sciendo
Control and Cybernetics
vol. 51 (2022) No. 1
pages: 31-42
DOI: 10.2478/candc-2022-0003

Some examples of solutions to an inverse problem for the first-passage place of a jump-diffusion process*

by

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Abstract: We report some additional examples of explicit solutions to an inverse first-passage place problem for one-dimensional diffusions with jumps, introduced in a previous paper. If $X(t)$ is a one-dimensional diffusion with jumps, starting from a random position $\eta \in [a,b]$, let be $\tau_{a,b}$ the time at which $X(t)$ first exits the interval (a,b), and $\pi_a = P(X(\tau_{a,b}) \leq a)$ the probability of exit from the left of (a,b). Given a probability $q \in (0,1)$, the problem consists in finding the density g of η (if it exists) such that $\pi_a = q$; it can be seen as a problem of optimization.

Keywords: jump-diffusion process, first-passage place, inverse first-passage place problem

1. Introduction and preliminary results

This short note is a continuation of the paper by the present author, Abundo (2020), in which we have studied an inverse first-passage place problem (IFPP) for a one-dimensional jump-diffusion process. While for simple-diffusions (i.e. without jumps) a number of examples were reported in Abundo (2020), in that article we were able to present only one example, concerning diffusions with jumps. Therefore, in this paper we report some additional examples of explicit solutions to the IFPP problem for diffusions with jumps.

The IFPP problem, as well as the analogous inverse first-passage time problem, have interesting applications in Mathematical Finance, in particular in credit risk modeling, where the first-passage time represents a default event of an obligor (see, e.g., Jackson, Kreinin and Zhang, 2009), in Biology, in the

*Submitted: November 2020; Accepted: February 2022

scope of diffusion models for neural activity (see, e.g., Lanska and Smiths, 1989), Engineering, and many other fields; for more about inverse first-passage time problems, see, e.g., Abundo (2012, 2013a,b,c, 2014, 2015, 2018, 2019), and Jackson, Kreinin and Zhang (2009). As regards the direct first-passage time problem for jump-diffusions, see, e.g., Abundo (2000, 2010), Kou and Wang (2003), or Tuckwell (1976); as for the direct first-passage place problem, few results are known: it was studied by Lefebvre (2019a,b, 2020), and Kou and Wang (2003), where equations for the moments of first-passage places were established; in a particular case Lefebvre found exact formulae for the k-th moments of the first-passage place, providing also an approximate analytical expression for it.

We recall the terms of the IFPP problem. Let

$$X(t) = \eta + \int_0^t \mu(X(s))ds + \int_0^t \sigma(X(s))dB_s + \sum_{i=1}^{N_1(t)} \varepsilon_i(X(t)) + \sum_{i=1}^{N_2(t)} \Delta_i(X(t)), \quad (1)$$

be a one-dimensional, time-homogeneous jump-diffusion process starting from a random position $\eta \in [a,b]$, where B_t is standard Brownian motion, $\mu(\cdot)$ and $\sigma(\cdot)$ are smooth enough deterministic functions, and $\{N_k(t)\}$ is a time-homogeneous Poisson process with rate $\lambda_k > 0$, for $k = 1,2$. The three stochastic processes $B_t,\ N_1(t)$ and $N_2(t)$ are assumed to be independent, and the r.v. η is independent of them; moreover, the state-dependent random variables $\varepsilon_i(X(t)) > 0$ and $\Delta_i(X(t)) < 0,\ i = 1, 2, \dots,$ are independent and identically distributed, and are independent between themselves. Note that Eq. (1) is slightly different from the analogous equation in Abundo (2020), which defines the jump-diffusion process $X(t)$, there considered; in fact, the representation of $X(t)$ by means of (1) allows to point out explicitly the positive and negative jumps; this formulation was inspired by Lefebvre's paper, Lefebvre (2020).

We suppose that the first-exit time of $X(t)$ from the interval (a,b), namely

$$\tau_{a,b} = \inf\{t \geq 0 : X(t) \notin (a,b)\}, \quad (2)$$

is finite with probability one, and let $X(\tau_{a,b})$ be the first-passage place of $X(t)$ at time $\tau_{a,b}$. By assumption, one has $X(\tau_{a,b}) \leq a$ or $X(\tau_{a,b}) \geq b$; we denote by $\pi_a = P(X(\tau_{a,b}) \leq a)$ the probability that the process $X(t)$ first exits the interval (a,b) from the left, and by $\pi_b = 1 - \pi_a = P(X(\tau_{a,b}) \geq b)$ the probability that $X(t)$ first exits from the right.

