed time interval $[0, 1]$ falls into the category of Lagrange control problems treated in many books (e.g. Fleming and Rishel, 1975). Thus, we define for problem ($\tilde{\mathrm{P}}$) the optimal value

$$S = \inf_{\tilde{\phi}^Q \in \tilde{\Phi}} J(\tilde{\phi}^Q).$$

We have to notice that problem ($\tilde{\mathrm{P}}$) does not admit perturbation, i.e. the optimal value S can not be defined for any particular point of $\tilde{\Phi}$ or at least a starting point of $\tilde{\phi}^Q$, which could be perturbed. That is the reason why we can not apply dynamic programming directly. We need a different approach to dynamic programming, which would allow for treating the problem of type ($\tilde{\mathrm{P}}$). We will develop for that problem ideas described in Nowakowski (1992).

3.3. Dual dynamic programming approach to problem $(\tilde{\mathrm{P}})$

In the classical dynamic programming (i.e. in the one dimensional case) we have a value function $S(t,z)$ depending on time t and state variable z. Having the possibility to perturb a given point (t,z), we are able to calculate the full derivative of $S(t,z) : S_t(t,z)+S_z(t,z)\dot{z}$ and using some properties of the value function we can derive the Hamilton-Jacobi equation. The essential point in that approach is that we can perturb $S(t,z)$ at each point of the open domain of definition of S. As we mentioned in former section, the problem $(\tilde{\mathrm{P}})$ does not admit perturbation. That is why we have to develop, basing on the ideas of Nowakowski (1992, 2013), a new approach to dynamic programming so called dual dynamic programming. Thus, instead of considering notions of dynamic programming, such as value function $S(t,z)$, or the Hamilton-Jacobi equation in the space (t,z), a new space – the dual space is proposed and new notions of dual dynamic programming are defined: an auxiliary function, a dual optimal value, and a dual Hamilton-Jacobi equation which the auxiliary function should satisfy. The dual space in Nowakowski (1992 and 2013) is, in fact, defined by conjugate (dual) functions (variables) which appear in Pontryagin maximum principle. It turns out that this approach works also in control problems of type $(\tilde{\mathrm{P}})$. That means: in dual approach to dynamic programming the perturbation of optimal value is not needed – instead, we deal with an auxiliary function. However, there is a price to be paid for that, as we have to impose on the auxiliary function some additional condition, called the transversality condition. We need to define the dual notions in some dual space. Thus, let $P \subset \mathbb{R}^{1+n+2}$ be an open (dual) set of the variables $(\tau,x,p)=(\tau,x,y^0,y)$, $(\tau,x)\in[0,1]\times\Omega$, $y^0 \le 0$, $y\in\mathbb{R}$. We shall also use the subset of P

$$P_1 = \{(x,p) : \ (1,x,p)\in P\}\,.$$

Let $V(x,p)$ of $W^{2,2}(P)$ be an (auxiliary) function defined on P and satisfying the following condition:

$$V(\tau,x,p) = y^0 V_{y^0}(\tau,x,p) + yV_y(\tau,x,p) = pV_p(\tau,x,p), \tag{24}$$

for $(\tau,x,p)\in P$. Here, V_{y^0}, V_y, and V_p denote the partial derivatives and the gradient with respect to the dual variables y^0, y, and $p=(y^0,y)$, respectively.

Now, we denote by $p(\tau,x) = (y^0, y(\tau,x))$, $(\tau,x)\in[0,1]\times\Omega$, the dual deformation, while $\tilde{\phi}^Q((\tau,x)$, $(\tau,x)\in[0,1]\times\Omega$ stands for the primal deformation (we should have in mind the convention $\tilde{\phi}^Q(\tau,x)=\tilde{\phi}^Q(\tau\cdot t,x)$ and this concerns also the dual deformation, i.e., $p(\tau,x)=p(\tau\cdot t,x)$, but not the auxiliary function $V(\tau,x,p)$!). Let us put

$$\phi(\tau,x,p) = -V_y(\tau,x,p), \text{ for } (\tau,x,p)\in P. \tag{25}$$

Using the function ϕ it is possible to come back from the dual deformations $p(\tau,x)$, $(\tau,x)\in[0,1]\times\Omega$, lying in P, to the primal deformations $\tilde{\phi}^Q((\tau,x)$, $(\tau,x)\in[0,1]\times\Omega$. The way to find V_y is described below (V is a solution to (26), (27)). Further, we confine ourselves only to those admissible deformations $\tilde{\phi}^Q(\cdot)$, for which there exist functions $p(\tau,x) = (y^0, y(\tau,x))$, $(\tau,x,p(\tau,x)) \in P$, $y(\cdot) \in W^{1,2}(P)$, such that $\phi^Q((\tau,x) = \phi(\tau,x,p(\tau,x))$ for $(\tau,x) \in [0,1]\times\Omega$. Thus, for any given $\overline{y}^0 < 0$, denote

$$\Phi_\phi = \Big\{ \tilde{\phi}^Q(\cdot) \in \tilde{\Phi}\text{: there is } p(\tau,x) = (\overline{y}^0, y(\tau,x)), \mathrm{y}(\cdot) \in W^{1,2}(P),\ (\tau,x,p(\tau,x)) \in P,$$
$$(\tau,x) \in [0,1]\times\Omega \text{ and } \psi : \mathbb{R}^n \mapsto \mathbb{R},\ y(0,x) = \psi(x),\ \phi(0,x,\overline{y}^0,\psi(x)) = \Psi(x),$$
$$\text{such that } \phi^Q((\tau,x) = \phi(\tau,x,p(\tau,x)),\ (\tau,x) \in [0,1]\times\Omega\,\}.$$

Actually, this means that we are going to study problem $(\tilde{\mathrm{P}})$ possibly in some smaller set Φ_ϕ, which is determined by the function (25).

