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Average case analysis of the set packing problem*

by

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Abstract: The paper deals with the well known set packing problem and its special case, when the number of subsets is maximized. It is assumed that some of the problem coefficients are realizations of mutually independent random variables. Average case (i.e. asymptotical probabilistic) properties of selected problem characteristics are investigated for the variety of possible instances of the problem. The important results of the paper are:

- Behavior of the optimal solution values of the set packing problem is presented for the special asymptotic case, where mutual asymptotical relation between m (number of elements of the packed set) and n (number of sets provided) is playing an essential role.
- Probability of reaching feasible solution is reasonably high (i.e. $\geqslant 2/e, 2/e \approx 0.736$); moreover, it may be set arbitrarily close to 1 (e.g. 0.999), although the deterioration in the quality of approximation of the behavior of the optimal solution values may be substantial.
- Some relations between the general case of the set packing problem and its maximization for the special case are investigated.

1. Introduction

Let us consider an m element set M and Φ a collection of n subsets M_i, $i = 1, \ldots, n$, of the set M, $\Phi = \{M_1, M_2, \ldots, M_n\}$. *The set packing problem* consists in finding a set of disjoint subsets Ψ in Φ, $\Psi \subseteq \Phi$, where, $M_i, M_k \in \Psi$ if and only if $M_i \cap M_k = \emptyset$, for every i, k, $i \neq k$, $i, k \in \{1, \ldots, n\}$. The set packing

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problem is often formulated as the binary multiconstraint knapsack problem, see Nemhauser and Wolsey (1988):

$$\begin{array}{ll} z_{OPT}(n) = \max \sum\limits_{i=1}^{n} c_i \cdot x_i & \\ \text{subject to} \quad \sum\limits_{i=1}^{n} a_{ji} \cdot x_i \leqslant 1 & \\ \text{where} \quad j = 1, ..., m, \quad x_i = 0 \text{ or } 1. & \end{array} \tag{1}$$

It is assumed that:

$$c_i > 0,\ a_{ji} = 0 \text{ or } 1,\ i = 1, \ldots, n,\ j = 1, \ldots, m.$$

In fact, a_{ji}, $i = 1, \ldots, n$, $j = 1, \ldots, m$, $j \in M$, are defining Φ, as the set of subsets of M, namely M_i, $i = 1, \ldots, n$, $\Phi = \{M_1, M_2, \ldots, M_n\}$, in the following way

$$a_{ji} = \begin{cases} 1 & \text{if } j \in M_i \\ 0 & \text{if } j \notin M_i \end{cases},$$

where c_i is the value expressing the preference assigned to the set M_i. Let us observe that the definition of the sets M_i, $i = 1, \ldots, n$, does not require them to be disjoint. Namely, if there exists $j \in \{1, \ldots, m\}$, $k \neq l$, $k, l \in \{1, \ldots, n\}$, such that $a_{jk} = a_{jl} = 1$, then $j \in M$ belongs to both M_k and M_l, i.e. $M_k \cap M_l \neq \emptyset$. The choice of x_i, fulfilling the constraints imposed in (1), is defining the packing of the set M into disjoint subsets M_i, $M_i \in \Psi$, where $M_i \cap M_k = \emptyset$ $i \neq k$, $i, k \in \{1, \ldots, n\}$, for every M_i, $M_k \in \Psi$. Namely, in (1)

$$\forall\ k,\ k \in \{1, \ldots, n\},\ M_k \in \Psi,\ \text{if and only if } \exists\ j \in M_k : a_{jk} \cdot x_k = 1. \tag{2}$$

Each of the constraints $\sum_{i=1}^{n} a_{ji} \cdot x_i \leqslant 1$, $j = 1, \ldots, m$ is guaranteeing that each of the items j of the set M is assigned to at most one of the subsets M_i, $M_i \in \Psi$. Optimisation criterion in (1) is securing the choice of the best possible packing, according to preferences expressed by c_i, $i = 1, \ldots, n$. If $c_i = c$, $i = 1, \ldots, n$, c - constant (e.g. $c = 1$), then the optimisation problem consists in seeking the maximum amount of subsets M_i to pack the set M, known as the *Maximum Set Packing Problem*. The maximum set packing problem maybe also formulated as the binary multiconstraint knapsack problem, similarly to (1), namely:

$$\begin{array}{ll} z_{OPT}(n) = \max \sum\limits_{i=1}^{n} x_i & \\ \text{subject to} \quad \sum\limits_{i=1}^{n} a_{ji} \cdot x_i \leqslant 1 & \\ \text{where} \quad j = 1, ..., m, \quad x_i = 0 \text{ or } 1. & \end{array} \tag{3}$$

Set packing problems arise in partitioning applications, where there is strong requirement that no elements of the set M be permitted to be included in more than one subset M_i. The set packing problem (1) is well known to be an $\mathcal{NP}$ hard combinatorial optimisation problem, see Garey and Johnson (1979). Moreover, the Set Packing Problem is one of the 21 first Karp's $\mathcal{NP}$ complete problems, see Karp (1972). There are also two closely related combinatorial problems, namely the *set covering problem* and the *set partitioning problem* (also known as *exact covering*), where in both of them one is looking for the subsets M_{kj}, $j = 1, ..., r$, of the collection Φ of n subsets of M_i, $i = 1, \ldots, n$, where demand $\bigcup_{j=1}^{r} M_{kj} = M$ holds, moreover, in the set partitioning problem there is an additional demand, namely that all M_{kj} be pairwise disjoint, i.e. $M_{k_j} \cap M_{k_l} = \emptyset$, for every $k_j, k_l,\ k_j \neq k_l,\ j, l \in \{1, \ldots, r\}$. Both problems may be also formulated as special cases of the binary multiconstraint knapsack problem, see Nemhauser and Wolsey (1988). The maximum set packing problem is also known as the *Maximum Hypergraph Matching*. The latter, under certain additional conditions, is equivalent to the well known *Maximum Clique problem*, see Ausiello, D'Atri and Protasi (1980). Another example of the application of the set packing problem in the graph theory is the so called *independent set*, i.e. the set of graph vertices having no common edges.

Scheduling of an airline flight crews with respect to airplanes is a good example of a practical application of the set packing problem. Each airplane must have a crew assigned to it, consisting of a pilot, a copilot, and a navigator. There is a set of possible crew members, based on their training in operating the relevant types of airplanes, as well as any personality conflicts. Considering all possible crews and airplane combinations, each represented by a subset of items, our goal is to find such an assignment of crews to airplanes that each airplane and each crew member is in exactly one selected combination. From the mathematical point of view, one is looking for a set packing, taking into account subset constraints. Simply, in the considered time period the same crew members cannot be on two different airplanes and every airplane must have a crew, but not all of the crew members must be assigned. In the case of the set partitioning problem all of the crew members must be assigned and in the case of the set covering problem some crew members may be assigned to multiple airplanes.

