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Equilibrium reinsurance-investment strategy for mean-variance insurers under state dependent risk aversion*

by

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Abstract: In this work, we study the equilibrium reinsurance/ new business and investment strategy for mean-variance insurers, under the assumption that the risk aversion is a function of current wealth level. The surplus of the agents is represented by a sum of a compound process and a linear premium perturbed with a Brownian component. The financial market consists of one riskless asset and a multiple risky assets whose price processes are driven by Poisson random measures and independent Brownian motions. We characterize explicit expressions for the time-consistent Nash equilibrium strategy and the equilibrium value function via a forward-backward stochastic system and an equilibrium condition. An interesting feature of these FBSDEs is that a time parameter is involved, so that they form a flow of FBSDEs. Furthermore, a feedback representation of an equilibrium solution is derived. This solution provides a tool for comparing the equilibrium strategy with those derived in other papers, where some special cases were studied by the dynamic programming argument.

Keywords: time inconsistency, mean-variance criterion, investment-reinsurance strategy, insurer, equilibrium strategy, forward–backward stochastic differential equation

1. Introduction

It is well known that some business activities, such as investing in a financial market, purchasing reinsurance, and acquiring new business (acting as a reinsurer for other insurers) are effective ways to control risk exposure for insurance companies. Therefore, many problems with various objectives in insurance risk management have been extensively investigated in the literature. For instance,

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Hipp and Plum (2000), Promislow and Young (2005) investigated the optimal reinsurance and investment problem for an insurer in the sense of minimizing the ruin probability. Browne (1995), Yang and Zhang (2005), Xu, Wang and Yao (2008), and Gu et al. (2010) studied optimal investment–reinsurance policies for an insurer who maximizes expected utility of terminal wealth in different situations.

Besides the ruin probability minimization and the expected utility maximization, mean–variance criterion is another important objective function. The optimization problems under mean-variance criterion were initiated by Markowitz (1952), who considered a single period (one time step) model for the portfolio optimization problem. The interesting feature of the mean-variance model is that it allows decision makers to be risk averse. Decisions are therefore made following two different objectives: to maximize the expected return and to minimize the risk. The two conflicting objectives can be combined, decisions are made so as to maximize the difference between expectation and variance of the random quantity representing the state at terminal time. After Markowitz's work hundreds of papers have been published on this topic. Among others, Li and Ng (2000) and Zhou and Li (2000) extended the model to multi-period and continuous time settings, respectively, by using tools from stochastic LQ control theory. Moreover, optimal investment–reinsurance problems for insurance companies under the mean–variance criterion have recently gained a lot of attention. See, for examples, Bäuerle (2005), Delong and Gerrard (2000), Bai and Zhang (2008), Zeng, Li and Liu (2011), Zeng and Li (2012), and Li and Li (2013).

Due to the existence of a non-linear function of the expectation in the objective functional, the mean-variance criterion lacks the iterated expectation property. Consequently, continuous-time and multi-period mean-variance problems are time-inconsistent in the sense that the Bellman's principle of optimality does not hold; which means that a control, optimizing the mean–variance utility at time zero may not be optimal for mean–variance utility at later time. One way to get around the time-inconsistency issue is to consider only pre-committed controls i.e. the controls that are optimal only when viewed at the initial time instant. This is what all of the literature mentioned above have considered. However, the study of time inconsistency by economists goes back to Strotz (1955) who was the first to propose another way to handle the time-inconsistent problem: the formulation of a time-inconsistent decision problem as a noncoperative game between incarnations of the controller at different instants of time; Nash equilibrium of these strategies was then considered to define the new concept of solution of the original problem. Although the game formulation is very easy to understand when the time setting is discrete, in a continuous-time setting, the formulation is considerably more delicate and the concept of equilibrium solution can be presented in different ways. Ekeland

and Pirvu (2008) first provided a precise definition of equilibrium concept in continuous time. Further extensions of Ekeland and Pirvu's work can be found in Björk and Murgoci (2008), and Ekeland, Mbodji and Pirvu (2012). Recently, Basak and Chabakauri (2010) considered a continuous time mean–variance portfolio problem and derived the closed-form expression for its time-consistent, or equilibrium, strategy via some extended Hamilton-Jacobi-Bellman (HJB) equations. Björk, Murgoci and Zhou (2014) introduced the mean–variance problem with state-dependent risk aversion. Then, by using the extended HJB equation, obtained in Björk and Murgoci (2008), they obtained equilibrium solutions via some well posed integral equations. In order to study the mean-variance portfolio problem with state-dependent risk aversion and stochastic coefficients, Hu, Jin and Zhou (2012) provided a precise definition of open loop Nash equilibrium controls, in continuous time setting, which is different from the feedback one, given in Björk and Murgoci (2008) and Ekeland and Privu (2008).