In order to prove the verification theorem we require the function $V(\tau,x,p)$ to satisfy the first order partial differential equation in the dynamic programming form:

$$V_\tau(\tau,x,p) = \inf_{t\in[0,1],\ Q\in F} \{-t\,|\nabla_x V(\tau,x,p)|\,Q(x) \tag{26}$$
$$+y^0(\mathrm{div})^{-1}(\tilde{L}(x,\tilde{u}^{\tilde{\phi}^Q}(x),\nabla\tilde{u}^{\tilde{\phi}^Q}(x)))\frac{d}{d\tau}H(V_y(\tau,x,p)))\Big\},\ \ (\tau,x,p) \in P$$

with the terminal condition

$$V_{y^0}(1,x,p) = 0,\ (x,p) \in P_1, \tag{27}$$

where $\tilde{u}^{\tilde{\phi}^Q}$ satisfies (23) for $\tilde{\Omega}(\tilde{\phi}^Q)$ and $(\mathrm{div})^{-1}$ is an inverse of divergence operator acting on the space $L^2(\Omega)$. In terms of the dual feedback speed $Q(\tau,x,p)$, (26) has the form

$$V_\tau(\tau,x,p) = -t\,|\nabla_x V(\tau,x,p)|\,Q(\tau,x,p)$$
$$+y^0(\mathrm{div})^{-1}(\tilde{L}(x,\tilde{u}^{\tilde{\phi}^Q}(x),\nabla\tilde{u}^{\tilde{\phi}^Q}(x)))\frac{d}{d\tau}H(V_y(\tau,x,p))),\ \ (\tau,x,p) \in P.$$

We shall not discuss here the question of existence of solution to (26) and satisfying the condition (24). We simply assume in the verification theorem (given in the next section) that such a function exists. We define a dual optimal value S_D^ϕ for the problem $(\tilde{\mathrm{P}})$ by the formula

$$S_D^\phi = \inf_{\tilde{\phi}^Q\in\Phi_\phi} \int_D \tilde{L}(x,\tilde{u}^{\tilde{\phi}^Q}(x),\nabla\tilde{u}^{\tilde{\phi}^Q}(x))\tilde{H}(\tilde{\phi}^Q(x))dx, \tag{28}$$

where $\tilde{u}^{\tilde{\phi}^Q}$ satisfies (23) in $\tilde{\Omega}(\tilde{\phi}^Q)$.

3.4. The verification theorem

In this section we formulate and prove one of our main theorems, called "verification theorem", which provides the sufficient optimality conditions for $(\tilde{\mathrm{P}})$ in terms of a solution $V(\tau,x,p)$ to the first order partial differential equation of dynamic programming (26).

Theorem 1 *Assume that there exists a* $W^{2,2}(P)$ *solution* $V(\tau,x,p)$ *of* (26) *on* P *with terminal condition (27) such that (24) holds and let* $\overline{\phi}(\tau,x,p) = -V_y(\tau,x,p)$, $(\tau,x,p) \in P$. *Let* $\tilde{\phi}^{\bar{Q}}(\cdot) \in \Phi_{\bar{\phi}}$, *and* $\overline{p}(\tau,x) = (\overline{y}^0, \overline{y}(\tau,x))$, $\bar{y}(\cdot) \in W^{1,2}([0,1] \times \Omega)$, $(\tau,x,\bar{p}(\tau,x)) \in P$, $\overline{y}(0,x) = \psi(x)$, $x \in \Omega$, $\overline{y}^0 < 0$, *be a function such that* $\tilde{\phi}^{\bar{Q}}(\tau,x) = -V_y(\tau,x,\overline{p}(\tau,x))$ *for* $(\tau,x) \in [0,1] \times \Omega$. *Suppose that, for* $(\tau,x) \in [0,1] \times \Omega$, *and some* $\bar{t} \in [0,1]$

$$V_\tau(\tau,x,\overline{p}(\tau,x)) = -\bar{t}\,|\nabla_x V(\tau,x,\overline{p}(\tau,x))|\, Q(\tau,x,\overline{p}(\tau,x)), \tag{29}$$

$$+\overline{y}^0(\mathrm{div})^{-1}(\tilde{L}(x,\tilde{u}^{\tilde{\phi}^{\bar{Q}}}(x),\nabla\tilde{u}^{\tilde{\phi}^{\bar{Q}}}(x)))\frac{d}{d\tau}H(V_y(\tau,x,\overline{p}(\tau,x)))),$$

with

$$V_{y^0}(1,x,\overline{p}(\bar{t},x)) = 0,$$

where $Q(\tau,x,p)$ *is a dual feedback speed. Then* $\tilde{\phi}^{\bar{Q}}(\cdot)$ *is an optimal deformation relative to all* $\tilde{\phi}^{Q}(\cdot) \in \Phi_{\bar{\phi}}$.

Proof Let us take a $W^{2,2}(P)$ solution $V(\tau,x,p)$ of (26) on P with terminal condition (27) such that (24) holds. Fix any $\tilde{\phi}^Q(\cdot) \in \Phi_{\bar{\phi}}$ corresponding to some $Q(\cdot)$ and t, and take any $p(\tau,x) = (\overline{y}^0, y(\tau,x))$, $y(\cdot) \in W^{1,2}(P)$, $(\tau,x,p(\tau,x)) \in P$, such that $\tilde{\phi}^Q(\tau,x) = V_y(\tau,x,p(\tau,x))$ for $(\tau,x) \in [0,1] \times \Omega$. Let $x(\tau,x_0,p)$, $x_0 \in \Gamma_0(\tilde{\phi}^Q)$, be a trajectory generated by the velocity Q and normal

$$n = \frac{\nabla_x V_y(\tau,x,p)}{|\nabla_x V_y(\tau,x,p)|},$$

i.e. $Q(x(\tau,x_0,p)) = x'(\tau,x_0,p)n$. From the transversality condition (24), we see that for $(\tau,x) \in [0,1] \times \Omega$,

$$\begin{aligned} &V_\tau(\tau,x,p(\tau,x)) + t\,|\nabla_x V(\tau,x,p(\tau,x))|\, Q(x) \\ &= \overline{y}^0\left((d/d\tau)\, V_{y^0}(\tau,x,p(\tau,x)) + t\left|\nabla_x V_{y^0}(\tau,x,p(\tau,x))\right| x'(\tau,x_0,p(\tau,x))\right) \\ &+y(\tau,x)\left((d/d\tau)\, V_y(\tau,x,p(\tau,x)) + t\,|\nabla_x V_y(\tau,x,p(\tau,x))|\, Q(x)\right). \end{aligned} \tag{30}$$

