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Berge equilibrium in discontinuous games*

by

Messaoud Deghdak

Département de Mathématiques, Faculté des Sciences Exactes,
Université de Constantine 1, Algeria
deghdak.messaoud@gmail.com

Abstract: In this paper we consider some classes of abstract discontinuous games, for which the games possessing essential Berge equilibrium are the generic case. We extend the essential Berge equilibrium result from Deghdak (2014) to general abstract games.

Keywords: Berge equilibrium, game theory, discontinuous games, abstract games

1. Introduction

The concept of Berge equilibrium for a non cooperative game with a finite number of persons goes back to the book of Berge (1957). This equilibrium means that if each person plays an own strategy at a Berge equilibrium, then this person obtains the maximum payoff if all the remaining players play their strategy in the Berge equilibrium. It is worth noticing that the Berge equilibrium is generally different from the Nash equilibrium (see Nash, 1951), since the Nash equilibrium is stable with respect to the deviation of any unique player.

The existence of Berge equilibrium has been studied by Abalo and Kostreva (2004, 2005), Nessah, Larbani and Tazdait (2007), and Larbani and Nessah (2008). In their paper, Larbani and Nessah (2008) showed that a Berge equilibrium could also be considered as a Nash equilibrium (called Berge-Nash equilibrium) under certain assumptions. Later, Colman et al. (2011) have proven the existence of Berge equilibrium by establishing a correspondence with Nash equilibrium. Recently, Musy, Pottier and Tazdait (2012) have established the existence of Berge equilibrium without using the notion of Nash equilibrium. In all the previously mentioned works, the authors have assumed that payoffs of players are continuous.

The most studied solution concept in game theory is the Nash equilibrium. The problem with the Nash equilibrium is that it cannot be applied in many

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of the real life situations. This occurs, in particular, when there are several Nash equilibria. The selection problem can appear, which equilibrium to choose out of the set of the admissible ones. For this purpose, several refinements are introduced to solve this problem. For instance, the Berge's strong equilibrium introduced by Berge (1957) is one of these refinements. This latter equilibrium verifies stronger stability than the Nash equilibrium (it is stable with respect to the deviation of all players except one). For more details, see Deghdak (2011).

The Berge equilibrium concept, which is totally different from Berge's strong equilibrium, is particularly interesting as a complement to Nash equilibrium in the case of prisoner's dilemma (see Colman et al., 2011) and in chicken game (see Musy, Pottier and Tazdait, 2012).

Among refinements, Yu (1999) has introduced the notion of essential Nash equilibrium, which means that if two games are "close" then they possess two Nash equilibria which are also "close". This requiers a topological structure on the space of games and issues. Then, because upper semicontinuity of Nash equilibria correspondence implies lower semicontinuity on the subsets of Baire category and essential games are by definition equivalent to lower semicontinuity, Yu (1999) proves via upper semicontinuity that most of these games are essential in the sense of Baire category.

However, several authors have studied the existence of the Nash equilibrium and the essential Nash equilibrium for the case, when payoffs are not necessarily continuous (discontinuous games). Let us quote, for example, Reny (1999), Morgan and Scalso (2007), Scalso (2009, 2013), and Prokopovych (2013). Remarkable examples of such discontinuous games are constituted by the Bertrand oligopoly (see Bertrand, 1883) and the Hotelling's linear city model (see Hotelling, 1929).

The present author has proven in Deghdak (2014) the existence of the Berge equilibrium and the essential Berge equilibrium in discontinuous games. More precisely, the paper quoted considers a particular case of general games called abstract economies (see Borglin and Keiding, 1976); the strategies of players are represented by the constant correspondences defined from the space of issues into itself. The payoff functions of players are pseudocontinuous (see Morgan and Scalso, 2007). The focus of our paper is to extend the previous results to the context of abstract economies, in which the strategies of players are represented by feasible correspondences.