Concerning equilibrium strategies for optimal investment-reinsurance problems under the mean-variance criterion, Zeng and Li (2011) were the first to investigate Nash equilibrium strategies, where the surplus of insurers is modeled by the diffusion model and the price process of the risky asset is driven by geometric Brownian motion. Zeng and Li (2012) studied equilibrium investment–reinsurance strategy for mean–variance insurers, where the surplus process, as well as the price process of the risky asset, are modeled by geometric Levy process. Recently, Li, Zheng and Lai (2012) investigated equilibrium investment and reinsurance strategies for insurers under Heston's SV model. The work of Li, Rong and Zhao (2015) investigated equilibrium reinsurance and investment strategies for an insurer and a reinsurer with mean–variance criterion under the CEV (Constant Elasticity of Variance) model. However, in all those papers the authors considerd the problems with constant risk aversion. This assumption of a constant risk aversion parameter leads to some (deterministic) equilibrium solutions. Especially, the dollar amount invested in the risky asset is independent of current wealth, which turns out be unrealistic from an economic point of view, the reason having been elaborated in Björk, Murgoci and Zhou (2014). Moreover, in order to be economically reasonable, Björk, Murgoci and Zhou (2014) suggest that it is more rational to allow the risk aversion to depend on current wealth. Unfortunately, as far as we know, there is little work in the literature concerning equilibrium strategies for optimal investment-reinsurance problems under the mean-variance criterion with state-dependent risk aversion. Li and Li (2013) were the first to investigate the case, where the surplus of insurers is modeled by the diffusion model and the price processes of the risky assets are only driven by geometric Brownian motions. Following the approach developed by Björk, Murgoci and Zhou (2012, 2014), Li and Li (2013) have derived feedback equilibrium strategies via some class of well posed integral equations.

The purpose of this paper is to develop on the existing theory concerning the study of equilibrium solutions to investment and reinsurance strategies for mean–variance insurers, in which the state dependent on the risk aversion coefficient is taken into consideration, this case being more reasonable that the one with constant risk aversion, since the equilibrium strategies obtained are proportional to current wealth. Another main contribution of this work is that, following the idea of Hu, Jin and Zhou (2012), we consider the definition of equilibrium strategies in the sense of open-loop one, then by means of a variational method, we characterize an explicit representation of the equilibrium investment and reinsurance strategies in the setting where the surplus process is assumed to follow a geometric Lévy process, while in Li and Li (2013) the surplus process is approximated by a diffusion process in the absence of Poisson jumps.

We want to point out also that, in distinction from Li and Li (2013), where some feedback equilibrium strategies are derived via several very complicated highly nonlinear integro-differential equations, in this work we give an explicit representation of the equilibrium strategies via simple ODEs. This is essentially due to the difference between the two definitions of equilibria (open-loop and feedback).

2. The model and problem formulation

Throughout this paper, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space such that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets, $\mathcal{F}_T = \mathcal{F}$ for an arbitrarily fixed finite time horizon $T > 0$, and $(\mathcal{F}_t)_{t\in[0,T]}$ satisfy the usual conditions. $\mathcal{F}_t$ stands for the information available up to time t and any decision made at time t is based on this information. We also assume that all processes and random variables are well defined and adapted in this filtered probability space.

2.1. Notations

We use $C^\top$ to denote the transpose of any vector or matrix C, $\text{diag}(C)$ stands for the diagonal matrix with the elements of a vector C on the diagonal. In addition, for some Euclidean space $\mathbb{R}^m$ with the inner product $\langle .,. \rangle$ and Frobenius norm $|.|$, we denote by $\mathbf{0}_{\mathbb{R}^m}$ the null vector and we use the standard notations; furthermore, for all $t \in [0,T]$ we let

1. $\mathbb{L}^p(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^m)$: the set of random variables ξ, with $\mathbb{E}[|\xi|^p] < \infty$, for any $p \geq 1$.

2. $\mathcal{S}^2_{\mathcal{F}}(t,T;\mathbb{R}^m)$: the space of $\mathbb{R}^m-$valued, $(\mathcal{F}_s)_{s\in[t,T]}-$adapted càdlàg pro-

cesses $Y(\cdot)$, with

$$\|Y(\cdot)\|^2_{\mathcal{S}^2_{\mathcal{F}}(t,T;\mathbb{R}^m)} = \mathbb{E}\left[\sup_{s\in[t,T]} |Y(s)|^2 \, ds\right] < \infty.$$

3. $\mathcal{L}^2_{\mathcal{F}}(t,T;\mathbb{R}^m)$: the space of $\mathbb{R}^m-$valued, $(\mathcal{F}_s)_{s\in[t,T]}$ $-$adapted processes $Z(\cdot)$, with

$$\|Z(\cdot)\|^2_{\mathcal{L}^2_{\mathcal{F}}(t,T;\mathbb{R}^m)} = \mathbb{E}\left[\int_t^T |Z(s)|^2 \, ds\right] < \infty.$$

4. $\mathcal{L}^2_{\mathcal{F},p}(t,T;\mathbb{R}^m)$: the space of $\mathbb{R}^m-$valued, $(\mathcal{F}_s)_{s\in[t,T]}$ $-$predictable processes $u(\cdot)$, with

$$\|u(\cdot)\|^2_{\mathcal{L}^2_{\mathcal{F},p}(t,T;\mathbb{R}^m)} = \mathbb{E}\left[\int_t^T |u(s)|^2 \, ds\right] < \infty.$$