Since $\phi^Q(\tau,x) = -V_y(\tau,x,p(\tau,x))$ for $(\tau,x) \in [0,1] \times \Omega$, (13) shows that for $(\tau,x) \in [0,1] \times \Omega$,

$$(d/d\tau)\, V_y(\tau,x,p(\tau,x)) + t\,|\nabla_x V_y(\tau,x,p(\tau,x))|\, Q(x) = 0. \tag{31}$$

Now, define a function $[0,1] \times \Omega \ni (\tau,x) \mapsto W(\tau,x,p(\tau,x))$ by

$$\begin{aligned} &W(\tau,x,p(\tau,x)) = \\ &\overline{y}^0\Big[-(d/d\tau)\, V_{y^0}(\tau,x,p(\tau,x)) - t\left|\nabla_x V_{y^0}(\tau,x,p(\tau,x))\right| x'(\tau,x_0,p(\tau,x)) \\ &+(\mathrm{div})^{-1}(\tilde{L}(x,\tilde{u}^{\tilde{\phi}^Q}(x),\nabla\tilde{u}^{\tilde{\phi}^Q}(x)))\frac{d}{d\tau}H(V_y(\tau,x,p)))\Big]. \end{aligned} \tag{32}$$

We conclude from (30)–(32) that

$$W(\tau,x,p(\tau,x)) = V_\tau(\tau,x,p(\tau,x)) + t\,|\nabla_x V(\tau,x,p(\tau,x))|\,Q(x)$$
$$+\overline{y}^0(\mathrm{div})^{-1}(\tilde{L}(x,\tilde{u}^{\tilde{\phi}^Q}(x),\nabla\tilde{u}^{\tilde{\phi}^Q}(x)))\tfrac{d}{d\tau}H(V_y(\tau,x,p))),\ (\tau,x)\in[0,1]\times\Omega. \quad (33)$$

Hence, (26) and (33) imply

$$W(\tau,x,p(\tau,x)) \le 0 \ \text{ for } \ (\tau,x)\in[0,1]\times\Omega. \quad (34)$$

By integrating (34) along $x(\tau,x_0,p)$, $\tau\in[0,1]$ (we remember that in $x(\tau,x_0,p)$, $\tau=\tau t$) for any fixed $x_0\in\Gamma_0(\tilde{\phi}^Q)$, and applying (32) and (27), and next integrating over $\Gamma_0(\tilde{\phi}^Q)$, we obtain

$$-\overline{y}^0\int_{\Gamma_0(\tilde{\phi}^Q)} V_{y^0}(0,x,\overline{y}^0,\psi(x))dx \le -\overline{y}^0\int_D L(x,\tilde{u}^{\tilde{\phi}^Q}(x),\nabla\tilde{u}^{\tilde{\phi}^Q}(x)))H(\phi^Q(t,x))dx. \quad (35)$$

By proceeding similarly as above, from (29) we get

$$-\overline{y}^0\int_{\Gamma_0(\tilde{\phi}^Q)} V_{y^0}(0,x,\overline{y}^0,\psi(x))dx = -\overline{y}^0\int_D \tilde{L}(x,\tilde{u}^{\tilde{\phi}^{\bar{Q}}}(x),\nabla\tilde{u}^{\tilde{\phi}^{\bar{Q}}}(x)))H(\phi^Q(\bar{t},x))dx.$$

Thus, from (35) and the last equality it follows that

$$\int_D \tilde{L}(x,\tilde{u}^{\tilde{\phi}^{\bar{Q}}}(x),\nabla\tilde{u}^{\tilde{\phi}^{\bar{Q}}}(x)))H(\phi^{\bar{Q}}(\bar{t},x))dx \quad (36)$$
$$\le \int_D \tilde{L}(x,\tilde{u}^{\tilde{\phi}^Q_t}(x),\nabla\tilde{u}^{\tilde{\phi}^Q}(x)))H(\phi^Q(t,x))dx,$$

which completes the proof. □

3.5. $\varepsilon-$optimality, the verification theorem

From the practical point of view, more important than optimality is $\varepsilon-$optimality and the possibility of verifying that a given value - calculated e.g. numerically, is $\varepsilon-$optimal. In this section we define the dual $\varepsilon-$optimal value and we prove the verification theorem for that value. Let the function $V(\tau,x,p)$ satisfy the first order partial differential inequality of dynamic programming form, $(\tau,x,p)\in P$:

$$0\le -V_\tau(\tau,x,p) + \inf_{t\in[0,1],\ Q\in F}\{-t\,|\nabla_x V(\tau,x,p)|\,Q(x) \quad (37)$$
$$\left. +y^0(\mathrm{div})^{-1}(\tilde{L}(x,\tilde{u}^{\tilde{\phi}^Q}(x),\nabla\tilde{u}^{\tilde{\phi}^Q}(x)))\frac{d}{d\tau}H(V_y(\tau,x,p)))\right\} \le -y^0_\varepsilon\varepsilon,$$

for any fixed $y^0_\varepsilon<0$, with initial condition

$$V_{y^0}(1,x,p)=0,\ (x,p)\in P_1, \quad (38)$$

where $\tilde{u}^{\tilde{\phi}^Q}$ satisfies (23) for $\tilde{\Omega}(\tilde{\phi}^Q)$ and $(\mathrm{div})^{-1}$ is an inverse of divergence operator acting on the space $L^2(\Omega)$.

Definition 1 *Let $\varepsilon > 0$ be fixed. A scalar $S^{\phi}_{\varepsilon D}$ is called a dual ε-optimal value for problem $(\tilde{P})$ if*

$$S^{\phi}_{D} \leq S^{\phi}_{\varepsilon D} \leq S^{\phi}_{D} - \varepsilon y^0_{\varepsilon} vol(\Omega) \tag{39}$$

for any fixed $y^0_{\varepsilon} < 0$.