Actually, we have considered in Abundo (2020) the following inverse first-passage place (IFPP) problem:

given a probability $q \in (0,1)$, find the density g of η (if it exists), for which there results $\pi_a = q$.

The function g is called a solution to the IFPP problem. In fact, the solution to the IFPP problem, if it exists, is not necessarily unique (see Abundo, 2020).

As we will see in Remark 3, the IFPP problem can be also seen as a problem of optimization.

We can also admit η to be a discrete random variable, taking values in a set $S \subset [a, b]$; in this case $g(x)$ turns out to be a discrete probability density. Some examples of solutions to the IFPP problem of this kind are contained in the paper Lefebvre (2022).

In the next section, we present some additional examples of explicit solutions to the IFPP problem for one-dimensional diffusions $X(t)$ with jumps. They also provide information about the corresponding direct first-passage time problem, since they involve the calculation of the exit probability of $X(t)$ from the left of the interval (a, b).

Let $f_\varepsilon(\epsilon)$ and $f_\Delta(\delta)$ be the probability density functions of the random variables $\varepsilon_i(\cdot) > 0$ and $\Delta_i(\cdot) < 0$, respectively; we suppose that the infinitesimal drift $\mu(x)$ and the infinitesimal diffusion coefficient $\sigma(x)$ of the process $X(t)$ are smooth enough deterministic functions, and we denote by $\tau_{a,b}(x)$ the first-exit time of $X(t)$ from the interval (a, b), with the condition that $\eta = x \in [a, b]$. Moreover, we set $\pi_a(x) = P(X(\tau_{a,b}(x)) \leq a)$ and $\pi_b(x) = P(X(\tau_{a,b}(x)) \geq b) = 1 - \pi_a(x)$.

We recall (see, e.g., Abundo, 2020; Lefebvre, 2019a, 2020) that the function $v(x) := \pi_a(x)$ satisfies the integro-differential problem with the outer conditions:

$$\begin{cases} \frac{1}{2}\sigma^2(x)v''(x) + \mu(x)v'(x) + \lambda_1 \int_{-\infty}^{+\infty}[v(x+\epsilon) - v(x)]f_\varepsilon(\epsilon)d\epsilon + \\ \quad +\lambda_2 \int_{-\infty}^{+\infty}[v(x+\delta) - v(x)]f_\Delta(\delta)d\delta = 0, \quad x \in (a, b) \\ v(x) = 1 \text{ if } x \leq a \text{ and } v(x) = 0 \text{ if } x \geq b. \end{cases} \tag{3}$$

If there are no jumps, that is, $f_\varepsilon(\epsilon)$ and $f_\Delta(\delta)$ are identically zero, then the process defined by (1) is a (continuous) simple-diffusion, and so the outer conditions in (3) become the boundary conditions $v(a) = 1,\ v(b) = 0$.

Returning back to the case when the jump-diffusion $X(t)$ starts from the random position $\eta \in [a, b]$, we suppose that η possesses a density $g(x)$; then the following holds (see Abundo, 2020):

Proposition 1 *Let $X(t)$ be the jump-diffusion process defined by* (1)*; with the previous notations, if a solution g exists to the IFPP problem for $X(t)$ and $q \in (0, 1)$, then the function g must satisfy the following equation:*

$$q = \int_a^b g(x)\pi_a(x)dx, \tag{4}$$

where $\pi_a(x)$ is the solution of (3). □

Remark 1 *For an assigned $q \in (0, 1)$, Eq.* (4) *is an integral equation in the unknown $g(x)$. Unfortunately, no method is available to solve analytically this equation, so any possible solution g to the IFPP problem must be found by making attempts (see also Remark 2.5 in Abundo, 2020).*

Remark 2 *By restricting the class of candidate densities, for instance, by looking for functions $g(x)$ of the form $g(x) = a_1 x + a_0$, where a_0, a_1 are suitable constants, one can easily find solutions to the Eq.* (4) *in analytical way; in certain cases, one even obtains a unique solution (see Example 6 in Section 2).*

Note, however, that we are mainly interested in getting the solutions to the Eq. (4) *that are important for the applications mentioned in the Introduction.*