Our proof on the existence of Berge equilibrium is similar to the proof of the existence of Nash equilibrium. We define with slight modification the best reply correspondence of each player, then by the Kakutani fixed point theorem we show that the abstract economy possesses a Berge equilibrium. The proof for the essential Berge equilibrium is inspired by Yu (1999), Deghdak (2011), and Scalso (2009). First, we establish that the correspondence of Berge equilibria is upper semicontinuous. Second, we deduce in our setting that most of games (in the sense of the first class subsets of Baire category) have the essential Berge equilibrium. The outline of the paper is as follows. In Section 2, we first introduce the definitions of the Berge equilibrium for games in normal form and abstract economy. Then, we recall the existence theorem of Berge equilibrium for abstract economy proved in Deghdak (2014) by giving a sketch of the proof, and finally we establish the existence of the essential Berge equilibrium for the abstract economy. We end our work by giving some concrete applications of the Berge equilibrium. In Section 3, we give a summary on the problem considered in our paper.

2. Formulation of the problem

Let us consider the following game in normal form:

$$G = (I, (X_i, u_i)_{i \in I}) \tag{1}$$

where $I = \{1, ..., n\}$ is a finite set of players, X_i is a set of strategies of player i, $X = \prod_{i=1}^{n} X_i$ is the set of issues of the game G, and u_i: $X \to \boldsymbol{R}$ is a payoff function of player i.

For each player i, we let $I \backslash \{i\} = \{1, ..., i-1, i+1, ..., n\}$ and we denote by $X_{-i} = \prod_{j \neq i} X_j$. For each $x \in X$, we denote by x_{-i} the element in X_{-i}.

In choosing a strategy, $x_i \in X_i$, the aim of each player in the game G is to maximize this player's payoff function. Recall that$z \in X$ is a Nash equilibrium of the game G if for every $i \in I$, for all $x_i \in X_i$, $u_i(z) \geqslant u_i(x_i,\, z_{-i})$. The following definition is due to Berge (1957).

2.1. Berge equilibrium of a noncooperative game

Definition 1 *A Berge equilibrium of the game G is an n-tuple of strategies $z \in X$ such that $\forall i \in I$, $\forall y_{-i} \in X_{-i} : u_i(z) \geqslant u_i(z_i,\, y_{-i})$.*

2.2. Berge equilibrium of an abstract economy

We now consider the following generalized game that we call abstract economy (see Borglin, 1976).

Definition 2 *An abstract economy H is described by:*

$$H = (I, (X_i, F_i, u_i)_{i \in I}) \tag{2}$$

where $I = \{\ 1, ..., n\ \}$ is a finite set of players, X_i is a set of strategies of player i, and if $X = \prod_{i=1}^{n} X_i$, then u_i: $X \to \boldsymbol{R}$ is a payoff function of player i, while F_i: $X \to X_i$ denotes a feasibility correspondence for the player i, given the strategies of the other agents. Now, we give an extended version of the Definition 1 for an abstract game H.

Definition 3 *A Berge equilibrium of H is an n-tuple of strategies z such that:*

$$\forall i \in I, z_i \in F_i(z) \tag{3}$$

and

$$\forall i \in I, \forall y_{-i} \in \prod\nolimits_{j \in I \backslash \{i\}} F_i(z), u_i(z) \geqslant u_i(z_i, y_{-i}). \tag{4}$$

Condition (3) stipulates that z belong to the set of feasible strategies.

2.3. Best reply correspondence and fixed point of correspondences

Definition 4 *We call best reply correspondence for the player i in the game G, the correspondence Γ_i: $X \to X$, defined by:*

$$\Gamma_i(x) = \{y \in X : u_i(x_i, y_{-i}) \geqslant u_i(x_i, t_{-i}) \ \forall\ t_{-I} \in X_{-i}\} \tag{5}$$

According to Definition 4 above, if we set for each $x \in X$:

$$\Gamma(x) = \cap_{i \in I} \Gamma_i(x) \tag{6}$$

then, using Definition 3, a Berge equilibrium of the game G is a fixed point of the correspondence Γ, that is, an n-tuple $x \in \Gamma(x)$.

