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Estimation of the partial order on the basis of pairwise comparisons*†

by

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Abstract: The problem of estimation of the partial order on the basis of multiple pairwise comparisons in binary and multivalent form, with random errors, is investigated. The estimators are based on the idea of the nearest adjoining order (see Slater, 1961; Klukowski 2011). Two approaches are examined: comparisons indicating the direction of preference (binary) and comparisons indicating the difference of ranks (multivalent) - both with possibility of existence of incomparable elements. The properties of estimators and the optimization problems formulated in order to obtain them are similar to those for the case of complete relation. However, the assumptions about the distributions of comparison errors are different – they comprise the case of incomparable elements.

Keywords: estimation of partial order, multiple pairwise comparisons with random errors, binary and multivalent comparisons

1. Introduction

The problem of estimation of the complete preference relation on the basis of multiple pairwise comparisons in binary and multivalent form, with random errors, has been considered in Klukowski (1994, 2011: Chapters 7 – 11). The same approach can be applied to the partial order – the main difference consisting in taking into account the fact of existence of incomparable elements. This fact implies the following modifications: equivalent elements are not taken into account, distributions of comparisons errors include probabilities related to incomparable elements, and the aggregation of comparisons of individual pairs with the use of the median (Klukowski, 2011) cannot be considered.

The idea of estimation consists in minimization of differences between the form of relation, expressed in a specified way, and the comparisons available (Slater, 1961). Thus, the estimates are obtained as the optimal solutions of the appropriate integer programming problems.

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The approach here presented rests on the statistical paradigm; therefore, it provides the properties of estimates and the possibility of verification of the results obtained. The main property is consistency, for the number of comparisons (for each pair) $N \to \infty$, under weak assumptions about comparison errors. In the case of binary comparisons it is assumed that probability of a correct comparison is greater than that of the incorrect one. In the case of multivalent comparisons, expressing the differences of ranks of the compared elements, it is assumed that distributions of comparison errors are unimodal, with mode and median equal zero. In the case of pairs composed of incomparable elements, it is assumed that the probability of correct recognition of incomparability is greater than 1/2. The estimators can be applied also in the case of unknown distributions of comparison errors (non-parametric approach), which have to satisfy the assumptions made.

In the earlier works of the author (Klukowski, 1994, 2008, 2011) two kinds of estimators have been considered: the first one based on the total sum of differences between the relation form and the comparisons, and the second - based on the sum of differences with regard to medians of comparisons of each pair. The second estimator imposes a lower computational burden, which is important for large N. In the case of partial orders, such an estimator can be applied, in a simple way, only for binary comparisons. Thus, the median case is omitted in the present work.

The idea of the estimators, for the case of binary comparisons and the complete relation, was presented in Slater (1961); some other ideas in the area of pairwise comparisons have been presented in: David (1988), Bradley (1984), as well as Flinger and Verducci (1993).

The paper consists of four sections and the appendix with the proof of the theorem from Section 3. The second section presents the definitions, notations and assumptions about comparison errors. The subsequent section considers the form of estimators, for both kinds of comparisons, and their properties. The last section summarizes the results.

2. Definitions, notations and assumptions about comparison errors

2.1. Definitions and notations

The problem of estimation of the partial order on the basis of pairwise comparisons can be stated as follows.

We are given a finite set of elements $X = \{x_1, \ldots, x_m\}$ $(3 \leqslant m < \infty)$. There exists in the set X a partial order relation $\mathbf{R}^{(p)}$. Each pair of elements (x_i, x_j) is ordered or incomparable; thus the set of pairs of indices:

$$R_m = \{< i, j > \mid i = 1, \ldots, m-1, \ j = i+1, \ldots, m\} \tag{1}$$

can be divided into two disjoint subsets, including comparable, I_0, and incomparable, I_n, pairs of indices, i.e.: $R_m = I_o \cup I_n$.