Remark 3 *The IFPP problem can be seen as a problem of optimization: indeed, let $\mathcal{G}$ be the set of probability densities on the interval (a, b), and consider the functional $\Psi : \mathcal{G} \longrightarrow \mathbb{R}^+$ defined, for any $g \in \mathcal{G}$, by*

$$\Psi(g) = \left(q - \int_a^b g(x)\pi_a(x)dx \right)^2 . \tag{5}$$

Then, a solution g to the IFPP problem, is characterized by

$$g = \arg\min_{g \in \mathcal{G}} \Psi(g). \tag{6}$$

Of course, if there exists more than one function $g \in \mathcal{G}$ at which $\Psi(g)$ attains the minimum, the solution of the IFPP problem is not unique.

Remark 4 *If one is looking for uniqueness of the solution to the IFPP problem, one must introduce constraints on the set $\mathcal{G}$ of probability densities of $\eta = X(0)$ on the interval (a, b) (see also Remark 2). For instance, the article by M. Lefebvre, Lefebvre (2022), contains several examples in which the solution to the IFPP problem for $X(t) = \mu t + B(t)$ in the interval $[0, 1]$ was uniquely found. They are obtained by restricting the class of candidate densities $g(x)$ of η, taking, for instance, exponential densities truncated at* 1, *and Gaussian densities truncated to $[0, 1]$, but also discrete densities, e.g., in the case when η takes only a finite number of different values in $[0, 1]$. These examples hold for the Wiener process $X(t) = \mu t + B(t)$ (i.e. a diffusion without jumps), but possibly they can be extended to some diffusions with jumps.*

2. Examples

Example 1

Let $X(t)$ be a jump-diffusion process of the form (1); we suppose that $\varepsilon_i(X(t))$, given that $X(t) = \xi$, is uniformly distributed on the interval $(0, \alpha_1 \xi)$, where $\alpha_1 > 0$; in analogous way, we assume that $\delta_i(X(t))$, given that $X(t) = \xi$, is uniformly distributed on the interval $(-\alpha_2 \xi, 0)$, with $0 < \alpha_2 \le 1$. Moreover, we suppose that the drift is $\mu(x) = \frac{1}{2}(\lambda_2 \alpha_2 - \lambda_1 \alpha_1)x$, while $\sigma(x)$ is any diffusion coefficient, and let be α, β positive constants; then, a solution g to the IFPP problem for $X(t)$ and $q = \frac{\beta}{\alpha+\beta}$ is the modified Beta density in the interval (a, b)

with parameters α and β, namely:

$$g(x) = \frac{1}{(b-a)^{\alpha+\beta-1}} \cdot \frac{(x-a)^{\alpha-1}(b-x)^{\beta-1}}{B(\alpha,\beta)} \cdot \mathbb{I}_{(a,b)}(x), \tag{7}$$

where $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ (the ordinary Beta density is obtained for $a = 0$ and $b = 1$).

In fact, from (3) the equation for $\pi_a(x)$ is (see also Lefebvre, 2020):

$$\frac{1}{2}\sigma^2(x)v''(x) + \mu(x)v'(x) - (\lambda_1+\lambda_2)v(x) + \frac{\lambda_1}{\alpha_1 x}\int_0^{\alpha_1 x} v(x+\epsilon)d\epsilon$$
$$+\frac{\lambda_2}{\alpha_2 x}\int_{-\alpha_2 x}^{0} v(x+\delta)d\delta = 0, \tag{8}$$

with the conditions

$$v(x) = 1,\ x \le a;\ v(x) = 0,\ x \ge b, \tag{9}$$

and it is satisfied by

$$\pi_a(x) = \begin{cases} 1 & \text{if } x \le a \\ \frac{b-x}{b-a} & \text{if } x \in (a,b) \\ 0 & \text{if } x \ge b, \end{cases} \tag{10}$$

irrespective of the diffusion coefficient $\sigma(x)$. Then, to verify that g, given by (7), is a solution to the IFPP problem, it suffices to substitute $g,\ q$ and $\pi_a(x)$ into Eq. (4) (in the calculation of the integral, one can use the fact that the mean of the r.v. η with density g is $(a\beta + b\alpha)/(\alpha+\beta)$; in fact, one has $\eta = a + (b-a)U$, U being a r.v. with Beta density).