5. $\mathcal{L}^{\mu,2}_{\mathcal{F},p}([t,T]\times\mathbb{R}^*;\mathbb{R}^m)$, where $\mathbb{R}^*\equiv\mathbb{R}-\{0\}$: the space of $\mathbb{R}^m-$valued,

   $(\mathcal{F}_s)_{s\in[t,T]}$ $-$predictable processes $R(\cdot,\cdot)$, with

$$\|R(\cdot,\cdot)\|^2_{\mathcal{L}^{\mu,2}_{\mathcal{F},p}([t,T]\times\mathbb{R}^*;\mathbb{R}^m)} = \mathbb{E}\left[\int_t^T \int_{\mathbb{R}^*} R(s,z)^\top \operatorname{diag}(\nu(dz))\, R(s,z)\, ds\right] < \infty,$$

   for any positive and $\sigma-$finite measure $\nu(dz) = (\nu_1(dz), \nu_2(dz), ..., \nu_m(dz))^\top$.

2.2. Surplus process and the financial market

In this section, we first present the dynamics of the financial market and insurance risk model and we formulate the optimal investment–reinsurance problem for an insurer under the mean–variance criterion with state dependent risk aversion. We use the standard assumptions of continuous-time financial models: continuous trading is allowed, no transaction cost or tax is involved in trading and all assets are infinitely divisible.

2.2.1. Surplus process

We consider the classical compound Poisson risk model perturbed by a diffusion, which is generated by the randomness of claim sizes and claim occurrence times. A classical model is the Lundberg model for the risk process, which uses a compound Poisson process for the claims. The surplus process of the insurance company is described by

$$dR(s) = c ds + \sigma_0 dW_0(s) - d\left\{\sum_{j=1}^{N(s)} Z_j\right\}, \tag{2.1}$$

where $c > 0$ denotes the premium rate per unit of time, σ_0 is a positive constant representing the diffusion volatility parameter, Z_j for $j = 1, 2, ...$ is a sequence of independent and identically distributed nonnegative random variables with a common distribution $\mathbb{P}_Z$, finite first and finite second moments $\mu_Z = \int_0^{+\infty} z\mathbb{P}_Z(dz)$ and $\sigma_Z = \int_0^{+\infty} z^2\mathbb{P}_Z(dz)$, respectively. Note that Z_j denotes the amount of the j-th claim and $N(s)$ represents the number of claims occurring within the time horizon $[0, s]$. We assume that $N(\cdot)$ is a time homogeneous Poisson process with intensity $\lambda > 0$ and $W_0(\cdot)$ is a standard Brownian motion. In addition, we assume that $(Z_j)_{j\geq 1}$, $N(\cdot)$ and $W_0(\cdot)$ are mutually independent. The premium rate c is assumed to be calculated via the expected value principle, i.e. $c = (1 + \eta)\lambda\mu_Z$ with safety loading $\eta > 0$. Let us note at this point that the classical Lundberg model is given by (2.1) without reinsurance and investment.

Now, we assume that the insurer can control its insurance risk by purchasing proportional reinsurance or acquiring new business, for example, acting as a reinsurer of other insurers, see e.g. Bäuerle (2005). Let $u_R(s)$ denote the retention level of reinsurance or new business acquired at time $s \in [0, T]$. When $u_R(s) \in [0, 1]$, it corresponds to a proportional reinsurance cover and shows that the cedent should divert part of the premium to the reinsurer at the rate of $(1 - u_R(s))(\theta_0 + 1)\lambda\mu_Z$, where θ_0 is the relative safety loading of the reinsurer, satisfying $\theta_0 \geq \eta$. Meanwhile, for each claim occurring at time s, the reinsurer pays $100(1 - u_R(s))\%$ of the claim, while the insurer pays the rest. The case, where $u_R(s) \in (1, +\infty)$, corresponds to acquiring new business. The process $u_R(\cdot)$ is called a reinsurance strategy. Incorporating purchasing proportional reinsurance and acquiring new business into the surplus process, changes the equation (2.1) to

$$dR^{u_R(s)}(s) =$$
$$\{\eta - \theta_0 + (1 + \theta_0)u_R(s)\}\lambda\mu_Z ds + \sigma_0 u_R(s)dW_0(s) - u_R(s)d\left\{\sum_{j=1}^{N(s)} Z_j\right\}.$$

We refer the readers, for example, to Zheng, Li and Lai (2013) and references therein for more information about the above model.