Definition 2 *Let $\varepsilon > 0$ be fixed and let $V(\tau, x, p)$ be a given $W^{2,2}(P)$ function satisfying (37), (38) and let $\phi_{\varepsilon}(\tau, x, p) = V_y(\tau, x, p)$, $(\tau, x, p) \in P$. Let $\tilde{\phi}^Q_{\varepsilon}(\cdot) = (\phi^Q_{\varepsilon}(\cdot), t_{\varepsilon}) \in \Phi_{\phi_{\varepsilon}}$ and let $p_{\varepsilon}(\tau, x) = (y^0_{\varepsilon}, y_{\varepsilon}(\tau, x))$, $y_{\varepsilon}(\tau, x) \in W^{1,2}([0,1] \times \Omega)$, $(\tau, x, p_{\varepsilon}(\tau, x)) \in P$, $y_{\varepsilon}(0, x) = \psi(x)$, $x \in \Omega$, $y^0_{\varepsilon} < 0$, be such a function that*

$$\phi^Q_{\varepsilon}(\tau, x) = V_y(\tau, x, p_{\varepsilon}(\tau, x)) \text{ for } (\tau, x) \in [0,1] \times \Omega. \tag{40}$$

The deformations $\tilde{\phi}^Q_{\varepsilon}(\cdot)$ is called an ε-optimal deformation relative to all admissible deformation $\tilde{\phi}^Q(\cdot) \in \Phi_{\phi_{\varepsilon}}$ if

$$\begin{aligned} & - y^0_{\varepsilon} \int_D \tilde{L}(x, \tilde{u}^{\tilde{\phi}^Q_{\varepsilon}}(x), \nabla \tilde{u}^{\tilde{\phi}^Q_{\varepsilon}}(x))) H(\phi^Q_{\varepsilon}(t_{\varepsilon}, x)) dx d\tau \leq \\ & - y^0_{\varepsilon} \int_D \tilde{L}(x, \tilde{u}^{\tilde{\phi}^Q}(x), \nabla \tilde{u}^{\tilde{\phi}^Q}(x))) H(\phi^Q(t, x)) dx d\tau - y^0_{\varepsilon} \varepsilon vol(\Omega), \end{aligned} \tag{41}$$

where $\tilde{u}^{\tilde{\phi}^Q_{\varepsilon}}$, $\tilde{u}^{\tilde{\phi}^Q}$ satisfies (23) for $\tilde{\Omega}(\tilde{\phi}^Q_{\varepsilon})$, $\tilde{\Omega}(\tilde{\phi}^Q)$, respectively and $vol(\Omega) = \int_{\Omega} dx$.

Now we formulate and prove the ε-version of the verification theorem which, as it appears, could be applied in constructing the numerical methods for computing an optimal value in $(\tilde{\mathrm{P}})$.

Theorem 2 *Assume that there exists a $W^{2,2}(P)$ solution $V(\tau, x, p)$ of (37) on P with terminal condition (38) such that (24) holds and let $\phi_{\varepsilon}(\tau, x, p) = V_y(\tau, x, p)$, $(\tau, x, p) \in P$. Let $\tilde{\phi}^Q_{\varepsilon}(\cdot) = (\phi^Q_{\varepsilon}(\cdot), t_{\varepsilon}) \in \Phi_{\phi_{\varepsilon}}$, and $p_{\varepsilon}(\tau, x) = (y^0_{\varepsilon}, y_{\varepsilon}(\tau, x))$, $y_{\varepsilon}(\cdot) \in W^{1,2}([0,1] \times \Omega)$, $(\tau, x, p_{\varepsilon}(\tau, x)) \in P$, $y_{\varepsilon}(0, x) = \psi(x)$, $x \in \Omega$, be a function such that $\phi^Q_{\varepsilon}(\tau, x) = V_y(\tau, x, p_{\varepsilon}(\tau, x))$ for $(\tau, x) \in [0,1] \times \Omega$. Suppose that, for $(\tau, x) \in [0,1] \times \Omega$*

$$\begin{aligned} 0 \leq & -V_{\tau}(\tau, x, p_{\varepsilon}(\tau, x)) - t_{\varepsilon} \left| \nabla_x V(\tau, x, p_{\varepsilon}(\tau, x)) \right| Q(\tau, x, p_{\varepsilon}(\tau, x)) \\ & + y^0_{\varepsilon} (\operatorname{div})^{-1} (L(x, \tilde{u}^{\tilde{\phi}^Q_{\varepsilon}}(x), \nabla \tilde{u}^{\tilde{\phi}^Q_{\varepsilon}}(x))) \frac{d}{d\tau} H(V_y(\tau, x, p_{\varepsilon}(\tau, x)))) \leq -y^0_{\varepsilon} \varepsilon, \end{aligned} \tag{42}$$

where $Q(\tau, x, p)$ is a dual feedback speed. Then $\tilde{\phi}^Q_{\varepsilon}(\cdot)$ is an $\varepsilon-$optimal deformation relative to all $\tilde{\phi}^Q(\cdot) \in \Phi_{\phi_{\varepsilon}}$.

Proof Let us take a $W^{2,2}(P)$ solution $V(\tau, x, p)$ to (26) on P with terminal condition (27) such that (24) holds. Fix any $\tilde{\phi}^Q(\cdot) \in \Phi_{\phi_{\varepsilon}}$ corresponding to some $Q(\cdot)$ and t, and take any $p(\tau, x) = (y^0_{\varepsilon}, y(\tau, x))$, $y(\cdot) \in W^{1,2}(P)$, $(\tau, x, p(\tau, x)) \in$

P, such that $\phi^Q(\tau,x) = V_y(\tau,x,p(\tau,x))$ for $(\tau,x) \in [0,1]\times\Omega$. Let $x(\tau,x_0,p)$, $x_0 \in \Gamma_0(\tilde{\phi}^Q)$, be a trajectory generated by the velocity Q and normal

$$n = \frac{\nabla_x V_y(\tau,x,p)}{|\nabla_x V_y(\tau,x,p)|},$$

i.e. $Q(x(\tau,x_0,p)) = x'(\tau,x_0,p)n$. From transversality condition (24), we see that for $(\tau,x)\in[0,1]\times\Omega$,

$$\begin{aligned} &V_\tau(\tau,x,p(\tau,x)) + t\,|\nabla_x V(\tau,x,p(\tau,x))|\,Q(x)\\ =\,& y_\varepsilon^0\left((d/d\tau)\,V_{y^0}(\tau,x,p(\tau,x)) + t\left|\nabla_x V_{y^0}(\tau,x,p(\tau,x))\right| x'(\tau,x_0,p(\tau,x))\right)\\ &+y(\tau,x)\left((d/d\tau)\,V_y(\tau,x,p(\tau,x)) + t\,|\nabla_x V_y(\tau,x,p(\tau,x))|\,Q(x)\right). \end{aligned} \tag{43}$$