Note that for $\beta > \alpha$ it results that $q > 1/2$, if $\beta = \alpha$ one has $q = 1/2$, while for $\beta < \alpha$ one has $q < 1/2$. For $\alpha = \beta = 1$, g turns out to be the uniform density in the interval (a,b).

We remark that the simple-diffusion process $\widetilde{X}(t)$ obtained from $X(t)$ by disregarding the jumps (that is, by setting $f_\varepsilon(\epsilon) = f_\Delta(\delta) = 0$), is driven by the SDE

$$d\widetilde{X}(t) = \frac{1}{2}(\lambda_2\alpha_2 - \lambda_1\alpha_1)\widetilde{X}(t)dt + \sigma(\widetilde{X}(t))dB_t. \tag{11}$$

Thus, $\widetilde{X}(t)$ is (see also Lefebvre, 2020):

- Brownian motion, if $\lambda_2\alpha_2 = \lambda_1\alpha_1$ and $\sigma(x) = 1$,
- Ornstein-Uhlenbeck process, if $\lambda_2\alpha_2 < \lambda_1\alpha_1$ and $\sigma(x) = const.$,
- Geometric Brownian motion, if $\lambda_2\alpha_2 > \lambda_1\alpha_1$ and $\sigma(x) = cx$, with c a positive constant,

• the CIR-like model in mathematical finance, if $\sigma(x) = \sqrt{x \vee 0}$,

• the Wright&Fisher-like process, if $\sigma(x) = \sqrt{x(1-x) \vee 0}$ (see, e.g., Abundo, 2020).

Note that $\frac{b-x}{b-a}$, $x \in (a,b)$, is nothing but the exit probability of $W(t) := x + B_t$ at the left of the interval (a,b); thus, the function g, given by (7), is also a solution to the IFPP problem for Brownian motion and $q = \frac{\beta}{\alpha+\beta}$, $\alpha,\ \beta > 0$ (see Example 3 of Abundo, 2020).

Example 2

Take $a = 0,\ b = 1,\ \gamma > 0$, and suppose that $X(t)$ is the jump-diffusion (1) with diffusion coefficient $\sigma(x) = \sqrt{x \vee 0}$ and linear drift $\mu(x) = Ax + B$, where

$$A = \frac{1}{\gamma}\left[\lambda_1 + \lambda_2 + \frac{1}{\gamma+1}\left(\frac{\lambda_1}{\alpha_1}(1-(1+\alpha_1)^{\gamma+1}) + \frac{\lambda_2}{\alpha_2}((1-\alpha_2)^{\gamma+1}-1)\right)\right],$$
$$B = -\frac{1}{2}(\gamma - 1).$$

We assume that the functions $f_\varepsilon(\epsilon),\ f_\Delta(\delta)$ are the same ones, as in Example 1. Then, for positive $\alpha,\ \beta$, a solution g to the IFPP problem for $X(t)$ and

$$q = 1 - \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+\gamma)}$$

is the Beta density in $(0,1)$ with parameters α, β, that is,

$$g(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x^{\alpha-1}(1-x)^{\beta-1} \cdot \mathbb{I}_{(0,1)}(x).$$

In fact, for the above infinitesimal coefficients $\mu(x)$ and $\sigma(x)$, it is easy to see that Eq. (8) is satisfied by $\pi_0(x) = 1 - x^\gamma$ for $x \in (0,1)$, and so to verify that g is solution to the IFPP problem, it is enough to substitute g, q and $\pi_0(x)$ into Eq. (4) (now, the integral in (4) is nothing but $1 - E(Z^\gamma)$, where Z is a r.v. with Beta density; thus, in the calculation it is convenient to use the fact that

$$E(Z^\gamma) = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+\gamma)}{\Gamma(\alpha)\Gamma(\alpha+\beta+\gamma)}$$

(see, e.g., Gupta and Nadarajah, 2004).

For instance, if $\gamma = 2$, then $\pi_0(x) = 1 - x^2$, and

$$q = \frac{\beta(\beta+2\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}.$$

Notice that the simple-diffusion process $\widetilde{X}(t)$, obtained by $X(t)$ disregarding the jumps, satisfies the SDE:

$$d\widetilde{X}(t) = (A\widetilde{X}(t) + B)dt + \sqrt{\widetilde{X}(t) \vee 0}\, dB_t, \tag{12}$$

which provides a special case of the CIR model.