Following Øksendal and Sulem (2007), the compound Poisson process $\sum_{j=1}^{N(\cdot)} Z_j$ can also be defined through a finite Poisson random measure $\gamma_0(\cdot, \cdot)$ with a compensator having the form $\nu_0(dz)ds = \lambda\mathbb{P}_Z(dz)ds$, on the measurable space $([0, T] \times \mathbb{R}^*, \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^*))$, as follows

$$\sum_{j=1}^{N(s)} Z_j = \int_0^s \int_{\mathbb{R}^*} z\gamma_0(dr, dz).$$

We use the notation $\widetilde{\gamma}_0(dr,dz) = \gamma_0(dr,dz) - \nu_0(dz)\,dr$ for the compensated jump martingale measure of $\gamma_0(dr,dz)$. Obviously, we have

$$\int_{\mathbb{R}^*} z\nu_0(dz)\,ds = \lambda \int_{\mathbb{R}^*} z\mathbb{P}_Z(dz)\,ds = \lambda\mu_Z ds.$$

Hence, the dynamics for the surplus process becomes

$$\begin{aligned} dR^{u_R(s)}(s) = &\left\{ (\eta - \theta_0 + \theta_0 u_R(s))\,\lambda\mu_Z + u_R(s) \int_{\mathbb{R}^*} z\nu_0(dz) \right\} ds \\ &+ \sigma_0 u_R(s)\,dW_0(s) \\ &- u_R(s) \int_{\mathbb{R}^*} z\gamma_0(dr,dz), \end{aligned}$$

equivalently, we obtain

$$\begin{aligned} &dR^{u_R(s)}(s) = \\ &(\eta - \theta_0 + \theta_0 u_R(s))\,\lambda\mu_Z ds + \sigma_0 u_R(s)\,dW_0(s) - u_R(s) \int_{\mathbb{R}^*} z\tilde{\gamma}_0(dr,dz). \end{aligned} \tag{2.2}$$

2.2.2. Financial market

Besides taking reinsurance strategy, the insurer can also invest in financial market, in which $n+1$ assets (or securities) are traded continuously. One of them is a bond, with price $P_0(s)$ at time $s \in [0,T]$ governed by

$$dP_0(s) = r_0(s)\,P_0(s)\,ds,\ \ P_0(0) = p_0 > 0, \tag{2.3}$$

where $r_0 : [0,T] \to (0,+\infty)$ is a deterministic function, which represents the risk-free rate. The other n assets are called risky stocks, whose price processes $P_i(\cdot)$, for $i = 1,2,...,n$, satisfy the following jump-diffusion stochastic differential equations

$$\begin{cases} dP_i(s) = P_i(s-) \left(r_i(s)\,ds + \sum\limits_{j=1}^{n} \sigma_{ij}(s)\,dW_j(s) + \sum\limits_{j=1}^{n} \int_{\mathbb{R}} \phi_{ij}(s,z)\,\tilde{\gamma}_j(ds,dz) \right), \\ P_i(0) = p_i > 0. \end{cases} \tag{2.4}$$

For $(i,j) \in \{1,2,...,n\}^2$, $r_i : [0,T] \to \mathbb{R}$, $\sigma_{ij} : [0,T] \to \mathbb{R}$ and $\phi_{ij} : [0,T] \times \mathbb{R}^* \to \mathbb{R}$ are assumed be deterministic functions, such that $\forall s \in [0,T]$, $r_i(s) \geq r_0(s)$. The process $W(\cdot) = (W_1(\cdot),...,W_n(\cdot))^\top$ is an n-dimensional standard Brownian motion, and let $\gamma(\cdot,\cdot) = (\gamma_1(\cdot,\cdot),...,\gamma_n(\cdot,\cdot))^\top$ be an n-dimensional Poisson random measure on the measurable space $([0,T] \times \mathbb{R}^*, \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^*))$.

For $i=1,2,...,n$, we assume that the compensator of $\gamma_i(ds,dz)$ has the form $\mu_i(ds,dz)=\nu_i(dz)\,ds$, for some positive and $\sigma-$finite Lévy measure $\nu_i(dz)$ on $\mathbb{R}^*$, such that $\int_{\mathbb{R}^*} 1\wedge z^2\nu_i(dz)<\infty$. Denote by $\nu(dz)=(\nu_1(dz),...,\nu_n(dz))^\top$ the n-dimensional Lévy measure. We assume that $W(\cdot)$, $\gamma(ds,dz)$ and $\sum_{j=1}^{N(\cdot)} Z_j$ are independent and write $\widetilde{\gamma}_i(\cdot,\cdot)=\gamma_i(\cdot,\cdot)-\mu_i(\cdot,\cdot)$ for the compensated jump random measure of $\gamma_i(\cdot,\cdot)$.

2.2.3. Wealth process

Starting from an initial capital $x_0>0$ at time 0, the insurer is allowed to dynamically purchase proportional reinsurance, acquire new business and invest in the financial market during the time horizon $[0,T]$. A reinsurance-investment strategy is described by an $(n+1)$-dimensional stochastic process $u(\cdot)=(u_R(\cdot),u_1(\cdot),...,u_n(\cdot))^\top$. The process $u_R(s)$ is the retention level of reinsurance or new business acquired at time $s\in[0,T]$ and $u_i(\cdot)$ for $i=1,2,...,n$, represents the amount invested in the i-th risky stock at time $s\in[0,T]$. The vector $u_I(\cdot)=(u_1(\cdot),...,u_n(\cdot))^\top$ is called an investment strategy. The dollar amount invested in the bond at time s is $X^{x_0,u(\cdot)}(s)-\sum_{j=1}^{n} u_i(\cdot)$, where $X^{x_0,u(\cdot)}(\cdot)$ is the wealth process associated with the strategy $u(\cdot)$ and the initial capital x_0. The evolution of $X^{x_0,u(\cdot)}(\cdot)$ can be described as