As it was already mentioned, the set packing problem is often formulated as the binary multiconstraint knapsack problem, see (1) and (3). However, the above formulations constitute a rather special case of it, see Martello and Toth (1990). Its peculiarity consists in the following facts:

- All the coefficients of the left hand sides of constraints are equal either to 1 or to 0:

$$a_{ji} = 0 \text{ or } 1,\ i = 1, \ldots, n,\ j = 1, \ldots, m.$$

- All of the coefficients of the right hand sides of constraints are equal to 1.
- The number of constraints m maybe arbitrarily big in comparison with n (number of decision variables).

In the general formulation of the binary multiconstraint knapsack problem it is only required that all of the knapsack problem coefficients, i.e. goal function, constraints' left and right hand sides, be non-negative or, in order to avoid unclear interpretations, strictly positive. The latter especially applies to goal function and coefficients of the right hand sides of constraints. It is usually also assumed that m (the number of constraints) is not large with respect to the number of decision variables n.

This does mean that the results obtained for the general knapsack problem, e.g. in the case of Lagrange and dual estimations or asymptotic probabilistic analysis of the optimal solution value behavior, may not be valid in the case of the set packing problem specific formulations provided in (1) or (3). In the present paper, the set packing problem (1) The specific Lagrange and dual estimations are provided. Then, for the random model of the problem (1) interesting results concerning the feasibility of the obtained solutions and asymptotical growth of the optimal solution values $z_{OPT}(n)$, when $n \to \infty$, are provided.

2. Definitions

The following definitions are necessary for further presentation:

Definition 1 *We denote* $V_n \approx Y_n$, *where* $n \to \infty$, *if*

$$Y_n \cdot (1 - o(1)) \leqslant V_n \leqslant Y_n \cdot (1 + o(1))$$

when V_n, Y_n *are sequences of numbers, or*

$$\lim_{n\to\infty} P\{Y_n \cdot (1 - o(1)) \leqslant V_n \leqslant Y_n \cdot (1 + o(1))\} = 1$$

when V_n *is a sequence of random variables and* Y_n *is a sequence of numbers or random variables, where* $\lim_{n\to\infty} o(1) = 0$ *as it is usually presumed.*

Definition 2 *We denote* $V_n \preceq Y_n (V_n \succeq W_n)$ *if*

$$V_n \leqslant (1 + o(1)) \cdot Y_n \ (V_n \geqslant (1 - o(1)) \cdot W_n)$$

when V_n, Y_n (W_n) *are sequences of numbers, or*

$$\lim_{n\to\infty} P\{V_n \leqslant 1 + o(1) \cdot Y_n\} = 1 \ \lim_{n\to\infty} P\{V_n \geqslant 1 - o(1) \cdot W_n\} = 1$$

when V_n *is a sequence of random variables and* Y_n (W_n) *is a sequence of numbers or random variables, where* $\lim_{n\to\infty} o(1) = 0$.

Definition 3 *We denote* $V_n \approxeq Y_n$ *if there exist constants* $c'' \geqslant c' > 0$ *such that*

$$c' \cdot Y_n \preceq V_n \preceq c'' \cdot Y_n$$

where Y_n, V_n *are sequences of numbers or random variables.*

The following random model of (1) will be considered in the paper:

- m, n, $0 < n \leqslant m!$, are arbitrary positive integers and, moreover, $n \to \infty$.
- c_i, a_{ji}, $i = 1, \ldots, n$, $j = 1, \ldots, m$, are realizations of mutually independent random variables and, moreover, c_i are uniformly distributed over $(0, 1]$ and $P\{a_{ji} = 1\} = p$, where $0 < p \leq 1$.

Let us observe that asymptotical relations $0 < n \leqslant m!$ and $n \to \infty$ require that also $m \to \infty$. As the matter of fact, mutual asymptotic relation of the values of m and n may vary between two extreme cases: $n/m \approx 0$ or $n \approx m!$ as $n \to \infty$

Under the assumptions made about c_i, a_{ji}, and taking into account model, (1) the following always holds

$$0 \leqslant z_{OPT}(n) \leqslant \sum_{i=1}^{n} c_i \leqslant n. \tag{4}$$

Moreover, from the strong law of large numbers it follows that

$$\sum_{i=1}^{n} c_i \approx E(c_1) \cdot n = n/2, \ \sum_{i=1}^{n} a_{ji} \approx p \cdot n, \ \sum_{j=1}^{m} a_{ji} \approx p \cdot m. \tag{5}$$

Therefore, it is justified to enhance formulas (4) and (5) in the following way:

$$0 \leqslant z_{OPT}(n) \preceq n/2, \ \sum_{i=1}^{n} a_{ji} \preceq 1, \text{ if } p < \frac{1}{n} \text{ or } \sum_{i=1}^{n} a_{ji} \succeq 1 \text{ when } p > \frac{1}{n}. \tag{6}$$

Formula (6) shows that the random model of the set packing problem (1) is complete in the sense that nearly all possible instances of the problem are considered.

The growth of $z_{OPT}(n)$, i.e. value of the optimal solution of the problem (1) may be influenced by the problem coefficients, namely:

$$n, \ m, \ c_i, \ a_{ji}, \text{ where } i = 1, \ldots, n, \ j = 1, \ldots, m.$$

We have assumed that c_i, a_{ji} are realizations of the random variables and therefore their impact on the $z_{OPT}(n)$ growth is in this case indirect. Moreover, we have also assumed that m, n are arbitrary positive integers and $n \to \infty$.

The main aim of the present paper is to perform probabilistic analysis of the considered class of random set packing problems in the asymptotical case, i.e. when $n \to \infty$. For the considered random model, the probabilistic analysis has two strategic goals, namely:

- To examine the existence of the feasible solutions.
- To investigate the asymptotic behavior of $z_{OPT}(n)$.

Existence of the feasible solution, provided by $x_1, \ldots, x_n$, means that $\sum_{i=1}^{n} a_{ji} \cdot x_i \leqslant 1$ for all $j = 1, \ldots .m$. If any of the constraints is violated, i.e. $\exists \ j'$ such that $\sum_{i=1}^{n} a_{j'i} \cdot x_i \geqslant 1$, then solution, provided by $x_1, \ldots, x_n$, is not feasible.