Since $\phi^Q(\tau,x) = V_y(\tau,x,p(\tau,x))$ for $(\tau,x)\in[0,1]\times\Omega$, (13) shows that for $(\tau,x)\in[0,1]\times\Omega$,

$$(d/d\tau)\,V_y(\tau,x,p(\tau,x)) + t\,|\nabla_x V_y(\tau,x,p(\tau,x))|\,Q(x) = 0.$$

Now, define a function $[0,1]\times\Omega \ni (\tau,x)\mapsto W(\tau,x,p(\tau,x))$ by

$$\begin{aligned} &W(\tau,x,p(\tau,x)) =\\ &y_\varepsilon^0\Big[-(d/d\tau)\,V_{y^0}(\tau,x,p(\tau,x)) - t\left|\nabla_x V_{y^0}(\tau,x,p(\tau,x))\right| x'(\tau,x_0,p(\tau,x))\\ &+(\mathrm{div})^{-1}(\tilde{L}(x,\tilde{u}^{\tilde{\phi}^Q}(x),\nabla\tilde{u}^{\tilde{\phi}^Q}(x)))\frac{d}{d\tau}H(V_y(\tau,x,p)))\Big]. \end{aligned} \tag{44}$$

We conclude from (43)–(44) that

$$\begin{aligned} W(\tau,x,p(\tau,x)) = V_\tau(\tau,x,p(\tau,x)) + t\,|\nabla_x V(\tau,x,p(\tau,x))|\,Q(x)\\ +y_\varepsilon^0(\mathrm{div})^{-1}(\tilde{L}(x,\tilde{u}^{\tilde{\phi}^Q}(x),\nabla\tilde{u}^{\tilde{\phi}^Q}(x)))\tfrac{d}{d\tau}H(V_y(\tau,x,p))),\ (\tau,x)\in[0,1]\times\Omega. \end{aligned} \tag{45}$$

The inequality (37), together with (45) imply

$$-y_\varepsilon^0\varepsilon \le W(\tau,x,p(\tau,x)) \le 0 \ \text{ for } \ (\tau,x)\in[0,1]\times\Omega. \tag{46}$$

By integrating (46) along $x(\tau,x_0,p)$, $\tau\in[0,1]$ (we remember that in $x(\tau,x_0,p)$, $\tau=\tau t$) for any fixed $x_0\in\Gamma_0(\tilde{\phi}^{V_n})$ and applying (44) and (38), and next integrating over $\Gamma_0(\tilde{\phi}^{V_n})$, we obtain

$$-y_\varepsilon^0\int_{\Gamma_0(\tilde{\phi}^Q)} V_{y^0}(0,x,y_\varepsilon^0,\psi(x))dx \le -y_\varepsilon^0\int_D \tilde{L}(x,\tilde{u}^{\tilde{\phi}^Q}(x),\nabla\tilde{u}^{\tilde{\phi}^Q}(x)))H(\phi^Q(t,x))dx. \tag{47}$$

Following, similarly as above, from (42) we get

$$\begin{aligned} &-y_\varepsilon^0\int_{\Gamma_0(\tilde{\phi}^Q)} V_{y^0}(0,x,\psi(x))dx \ge\\ &-y_\varepsilon^0\int_D \tilde{L}(x,\tilde{u}^{\tilde{\phi}_\varepsilon^Q}(x),\nabla\tilde{u}^{\tilde{\phi}_\varepsilon^Q}(x)))H(\phi_\varepsilon^Q(t_\varepsilon,x))dxd\tau - y_\varepsilon^0\varepsilon vol(\Omega). \end{aligned}$$

Thus, from (47) and the last equality it follows that

$$
\begin{aligned}
&- y_\varepsilon^0 \int_D \tilde{L}(x, \tilde{u}^{\tilde{\phi}_\varepsilon^Q}(x), \nabla \tilde{u}^{\tilde{\phi}_\varepsilon^Q}(x))) H(\phi_\varepsilon^Q(t_\varepsilon, x)) dx d\tau \\
\leq &- y_\varepsilon^0 \int_D \tilde{L}(x, \tilde{u}^{\tilde{\phi}^Q}(x), \nabla \tilde{u}^{\tilde{\phi}^Q}(x))) H(\phi^Q(t, x)) dx d\tau - y_\varepsilon^0 \varepsilon vol(\Omega),
\end{aligned}
$$

which completes the proof. □

4. Algorithm

4.1. General description

Recall the previously introduced formula (see Theorem 2)

$$
\begin{aligned}
0 \leq -V_\tau(\tau, x, p) + &\inf_{t \in [0,1],\ Q \in F} \{-t \left|\nabla_x V(\tau, x, p)\right| Q(x) \\
&+ y_\varepsilon^0 (\mathrm{div})^{-1} \tilde{L}(x, \tilde{u}^{\tilde{\phi}^Q}(x), \nabla \tilde{u}^{\tilde{\phi}^Q}(x))) H(V_y(\tau, x, p))\Big\} \leq -y_\varepsilon^0 \varepsilon, \quad (\tau, x, p) \in P.
\end{aligned} \tag{48}
$$

Our first step is to find an auxiliary function V. To this effect, we base on the traditional dynamic programming theory - an auxiliary function V can be computed by iteration, i.e. to start from the initial V^0 and update iteratively according to

$$
\begin{aligned}
W^{k+1}(x, p) = V^k(x, p) + &\inf_{t \in [0, k \cdot \Delta\tau],\ Q \in F} \Big\{-t \left|\nabla_x V^k(x, p)\right| Q(x) \\
&+ y_\varepsilon^0 (\mathrm{div})^{-1} \tilde{L}(x, \tilde{u}^{\tilde{\phi}^Q}(x), \nabla \tilde{u}^{\tilde{\phi}^Q}(x))) H(V_y(\tau, x, p))\Big\}, \\
&(k \cdot \Delta\tau, x, p) \in P
\end{aligned} \tag{49}
$$

$$
V^{k+1}(x, p) = F(W^{k+1})(x, p)
$$

where $F(W^{k+1})(x, p)$ is a formula, which depends on W^{k+1}, and will be defined later. An initial function V^0 should satisfy the condition (24)

$$
V^0(x, p) = y^0 V_{y^0}^0(x, p) + y V_y^0(x, p)
$$

and (25)