The partial order relation can be expressed in binary and multivalent form. The binary description $T_b(x_i, x_j)$ $(< i, j >\in R_m)$, expresses either the direction of preference in a pair of elements, or their incomparability; this fact can be represented with three values: two possible directions of preference (values of ± 1) and incomparability (value of 2). Such a description assumes the form:

$$T_b(x_i, x_j) = \begin{cases} -1 \; if \;\; x_i \;\; precedes \;\; x_j, \\ 1 \;\; if \;\; x_j \;\; precedes \;\; x_i; \\ 2 \; if \;\;\; x_i \;\; and \;\; x_j \;\; incomparable. \end{cases} \quad (2)$$

The multivalent description $T_\mu(x_i, x_j)$ expresses the difference of ranks of the comparable elements, denoted d_{ij}, or their incomparability; $d_{ij} = r - s$ is the distance between the elements, where r is the rank of x_i, and s is the rank of x_j. The distance can be presented through a digraph – it is, namely, the number of edges connecting the elements of a pair, and, of course, it must be lower than m. This description assumes the form:

$$T_\mu(x_i, x_j) = \begin{cases} d_{ij} \;\; if \; elements \;\; x_i \;\;\; and \;\; x_j \;\; are \; comparable, \\ m \; if \; elements \;\; x_i \;\; and \;\; x_j \;\; are \; not \; comparable. \end{cases} \quad (3)$$

The values of the binary description are included in the set $\{-1,\ 1\} \cup \{2\}$, the values of multivalent description - in the set $\{-(m-2),\ ...,\ -1,\ 1,\ ...,\ m-2\} \cup \{m\}$. The sets of "comparable values" – binary and multivalent $T_\upsilon(x_i, x_j)$ $(\upsilon \in \{b, \mu\})$ will be denoted, respectively, $\wp_b$ and $\wp_\mu$:

$$\wp_b = \{-1,\ 1\},\ \ \wp_\mu = \{-(m-2),\ ...,\ -1,\ 1,\ ...,\ m-2\}. \quad (4)$$

Examples of the values $T_\upsilon(x_i, x_j)$ $(\upsilon \in \{b, \mu\})$.

The relation form – the following partial order,i.e.:

x_1 precedes x_2, x_1 precedes x_3, x_2 and x_3 incomparable, x_2 precedes x_4, x_3 precedes x_4, x_4 precedes x_5,

$I_o = \{< 1, 2 >, < 1, 3 >, < 1, 4 >, < 1, 5 >, < 2, 4 >, < 2, 5 >, < 3, 4 >, < 3, 5 >, < 4, 5 >\}$,

$I_n = \{< 2, 3 >\}$.

The values of $T_\upsilon(x_i, x_j)$ $(\upsilon \in \{b, \mu\})$ assume the form:

$$T_b(x_i, x_j) = \begin{bmatrix} \times & -1 & -1 & -1 & -1 \\ & \times & 2 & -1 & -1 \\ & & \times & -1 & -1 \\ & & & \times & -1 \\ & & & & \times \end{bmatrix},\ T_\mu(x_i, x_j) = \begin{bmatrix} \times & -1 & -1 & -2 & -3 \\ & \times & 5 & -1 & -2 \\ & & \times & -1 & -2 \\ & & & \times & -1 \\ & & & & \times \end{bmatrix}.$$

2.2. Assumptions regarding the comparison errors

The relation form, expressed by $T_b(x_i, x_j)$ or $T_\mu(x_i, x_j)$, has to be determined (estimated) on the basis of N $(N \geqslant 1)$ comparisons of each pair (x_i, x_j) $(< i, j >\in R_m)$, in binary form or in multivalent form, under disturbance by random errors. The form of $T_\upsilon(x_i, x_j)$ $(\upsilon \in \{b, \mu\})$ has to be compatible with comparisons; they will be denoted – respectively - $g_{bk}(x_i, x_j)$ and $g_{\mu k}(x_i, x_j)$ $(k = 1, ..., N)$. The comparison errors - respectively $\varphi^*_{bk}(x_i, x_j)$ (binary) or $\varphi^*_{\mu k}(x_i, x_j)$ (multivalent) can be expressed in the following form:

$$\varphi^*_{bk}(x_i, x_j) = \begin{cases} 0\ if\ \ g_{bk}(x_i, x_j)\ and\ T_b(x_i, x_j)\ are\ the\ same, \\ 1\ f\ \ g_{bk}(x_i, x_j)\ and\ T_b(x_i, x_j)\ are\ not\ the\ same, \end{cases} \tag{5}$$

$$\varphi^*_{\mu k}(x_i, x_j) = \begin{cases} 0\ if\ \ g_{\mu k}(x_i, x_j) = m\ and\ T_\mu(x_i, x_j) = m, \\ g_{\mu k}(x_i, x_j) - T_\mu(x_i, x_j)\ if\ \ g_{\mu k}(x_i, x_j),\ T_\mu(x_i, x_j) \neq m, \\ 2m - 1\ in\ other\ cases. \end{cases} \tag{6}$$

The distributions of comparison errors have to satisfy the following assumptions.