3. The Lagrange and dual estimations

When the general knapsack type problem, with one or many constraints, is considered, then the Lagrange function and the corresponding dual problems, see Averbakh (1994), Meanti et al. (1990), Szkatuła (1994, 1997) are very useful tools to perform various kinds of analyses of the original problem. In the specific case of the set packing problem (i.e. all of the coefficients of the right hand sides of constraints are equal to 1) the Lagrange function of the problem (1) may be formulated as follows:

$$\begin{aligned} L_n(x) &= \sum_{i=1}^{n} c_i \cdot x_i + \sum_{j=1}^{m} \lambda_j \cdot \left(1 - \sum_{i=1}^{n} a_{ji} \cdot x_i\right) = \\ &= \sum_{j=1}^{m} \lambda_j + \sum_{i=1}^{n} \left(c_i - \sum_{j=1}^{m} \lambda_j \cdot a_{ji}\right) \cdot x_i \end{aligned}$$

where $x = [x_1, \ldots, x_n]$ and $\Lambda = [\lambda_1, \ldots, \lambda_m]$ - vector of Lagrange multipliers. Moreover, let for every Λ, $\lambda_j \geq 0$, $j = 1, \ldots, m$:

$$\phi_n(\Lambda) = \max_{x \in \{0,1\}^n} L_n(x, \Lambda) = \max_{x \in \{0,1\}^n} \left\{ \sum_{j=1}^{m} \lambda_j + \sum_{i=1}^{n} \left(c_i - \sum_{j=1}^{m} \lambda_j a_{ji}\right) x_i \right\}.$$

Taking the following notation:

$$x_i(\Lambda) = \begin{cases} 1 & \text{if } c_i - \sum\limits_{j=1}^{m} \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

$$c_i(\Lambda) = \begin{cases} c_i & \text{if } c_i - \sum\limits_{j=1}^{m} \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$a_{ji}(\Lambda) = \begin{cases} a_{ji} & \text{if } c_i - \sum\limits_{j=1}^{m} \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

we have for every Λ, $\lambda_j \geq 0$, $j = 1, \ldots, m$:

$$\begin{aligned} \phi_n(\Lambda) &= \sum_{j=1}^{m} \lambda_j + \sum_{i=1}^{n} \left(c_i - \sum_{j=1}^{m} \lambda_j \cdot a_{ji}\right) \cdot x_i(\Lambda) = \\ &= \sum_{j=1}^{m} \lambda_j + \sum_{i=1}^{n} \left(c_i(\Lambda) - \sum_{j=1}^{m} \lambda_j \cdot a_{ji}(\Lambda)\right). \end{aligned}$$

Obviously, for $i = 1, \ldots, n,\ j = 1, \ldots, m,$

$$c_i(\Lambda) = c_i \cdot x_i(\Lambda), \quad a_{ji}(\Lambda) = a_{ji} \cdot x_i(\Lambda).$$

The dual problem to the set packing problem (1) may be formulated as follows:

$$\Phi_n^* = \min_{\Lambda \geq 0} \phi_n(\Lambda). \tag{8}$$

For every $\Lambda \geq 0$ the following holds:

$$z_{OPT}(n) \leq \Phi_n^* \leq \phi_n(\Lambda) = z_n(\Lambda) + \sum_{j=1}^{m} \lambda_j (1 - s_j(\Lambda)). \tag{9}$$

Let us denote:

$$\begin{aligned} z_n(\Lambda) &= \sum_{i=1}^{n} c_i \cdot x_i(\Lambda) = \sum_{i=1}^{n} c_i(\Lambda), s_j(\Lambda) = \sum_{i=1}^{n} a_{ji} \cdot x_i(\Lambda) = \sum_{i=1}^{n} a_{ji}(\Lambda), \\ S_{nm}(\Lambda) &= \sum_{j=1}^{m} \lambda_j \cdot s_j(\Lambda),\ \tilde{\Lambda}(m) = \sum_{j=1}^{m} \lambda_j. \end{aligned}$$

By definition of $c_i(\Lambda)$ and $a_{ji}(\Lambda)$, see also (7), we have:

$$c_i(\Lambda) \geq \sum_{j=1}^{m} \lambda_j \cdot a_{ji}(\Lambda),\ i = 1, \ldots, n,$$

and therefore

$$z_n(\Lambda) \geq S_{nm}(\Lambda). \tag{10}$$

For certain Λ, $x_i(\Lambda)$, given by (7), may provide feasible solution of (1), i.e.:

$$s_j(\Lambda) \leq 1 \quad \text{for every} \quad j = 1, \ldots, m. \tag{11}$$

Then:

$$z_n(\Lambda) \leq z_{OPT}(n) \leq \Phi_n^* \leq \phi_n(\Lambda) = z_n(\Lambda) + \tilde{\Lambda}(m) - S_{nm}(\Lambda). \tag{12}$$

If (11) holds, then the inequality below also holds:

$$\tilde{\Lambda}(m) - S_{nm}(\Lambda) \geq 0.$$

From (10) we get:

$$\frac{\phi_n(\Lambda)}{z_n(\Lambda)} = \frac{z_n(\Lambda)}{z_n(\Lambda)} + \frac{\tilde{\Lambda}(m) - S_{nm}(\Lambda)}{z_n(\Lambda)} \leq 1 + \frac{\tilde{\Lambda}(m) - S_{nm}(\Lambda)}{S_{nm}(\Lambda)}.$$

Therefore, if (11) holds, then the following inequality also holds:

$$1 \leq \frac{z_{OPT}(n)}{z_n(\Lambda)} \leq \frac{\Phi_n^*}{z_n(\Lambda)} \leq \frac{\phi_n(\Lambda)}{z_n(\Lambda)} \leq \frac{\tilde{\Lambda}(m)}{S_{nm}(\Lambda)}. \tag{13}$$

Formula (13) shows that if there exists such a set of the Lagrange multipliers $\Lambda(n)$ which fulfills the formula (11) and if the formula below holds:

$$\lim_{n\to\infty} \frac{\tilde{\Lambda}(m)}{S_{nm}(\Lambda(n))} = 1 \tag{14}$$

then, due to (13), $\lim_{n\to\infty} \frac{z_{OPT}(n)}{z_n(\Lambda)} = 1$, and, therefore, $x_i(\Lambda(n))$, $i = 1, \ldots, n$, given by (7), is the asymptotically sub-optimal solution of the set packing problem (1). Moreover, the value of $z_n(\Lambda(n))$ is an asymptotical approximation of the optimal solution value of the set packing problem, i.e. $z_{OPT}(n)$.