$$
\phi(0, x, p) = -V_y^0(x, p), \text{ for } (0, x, p) \in P.
$$

Because at $k = 0$ we have that

$$
\phi(0, x, p) = \Psi(x)
$$

so

$$
-V_y^0(x, p) = \Psi(x), \text{ for } (0, x, p) \in P.
$$

The algorithm consists of the following steps

1. Define two connected domains D, Ω, $\bar{\Omega} \subset D$, Ω with Lipschitz boundary.
2. In D define a Lipschitz continuous function Ψ such that

$$\Psi(x) < 0 \text{ on } \Omega,\ \Psi(x) = 0 \text{ on } \partial\Omega,$$

3. In D introduce the rectangular mesh M and use it to discretise the function Ψ.
4. Define P as a set of points (τ, x, p), where $p = (y^0, y)$,

$$P = \{[0,1] \times \Omega \times [a_1, 0] \times [a_2, a_3]\},$$

where a_1, a_2 and a_3 are constant parameters of the algorithm, and discretise the set P with $\Delta\tau$, Δy^0 and Δy such that $1 = \Delta\tau \cdot c_\tau$, $0 = a_1 + \Delta y^0 \cdot c_{y^0}$ and $a_3 = a_2 + \Delta y \cdot c_y$, where a_1, a_2, a_2 are real numbers, c_τ, c_{y^0} and c_y are natural numbers, all at the beginning arbitrarily chosen (we change them if the calculated V^{k+1} does not satisfy (48)). Denote by P_d the discretization of P.
5. Take $\digamma$ as a finite set of functions Q

$$\digamma = \{Q : \ Q \in Lips(\mathbb{R}^n), \ -K \le Q(x) \le K,\, x \in D\},$$

where $K > 0$ is arbitrarily chosen (we do not relate Q, K to the gradient of the functional).
6. Calculate function V^0 in every node of M.
An initial function V^0 should satisfy condition (24)

$$V^0(x,p) = y^0 V^0_{y^0}(x,p) + y V^0_y(x,p)$$

and (25)

$$\phi(0,x,p) = -V^0_y(x,p), \text{ for } (0,x,p) \in P.$$

Because at $k = 0$ we have that

$$\phi(0,x,p) = \Psi(x)$$

so

$$-V^0_y(x,p) = \Psi(x), \text{ for } (0,x,p) \in P.$$

Define V^0 as

$$V^0(x,p) = y^0 - y\Psi(x).$$

7. Calculate W^1. To do this
   (a) i. For all triples $(0,x,p) \in P_d$ do
       ii. find a minimum of

$$y^0(\mathrm{div})^{-1}\tilde{L}(x, \tilde{u}^{\tilde{\phi}^Q}(x), \nabla\tilde{u}^{\tilde{\phi}^Q}(x))H(V^0_y(x,p)),$$

over all $Q \in \digamma$, and denote it by $M^0(x,p)$:

A. Find $\tilde{u}$ using software FreeFEM (or any other solver of PDE) from equation

$$\begin{aligned} A\tilde{u}(x) &= f(x,\tilde{u}(x)) \text{ a.e. on } \Omega, \\ \tilde{u}(x) &= \varphi(x) \text{ on } \partial\Omega\backslash\tilde{\Gamma}, \\ B\tilde{u}(x) &= \Upsilon(x) \text{ on } \tilde{\Gamma}. \end{aligned}$$

B. Calculate

$$y_\varepsilon^0(\mathrm{div})^{-1}\tilde{L}(x,\tilde{u}^{\tilde{\phi}^Q}(x),\nabla\tilde{u}^{\tilde{\phi}^Q}(x))H(V_y^0(x,p)).$$

iii. Put $W^1(x,p) = V^0(x,p) + M^0(x,p)$.

(b) To ensure that V^1 satisfies condition (24) define it as

$$V^1(x,p) = F(W^1)(x,p) = \frac{1}{2}y^0 y W^1(x,p).$$

8. For $k = 1,\ldots,c_\tau - 1$, based on W^{k+1}, calculate V^{k+1}. To do this

(a) For all triples $(k\cdot\Delta\tau, x, p) \in P_d$ do

i. find a minimum of

$$-t\left|\nabla_x V^k(x,p)\right| Q(x) + y^0(\mathrm{div})^{-1}\tilde{L}(x,\tilde{u}^{\tilde{\phi}^Q}(x),\nabla\tilde{u}^{\tilde{\phi}^Q}(x))H(V_y^k(x,p)),$$

over all discrete $t \in [0, k\cdot\Delta\tau]$ and $Q \in \digamma$, and denote it by $M^k(x,p)$:

A. Calculate $\tilde{\phi}^Q$ solving the H-J equation for the given Q.

B. Calculate the domain $\tilde{\Omega}_t(\tilde{\phi}^Q)$ in Ω.

C. Find $\tilde{u}^{\tilde{\phi}^Q}$ using software FreeFEM (or any other solver of PDE) from equation

$$\begin{aligned} &A\tilde{u}(x) = \tilde{f}(x,\tilde{u}(x)),\ x \in \tilde{\Omega}_t(\tilde{\phi}^Q), \\ &\tilde{u}(x) = \varphi(x) \quad \text{on} \quad \partial\tilde{\Omega}_t(\tilde{\phi}^Q)\backslash\tilde{\Gamma}_{\tilde{\phi}^Q}, \\ &B\tilde{u}(x) = \Upsilon(x) \text{ on } \tilde{\Gamma}_{\tilde{\phi}^Q}. \end{aligned}$$

D. Calculate

$$-t\left|\nabla_x V^k(x,p)\right| Q(x) + y^0(\mathrm{div})^{-1}\tilde{L}(x,\tilde{u}^{\tilde{\phi}^Q}(x),\nabla\tilde{u}^{\tilde{\phi}^Q}(x))H(V_y^k(x,p)).$$

E. Minimum over t is denoted by t_ε^k.

ii. Put $W^{k+1}(x,p) = V^k(x,p) + M^k(x,p)$.