A1. Any comparison $g_{\upsilon k}(x_i, x_j)$ $(\upsilon \in \{b,\ \mu\};\ k = 1,\ ...,\ N;\ \ < i, j > \in R_m)$, is an evaluation of the value $T_\upsilon(x_i, x_j)$; the probabilities of errors

$$P(\varphi^*_{bk}(x_i, x_j) = l)\,(l \in \{0,\ 1\})$$

and

$$P(\varphi^*_{\mu k}(x_i,\ x_j) = l)\,(l \in \{-2(m-1), \ldots, 0, \ldots, 2(m-1), 2m-1\})$$

have to satisfy the following assumptions:

$$P(\varphi^*_{bk}(x_i, x_j) = 0) \geqslant 1 - \delta \quad (\delta \in (0,\ 1/2)), \tag{7}$$

$$P(\varphi^*_{bk}(x_i, x_j) = 0) + P(\varphi^*_{bk}(x_i, x_j) = 1) = 1, \tag{8}$$

$$\sum_{l \leqslant 0} P(\varphi^*_{\mu k}(x_i, x_j) = l \mid g_{\mu k}(x_i, x_j), T_\mu(x_i, x_j) \neq m) \geqslant 1-\delta,\ \ (\delta \in (0,\ 1/2)), \tag{9}$$

$$\sum_{l \geqslant 0} P(\varphi^*_{\mu k}(x_i, x_j) = l \mid g_{\mu k}(x_i, x_j), T_\mu(x_i, x_j) \neq m) \geqslant 1-\delta,\ \ (\delta \in (0,\ 1/2)), \tag{10}$$

$$\begin{aligned} &P(\varphi^*_{\mu k}(x_i, x_j) = l \mid g_{\mu k}(x_i, x_j), T_\mu(x_i, x_j) \neq m,\ l \geqslant 0) \geqslant \\ &\geqslant P(\varphi_{\mu k}(x_i, x_j) = l + 1 \mid g_{\mu k}(x_i, x_j), T_\mu(x_i, x_j) \neq m,\ l \geqslant 0), \end{aligned} \tag{11}$$

$$\begin{aligned} &P(\varphi^*_{\mu k}(x_i, x_j) = l \mid g_{\mu k}(x_i, x_j), T_\mu(x_i, x_j) \neq m,\ l \leqslant 0) \geqslant \\ &\geqslant P(\varphi_{\mu k}(x_i, x_j) = l - 1 \mid g_{\mu k}(x_i, x_j), T_\mu(x_i, x_j) \neq m,\ l \leqslant 0), \end{aligned} \tag{12}$$

$$\begin{aligned} &\sum_{l \neq 2m-1} P(\varphi^*_{\mu k}(x_i, x_j) = l \mid g_{\mu k}(x_i, x_j), T_\mu(x_i, x_j) \neq m) + \\ &+ P(\varphi^*_{\mu k}(x_i, x_j) = 2m - 1 \mid T_\mu(x_i, x_j) \neq m) = 1. \end{aligned} \tag{13}$$

$$P(\varphi^*_{\mu k}(x_i, x_j) = 0 \mid T_\mu(x_i, x_j) = m) \geqslant 1 - \delta \quad (\delta \in (0,\ 1/2)), \tag{14}$$

$$\begin{aligned} &P(\varphi^*_{\mu k}(x_i, x_j) = 0 \mid T_\mu(x_i, x_j) = m)+ \\ &+P(\varphi^*_{\mu k}(x_i, x_j) = 2m - 1 \mid T_\mu(x_i, x_j) = m) = 1, \end{aligned} \tag{15}$$

$$\begin{aligned} &P(\varphi^*_{\mu k}(x_i, x_j) = 2m - 1 \mid T_\mu(x_i, x_j) \in \wp_\mu) \leqslant \\ &\leqslant \min_l \{P(\varphi^*_{\mu k}(x_i, x_j) = l \mid l \neq 2m - 1,\ T_\mu(x_i, x_j) \in \wp_\mu)\}, \end{aligned} \tag{16}$$

$$\begin{aligned} &P(\varphi^*_{\mu k}(x_i, x_j) = 2m - 1 \mid T_\mu(x_i, x_j) \in \wp_\mu)+ \\ &+ \sum_{l \neq 2m-1} (\varphi^*_{\mu k}(x_i, x_j) = l \mid T_\mu(x_i, x_j) \in \wp_\mu) = 1. \end{aligned} \tag{17}$$