In the case of the maximum set packing problem (3) $c_i \equiv 1$, $i = 1, \ldots, n$, and, moreover, c_i are no longer realizations of the random variables. Therefore, in the case of the maximum set packing problem (3) in the above formulas c_i should be replaced with 1. As the consequence, the formulas where c_i was involved will look differently, e.g. in (7) $c_i - \sum_{j=1}^m \lambda_j \cdot a_{ji} > 0$ should be replaced by $1 - \sum_{j=1}^m \lambda_j \cdot a_{ji} > 0$. So, we obtain:

$$c_i(\Lambda) = x_i(\Lambda) = \begin{cases} 1 & \text{if } 1 - \sum\limits_{j=1}^m \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

$$a_{ji}(\Lambda) = \begin{cases} a_{ji} & \text{if } 1 - \sum\limits_{j=1}^m \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise.} \end{cases}.$$

In turn, this means that $c_i(\Lambda) \equiv x_i(\Lambda)$, $i = 1, \ldots, n$, and therefore $z_n(\Lambda) = \sum_{i=1}^n x_i(\Lambda)$.

In either case, according to (2), $a_{ji}(\Lambda) = 1$ is guaranteeing that item j is assigned to set M_i. Obviously, this also implies that $s_j(\Lambda) = 1$.

4. Probabilistic analysis

In the present section of the paper some probabilistic properties of the set packing problems (1) and (3) will be investigated. In the paper by Vercellis (1986) some results of the probabilistic analysis of the set packing problems were presented. In the present paper a different approach is exploited. The random model of the specific knapsack problems (1) and (3) is significantly different from the one considered in the case of the general knapsack problem in the earlier papers, see Szkatuła (1994, 1997). Namely, coefficients of the left hand sides of constraints a_{ji}, $i = 1, \ldots, n$, $j = 1, \ldots, m$, have in the present case discrete probability distribution, while in the general case they have uniform (continuous) distribution. Moreover, all of the coefficients of the left hand sides of constraints are equal to 1 and m may be arbitrarily large in comparison with n. Therefore, probabilistic analysis of the set packing problem (1) requires a specific approach.

Let us first observe that due to the assumptions made, the following holds, for $j = 1, \ldots, m$:

$$\begin{aligned} P\{a_{ji} &= 1\} = p,\ P\{a_{ji} = 0\} = 1 - p,\ P\{a_{ji}(\Lambda) = 1\} = 1 - P\{a_{ji}(\Lambda) = 0\}, \\ P(c_i &< x) = \begin{cases} 0 & \text{when } x \leqslant 0 \\ x & \text{when } 0 < x \leqslant 1 \\ 1 & \text{when } x \geqslant 1 \end{cases} . \end{aligned} \tag{16}$$

Moreover, for the random variable $\sum_{k=1,k\neq j}^{m} a_{ji}$, due to the binomial distribution, the following holds for every integer r, $0 \leqslant r \leqslant m-1$:

$$P\left\{ \sum_{k=1,k\neq j}^{m} a_{ki} = r \right\} = \binom{m-1}{r} \cdot p^r \cdot (1-p)^{m-r-1}. \tag{17}$$

Let us also assume that

$$\Lambda = \{\lambda, \cdots, \lambda\},\ \text{i.e. } \lambda_j = \lambda,\ \lambda \geqslant 0,\ j = 1, \cdots, m.$$

In the case of the set packing problem (1) the following results hold.

Lemma 1 *If a_{ji} are realizations of mutually independent random variables where $P\{a_{ji} = 1\} = p$, $0 < p \leq 1$, then*

$$P\{a_{ji}(\Lambda) = 1\} = p - p \sum_{r=0}^{m-1} \binom{m-1}{r} \cdot p^r \cdot (1-p)^{m-r-1} \min\{1, \lambda(r+1)\}.$$

Moreover, if $\lambda \leqslant 1/m$ then:

$$P\{a_{ji}(\Lambda) = 1\} = p \cdot (1 - \lambda \cdot (m \cdot p + 1 - p)).$$

Proof. From (7), (16) and (17), and taking into account the fact that random variable $\sum_{k=1,k\neq j}^{m} a_{ji}$ may take any integer value r from the range $[0, m-1]$ with the probability given in (17) it follows that:

$$
\begin{aligned}
P\{a_{ji}(\Lambda)=0\} &= P\left\{a_{ji}=0 \cup a_{ji}=1 \cap c_i < \lambda \cdot \left(\sum_{k=1,k\neq j}^{m} a_{ki}+1\right)\right\} = \\
&= 1-p+p\cdot P\left\{c_i < \lambda \cdot \left(\sum_{k=1,k\neq j}^{m} a_{ki}+1\right)\right\} = \\
&= 1-p+p\sum_{r=0}^{m-1}\binom{m-1}{r}\cdot p^r\cdot(1-p)^{m-r-1}\min\{1,\lambda(r+1)\}.
\end{aligned}
$$

Due to (16) the first formula of the Lemma is proven. Because

$$
\binom{m-1}{r}=\frac{(m-1)!}{r!\cdot(m-1-r)!},
$$

then, when $\lambda \leqslant 1/m$, the following holds

$$
P\{a_{ji}(\Lambda)=0\}=1-p+\lambda\sum_{r=0}^{m-1}\frac{(m-1)!\cdot(r+1)}{r!\cdot(m-1-r)!}\cdot p^{r+1}\cdot(1-p)^{m-r-1}. \quad (18)
$$

Let us observe that for every integers l, m, $l, > 1$, $m \geqslant 2$, and $0 \leqslant p \leqslant 1$ the following hold

$$
\begin{aligned}
\sum_{k=0}^{l}\binom{l}{k}\cdot p^k\cdot(1-p)^{l-k} &= (p+1-p)^l=1 \\
r+1 &= m-(m-1-r).
\end{aligned}
$$

Using the above mentioned, the formulas (18) may be rewritten as:

$$
\begin{aligned}
P\{a_{ji}(\Lambda)=0\} &= 1-p+\lambda\cdot p\left(\sum_{r=0}^{m-1}\frac{(m-1)!\cdot m}{r!\cdot(m-1-r)!}\cdot p^r\cdot(1-p)^{m-1-r}-\right. \\
&\qquad \left. -\sum_{r=0}^{m-1}\frac{(m-1)!\cdot(m-1-r)}{r!\cdot(m-1-r)!}\cdot p^r\cdot(1-p)^{m-1-r}\right) = \\
&= 1-p+\lambda\cdot p\left(m\sum_{r=0}^{m-1}\binom{m-1}{r}\cdot p^r\cdot(1-p)^{m-1-r}-\right. \\
&\qquad \left. -p\cdot(m-1)\cdot(1-p)\sum_{r=0}^{m-2}\binom{m-2}{r}\cdot p^r\cdot(1-p)^{m-2-r}\right) = \\
&= 1-p+\lambda\cdot p\cdot(m-(m-1)\cdot(1-p)) = \\
&= 1-p+\lambda\cdot p\cdot(m\cdot p+1-p).
\end{aligned}
$$