Remark 1. The following definition of the best reply correspondence of player i for the abstract game H is similar to Definition 4.

Definition 5 *We call best reply correspondence for the player i in the game H, the correspondence Γ_i: $X \to X$, defined by:*

$$\Gamma_i(x) = \{y \in X : y_{-i} \in \prod\nolimits_{j \in I \setminus \{i\}} F_j(x), u_i(x_i, y_{-i}) \geqslant \\ u_i(x_i, t_{-i}), \forall y_{-I} \in \prod\nolimits_{j \in I \setminus \{i\}} F_j(x)\}. \tag{7}$$

Then the Berge equilibrium of the abstract economy H is a fixed point of the correspondence $\Gamma(x) = \cap_{i \in I} \Gamma_i(x)$.

2.4. Existence of Berge equilibrium in discontinuous games

In this subsection we prove the existence of Berge equilibrium in the case where the payoff of each player i is not continuous. More precisely, we consider the class of pseudocontinuous payoffs (see Morgan and Scalso, 2007). In the following definitions, we introduce the notion of pseudocontinuity of functions.

Definition 6 *Let f be a real valued function defined on a topological vector space E. The function f is said to be upper pseudocontinuous at x_0 if for all $x \in E$ such that $f(x_0) < f(x)$ it follows that:*

$$\lim_{y \to x_0} \sup f(y) < f(x). \tag{8}$$

The function f is said to be upper pseudocontinuous on E if it is upper pseudocontinuous at all $x_0 \in E$.

Definition 7 *Let f be a real valued function defined on a topological vector space E. The function f is said to be lower pseudocontinuous at x_0 if -f is upper pseudocontinuous at x_0 and the function f is said to be lower pseudocontinuous on E if it is lower pseudocontinuous at all $x_0 \in E$.*

Definition 8 *Let f be a real valued function defined on a topological vector space E. The function f is said to be pseudocontinuous on E if it is both upper and lower pseudocontinuous on E.*

Remark 2. Any upper (respectively: lower) semicontinuous function f is upper (respectively: lower) pseudocontinuous but the converse is not true (see Example 4.1, Morgan and Scalso, 2007). Moreover, any pseudocontinuous function verifies the following better reply secure assumption given in Reny (1999) by the following definition (see Proposition 4.1, Morgan and Scalso, 2007).

Definition 9 *A game $G = (I, (X_i, u_i)_{i\in I}$ is better reply secure if for every non Nash equilibrium x of G and for every vector v such that (x,v) belongs to the closure of the graph of the vector $u = (u_1,\ldots,u_n)$, there exists a player i with strategy z_i such that $u_i(z_i,y_{-i}) > v_i + \varepsilon$ for all y_i with a suitable $\varepsilon > 0$.*

Pseudocontinuity on payoffs is a sufficient condition, explicit on any data, for the better reply secure assumption, and it is independent of payoff security and reciprocal upper semi continuity given in Reny (1999), see Example 4.1 of Morgan and Scalso (2007).

The following theorem on the existence of the Berge equilibrium in abstract economy H has been proved in details in Deghdak (2014). For completeness, we give here a sketch of the proof.

Theorem 1 *Assume the following assumptions on the game H:*

1. *$\forall i \in I$, X_i, is a nonempty, convex and compact subset of a locally convex topological vector space E_i;*
2. *$\forall i \in I$, $\forall x_i \in X_i$, the function $y_{-i} \to u_i(x_i,y_{-i})$ is quasi concave on X_{-i};*
3. *$\forall i \in I$, the function u_i is pseudocontinuous on X;*
4. *$\forall i \in I$, the correspondence F_i: $X \to X_i$ is continuous with nonempty, convex and compact values;*
5. *$\forall x \in X$, $\cap_{i\in I}\Gamma_i(x) \neq \emptyset$.*

Then the game H has a Berge equilibrium.