A2. The comparisons: $g_{bk}(x_i, x_j) \quad (k = 1,\ ...,\ N;\ < i, j > \in R_m)$ are independent random variables, and the comparisons:

$$g_{\mu k}(x_i, x_j) \quad (k = 1,\ ...,\ N;\ < i, j > \in R_m)$$

are also independent random variables.

The assumptions about comparisons reflect the following facts.

In the case of binary comparisons the probability of a correct comparison is greater than of an incorrect one (see assumptions (7), (8)).

In the case of multivalent comparisons the following properties hold true. The probability of the correct detection of an incomparable pair is greater than 1/2 (see relationships (14), (15)). The distribution of the error, in the case of a comparable pair, is unimodal with mode and median equal zero (see relationships (10) – (13)). The probability of an incorrect detection of an incomparable pair is not greater than any probability of any incorrect difference of ranks (see (16)).

The assumption about independence of comparisons can be relaxed in such a way that (multiple) comparisons of the same pair are independent and comparisons of pairs comprising different elements are independent.

The random variables $\varphi_{bk}(x_i, x_j)$ and $\varphi_{\mu k}(x_i, x_j)$, corresponding to any partial order in the set X, denoted, respectively, by $t_b(x_i, x_j)$ and $t_\mu(x_i, x_j)$, can be expressed in the following form:

$$\varphi_{bk}(x_i, x_j) = \begin{cases} 0\, if\ \ g_{bk}(x_i, x_j)\ and\ \ t_b(x_i, x_j)\ are\ the\ same, \\ 1\, if\ \ g_{bk}(x_i, x_j)\ and\ \ t_b(x_i, x_j)\ are\ not\ the\ same, \end{cases}$$

$$\varphi_{\mu k}(x_i,x_j)=\begin{cases} 0\ if\ \ g_{\mu k}(x_i,x_j)=m\ and\ \ t_\mu(x_i,x_j)=m,\\ g_{\mu k}(x_i,x_j)-t_\mu(x_i,x_j)\ if\ \ g_{\mu k}(x_i,x_j),\ t_\mu(x_i,x_j)\neq m,\\ 2m-1\ in\ other\ cases.\end{cases}$$

3. Estimation problems and properties of estimates

The idea of the nearest adjoining order estimators is to minimize the differences between the comparisons, expressed in binary or multivalent form, on the one hand, and the relation, expressed in a "compatible" way, on the other hand. Thus, the estimates $\hat{T}_b(x_i,x_j)$ or $\hat{T}_\mu(x_i,x_j)\quad (<i,j>\in R_m)$ are the optimal solutions of the discrete programming problems – respectively:

$$\min_{F_X}\{\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\varphi_{bk}(x_i,x_j)\},\tag{18}$$

$$\min_{F_X}\{\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\varphi_{\mu k}(x_i,x_j)\},\tag{19}$$

where:

F_X – feasible set, i.e. family of all partial orders in the set X,

$\varphi_{vk}(x_i,x_j)\quad (v\in\{b,\mu\})$ – differences between comparisons and any relation from the family F_X.

In the book of Klukowski (2011), the consistency of such estimates was demonstrated, for $N\to\infty$, in the case of the complete preference relation. The proofs of the consistency are based on the following facts.