Finally, the above formulas can be summarized as:

$$
P\{a_{ji}(\Lambda)=0\}=1-p+\lambda\cdot p\cdot(m\cdot p+1-p). \quad (19)
$$

Due to the formulas (16) and (19) we have

$$\begin{aligned} P\{a_{ji}(\Lambda)=1\} &= 1-P\{a_{ji}(\Lambda)=0\}= \\ &= p-\lambda\cdot p\cdot(m\cdot p+1-p)=p\cdot(1-\lambda\cdot(m\cdot p+1-p)). \end{aligned}$$

■

As the direct consequence of the above formulas we have

$$E(a_{ji}(\Lambda))=1\cdot P\{a_{ji}(\Lambda)=1\}+0\cdot P\{a_{ji}(\Lambda)=0\}=P\{a_{ji}(\Lambda)=1\}. \quad (20)$$

Now, instead of Λ we will consider $\Lambda(n)$. This means that for every value of integer n, we may consider different vectors $\Lambda(n)=\{\lambda(n),\cdots,\lambda(n)\},\ \lambda(n)\geqslant 0$. For every j, $j=1,\cdots,m$, we have:

$$\begin{aligned} E(s_j(\Lambda(n))) &= \sum_{i=1}^{n} E(a_{ji}(\Lambda(n)))=n\cdot P\{a_{ji}(\Lambda(n))=1\}= \\ &= n\cdot p(1-\lambda(n)\cdot(m\cdot p+1-p)). \end{aligned} \quad (21)$$

The above equation, (21), provides the opportunity to determine $\lambda(n)$ by solving $E(s_j(\Lambda(n))=\alpha$, where $\alpha>0$. When $\alpha=1$, then $\lambda(n)$ is solving all of the constraints in (1) as equations, in the sense of average (mean) values, $E(\sum_{i=1}^{n} a_{ji}\cdot x_i(\Lambda(n)))=1$ for all $j=1,\ldots .m$. Unfortunately, there is no guarantee that the solution obtained is feasible, i.e. $\sum_{i=1}^{n} a_{ji}\cdot x_i(\Lambda(n))\leqslant 1$, for all $j=1,\ldots .m$. Therefore, one may try to consider smaller values of α, $0<\alpha\leqslant 1$, in order to increase the chance of obtaining the feasible solution of the set packing problem (1). Below, those ideas are considered in a formalized manner.

Lemma 2 *For every α, $\alpha>0$ there exist m', $n'>1$ such that for every $m\geqslant m'$ and $n\geqslant n'$, the following choice of $\lambda(n)$:*

$$\lambda(n)=\frac{1-\alpha/(n\cdot p)}{m\cdot p+1-p}$$

is solving the equations $E(s_j(\Lambda(n)))=\alpha$.

Corollary 1 *If $E(s_j(\Lambda(n)))=\alpha$, then $P\{a_{ji}(\Lambda(n))=1\}=\alpha/n$.*

Proof. Proof of Lemma and Corollary follows immediately from formulas (20) and (21) and the following fact, namely that for all $m\geqslant m'$ and $n\geqslant n'$:

$$\lambda(n)\leqslant\frac{1}{m}.$$

■

Solution of the set packing problem (1) given by formula (7) is feasible (provides packing of the set M) if and only if the formula (11) holds.

Theorem 1 *For every* α, $0 < \alpha \leqslant 1$ *there exist* m', n', m', $n' > 1$, *such that for* $\Lambda(n)$, *providing* $E(s_j(\Lambda(n))) = \alpha$, *the following hold*

$$P\{s_j(\Lambda(n)) \leqslant 1\} = \left(1 - \frac{\alpha}{n}\right)^{n-1} \cdot (1 + \alpha - \frac{\alpha}{n}).$$

Moreover, for every fixed value of $\alpha, \alpha > 0$, *we have*

$$\lim_{n\to\infty} P\{s_j(\Lambda(n)) \leqslant 1\} = \frac{1+\alpha}{e^{\alpha}}.$$

Proof. As it was already mentioned, the solution of the problem (1), given by formula (7) is feasible if and only if formula (11) holds i.e. $s_j(\Lambda(n)) = 0$ or $s_j(\Lambda(n)) = 1$. For every $\Lambda(n)$, random variable $s_j(\Lambda(n)) = \sum_{i=1}^{n} a_{ji}(\Lambda(n))$ may take any integer value r from the range $[0, n]$ with the probability given by the following formula:

$$P\left\{\sum_{i=1}^{n} a_{ji}(\Lambda(n)) = r\right\} = \binom{n}{r} \cdot \tilde{p}^r \cdot (1-\tilde{p})^{n-r}, \text{ where } \tilde{p} = P\{a_{ji}(\Lambda(n)) = 1\}.$$

From the above formula and Corollary 1 it follows that

$$\begin{aligned} P\{s_j(\Lambda(n)) \leqslant 1\} &= P\left\{\sum_{i=1}^{n} a_{ji}(\Lambda(n)) = 0 \cup \sum_{i=1}^{n} a_{ji}(\Lambda(n)) = 1\right\} = \qquad (22) \\ &= \left(1 - \frac{\alpha}{n}\right)^{n} + \alpha\left(1 - \frac{\alpha}{n}\right)^{n-1} = \left(1 - \frac{\alpha}{n}\right)^{n-1} \cdot (1 + \alpha - \frac{\alpha}{n}). \end{aligned}$$

The proof is finished by observing that $\lim_{n\to\infty} \left(1 - \frac{\alpha}{n}\right)^{n-1} = e^{-\alpha}$ and $\lim_{n\to\infty} \frac{\alpha}{n} = 0$. ∎

Corollary 2 $P\{s_j(\Lambda(n)) \leqslant 1\} = 1$ *if and only if* $n = 1$. *When* $\alpha \to 0$ *as* $n \to \infty$ *then*

$$\lim_{n\to\infty} P\{s_j(\Lambda(n)) \leqslant 1\} = 1.$$

However, if α, $\alpha > 0$, *is a constant, then:*

$$\lim_{n\to\infty} P\{s_j(\Lambda(n)) \leqslant 1\} < 1. \qquad (23)$$

Proof. Formula (23) follows immediately from Theorem 1. ∎

The above Theorem 1 and Corollary 2 to it have interesting interpretation, which may be observed on few examples presented below.