$$\begin{cases} dX^{x_0,u(\cdot)}(s)=dR^{u_R(s)}(s)+\left\{X^{x_0,u(\cdot)}(s)-\sum_{i=1}^{n}u_i(s)\right\}\dfrac{dP_0(s)}{P_0(s)} \\ \qquad\qquad +\sum_{i=1}^{n}u_i(s)\dfrac{dP_i(s)}{P_i(s-)},\ s\in[0,T], \\ X^{x_0,u(\cdot)}(0)=x_0. \end{cases} \tag{2.5}$$

Accordingly, the wealth process solves the following SDE with jumps

$$\begin{cases} dX^{x_0,u(\cdot)}(s)= \\ \qquad \left\{r_0(s)X^{x_0,u(\cdot)}(s)+(\delta+\theta_0u_R(s))\lambda\mu_Z+\sum_{i=1}^{n}u_i(s)(r_i(s)-r_0(s))\right\}ds \\ \qquad +\sigma_0u_0(s)\,dW_0(s)+\sum_{i,j=1}^{n}u_i(s)\sigma_{ij}(s)\,dW_j(s) \\ \qquad -\int_{\mathbb{R}^*}u_0(s)z\widetilde{\gamma}_0(ds,dz)+\sum_{i,j=1}^{n}\int_{\mathbb{R}^*}u_i(s)\phi_{ij}(s,z)\widetilde{\gamma}_j(ds,dz), \\ \qquad\qquad \text{for } s\in[0,T], \\ X^{x_0,u(\cdot)}(0)=x_0, \end{cases}$$

where $\delta=\eta-\theta_0$. To simplify our notation, we shall write

$$r(s)=(r_1(s)-r_0(s),...,r_n(s)-r_0(s))^\top,$$

$$\sigma(s) = (\sigma_{ij}(s))_{1\leq i,j\leq n} \in \mathbb{R}^{n\times n},$$

$$\phi(s,z) = (\phi_{ij}(s,z))_{1\leq i,j\leq n} \in \mathbb{R}^{n\times n}, \quad B(s) = (\lambda\mu_Z\theta_0, r(s))^\top, \quad \kappa = \delta\lambda\mu_Z,$$

$$D(s) = \begin{pmatrix} \sigma_0 & \mathbf{0}_{\mathbb{R}^n}^\top \\ \mathbf{0}_{\mathbb{R}^n} & \sigma(s) \end{pmatrix},$$

and

$$F(s,z) = \begin{pmatrix} -z & \mathbf{0}_{\mathbb{R}^n}^\top \\ \mathbf{0}_{\mathbb{R}^n} & \phi(s,z) \end{pmatrix}.$$

Then, if we define the processes $W^\star(\cdot)$ and $\widetilde{\gamma}^\star(\cdot,\cdot)$ by

$$W^\star(\cdot) = (W_0(\cdot), W_1(\cdot), .., W_n(\cdot))^\top$$

and

$$\widetilde{\gamma}^\star(\cdot,\cdot) = (\widetilde{\gamma}_0(\cdot,\cdot), \widetilde{\gamma}_1(\cdot,\cdot), .., \widetilde{\gamma}_n(\cdot,\cdot))^\top,$$

respectively, the state equation (2.5) admits the following representation

$$\begin{cases} dX^{x_0,u(\cdot)} = \Big(r_0(s) X^{x_0,u(\cdot)} + u(s)^\top B(s) + \kappa\Big) ds + u(s)^\top D(s) dW^\star(s) \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^\star} u(s)^\top F(s,z) \widetilde{\gamma}^\star(ds,dz), \text{ for } s\in[0,T], \\ X^{x_0,u(\cdot)}(0) = x_0. \end{cases} \tag{2.6}$$

As time evolves, we need to consider the controlled stochastic differential equation parameterized by $(t,x_t) \in [0,T]\times\mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R})$ and satisfied by $X(\cdot) = X^{t,x_t}(\cdot)$

$$\begin{cases} dX(s) = \Big(r_0(s) X(s) + u(s)^\top B(s) + \kappa\Big) ds + u(s)^\top D(s) dW^\star(s) \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^*} u(s)^\top F(s,z) \widetilde{\gamma}^\star(ds,dz), \text{ for } s\in[t,T], \\ X(t) = x_t. \end{cases} \tag{2.7}$$

In this paper, a trading strategy $u(\cdot) = (u_R(\cdot), u_I(\cdot))$ is said to be admissible over a time interval $[t,T]$ if $u(\cdot)\in\mathcal{L}^2_{\mathcal{F},p}\left(t,T;\mathbb{R}^{n+1}\right)$. Note that here we use the convention $(u_R(\cdot), u_I(\cdot)) = (u_R(\cdot), (u_1(\cdot), .., u_n(\cdot)))$.