Firstly, the expected values of the random variables:

$$W_b^*=\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\varphi_{bk}^*(x_i,x_j),\tag{20}$$

$$W_\mu^*=\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\left|\varphi_{\mu k}^*(x_i,x_j)\right|,\tag{21}$$

expressing the differences between the comparisons and the actual relation, ($T_b(x_i,x_j)$ or $T_\mu(x_i,x_j)\quad (<i,j>\in R_m)$), are lower than the expected values of the variables:

$$\tilde{W}_b=\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\tilde{\varphi}_{bk}(x_i,x_j),\tag{22}$$

$$\tilde{W}_\mu=\sum_{<i,j>\in R_m}\sum_{k=1}^{N}\left|\tilde{\varphi}_{\mu k}(x_i,x_j)\right|,\tag{23}$$

expressing the differences between comparisons and any other relation, denoted by $\tilde{T}_b(x_i,x_j)$ or $\tilde{T}_\mu(x_i,x_j)$.

Secondly, the variances of these variables, i.e.: $Var(\frac{1}{N}W_b^*)$, $Var(\frac{1}{N}W_\mu^*)$, $Var(\frac{1}{N}\tilde{W}_b)$, $Var(\frac{1}{N}\tilde{W}_\mu)$, converge to zero, as $N \to \infty$.

Thirdly, the probabilities: $P(W_b^* < \tilde{W}_b)$ and $P(W_\mu^* < \tilde{W}_\mu)$ converge to one, as $N \to \infty$; the speed of convergence is determined by the exponential subtrahend. These relationships can be formulated shortly in the following

Theorem 1 *The following relationships hold true:*

$$E(W_b^*) < E(\tilde{W}_b), \tag{24}$$

$$E(W_\mu^*) < E(\tilde{W}_\mu), \tag{25}$$

$$\lim_{N\to\infty} Var(\frac{1}{N}W_b^*) = 0, \quad \lim_{N\to\infty} Var(\frac{1}{N}\tilde{W}_b) = 0, \tag{26}$$

$$\lim_{N\to\infty} Var(\tfrac{1}{N}W_\mu^*) = 0, \quad \lim_{N\to\infty} Var(\tfrac{1}{N}\tilde{W}_\mu) = 0, \tag{27}$$

$$P(W_b^* < \tilde{W}_b) \geqslant 1 - \exp\{-2N(1/2 - \delta)^2\}, \tag{28}$$

$$P(W_\mu^* < \tilde{W}_\mu) \geqslant 1 - \exp\{-2N\tilde{\theta}^2\}, \tag{29}$$

where: $\tilde{\theta}$ *- positive constant, depending on* $\tilde{T}_\mu(x_i, x_j)$ *and the distribution of* $\tilde{W}_\mu(x_i, x_j)$.

Proof of the relationships (24) – (25) is similar to that for the case of complete relation (see Klukowski, 1994, 2008, 2011: Chapters 7, 8), the respective idea is presented in the Appendix.

The relationships (24) – (29) are the theoretical basis for establishing the estimators $\hat{T}_b(x_i, x_j)$ and $\hat{T}_\mu(x_i, x_j)$ - implying their consistency. This is so, because the random variables $\frac{1}{N}W_b^*$ and $\frac{1}{N}W_\mu^*$, corresponding to the actual relation, have minimal expected values in the family F_X and variances converging to zero. The optimal solutions of the problems (18) and (19), determining relations featuring the minimum values of differences with respect to comparisons, indicate such relation with probability converging to one. The approach can be applied also in the case of unknown probabilities of comparison errors. This is especially important in the case of multivalent comparisons. In such a case and for the number N of at least several, the distributions can be estimated in a way similar to that applied for the complete relation (see Klukowski, 2011, Chapter 9).

Minimization of functions (18), (19) is not an easy problem to solve. For a low number m of elements of the set X, i.e. just a couple, the minimization can be performed simply by means of complete enumeration. For the moderate values of m, the problem with binary comparisons can be solved with the use of known algorithms (David, 1988). In the remaining cases, heuristic algorithms are necessary (see also Hansen, Jaumard, Sanlaville, 1994).

4. Concluding remarks

The paper presents the estimators of the partial order relation, which are based on the pairwise comparisons in the binary and multivalent forms. They have similar properties to those of the estimators of the complete relation, in particular – consistency and speed of convergence. The case of binary comparisons is similar to the complete case, concerning the assumptions about comparison errors and the form of the estimators. In the case of multivalent comparisons, the assumptions about distributions of errors are more complex and lead to the elimination of the approach based on medians from comparisons. The estimators can be applied also to other structures of data, especially trees.

Appendix

The idea of the proof of the Theorem (relationships (24) – (29)).