Example 1

$$\begin{aligned} \textit{When } \alpha = 0.01 \textit{ then } \lim_{n\to\infty} P\{s_j(\Lambda(n)) \leqslant 1\} &= 0.999 \\ \textit{When } \alpha = 0.1 \textit{ then } \lim_{n\to\infty} P\{s_j(\Lambda(n)) \leqslant 1\} &= 0.995 \\ \textit{When } \alpha = 0.5 \textit{ then } \lim_{n\to\infty} P\{s_j(\Lambda(n)) \leqslant 1\} &= 0.9098 \\ \textit{When } \alpha = 1 \textit{ then } \lim_{n\to\infty} P\{s_j(\Lambda(n)) \leqslant 1\} &= \frac{2}{e} \approx 0.736. \end{aligned}$$

Interpretation of the above examples is as follows. The closer the value of α is to 1, the better approximation of the optimal solution values may be provided, with, however, less satisfactory value of the $\lim_{n\to\infty} P\{s_j(\Lambda(n)) \leqslant 1\}$. Yet, for any value of α, $0 < \alpha \leqslant 1$, $\lim_{n\to\infty} P\{s_j(\Lambda(n)) \leqslant 1\} \succeq 2/e$, where $2/e \approx 0.736$. Due approximations of the optimal solution values are provided in the next section.

In the case of the maximum set packing problem (3) the situation is significantly different. Namely, according to (17), where $\gamma = \frac{1}{\lambda} - 1, \lambda = \frac{1}{\gamma+1}$:

$$\begin{aligned} P\{a_{ji}(\Lambda) = 1\} &= P\left\{a_{ji} = 1 \cap \lambda \cdot \left(\sum_{k=1,k\neq j}^{m} a_{ki} + 1\right) \leq 1\right\} = \qquad (24) \\ &= p \cdot P\left\{\lambda \cdot \left(\sum_{k=1,k\neq j}^{m} a_{ki} + 1\right) \leq 1\right\} = \\ &= p \cdot P\left\{\sum_{k=1,k\neq j}^{m} a_{ki} \leq \frac{1}{\lambda} - 1\right\} = \\ &= p \cdot \sum_{r=0}^{\lfloor\gamma\rfloor} P\left\{\sum_{k=1,k\neq j}^{m} a_{ki} = r\right\} = \\ &= p \cdot \sum_{r=0}^{\lfloor\gamma\rfloor} \binom{m-1}{r} \cdot p^r \cdot (1-p)^{m-r-1}. \end{aligned}$$

It is pretty obvious that only m values of γ (and respectively λ) should be considered, namely $\gamma = 0, 1, \ldots, m-1$, $(\lambda = \frac{1}{m}, \frac{1}{m-1}, \ldots, 1)$ because

$$P\left\{\sum_{k=1,k\neq j}^{m} a_{ki} = r\right\} = 0 \text{ for } r < 0 \text{ and } r > m-1.$$

The above facts have very serious consequences for the probabilistic analysis of the maximum set packing problem (3). Namely, using formula (24) with $\gamma = 0$ and $\gamma = m-1$ ($\lambda = \frac{1}{m}$ and $\lambda = 1$) and taking into account (20) we conclude that

$$p \cdot (1-p)^{m-1} \leq E(a_{ji}(\Lambda)) = P\{a_{ji}(\Lambda) = 1\} \leq 1.$$

The latter means that, when considering $\Lambda(n)$, $n \to \infty$, in order to solve

$$E(s_j(\Lambda(n))) = \alpha \text{ or } P\{a_{ji}(\Lambda(n)) = 1\} = \frac{\alpha}{n}, i = 1, \ldots, n,\ j = 1, \ldots, m \quad (25)$$

the following condition should hold:

$$n \leq \frac{\alpha}{p \cdot (1-p)^{m-1}}.$$

As the matter of fact, (25) is implying asymptotic relations between n, m, p and α. It may be difficult to obtain exact solution of (25) due to the finiteness of the set of values of the Lagrange multipliers λ ($\lambda = \frac{1}{m}, \frac{1}{m-1}, \ldots, 1$) and the formula (24). Frequently, there may exist only approximate solutions of (25).

5. Behavior of the optimal solution values

The main goal of this paper is to analyze the behavior of the optimal solution value of the set packing problem (1) in the asymptotical probabilistic case. Moreover, it was the author's intention to use a simple and easy to follow probabilistic apparatus. In order to proceed with this analysis one may need to exploit the probabilistic properties of the random variables $c_i(\Lambda(n))$, $i = 1, \cdots, n$. The construction of the random variables $c_i(\Lambda(n))$ is defined by formulas (7) and (16), respectively. Distribution functions of the random variables $c_i(\Lambda(n))$, $i = 1, \cdots, n$ are given by the following formulas, where $0 < x \leq 1$:

$$P\{c_i(\Lambda(n)) < x\} = P\{c_i < x \cup c_i \geq x \cap c_i \leq \Lambda(n) \cdot \sum_{j=1}^{m} a_{ji}\} = \tag{26}$$

$$= x + P\{x \leq c_i \leq \Lambda(n) \cdot \sum_{j=1}^{m} a_{ji}\}.$$

Let us observe that $P\{x \leq c_i \leq \Lambda(n) \cdot \sum_{i=1}^{n} a_{ji}\}$ is by definition equal to zero if $c_i < x$ or $c_i > \Lambda(n) \cdot \sum_{i=1}^{n} a_{ji}$. Therefore, (26) may be rewritten as

$$P\{c_i(\Lambda(n)) < x\} = x + \sum_{r=1}^{m} P\{x \leq c_i \leq \Lambda(n) \cdot r \cap \sum_{j=1}^{m} a_{ji} = r\} = \tag{27}$$

$$= x + \sum_{r=1}^{m} (r\Lambda(n) - x)_+ P\{\sum_{j=1}^{m} a_{ji} = r\}. \tag{28}$$

The above formula may enable us to calculate the mean value of the random variables $c_i(\Lambda(n))$, $i = 1, \cdots, n$. Namely:

$$E(c_i(\Lambda(n))) = \int_0^1 x \cdot d(P\{c_i(\Lambda(n)) < x\}) = \tag{29}$$

$$= \frac{1}{2} + \int_0^{\Lambda(n)\cdot m} x \cdot \left(\sum_{r=1}^{m} (r\Lambda(n) - x)'_+ \cdot P\{\sum_{j=1}^{m} a_{ji} = r\} \right) =$$

$$= \frac{1}{2} + \sum_{k=1}^{m} \int_{\Lambda(n)\cdot(k-1)}^{\Lambda(n)\cdot k} x \left(\sum_{r=k}^{m} (r\Lambda(n) - x)'_+ \cdot P\{\sum_{j=1}^{m} a_{ji} = r\} \right) dx =$$

$$= \frac{1}{2} - \sum_{k=1}^{m} \int_{\Lambda(n)\cdot(k-1)}^{\Lambda(n)\cdot k} x \cdot P\{\sum_{j=1}^{m} a_{ji} = r\} dx.$$