2.3. Assumptions on the coefficients

We impose the following assumptions about the coefficients of the state equation
(H1) The functions $r_0(\cdot), r(\cdot), \sigma(\cdot)$ and $\phi(\cdot,\cdot)$ are continuous and such that

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^*} tr\left[\phi(t,z)^\top \operatorname{diag}(\nu(dz))\,\phi(t,z)\right] < +\infty.$$

(H2) We assume a uniform ellipticity condition as follows

$$\sigma(s)\,\sigma(s)^\top + \int_{\mathbb{R}^*}\phi(s,z)\operatorname{diag}(\nu(dz))\,\phi(s,z)^\top \geq \varepsilon I_{n\times n},\ a.e.$$

for some $\varepsilon > 0$, where $I_{n\times n}$ denotes the identity matrice of $\mathbb{R}^{n\times n}$.

Under **(H1)**, for any $(t, x_t, u(\cdot)) \in [0,T] \times \mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F},p}\left(t,T;\mathbb{R}^{n+1}\right)$, the state equation (2.7) has a unique solution $X(\cdot) \in \mathcal{S}^2_{\mathcal{F}}(t,T;\mathbb{R})$, see e.g. Meng (2014). We also have the following estimate

$$\mathbb{E}\left[\sup_{t\leq s\leq T}|X(s)|^2\right] \leq K\left(1+\mathbb{E}\left[|x_t|^2\right]\right), \tag{2.8}$$

for some positive constant K. In particular for $t = 0$ and $u(\cdot) \in \mathcal{L}^2_{\mathcal{F},p}\left(0,T;\mathbb{R}^{n+1}\right)$, the state equation (2.6) has a unique solution $X(\cdot) \in \mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R})$ with the following estimate holds

$$\mathbb{E}\left[\sup_{0\leq s\leq T}|X(s)|^2\right] \leq K\left(1+|x_0|^2\right).$$

2.4. Mean–variance criterion with state dependent risk aversion

For any fixed $(t,x_t) \in [0,T] \times \mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R})$, the objective of the insurer is to choose a reinsurance-investment strategy $u(\cdot)$ in order to maximize the conditional expectation of terminal wealth $\mathbb{E}^t[X(T)]$ over time interval $[t,T]$, while trying at the same time to minimize financial risk. Interpreting risk as the conditional variance $\operatorname{Var}^t[X(T)] = \mathbb{E}^t\left[X(T)^2\right] - \mathbb{E}^t[X(T)]^2$ and switching from gains to be maximized to costs to be minimized, the optimization problem becomes therefore the one to minimize

$$\begin{aligned} J(t,x_t,u(\cdot)) &\doteq \frac{1}{2}\operatorname{Var}^t[X(T)] - \gamma(x_t)\,\mathbb{E}^t[X(T)], \\ &= \frac{1}{2}\left\{\mathbb{E}^t\left[X(T)^2\right] - \mathbb{E}^t[X(T)]^2\right\} - \gamma(x_t)\,\mathbb{E}^t[X(T)], \end{aligned} \tag{2.9}$$

subject to $X(\cdot) = X^{t,x_t}(\cdot)$ satisfying (2.7) and over $u(\cdot) \in \mathcal{L}^2_{\mathcal{F},p}\left(t,T;\mathbb{R}^{n+1}\right)$. Here, $\mathbb{E}^t[\cdot] = \mathbb{E}[\cdot\,|\mathcal{F}_t]$, $\operatorname{Var}^t[\cdot] = \operatorname{Var}[\cdot\,|\mathcal{F}_t]$ and $\gamma(x_t) > 0$ is the risk aversion coefficient, which is assumed be dependent on the current wealth x_t, for an economic motivation of this choise, we refer the readers to Björk, Murgoci and Zhou (2014). Noting that, besides the constant risk aversion parameter, there are several state-dependent risk aversion parameters. Björk, Murgoci and Zhou (2014) and Wu (2013) proposed, respectively, in continuous time setting and multi-period setting, that risk aversion parameter takes a fractional form of current wealth level. Hu, Jin and Zhou (2012) proposed that the risk aversion parameter takes the form of a linear function of current wealth level. In our studies we use the risk aversion function $\gamma(x_t) = \dfrac{x_t}{\mu}$, with $\mu > 0$.

3. Characterization of the equilibrium strategies

It is well known that the stochastic control problem, described above, is not separable in the sense of dynamic programming theory. More specifically, the problem does not satisfy the Bellman principle and cannot be solved directly by the dynamic programming principle, because it evolves a nonlinear function of the expectation term in the variance. Many researchers have developed the pre-commitment solutions for the classical Markowitz's model in some cases. The basic idea, presented in Zhou and Li (2000) is to embed the problem into a stochastic LQ control problem. Such an approach establishes a natural connection of the portfolio selection problems and the standard stochastic control models. The paper by Chighoub and Mezerdi (2013) shows how to solve this optimization problem by applying a verification result for the stochastic maximum principle. However, since what is optimal for the t-agent, will not be optimal (in general) for the future s-agents, $s > t$, the concept of optimality plays no role here, and the optimal pre-commitment solution of the problem is not time-consistent. The most widely used approach is to reformulate the problem into a game problem and then apply the backward induction method from Björk, Murgoci and Zhou (2014), where the optimal strategy can be interpreted as the outcome of the sub-game perfect Nash equilibria in an interpersonal game, in which current and future selves of a hedger are different players.