The inequality (24), i.e. $E(W_b^*) < E(\tilde{W}_b)$ can be proved in similar way as the inequality (32) in Klukowski (1994).

The expected value of the difference $W_b^* - \tilde{W}_b$ assumes the form:

$$E(W_b^* - \tilde{W}_b) =$$
$$E(\sum_{<i,j>\in R_m} \sum_{k=1}^{N} \varphi_{bk}^*(x_i, x_j) - \sum_{<i,j>\in R_m} \sum_{k=1}^{N} \tilde{\varphi}_{bk}(x_i, x_j)) =$$
$$\sum_{k=1}^{N} \sum_{<i,j>\in R_m} E(\varphi_{bk}^*(x_i, x_j) - \tilde{\varphi}_{bk}(x_i, x_j)).$$

It is clear that each component $E(\varphi_{bk}^*(x_i, x_j) - \tilde{\varphi}_{bk}(x_i, x_j))$ can be either zero or negative; the value of zero corresponds to the case of $T_b(x_i, x_j) = \tilde{T}_b(x_i, x_j)$, while the negative – to the case of $T_b(x_i, x_j) \neq \tilde{T}_b(x_i, x_j)$, because the probability of correct comparison is greater than 1/2. This fact is sufficient for proving the inequality (24).

The inequality (25) can be proved in similar way, even though this is more cumbersome; the case of complete relation having been presented in Klukowski (2008). Let us consider firstly two cases: $T_\mu(x_i, x_j) = m$, $\tilde{T}_\mu(x_i, x_j) \neq m$ and the opposite ones $T_\mu(x_i, x_j) \neq m$, $\tilde{T}_\mu(x_i, x_j) = m$.

In the first case:

$$E(|\varphi_{\mu k}^*(x_i, x_j)| - |\tilde{\varphi}_{\mu k}(x_i, x_j)| \ ; \ T_\mu(x_i, x_j) = m) =$$
$$E(|\varphi_{\mu k}^*(x_i, x_j)| \ ; \ T_\mu(x_i, x_j) = m) - E\ (|\tilde{\varphi}_{\mu k}(x_i, x_j)| \ ; \ T_\mu(x_i, x_j) = m),$$

and:

$$E\ (|\varphi_{\mu k}^*(x_i, x_j)| \ ; \ T_\mu(x_i, x_j) = m) =$$
$$(2m-1)P(\varphi_{\mu k}^*(x_i, x_j) = 2m-1 \ ; \ T_\mu(x_i, x_j) = m), \tag{30}$$

$$E\,(|\,\tilde{\varphi}_{\mu k}(x_i,x_j)|\;\; T_\mu(x_i,x_j)=m)=E\,(|\,\tilde{\varphi}_{\mu k}(x_i,x_j)|\;\; \tilde{T}_\mu(x_i,x_j)\neq m)=$$
$$(2m-1)P\,(|\,\tilde{\varphi}_{\mu k}(x_i,x_j\,)|=2m-1\;;\;\tilde{T}_\mu(x_i,x_j)\neq m) \qquad (31)$$
$$+\sum_{l\in\wp_\mu}|l-\tilde{T}_\mu(x_i,x_j\,)|\;\;(P(g_{\mu k}(x_i,x_j)=l).$$

It is clear that the value of (30) is lower than that of (31), because

$$P(\varphi^*_{\mu k}(x_i,x_j)=2m-1\;;\;T_\mu(x_i,x_j)=m)<1/2$$

and

$$P(\tilde{\varphi}_{\mu k}(x_i,x_j)=2m-1\;\mid\tilde{T}_\mu(x_i,x_j)\neq m)>1/2.$$

In the second case:

$$E\,(|\,\varphi^*_{\mu k}(x_i,x_j)\,|\;;\;T_\mu(x_i,x_j)\neq m)=$$
$$(2m-1)P(\varphi^*_{\mu k}(x_i,x_j)=2m-1\;;\;T_\mu(x_i,x_j)\neq m)+$$
$$\sum_{l\in\wp_\mu}|\;l-T_\mu(x_i,x_j\,)|\;\;P(g_{\mu k}(x_i,x_j)=l;\;T_\mu(x_i,x_j)\neq m), \qquad (32)$$

$$E\,(|\,\tilde{\varphi}_{\mu k}(x_i,x_j)\,|\;;\;\tilde{T}_\mu(x_i,x_j)=m)=$$
$$(2m-1)\sum_{l\in\wp_\mu}P(g_{\mu k}(x_i,x_j)=l\quad|\;\tilde{T}_\mu(x_i,x_j)=m). \qquad (33)$$