Let us observe that, similarly to the formula (17), the random variable $\sum_{k=1}^{m} a_{ji}$, due to having binomial distribution, has the following distribution function for every integer r, $0 \leqslant r \leqslant m$:

$$P\left\{\sum_{k=1}^{m} a_{ki} = r\right\} = \binom{m}{r} \cdot p^r \cdot (1-p)^{m-r} \text{ and, moreover, } \left(\sum_{k=1}^{r}(2k-1)\right) = r^2.$$

Therefore, the formula (29) could be further simplified as follows:

$$\begin{aligned}
E(c_i(\Lambda(n))) &= \frac{1}{2} - \sum_{k=1}^{m}\left(\int\limits_{\Lambda(n)\cdot(k-1)}^{\Lambda(n)\cdot k} x dx\right) \cdot \left(\sum_{r=k}^{m}\binom{m}{r} \cdot p^r \cdot (1-p)^{m-r}\right) = \\
&= \frac{1}{2} - \frac{(\Lambda(n))^2}{2}\sum_{k=1}^{m}(2k-1) \cdot \left(\sum_{r=k}^{m}\binom{m}{r} \cdot p^r \cdot (1-p)^{m-r}\right) = \\
&= \frac{1}{2} - \frac{(\Lambda(n))^2}{2}\sum_{r=1}^{m}\left(\sum_{k=1}^{r}(2k-1)\right) \cdot \left(\binom{m}{r} \cdot p^r \cdot (1-p)^{m-r}\right) = \\
&= \frac{1}{2} - \frac{(\Lambda(n))^2}{2}\sum_{r=1}^{m} r^2 \cdot \left(\binom{m}{r} \cdot p^r \cdot (1-p)^{m-r}\right).
\end{aligned}$$

Let us observe that the following formula holds for $0 < p \leq 1$ and $m = 1, 2, \ldots$:

$$\sum_{r=1}^{m} r^2 \cdot \left(\binom{m}{r} \cdot p^r \cdot (1-p)^{m-r}\right) = m \cdot p \cdot (1 + p \cdot (m-1)).$$

From Lemma 2 (where $E(s_j(\Lambda(n))) = \alpha$, and $\lambda(n) = \frac{1-\alpha/(n\cdot p)}{m\cdot p+1-p}$) and due to the formula (9) we will therefore obtain

$$\begin{aligned}
E(z_n(\Lambda)) &= \frac{n}{2}\left(1 - \left(\frac{1-\alpha/(n\cdot p)}{m\cdot p+1-p}\right)^2 \cdot m \cdot p \cdot (m \cdot p + 1 - p)\right) = \\
&= \frac{n}{2}\left(1 - \frac{m \cdot p \cdot (1 - \dfrac{\alpha}{n\cdot p})^2}{m \cdot p + 1 - p}\right) = \frac{n}{2}\left(1 - \frac{(1-\dfrac{\alpha}{n\cdot p})^2}{1 + (1-p)/(m\cdot p)}\right).
\end{aligned}$$

If (11) holds then, due to the formulas (12) and (13), where $\tilde{\Lambda}(m) = \sum\limits_{j=1}^{m} \lambda_j(n) = m \cdot \lambda(n)$, $E(S_{nm}(\Lambda(n))) = \alpha \cdot m \cdot \lambda(n)$ and $\lambda(n) = \frac{1-\alpha/(n\cdot p)}{m\cdot p+1-p}$, one may obtain much stronger results for $0 < \alpha \leqslant 1$, namely:

$$1 \leqslant E\left(\frac{z_{OPT}(n)}{z_n(\Lambda(n))}\right) \leqslant \frac{1}{\alpha}, \text{ where } E\left(\frac{\tilde{\Lambda}(m,n)}{S_{nm}(\Lambda(n))}\right) = \frac{1}{\alpha} \text{ and} \tag{30}$$

$$E(z_n(\Lambda(n))) = \frac{n}{2} \cdot \left(1 - \frac{(1-\alpha/(n\cdot p))^2}{1+(1-p)/(m\cdot p)}\right). \tag{31}$$

Formulas (30) and (31) may provide us with some estimations in the increase of the set packing problem (1) optimal solution values $z_{OPT}(n)$, when $n \to \infty$. Corresponding to Example 1, the estimations of the $E\left(\frac{z_{OPT}(n)}{z_n(\Lambda(n))}\right)$ for the different values of α are provided in the Example 2 below, where appropriate value of $E(z_n(\Lambda(n)))$ is given in the formula (31):

Example 2

When $\alpha = 0.01$ then $1 \leqslant E\left(\frac{z_{OPT}(n)}{z_n(\Lambda(n))}\right) \leqslant 100$ with approximate probability 0.999

When $\alpha = 0.1$ then $1 \leqslant E\left(\frac{z_{OPT}(n)}{z_n(\Lambda(n))}\right) \leqslant 10$ with approximate probability 0.995

When $\alpha = 0.5$ then $1 \leqslant E\left(\frac{z_{OPT}(n)}{z_n(\Lambda(n))}\right) \leqslant 2$ with approximate probability 0.9098

When $\alpha = 1$then $E\left(\frac{z_{OPT}(n)}{z_n(\Lambda(n))}\right) = 1$ with approximate probability $\frac{2}{e} \approx 0.736$.

The smaller is the value of α, the higher is probability of providing a feasible solution of the set packing problem (1), but the quality of the approximation, provided by (30) and (31) is deteriorating. Obviously, approximation is not "strict" in the sense that, as α increases, only the upper bound on the expected value of the approximation quality increases. However, when α is very small, e.g. $\alpha = 0.01$ in the above example, then the expected values of all left hand side of constraints in (1) are very small either, i.e. $E(s_j(\Lambda(n))) = \alpha$, $j = 1,\ldots,m$. This, in turn may indicate that only the trivial solution like $x_i(\Lambda(n)) = 0$, $i = 1,\ldots,n$, of the original problem may be provided. Anyhow, moderate values of α, e.g. $\alpha = 0.5$ or $\alpha = 1$, in the example above are providing a reasonable compromise between quality of the approximation and the feasibility of the solution.