In this paper we follow an alternative method of the extended dynamic programing principle, following Hu, Jin and Zhou (2012), we adopt the concept of open loop Nash equilibrium solution, which is, for any $t \in [0, T]$, optimal "infinitesimally" via spike variation, and then we derive some general necessary and sufficient condition for equilibrium strategies, by using the second order expansion in the spike variation, in the same spirit of proving the stochastic Pontryagin's maximum principle for equilibriums as in Ju, Jin and Zhou (2012, 2017), where the authors studied the Brownian case only. We recall that the approach that we follow here is very distinctly different from the dynamic programming approach to the study of this problem as it appears in the existing literature.

Given an admissible strategy $\hat{u}(\cdot) \in \mathcal{L}^2_{\mathcal{F},p}\left(0,T;\mathbb{R}^{n+1}\right)$, for any $t \in [0,T]$ and for any $\varepsilon \in [0, T-t)$, define

$$u^{\varepsilon}(s) = \begin{cases} \hat{u}(s) + v, & \text{for } s \in [t, t+\varepsilon), \\ \hat{u}(s), & \text{for } s \in [t+\varepsilon, T], \end{cases} \tag{3.1}$$

where $v \in \mathbb{L}^2\left(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n+1}\right)$, and we have the following definition.

Definition 1 (Open-loop Nash equilibrium) *Let $\hat{u}(\cdot) \in \mathcal{L}^2_{\mathcal{F},p}\left(0,T;\mathbb{R}^{n+1}\right)$ be an admissible strategy and $\hat{X}(\cdot) \in \mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R})$ the corresponding wealth process. $\hat{u}(\cdot)$ is a Nash equilibrium strategy if*

$$\lim_{\varepsilon \downarrow 0} \inf \frac{1}{\varepsilon}\left\{ J\left(t, \hat{X}(t), u^{\varepsilon}(s)\right) - J\left(t, \hat{X}(t), \hat{u}(s)\right)\right\} \geq 0, \tag{3.2}$$

for any $t \in [0,T]$ and $v \in \mathbb{L}^2\left(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n+1}\right)$.

Remark 1 *The above definition of Nash equlibrium strategy is different from the one used in Björk and Murgoci (2008), Li and Li (2013), Zeng and Li (2011), Zeng, Li and Lai (2013), since an equilibrium strategy here is defined in the class of open-loop strategies, while in the most of the existing literature only feedback strategies are considered. In addition, in the above definition, the perturbation of the strategy $\hat{u}(\cdot)$ in $[t, t+\varepsilon)$ will not change $\hat{u}(\cdot)$ in $[t+\varepsilon, T]$, which is not the case with feedback strategies.*

In the rest of this paper, sometimes we simply call $\hat{u}(\cdot)$ an equilibrium strategy instead of open-loop Nash equilibrium strategy when there is no ambiguity.

3.1. The adjoint equations

First, motivated by Hu, Jin and Zhou (2012) nd Tang and Li (1994) we introduce the adjoint equations involved in the characterization of equilibrium strategies. Let $\hat{u}(\cdot) \in \mathcal{L}^2_{\mathcal{F},p}\left(0,T;\mathbb{R}^{n+1}\right)$ and $\hat{X}(\cdot) \in \mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R})$ be its corresponding wealth process. For each $t \in [0,T]$, we introduce the first order adjoint equation defined on the time interval $[t,T]$ and satisfied by the processes $(p(\cdot;\cdot), q(\cdot;\cdot), r(\cdot,\cdot;\cdot))$ as follows

$$\begin{cases} dp(s;t) = -r_0(s)\, p(s;t)\, ds + q(s;t)^{\top} dW^{\star}(s) + \displaystyle\int_{\mathbb{R}^{\star}} r(s,z;t)^{\top} \widetilde{\gamma}^{\star}(ds,dz), \\ \qquad\qquad s \in [t,T], \\ p(T;t) = -\hat{X}(T) + \mathbb{E}^t\left[\hat{X}(T)\right] + \dfrac{\hat{X}(t)}{\mu}, \end{cases} \tag{3.3}$$

where

$q(\cdot,\cdot) = (q_0(\cdot,\cdot), q_1(\cdot,\cdot), .., q_n(\cdot,\cdot))^{\top}$ and $r(\cdot,\cdot;\cdot) = (r_0(\cdot,\cdot;\cdot), r_1(\cdot,\cdot;\cdot), .., r_n(\cdot,\cdot;\cdot))^{\top}$.