Really, $\pi_0(x) = 1 - x^2$ is also the exit probability at the left of $(0,1)$ of the simple-diffusion driven by the SDE:

$$d\widetilde{X}(t) = -\frac{1}{2}dt + \sqrt{\widetilde{X}(t) \vee 0}\ dB_t. \tag{13}$$

Thus, the Beta density in $(0,1)$ is also solution to the IFPP problem for the diffusion driven by (13) and

$$q = \frac{\beta(\beta + 2\alpha + 1)}{(\alpha + \beta)(\alpha + \beta + 1)}.$$

Example 3

Take $a = 0,\ b = 1,\ \gamma > 0$ and suppose that $X(t)$ is the jump-diffusion (1) with $\sigma(x) = \sqrt{x(1-x) \vee 0}$, $\mu(x) = A'x + B$, with $A' = \frac{1}{2}(\gamma - 1) + A$, and the constant $A,\ B$, as well as the functions $f_\varepsilon(\epsilon),\ f_\Delta(\delta)$, are the same ones as in Example 2. Then, for positive $\alpha,\ \beta$, a solution g to the IFPP problem for $X(t)$ and

$$q = 1 - \frac{\Gamma(\alpha + \gamma)\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\alpha + \beta + \gamma)}$$

is the Beta density in $(0,1)$ with parameters α, β.

In fact, for the above infinitesimal coefficients Eq. (8) is satisfied by $\pi_0(x) = 1 - x^\gamma$ for $x \in (0,1)$, as in Example 2; thus to verify the result it is enough to substitute $g,\ q$ and $\pi_0(x)$ into Eq. (4).

Notice that the simple-diffusion process $\widetilde{X}(t)$, obtained by $X(t)$ disregarding the jumps, satisfies the SDE:

$$d\widetilde{X}(t) = (A'\widetilde{X}(t) + B)dt + \sqrt{\widetilde{X}(t)(1 - \widetilde{X}(t)) \vee 0}\ dB_t, \tag{14}$$

which provides the Wright-Fisher-like process (see, e.g., Abundo, 2020).

Really, $\pi_0(x) = 1 - x^2$ is also the exit probability at the left of $(0,1)$ of the simple-diffusion driven by the SDE:

$$d\widetilde{X}(t) = \frac{1}{2}(\widetilde{X}(t) - 1)dt + \sqrt{\widetilde{X}(t)(1 - \widetilde{X}(t) \vee 0}\ dB_t. \tag{15}$$

Thus, the Beta density in $(0,1)$ is also solution to the IFPP problem for the simple-diffusion driven by (15) and $q = \frac{\beta(\beta+2\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$.

Example 4

With the previous notations and assumptions on the Poisson processes $N_k(t)$, let be $\bar{\epsilon},\ \bar{\delta} > 0$, and suppose that, for $\eta \in [a,b]$:

$$X(t) = \eta + (\bar{\delta}\lambda_2 - \bar{\epsilon}\lambda_1)t + \int_0^t \sigma(X(s))dB_s + \bar{\epsilon}N_1(t) - \bar{\delta}N_2(t). \tag{16}$$

Then, a solution g to the IFPP problem for $X(t)$ and $q = \frac{\beta}{\alpha+\beta}$ $(\alpha,\ \beta > 0)$, is the modified Beta density in the interval (a, b), given by (7). In fact, now the equation for $v(x) = \pi_a(x)$ becomes:

$$\frac{1}{2}\sigma^2(x)v''(x) + (\bar{\delta}\lambda_2 - \bar{\epsilon}\lambda_1)v'(x) - (\lambda_1 + \lambda_2)v(x) + \lambda_1 v(x + \bar{\epsilon}) + \lambda_2 v(x - \bar{\delta}) = 0, \quad x \in (a, b), \tag{17}$$

which is satisfied by $v(x) = \pi_a(x) = \frac{b-x}{b-a},\ x \in (a, b)$, irrespective of $\sigma(x)$; thus, the assertion is soon verified, proceeding as in Example 1.