Since $n \leqslant m!$ and, moreover, $n \to \infty$, then, obviously also $m \to \infty$. According to formula (31), asymptotic growth of the $E(z_n(\Lambda(n)))$ may be influenced by both n and m. Let us consider the following mutual asymptotic dependence of both parameters:

$$n = \beta \cdot m^{\gamma}, \text{ where } \beta \text{ is constant, } 0 < \gamma \leqslant m,\ \beta > 0. \tag{32}$$

If $0 < \gamma \leqslant m$, then condition $n \leqslant m!$ is always fulfilled asymptotically since, due to the Stirling's formula, for every constant $\beta > 0$ there exists constant $m' \geqslant 1$ such that for all $m \geqslant m'$ the inequality $n \leqslant m!$ holds.

Under the above assumption, the following Lemma holds

Lemma 3 *If asymptotical dependence (32) holds then:*

$$E(z_n(\Lambda(n))) \approx \frac{2\cdot\alpha + \beta\cdot(1-p)\cdot m^{\gamma-1}}{2\cdot p} \text{ when } n \to \infty. \tag{33}$$

Proof. When (32) holds, then (31) may be reformulated as follows:

$$E(z_n(\Lambda(n))) = \frac{2m \cdot \alpha \cdot \beta \cdot p + m^{\gamma} \cdot \beta^2 \cdot p \cdot (1-p) - \alpha^2 \cdot m^{-\gamma+1}}{2\beta \cdot p \cdot (m \cdot p + 1 - p)}.$$

Taking into account the previously made assumptions on α, β, γ and p, the proof of the formula (33) is straightforward. ∎

Corollary 3 *Depending on the value of* γ, $0 < \gamma \leqslant m$, *the following cases of the asymptotical behavior of* $E(z_n(\Lambda(n)))$ *may be distinguished:*

$$\lim_{m\to\infty} E(z_n(\Lambda(n))) = \begin{cases} \dfrac{\alpha}{p} & \text{when } 0 < \gamma < 1 \\ \dfrac{2\alpha + \beta \cdot (1-p)}{2p} & \text{when } \gamma = 1 \\ \infty & \text{when } \gamma > 1. \end{cases} \tag{34}$$

Due to the formulas (13) and (30), $E(z_n(\Lambda(n)))$ is a reasonable asymptotic approximation of the optimal solution of the set packing problem (1), i.e. $E(z_{OPT}(n))$. The above Lemma and Corollary, especially formulas (33) and (34), provide interesting insight into the asymptotical behavior of the value of $E(z_n(\Lambda(n)))$. Namely:

$$\text{when } n = o(m) \text{ then } \lim_{m\to\infty} E(z_n(\Lambda(n))) = \frac{\alpha}{p}.$$

The above does mean that in this case the values of β and γ are negligible and so is the mutual asymptotic dependence of both n and m:

$$\text{when } n \approxeq m \text{ then } E(z_n(\Lambda(n))) \approx \frac{2\alpha + \beta \cdot (1-p)}{2p}.$$

In this case level of proximity of n and m is substantial and is expressed by the value β.

$$\text{When } m = o(n) \text{ then } E(z_n(\Lambda(n))) \approx \frac{\beta \cdot (1-p)}{2 \cdot p} \cdot m^{\gamma-1}.$$

In the latter case dependence on α is negligible, β and p are defining a constant multiplier.

In two first cases, where $\gamma \leqslant 1$, there is no asymptotical influence of the value of m (and therefore of n either) on the asymptotical value of $E(z_n(\Lambda(n)))$. However, in the case when $\gamma > 1$, there is very strong dependence on both m and γ.

On the other hand, the parameters α and p have substantial influence on the asymptotical behavior of $E(z_n(\Lambda(n)))$, when $\gamma \leqslant 1$. Namely the bigger is the value of α, $\alpha > 0$, and/or smaller is the value of p, $0 < p \leqslant 1$, the bigger is the value of $E(z_n(\Lambda(n)))$. The consequence of the above statement is as follows:

- The bigger is the value of α, the lower is the probability of feasibility of the corresponding solution of the set packing problem (1), see Theorem 1.
- The smaller the value of p, the sparser the initial subsets M_i, $i = 1, \cdots, n$, of the original set M may be.

In the case of the maximum set packing problem (3), the situation is different. Namely

$$\begin{aligned} P\{c_i(\Lambda) = 1\} &= P\{x_i(\Lambda) = 1\} = P\left\{\lambda \cdot \left(\sum_{k=1}^{m} a_{ki} + 1\right) \leq 1\right\} = \\ &= P\left\{\left(\sum_{k=1}^{m} a_{ki} + 1\right) \leq \frac{1}{\lambda}\right\} = \sum_{r=0}^{\lfloor 1/\lambda \rfloor} P\left\{\sum_{k=1, k\neq j}^{m} a_{ki} = r\right\} = \\ &= \sum_{r=0}^{\lfloor 1/\lambda \rfloor} \binom{m}{r} \cdot p^r \cdot (1-p)^{m-r} \text{ where } \lambda \in \left\{\frac{1}{m}, \frac{1}{m-1}, \ldots, 1\right\}. \end{aligned}$$

If there exist $\Lambda(n)$ and α solving (25), with sufficient level of accuracy, and assuring $s_j(\Lambda(n)) \leqslant 1$, $j = 1, \cdots, m$, then

$$E(z_n(\Lambda(n))) = n \cdot \sum_{r=0}^{\lfloor 1/\lambda(n) \rfloor} \binom{m}{r} \cdot p^r \cdot (1-p)^{m-r}$$

may serve as an appropriate approximation of the value of $z_{OPT}(n)$ as it was in the case of the set packing problem (1) before.

6. Concluding remarks

In the present paper some results describing the probabilistic properties of the set packing problem (1) and the maximum set packing problem (3) are summarized.

The distribution functions of the various random variables, representing important problem characteristics are presented. Moreover, some results concerning the feasibility of the obtained solutions and estimations of the asymptotic growth in the optimal solution values $z_{OPT}(n)$ of the set packing problem (1), when $n \to \infty$, are provided.

Examples 1 and 2 show that the higher is the accuracy of approximation of the optimal solution value, the lower is the probability of the feasibility of corresponding solution. For example, when $\alpha = 0.5$, the quality of approximation is pretty tolerable, with relatively high probability of the feasibility of the solution. Moreover, when $\alpha = 1$, the quality of approximation is very good with reasonable probability of the feasibility of the solution, approximately equal to 0.736. Lemma 3 shows the possible asymptotical behavior of the optimal solution values when there is certain mutual asymptotic dependence of the parameters n and m.

In the case of the*Maximum Set Packing Problem* there are some problem specific peculiarities, which have been preliminarily investigated in the present paper.

Some of the important avenues for the future research are the convergence of the approximate solutions to the optimal solution and possibility of investigating realistic approximations of their values.

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