(b) To ensure that V^{k+1} satisfies condition (24), define it as

$$V^{k+1}(x,p) = F(W^{k+1})(x,p) = \frac{1}{2}y^0 y W^{k+1}(x,p). \tag{50}$$

9. Calculate $V_y^{t_\varepsilon^{c_\tau}}(x,p)$ as

$$V_y^{t_\varepsilon^{c_\tau}}(x,p) = \frac{1}{2} y^0 W^{c_\tau}(x,p).$$

10. Calculate $p_\varepsilon(t_\varepsilon^{c_\tau}, x)$ from $\phi^Q(t_\varepsilon^{c_\tau}, x) = -V_y^{t_\varepsilon^{c_\tau}}(x, p_\varepsilon(t_\varepsilon^{c_\tau}, x))$ and check whether (42) is satisfied. If it is satisfied, then $\phi^Q(t_\varepsilon, x)$ is an $\varepsilon-$optimal deformation, if not, then go to step 6 and choose different $\digamma$ as a finite set of functions Q.

4.2. Example of implementation

We consider an implementation of our algorithm for a very simple example to focus on algorithm itself and not on technical details.

Put

- $\tilde{L_1} = 0$
- $\tilde{L_2} = x_1 x_2 (1 - x_1)(1 - x_2)$
- $\tilde{L} = \tilde{L_1} + \mathrm{div} \tilde{L_2}$
- $x_1 \in [0, 1]$
- $x_2 \in [0, 1]$
- $x = (x_1, x_2)$
- $\tau \in [0, 1]$
- $a_1 = -1$
- $y^0 \in [a_1, 0]$
- $a_2 = 0$
- $a_3 = 1$
- $y \in [a_2, a_3]$
- $y_\varepsilon^0 = -1$
- $\Delta_{x_1} = 0.1$, $c_{x_1} = 10$
- $\Delta_{x_2} = 0.1$, $c_{x_2} = 10$
- $\Delta_\tau = 0.1$, $c_\tau = 10$
- $\Delta_{y^0} = 0.1$, $c_{y^0} = 10$
- $\Delta_y = 0.1$, $c_y = 10$.

- Step 1. $\Omega = [0, 1] \times [0, 1]$ and $D = \Omega$.
- Step 2. In D define the Lipschitz continuous function Ψ as

$$\Psi(x) = -x_1 x_2 (1 - x_1)(1 - x_2).$$

- Step 3. In D we use a rectangular mesh M and discretise function Ψ in points from the set

$$(\Delta_{x_1} \cdot 0, \Delta_{x_1} \cdot 1, \ldots, \Delta_{x_1} \cdot c_{x_1}) \times (\Delta_{x_2} \cdot 0, \Delta_{x_2} \cdot 1, \ldots, \Delta_{x_2} \cdot c_{x_2}).$$

- Step 4. Put $P = [0,1] \times [0,1] \times [0,1] \times [-1,0] \times [0,1]$ and

$$\begin{aligned} P_d =& (\Delta_\tau \cdot 0, \Delta_\tau \cdot 1, \ldots, \Delta_\tau \cdot c_\tau) \\ & \times (\Delta_{x_1} \cdot 0, \Delta_{x_1} \cdot 1, \ldots, \Delta_{x_1} \cdot c_{x_1}) \\ & \times (\Delta_{x_2} \cdot 0, \Delta_{x_2} \cdot 1, \ldots, \Delta_{x_2} \cdot c_{x_2}) \\ & \times (a_1 + \Delta_{y^0} \cdot 0, a_1 + \Delta_{y^0} \cdot 1, \ldots, a_1 + \Delta_{y^0} \cdot c_{y^0}) \\ & \times (a_2 + \Delta_y \cdot 0, a_2 + \Delta_y \cdot 1, \ldots, a_2 + \Delta_y \cdot c_y). \end{aligned}$$

- Step 5. Define $\digamma$ as a finite set of functions Q

$$\digamma = \left\{ Q(x) = \; - k2^k \colon \; k = 0, 1, \ldots, 50, \; x \in D \right\}.$$

Now we can rewrite formulas (48) and (49) as

$$0 \le -V_\tau(\tau, x, p) + \inf_{t \in [0,1], \; Q \in \digamma} \{-t \left| \nabla_x V(\tau, x, p) \right| Q(x) - x_1 x_2 (1 - x_1)(1 - x_2)\} \le \varepsilon, \tag{51}$$

$$\begin{aligned} & W^{k+1}(x,p) = \\ & V^k(x,p) + \inf_{t \in [0, k\Delta\tau], \; Q \in \digamma} \left\{ -t \left| \nabla_x V^k(x,p) \right| Q(x) - x_1 x_2 (1 - x_1)(1 - x_2) \right\}. \end{aligned} \tag{52}$$

- Step 6. Define V^0 as

$$V^0(x,p) = y^0 + y x_1 x_2 (1 - x_1)(1 - x_2).$$

- For all triples $(0, x, p) \in P_d$ calculate
  - Step 7a i.

    $$M^0(x,p) = y^0 x_1 x_2 (1 - x_1)(1 - x_2).$$

  - Step 7a ii.

    $$W^1(x,p) = V^0(x,p) + M^0(x,p).$$

  - Step 7b.

    $$V^1(x,p) = Y \left(V^0(x,p) + M^0(x,p) \right),$$

    where $Y = \frac{1}{2} y^0 y$.

- Changing k from 1 to $c_\tau - 1$ for all triples $(k\Delta_\tau, x, p) \in P_d$ and $t_\varepsilon^k = k\Delta\tau$ calculate
  - Step 8a i.

    $$M^k(x,p) = M^0(x,p) = y^0 x_1 x_2 (1 - x_1)(1 - x_2).$$

– Step 8a ii.

$$W^{k+1} = V^k + M^0 = Y^k V^0 + M^0(Y^k + Y^{k-1} + ... + Y + 1).$$

– Step 8b.

$$V^{k+1} = YW^{k+1} = Y^{k+1}V^0 + M^0(Y^{k+1} + Y^k + ... + Y^2 + Y).$$

- Step 9. Calculate $V_y^{t_\varepsilon^{10}}$ with $t_\varepsilon^{10} = 1$ for all $(x, p) \in [0, 1]\times[0, 1]\times[-1, 0]\times[0, 1]$ as
$$V_y^{t_\varepsilon^{10}} = \frac{1}{2}y^0 W^{10} = \frac{1}{2}y^0 \left(Y^9 V^0 + M^0(Y^9 + Y^8 + ... + Y + 1)\right).$$
- Step 10. Notice that $V_y^{t_\varepsilon^k} > 0$, therefore $H(V_y^{t_\varepsilon^k}) = 0$. Moreover, the minimum in (52) is attained for $Q = 0$ and so $\phi^0(t_\varepsilon, x) = -x_1x_2(1-x_1)(1-x_2)+$ *const*, assume *const* $= 0$. Thus, $y = 0$ satisfies (51) with any $\varepsilon > 0$. Hence $\phi^0(t_\varepsilon, x) = -x_1x_2(1 - x_1)(1 - x_2)$ is an $\varepsilon-$optimal deformation.