The value of (32) is lower than that of (33), because:

- in (32) the component multiplied by the maximum value of the probability function $P(g_{\mu k}(x_i,x_j)=T_\mu(x_i,x_j)\;;\;T_\mu(x_i,x_j)\neq m)$ is equal to zero, the component $(2m-1)P(g_{\mu k}(x_i,x_j)=m\;;\;T_\mu(x_i,x_j)\neq m)$ is a product of (2m-1) and the minimal value of the probability function, and the remaining components are products including the probabilities lower than the maximum of the probability function and values lower than $2m-1$;
- in (33) the component with the minimum probability $P(g_{\mu k}(x_i,x_j)=m\;;\;T_\mu(x_i,x_j)\neq m)$ equals zero, while the remaining part of probability (i.e. $1-P(g_{\mu k}(x_i,x_j)=m\;;\;T_\mu(x_i,x_j)\neq m)$ is multiplied by $2m-1$.

The proof of the inequality (25), for the remaining values of $T_\mu(x_i,x_j)$, $\tilde{T}_\mu(x_i,x_j)\quad(<i,j>\in R_m)$, is similar.

The validity of the relationships (26) results from the following facts:

- each random variable:

$$\sum_{<i,j>\in R_m} \varphi^*_{bk}(x_i, x_j) \text{ and } \sum_{<i,j>\in R_m} \tilde{\varphi}_{bk}(x_i, x_j)\,(k = 1,\ ...,\ N)$$

  has finite, bounded expected value and variance,
- the variances of the variables:

$$\frac{1}{N}\sum_{k=1}^{N}\sum_{<i,j>\in R_m} \varphi^*_{bk}(x_i, x_j),\ \tfrac{1}{N}\sum_{k=1}^{N}\sum_{<i,j>\in R_m} \tilde{\varphi}_{bk}(x_i,\ x_j)$$

  are bounded (see(7)); their values will be denoted - respectively: $\frac{1}{N}\,V^*_b$ and $\frac{1}{N}\,\tilde{V}_b$, where: V^*_b and V_b - maximum of variances of the variables

$$\sum_{k=1}^{N}\sum_{<i,j>\in R_m} \varphi^*_{bk}(x_i, x_j) \quad \text{and} \quad \sum_{k=1}^{N}\sum_{<i,j>\in R_m} \tilde{\varphi}_{bk}(x_i,\ x_j),$$

  respectively,
- the values of $\frac{1}{N}\,V^*_b$ and $\frac{1}{N}\,\tilde{V}_b$ converge to zero, as $N \to \infty$.

The validity of the relationships (27) can be proved in similar way.

The validity of the inequalities (28), (29) can be proved on the basis of Hoeffding's (1963) inequalities for a sum of independent, bounded random variables. The inequality applied in the case under consideration assumes the form:

$$P(\sum_{k=1}^{N} Y_k - \sum_{k=1}^{N} E(Y_k) \geqslant Nt) \leqslant \exp(-2Nt^2/(b-a\,)^2), \qquad (*)$$

where:

$Y_1,\ ...,\ Y_N -$ independent random variables satisfying

$$P(a \leqslant Y_k \leqslant b) = 1, \quad a < b,$$

with:

a, b – finite constants,

t – positive constant.

The inequality (*) can be applied to the random variables

$$\sum_{k=1}^{N}\sum_{<i,j>\in R_m} (\varphi^*_{\upsilon k}(x_i, x_j) - \tilde{\varphi}_{\upsilon k}(x_i, x_j)) \ \ (\upsilon \in \{b, \mu\}),$$

after the following transformation:

$$P(\sum_{k=1}^{N} \sum_{<i,j>\in R_m} (\varphi^*_{\upsilon k}(x_i, x_j) \ - \tilde{\varphi}_{\upsilon k}(x_i, x_j)) < 0) =$$

$$1 - P(\sum_{k=1}^{N} \sum_{<i,j>\in R_m} (\varphi^*_{\upsilon k}(x_i, x_j) \ - \tilde{\varphi}_{\upsilon k}(x_i, x_j)) \geqslant 0) =$$