A variant is obtained by considering the jump-diffusion:

$$X(t) = \eta - \bar{\epsilon}\lambda_1 t + \int_0^t \sigma(X(s))dB_s + N_1(t); \tag{18}$$

then, a solution g to the IFPP problem for $X(t)$ and $q = \frac{\beta}{\alpha+\beta}$ $(\alpha,\ \beta > 0)$, is again the modified Beta density in the interval (a, b). It suffices to note that now the equation for $v(x) = \pi_a(x)$ is

$$\frac{1}{2}\sigma^2(x)v''(x) - \bar{\epsilon}\lambda_1 v'(x) - \lambda_1 v(x) + \lambda_1 v(x + \bar{\epsilon}) = 0,\ x \in (a, b), \tag{19}$$

and it is satisfied again by $\pi_a(x) = \frac{b-x}{b-a},\ x \in (a, b)$, irrespective of $\sigma(x)$.

Example 5

Take $a = 0,\ b = 1$; for $\bar{\epsilon},\ \bar{\delta} > 0$, suppose that:

$$dX(t) = \mu(X(t))dt + \sqrt{X(t)}dB_t + \bar{\epsilon}dN_1(t) - \bar{\delta}dN_2(t),\ X(0) = \eta \in [0, 1], \tag{20}$$

where

$$\mu(x) = \frac{1}{\ln 2}\left[-\frac{(\ln 2)^2 x}{2} - \lambda_1(2^{\bar{\epsilon}} - 1) + \lambda_2(1 - 2^{-\bar{\delta}})\right], \tag{21}$$

and $N_k(t)$ are Poisson processes with intensity $\lambda_k,\ k = 1, 2$ (we can write $\sqrt{X(t)}$ instead of $\sqrt{X(t) \vee 0}$, since $X(t)$ is ≥ 0 until the first-exit time of $X(t)$ from the interval $(0, 1)$). Then, for any $\alpha, \beta > 0$ a solution g to the IFPP problem for $X(t)$ and

$$q = 2 - \sum_{k=0}^{\infty} \frac{(\ln 2)^k}{k!}\ \frac{B(\alpha + k, \beta)}{B(\alpha, \beta)}, \tag{22}$$

is the Beta density in the interval $(0, 1)$ with parameters α and β (if e.g. $\alpha = \beta = 1$, one has $q = 2 - 1/\ln 2$ and g is the uniform density in $(0, 1)$, if $\alpha = \beta = 2$, then $q = 2 - 6 \cdot \frac{3\ln 2 - 2}{(\ln 2)^3}$ and $g(x) = 6x(1 - x),\ x \in (0, 1)$).

In fact, now the equation for $v(x) = \pi_0(x)$ becomes:

$$\frac{1}{2}xv''(x)+\mu(x)v'(x)-(\lambda_1+\lambda_2)v(x)+\lambda_1 v(x+\bar{\epsilon})+\lambda_2 v(x-\bar{\delta}) = 0,\ x \in (0,1), \quad (23)$$

which is satisfied by $v(x) = \pi_0(x) = 2 - 2^x,\ x \in (0,1)$, if $\mu(x)$ is given by (21); thus, to verify that

$$g(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

is solution to the IFPP problem, it is enough to substitute $g,\ q$ and $\pi_0(x)$ into Eq. (4) (to calculate the integral in (4) it is convenient to note that

$$\int_0^1 2^x g(x)\ dx = E\left(e^{(\ln 2)X}\right), \quad (24)$$

where

$$E\left(e^{tX}\right) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\ \frac{B(\alpha+k,\beta)}{B(\alpha,\beta)} \quad (25)$$

is the moment generating function of a r.v. having Beta density, with parameters α and β, see, e.g., Gupta and Nadarajah, 2004).

Example 6

Take $a = 0,\ b = 1$ and, for $\eta \in [0,1]$, consider the jump-diffusion:

$$X(t) = \eta - \frac{\pi}{4}\int_0^t \cos\left(\frac{\pi}{2}X(s)\right) ds + \int_0^t \sqrt{\sin\left(\frac{\pi}{2}X(s)\right)}\ dB_s + 4N_1(t), \quad (26)$$

where $N_1(t)$ is a Poisson Process with intensity λ_1 (note that the amplitude of jumps is 4 and, as soon as a jump occurs, the process exits $(0,1)$ from the right).

Now, we look for solutions to the IFPP problem for $X(t)$ of the form $g(x) = a_1x + a_0$, for suitable constants $a_0,\ a_1$; notice that $\int_0^1 g(x)dx = 1$ implies $a_1/2 + a_0 = 1$.