5. Conclusion

The advantage of the algorithm presented above is that after having calculated the result, the difference between this result and the exact solution can be precisely calculated. In other words, we know how far this result is from the real optimal value. As a disadvantage notice that solution is searched among the predefined elements of finite sets $\mathcal{F}$. If those sets are generated to be "dense" or "representative" for all solutions, the result we find is near to the real optimal value. The question how to generate the representative sets remains still open.

References

BEDNARCZUK, E., PIERRE, M., ROUY, E. and SOKOŁOWSKI, J. (2000) Tangent sets in some functional spaces. *Nonlinear Anal.,* 42*, Ser. A: Theory Methods*, 871–886.

BELLMAN, R. (1957) *Dynamic Programming.* Princeton University Press, Princeton.

BURGER, M., HACKL, B. and RING, W. (2004) Incorporating Topological Derivatives into Level Set Methods. *Journal of Computational Physics* **194** (1), 344–362.

DELFOUR, M. C. and ZOLESIO, J. P. (2001) *Shapes and Geometries - Analysis, Differential Calculus and Optimization. Advances in Design and Control*, SIAM.

FLEMING, W.H. and RISHEL, R.W. (1975) *Deterministic and Stochastic Optimal Control.* Springer–Verlag, New York.

FULMAŃSKI, P., LAURIN, A., SCHEID, J.F. and SOKOŁOWSKI, J. (2007) A Level Set Method in Shape and Topology Optimization for Variational Inequalities. *International Journal of Applied Mathematics and Computer Science*, 17, 413-430.

GALEWSKA, E. and NOWAKOWSKI, A. (2006) A dual dynamic programming for multidimensional elliptic optimal control problems. *Numer. Funct. Anal. Optim.*, 27, 279–289.

GARREAU, S., GUILLAUME, P. and MASMOUDI, M. (2001) The topological Asymptotic for PDE Systems: the Elasticity Case. *SIAM Journal on Control Optimization*, 39, 1756–1778.

HASLINGER, J. and MÄKINEN, R. (2003) *Introduction to Shape Optimization. Theory, Approximation and Computation.* SIAM Publications, Philadelphia.

HÜBER, S., STADLER, G. and WOHLMUTH, B. (2008) A Primal-Dual Active Set Algorithm for Three Dimensional Contact Problems with Coulomb Friction. *SIAM J. Sci. Comput*, **30** (2), 572-596.

MAURER, H., OBERLE, J. (2002) Second order sufficient conditions for optimal control problems with free final time: the Riccati approach. *SIAM J. Control Optim*, **41** (2) 380-403.

MYŚLIŃSKI, A. (2004) A Level Set Method for Shape Optimization of Contact Problems. In: P. Neittaanmäki ed. *CD-ROM Proceedings of European Congress on Computational Methods in Applied Sciences and Engineering*, Jyväskylä, Finland, 24-28 July 2004. WIT Press, Southampton–Boston.

MYŚLIŃSKI, A. (2005) Topology and Shape Optimization of Contact Problems using a Level Set Method. In: J. Herskovits, S. Mazorche, A. Canelas, eds., *Proceedings of VI World Congresses of Structural and Multidisciplinary Optimization*, Rio de Janeiro, Brazil, 30 May - 2 June 2005. CD-ROM: WCSMO6, International Society for Structural and Multidisciplinary Optimization.

MYŚLIŃSKI, A. (2010) Radial Basis Function Level Set Method for Structural Optimization. *Control Cybernet.* **39**, 3, 627–645.

NOWAKOWSKI, A. (2008) ε-Value Function and Dynamic Programming. *Journal of Optimization Theory and Applications*, **138**, 1, 85–93.

NOWAKOWSKI, A. (1992) The dual dynamic programming. *Proceedings of the American Mathematical Society*, 116, 1089–1096.

NOWAKOWSKI, A. (2013) ε–optimal value and approximate multidimensional dual dynamic programming. *Asian J. Control* **15**, 2, 444–452.

NOWAKOWSKI, A. and SOKOŁOWSKI J. (2012) On dual dynamic programming in shape control. *Commun. Pure Appl. Anal.* **11**, 6, 2473–2485.

SETHIAN, J. A. (1987) Numerical Methods for Propagating Fronts. In: P. Concus and R, Finn, eds., *Variational Methods for Free Surface Interfaces.* Springer–Verlag.

SETHIAN, J. A. (1996) A fast marching level set method for monotonically advancing fronts. *Proc. Nat. Acad. Sci.* **93** (4), 1591.

SETHIAN, J. A. and A. WIEGMANN (2000) Structural Boundary Design via Level Set and Immersed Interface Methods. *Journal of Computational Physics* 163, 489–528.

SOKOŁOWSKI, J. and ZOLESIO, J.P. (1992) *Introduction to Shape Optimiation.* Springer–Verlag.

SOKOLOWSKI, J. and ZOCHOWSKI, A. (1999) Topological derivative for elliptic problems. *Inverse Problems*, 15, 123–134.

SOKOŁOWSKI, J. and ŻOCHOWSKI, A. (2003) Optimality Conditions for Simultaneous Topology and Shape Optimization. *SIAM Journal of Control*, **42** (4), 1198–1221.

SOKOŁOWSKI, J. and ŻOCHOWSKI, A. (2004) On Topological Derivative in Shape Optimization. In: T. Lewiński, O. Sigmund, J. Sokołowski, A. Żochowski, eds., *Optimal Shape Design and Modelling*. Academic Printing House EXIT, Warsaw, Poland, 55–143.

SOKOŁOWSKI, J. and ŻOCHOWSKI, A. (2005) A Modeling of Topological Derivatives for Contact Problems. *Numerische Mathematik* **102** (1), 145–179.

STADLER, G. (2004) Semismooth Newton and Augmented Lagrangian methods for a Simplified Friction Problem. *SIAM Journal on Optimization* **15** (1), 39–62.