$$1 - P(\sum_{k=1}^{N} \sum_{<i,j>\in R_m} (\varphi^*_{\upsilon k}(x_i, x_j) \ - \tilde{\varphi}_{\upsilon k}(x_i, x_j)) -$$

$$E(\sum_{k=1}^{N} \sum_{<i,j>\in R_m} (\varphi^*_{\upsilon k}(x_i, x_j) \ - \tilde{\varphi}_{\upsilon k}(x_i, x_j))) \geqslant$$

$$\geqslant -E(\sum_{k=1}^{N} \sum_{<i,j>\in R_m} (\varphi^*_{\upsilon k}(x_i, x_j) \ - \tilde{\varphi}_{\upsilon k}(x_i, x_j))).$$

The probability subtracted from one can be evaluated on the basis of Hoeffding's inequality (*) using simple transformations. In the case of binary comparisons, the value $(b - a)^2$, in inequality (*), equals 4 and

$$t = 2(\frac{1}{2} - \delta).$$

Moreover, any component of the sum:

$$E(\sum_{k=1}^{N} \sum_{<i,j>\in R_m} (\varphi^*_{\upsilon k}(x_i, x_j) \ - \tilde{\varphi}_{\upsilon k}(x_i, x_j))$$

is negative or equal $\emptyset$ (for each (x_i, x_j) $(< i, j >\in R_m)$) and, after dividing the sum

$$\sum_{k=1}^{N} \sum_{<i,j>\in R_m} (\varphi^*_{\upsilon k}(x_i, x_j) \ - \tilde{\varphi}_{\upsilon k}(x_i, x_j)$$

by the number of non-zero elements (i.e. non-equal values of $\varphi^*_{\upsilon k}(x_i, x_j)$ and $\tilde{\varphi}_{\upsilon k}(x_i, x_j)$), can be evaluated by $-2N(1/2 - \delta)$. These facts are sufficient for proving the inequality (28).

The proof of the inequality (29) is similar, with such a difference that the value of $(b - a)^2$ is different than 4 and the value of t cannot be expressed on the basis of δ; more precisely, the value of t depends on the distributions of comparison errors and values of $\tilde{T}(x_i, x_j)$ $(< i, j >\in R_m)$. The value of $\tilde{\theta}^2$ in (29) can be determined in a similar way as in the case of the complete preference relation (Klukowski, 2008).

References

Bradley, R. A. (1984) Paired comparisons: some basic procedures and examples. In: P. R. Krishnaiah and P. K. Sen, eds., *Handbook of Statistics*, Vol. 4. Amsterdam: North-Holland, 299 – 326.

David H. A. (1988) *The Method of Paired Comparisons*, 2^{nd} ed. Ch. Griffin, London.

Flinger A. F., Verducci J. S., eds. (1993) *Probability Models and Statistical Analyses for Ranking Data. Lecture Notes in Statistics* **80**, Springer-Verlag, New York, Berlin, Heidelberg.

Hansen, P., Jaumard. B., Sanlaville E. (1994) Partitioning Problems in Cluster Analysis: A Review of Mathematical Programming Approaches. In: *Studies in Classification, Data Analysis, and Knowledge Organization.* Springer - Verlag.

Hoeffding, W. (1963) Probability inequalities for sums of bounded random variables. *JASA*, **58**, 13 – 30.

Klukowski L. (1994) Some probabilistic properties of the nearest adjoining order method and its extensions. *Annals of Operational Research*, **51**, 241 – 261.

Klukowski L. (2008) Estimation of the preference relation the basis of multiple pairwise comparisons in the form of differences of ranks. *Control and Cybernetics*, **37**, 711-729.

Klukowski, L. (2011) *Methods of Estimation of Relations of: Equivalence, Tolerance, and Preference in a Finite Set.* IBS PAN, Series: *Systems Research*, Volume 69, Warsaw.

Klukowski L. (2012) Properties of estimators of the preference relation based on pairwise comparisons – simulation survey. In: K. T. Atanassov et al., eds., *New Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics*, 75 - 90. *Volume II: Applications*, SRI PAS, Warsaw.

Slater P. (1961) Inconsistencies in a schedule of paired comparisons. *Biometrika*, **48**, 303 - 312.