The equation for $v(x) = \pi_0(x)$ is:

$$\frac{1}{2}\sin\left(\frac{\pi}{2}x\right)v''(x) - \frac{\pi}{4}\cos\left(\frac{\pi}{2}x\right)v'(x) - \lambda_1 v(x) + \lambda_1 v(x+4) = 0,\ x \in (0,1), \quad (27)$$

which is satisfied by $v(x) = \pi_0(x) = \cos\left(\frac{\pi}{2}x\right)$, for $x \in (0,1)$. Then, since

$$\int_0^1 g(x)\cos\left(\frac{\pi}{2}x\right) dx = \int_0^1 (a_1x + a_0)\cos\left(\frac{\pi}{2}x\right) dx = \frac{2a_0}{\pi} + \frac{2a_1}{\pi^2}(\pi - 2),$$

in the present case one obtains that Eq. (4) is equivalent to

$$\begin{cases} q = \frac{2a_0}{\pi} + \frac{2a_1}{\pi^2}(\pi - 2), \\ 1 = \frac{a_1}{2} + a_0, \end{cases}$$

that is,

$$q = \frac{2}{\pi} + \frac{a_1}{\pi}(1 - \frac{4}{\pi});$$

this is a linear equation that permits to find a_1 and $a_0 = 1 - a_1/2$, for any fixed $q \in (0, 1)$. In conclusion, by imposing $g(x)$ to be a polynomial of degree one, we find a unique solution g to the IFPP problem for $X(t)$. For instance, if $q = 2/\pi$, we obtain $a_1 = 0$ and $a_0 = 1$, namely the solution g to the IFPP problem turns out to be the uniform density in the interval $(0, 1)$; if $q = \frac{4}{\pi}(1 - \frac{2}{\pi})$, we get that the solution is $g(x) = 2x, \; x \in (0, 1)$.

Instead, by imposing $g(x)$ to be a polynomial of degree ≥ 2, one finds an infinite number of possible solutions to the Eq. (4).

Note that $\pi_0(x) = \cos\left(\frac{\pi}{2}x\right)$ is also the exit probability at the left of $(0, 1)$ of the simple-diffusion obtained from $X(t)$ disregarding the jumps.

Finally, we recall the following example, already presented in Abundo (2020), in which $a = 0, \; b = 2\epsilon$ (ϵ being a fixed positive number), and the exit probability, $\pi_0(x)$, from the left of the interval $(0, b)$ has a more complicated form, since it is not a polynomial, exponential-like, or trigonometric function.

Example 7

For $\epsilon > 0$, take $a = 0, \; b = 2\epsilon$, and let be $X(t) = \eta + B_t + \epsilon N_1(t)$, where the starting point η is random in $[0, 2\epsilon]$ and $N_1(t)$ is a time-homogeneous Poisson process with rate $\lambda_1 = 1$.

Let there be

$$q = \frac{1}{2\epsilon\sqrt{2}} \left[\frac{\gamma - (\alpha + \beta)\sqrt{2} + \delta}{\alpha\delta - \beta\gamma} + e^{\epsilon\sqrt{2}} \left(1 - \frac{e^{\epsilon\sqrt{2}}(\beta\sqrt{2} - \delta)}{\alpha\delta - \beta\gamma} \right) \right]$$

$$+ \frac{1}{2\epsilon\sqrt{2}} \left[\frac{\sqrt{2}e^{-\epsilon\sqrt{2}} \left(\sqrt{2} - \epsilon - \frac{1}{4}\right)(\gamma - \alpha\sqrt{2})}{\alpha\delta - \beta\gamma} \right]$$

$$+ \frac{1}{2\epsilon\sqrt{2}} \left[\frac{e^{-2\epsilon\sqrt{2}}(\gamma - \alpha\sqrt{2})}{\alpha\delta - \beta\gamma} + \frac{2e^{\epsilon\sqrt{2}}(\beta\sqrt{2} - \delta)}{\alpha\delta - \beta\gamma} + \frac{\gamma - \alpha\sqrt{2}}{2(\alpha\delta - \beta\gamma)} - 1 \right]. \quad (28)$$

Then, a solution g to the IFPP problem for $X(t)$ and the above value of q is the uniform density in $(0, 2\epsilon)$, i.e. $g(x) = \frac{1}{2\epsilon}\mathbf{1}_{(0,2\epsilon)}(x)$ (see Abundo, 2020, for details).

Acknowledgments

The author expresses particular thanks to the reviewer for his/her useful comments, leading to improved presentation.

The author acknowledges the MIUR Excellence Department Project, awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006

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