Proof. We prove that the correspondence $\Gamma(x) = \cap_{i\in I}\Gamma_i(x)$ verifies the Kakutani fixed point theorem (see Florenzano, 2003). Indeed, an easy adaptation of the Berge maximum theorem (see Theorem 3.1, Morgan and Scalso, 2007) shows that each correspondence Γ_i has a closed graph. Then, $\Gamma(x) = \cap_{i\in I}\Gamma_i(x)$ is a correspondence with nonempty, convex values and is upper semicontinuous (intersection of the correspondences Γ_i).

2.5. Essential Berge equilibrium in an abstract economy

In this subsection, we assume that each strategy space E_i is a normed space. Let us consider the games $(I,\ (F_i)_{i=1}^n, (u_i)_{i=1}^n)$ parameterized by the payoff profiles and feasibility strategies correspondences, which satisfy Theorem 1. Let denote by V the set of vectors of such games (see Deghdak, 2011) endowed with the metric $\rho : V \to \bar{R}$:

$$\rho(v^1, v^2) = \sum_{i=1}^{n} \sup_{x\in X} \left|u_i^1(x) - u_i^2(x\ )\right| + \sum_{i=1}^{n} \sup_{x\in X} H_i(F_i^1(x), F_i^2(x)) + \sum_{i=1}^{n} \sup_{x\in X} T_i(F_{-i}^1(x), F_{-i}^2(x)) \quad (9)$$

where H_i is the Hausdorff distance on the set of subsets of the normed space E_i and T_i is the Hausdorff distance on the set of the subsets of the normed space $E_i = \prod_{j\neq i} E_j$. Suppose that ρ takes on finite values, then V is a complete metric space. Now, we define the Berge equilibria correspondence J: $V \to X$, where for each $v \in V$, $J \subseteq X$ is the set of Berge equilibria of the game H. Then, we have the following theorem.

Theorem 2 *The Berge equilibria correspondence J is upper semicontiunous with nonempty and compact values.*

Proof. The correspondence J has nonempty values by virtue of Theorem 1. We prove that the correspondence J is closed. Let $(v^n,\ x^n)$ be a sequence of the graph of J such that $(v^n,\ x^n) \to (v, x) \in V \times X$. We have $lim_n v^n = v$ and $lim_n x^n = x$. Proceeding as in Theorem 4 from Deghdak (2011) , we obtain that $\forall i \in I,\ x_i \in F_i(x)$.

Suppose that $x \notin J(v)$, then there exists i_0 such that:

$$u_{i_0}(x_{i_0}, x_{-i_0}) < u_{i_0}(x_{i_0}, u^0_{-i_0}) \quad (10)$$

where

$$u^0_{-i_0} \in F_{-i_0}(x). \quad (11)$$

Since the function u_{i_0} is pseudocontinuous and the subset $\prod_{i=1}^n F_i(x)$ is connected (as a product of convex sets), then there exist x^0, x^1 (see Proposition 2.2 in Scalso, 2009) such that:

$$u_{i_0}(x_{i_0}, x_{-i_0}) < u_{i_0}(x^0_{i_0}, x^0_{-i_0}) < u_{i_0}(x^1_{i_0}, x^1_{-i_0}) < u_{i_0}(x_{i_0}, u^0_{-i_0}). \quad (12)$$

Using the pseudocontinuity of u_i at the point (x_{i_0}, x_{-i_0}) we obtain:

$$u_{i_0}(z_{i_0}, z_{-i_0}) < u_{i_0}(x^0_{i_0}, x^0_{-i_0}) < u_{i_0}(x^1_{i_0}, x^1_{-i_0}) < u_{i_0}(x_{i_0}, u^0_{-i_0}) \quad (13)$$

for all (z_{i_0}, z_{-i_0}) in the neighbourhood $V(x_{i_0}) \times V(x_{-i_0})$ of the point (x_{i_0}, x_{-i_0}).

Let $V(u^0_{-i_0})$ be an open subset of X_{-i_0} such that $u^0_{-i_0} \in F_{-i_0}(x)$.
Since

$$V(u^0_{-i_0}) \cap F_{-i_0}(x) \neq \emptyset \quad (14)$$

and $\rho(v^n,v)$ converges to zero, it follows from Theorem 4 in Deghdak (2011) that:

$$V(u^0_{-i_0}) \cap F^n_{-i_0}(x^n) \neq \emptyset \tag{15}$$

and

$$x^n \in V(x_{i_0}) \times V(x_{-i_0}). \tag{16}$$

Pick

$$u^n_{-i_0} \in V(u^0_{-i_0}) \cap F^n_{-i_0}(x^n). \tag{17}$$

It is obvious that:

$$u_{i_0}(z_{i_0}, z_{-i_0}) + \rho(v^n, v) < u_{i_0}(x^1_{i_0}, x^1_{-i_0}) < u_{i_0}(x_{i_0}, u^0_{-i_0}). \tag{18}$$

Then

$$u^n_{i_0}(x^n_{i_0}, x^n_{-i_0}) < u_{i_0}(x^1_{i_0}, x^1_{-i_0}) < u_{i_0}(x_{i_0}, u^0_{-i_0}). \tag{19}$$

Once again, from the pseudocontinuity of u_{i_0} at the point $(x_{i_0}, u^0_{-i_0})$ and the Proposition 3.1, given in Scalso (2009), we deduce that:

$$u^n_{i_0}(x^n_{i_0}, x^n_{-i_0}) < u^n_{i_0}(x^n_{i_0}, u^n_{-i_0}). \tag{20}$$

The inequality (20) contradicts the fact that x^n constitute a sequence of Berge equilibria.

Since X is compact and J is closed, then J is upper semicontiunous with compact values.

In the following definition we introduce the notion of the essential equilibrium (see Yu, 1999).

Definition 10 *Let M be a nonempty and closed subset of V and $y \in M$. An equilibrium $x \in J(y)$ is called essential for the game y with respect to M if for any $O \in V(x)$ there exists $W \in V(y)$ such that for each $y_1 \in M \cap W$, there exists $x_1 \in J(y_1)$.*

This definition means that if two games are "close", then they possess, respectively, two equilibria which are also "close" and this fact is equivalent to the lower semicontinuity of the correspondence J (see Theorem 4.1 in Yu, 1999).

In the following theorem, we establish that most of games (in the sense of the Baire category) in V possess essential equilibria.

Theorem 3 *Let V be the space of games defined in Subsection 2.5, then most of games in V possess the essential equilibria.*

Proof. The proof is similar to those in Deghdak (2011) and Yu (1999). Since the correspondence J is upper semicontinuous (see Theorem 2), it follows from Lemma 2.1 in Yu (1999) that J is lower semicontinuous on the first class subsets of the Baire category. Since the correspondence J is lower semicontinuous, we deduce that most of games in V are essential in the sense of the Baire category.

2.6. Applications

In this subsection we present some applications of our results, which illustrate the potential application of the Berge equilibrium. First, we show by an example that the Nash and Berge equilibria are perfectly complementary. The Nash equilibrium reflects the behaviour of players, who act in their own interest, while the Berge equilibrium reflects the behaviour of players, who act in a mutually supportive way. Then we consider an application of our results to the oligopoly markets, as given in Example 4.2 of Nessah and Larbani (2014).

Example 1. In this example, we consider a finite economic game given in Example 4.2 of Nessah and Larbani (2014), where two firms A and B are considered, which sell the same product. They have the same strategies, the regular price (RP) or the cut down price (CP). The payoff function of each firm is given by the Table 1.

Table 1. Payoffs for the companies from Example 1. RP - regular price, CP - cut down price.

	RP	CP
RP	(11,11)	(7,8)
CP	(12,7)	(8,8)

In this payoff table firm A is the row player and firm B is the column player. It is easy to check that the strategies (RP,RP) and (CP,CP) are, respectively, the Berge and Nash equilibria. It is clear that (RP,RP) is better than (CP,CP), therefore both firms would do better by moving away from the (CP,CP) strategy. Numerous examples in this context may be found in the existing literature, as pointed out in the introduction.

Example 2. Let us consider Example 4.2 of Nessah and Larbani (2014) for the oligopoly markets (Cournot model) given in the abstract form. First, we recall the basic model. A single good is produced by n firms, and [0, c] is the production set of each firm. The cost of the firm i to produce x_i units of the good is denoted by the values of the function $c_i(x_i)$. All the output is sold at the single price P determined by the demand and the total output of the firms. Since

$\sum\limits_{i=1}^{n} x_i$ and $P(\sum\limits_{i=1}^{n} x_i)$

are, respectively, the total output and the market price, then, the revenue of each of the firms i is

$x_i P(\sum\limits_{i=1}^{n} x_i)$

The payoff of each firm is defined by:

$$u_i(x_1, ..., x_n) = x_i P(\sum_{i=1}^{n} x_i) - c_i(x_i) \tag{21}$$

In Nessah and Larbani (2014), the market price is called the inverse demand function and is denoted by F. The function $F$and the cost function $c_i(x_i)$ are twice differentiable with, respectively, negative and positive second derivatives. For this continuous game model, the necessary and sufficient conditions are given for the existence of Berge equilibrium (see Proposition 4.3 in the reference quoted).

In order to adapt this application to our context of Berge equilibrium in discontinuous games, we need some necessary modifications regarding this example in order to verify conditions (1)-(5) in Theorem 1 of Subsection 2.4. For simplicity, we may take $n = 2$, and identically null cost functions $c_i(x_i)$. The payoff functions of the two players are defined as follows:

$$u_i(x_i, x_{-i}) = \begin{cases} l_i(x_i) & if \quad x_i < x_{-i} \\ \phi_i(x_i) & if \quad x_i = x_{-i} \\ m_i(x_i) & if \quad x_i > x_{-i} \end{cases}, \tag{22}$$

where $l_i(x_i) = \alpha x_i f(x_i)$, $\phi_i(x_i) = \beta x_i f(x_i)$, and $m_i(x_i) = \gamma x_i f(x_i)$, for $i = 1, 2$, such that α, β, $\gamma \in \boldsymbol{R}$, and $f(.)$ is a function defined to map $\boldsymbol{R}$ into itself. Because the strategy space is $\boldsymbol{R}^2$ and the strategy subsets of each player are [0,c], then condition (1) of Theorem 1 is satisfied. Condition (4) is also verified, since the game is in normal form. We can impose some conditions on α, β, γ and $f(.)$ such that conditions (2), (3) and (5) from Theorem 1 are satisfied. For instance, the function $l_i(x)$ is positive nondecreasing, and the function $\phi_i(x) \in co\{l_i(x), m_i(x)\}$, where co{,} denotes the convex hull. Quasi-concavity is then verified, which implies satisfaction of the condition (2) in Theorem 1. It is possible to choose accordingly the functions $l_i(x_i)$, $\phi_i(x_i)$ and $m_i(x_i)$ in such a way as to have the assumptions (3) and (5) also satisfied.

3. Summary

In this paper, we have proved that abstract games having essential Berge equilibria are the generic case in the space of discontinuous games. We have used the weakening of continuity called pseudocontinuity in Morgan and Scalso (2007). In the setting of games in normal form, Deghdak (2014) has shown that this hypothesis could be weakened by the better reply assumption from Reny (1999). However, in the present paper, this assumption could not be relaxed by better reply in the case of abstract economy (see Theorems 1 and 2). The maximum theorem of Berge in Morgan and Scalso (2007), the fixed point theorem of Kakutani and the Baire theorem have played the central role in the main results of the present paper.

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