Under **(H1)**, equation (3.3) is uniquely solvable, moreover, there exists a constant $K > 0$ such that the following estimate holds

$$\|p\,(.;t)\|_{\mathcal{S}^2_{\mathcal{F}}(t,T;\mathbb{R})}+\|q\,(.;t)\|_{\mathcal{L}^2_{\mathcal{F}}(t,T;\mathbb{R}^{n+1})}+\|r\,(.,.;t)\|_{\mathcal{L}^{\nu^*,2}_{\mathcal{F},p}([t,T]\times\mathbb{R}^*;\mathbb{R}^{n+1})} \leq K\,(1+x_0)\,. \tag{3.4}$$

Next, associated to the 5-tuple $\left(\hat{u}\,(\cdot)\,,\hat{X}\,(\cdot)\,,p\,(\cdot,\cdot)\,,q\,(\cdot,\cdot)\,,r\,(\cdot,\cdot;\cdot)\right)$ we define for any $t \in [0,T]$ and $s \in [t,T]$

$$H\,(s;t) = B\,(s)\,p\,(s;t)+D\,(s)\,q\,(s;t)+\int_{\mathbb{R}^\star} F\,(s,z)\,\mathrm{diag}\,(\nu^\star\,(dz))\,r\,(s,z;t)\,, \tag{3.5}$$

and

$$L\,(s) = -e^{\int_s^T 2r_0(\tau)d\tau}\left(D\,(s)\,D\,(s)^\top + \int_{\mathbb{R}^\star} F\,(s,z)\,\mathrm{diag}\,(\nu^\star\,(dz))\,F\,(s,z)^\top\right), \quad s \in [0,T]\,. \tag{3.6}$$

3.2. Necessary and sufficient condition for equilibriums

The following theorem is the first main result of this work. It provides a necessary and sufficient condition to characterize the open-loop Nash equilibrium controls for the minimization problem (2.9) subject to the dynamics (2.7).

THEOREM 1 *Let **(H1)-(H2)** hold. Given an admissible strategy $\hat{u}\,(\cdot) \in \mathcal{L}^2_{\mathcal{F},p}\left(0,T;\mathbb{R}^{n+1}\right)$, let for any $t \in [0,T]\,,$ $(p\,(\cdot,\cdot)\,,q\,(\cdot,\cdot)\,,r\,(\cdot,\cdot;\cdot))$ be the unique solution to the BSDE* (3.3)*. Then, $\hat{u}\,(\cdot)$ is an open-loop Nash equilibrium, if and only if, the following condition holds*

$$H\,(t;t) = 0,\; d\mathbb{P}\text{−}a.s.,\; dt - a.e., \tag{3.7}$$

where $H\,(\cdot,\cdot)$ is given by $(3.5)\,.$

3.2.1. Proof of the main result

Let $\hat{u}\,(\cdot) \in \mathcal{L}^2_{\mathcal{F},p}\left(0,T;\mathbb{R}^{n+1}\right)$ be an admissible strategy and $\hat{X}\,(\cdot)$ the corresponding controlled process. Consider the perturbed strategy $u^\varepsilon\,(\cdot)$ defined by the spike variation (3.1) for some fixed arbitrary $t \in [0,T]\,,$ $v \in \mathbb{L}^2\left(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^{n+1}\right)$ and $\varepsilon \in [0,T-t]\,.$ Denote by $\hat{X}^\varepsilon\,(\cdot)$ the solution of the state equation corresponding to $u^\varepsilon\,(\cdot)$. Since the coefficients of the controlled state equation are li-near, then by the standard perturbation approach, see, e.g., Tang and Li (1994), we have

$$\hat{X}^\varepsilon\,(s) - \hat{X}\,(s) = y^\varepsilon\,(s)\,,\; s \in [t,T]\,, \tag{3.8}$$

where $y^{\varepsilon}(\cdot)$ solves the following linear stochastic differential equation

$$\begin{cases} dy^{\varepsilon}(s) = \left\{ r_0(s)\, y^{\varepsilon}(s) + 1_{[t,t+\varepsilon)}(s)\, v^{\top} B(s) \right\} ds + 1_{[t,t+\varepsilon)}(s)\, v^{\top} D(s)\, dW^{\star}(s) \\ \qquad\qquad + 1_{[t,t+\varepsilon)}(s) \int_{\mathbb{R}^{\star}} v^{\top} F(s,z)\, \widetilde{\gamma}^{\star}(ds,dz),\ s \in [t,T], \\ y^{\varepsilon}(t) = 0, \end{cases} \tag{3.9}$$

The following two Lemmas play a fundamental role when establishing Theorem 1.

Lemma 1 *For any $t \in [0,T]$, $\varepsilon \in [0, T-t)$ and $v \in \mathbb{L}^2\left(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n+1}\right)$ define $u^{\varepsilon}(s)$ by* (3.1). *Under the assumption **(H1)**, we have the following estimates, for $k \geq 1$,*

$$\sup_{s \in [t,T]} \left| \mathbb{E}^t \left[y^{\varepsilon}(s) \right] \right|^{2k} = O\left(\varepsilon^{2k}\right), \tag{3.10}$$

$$\sup_{s \in [t,T]} \mathbb{E}^t \left[\left| y^{\varepsilon}(s) \right|^{2k} \right] = O\left(\varepsilon^{k}\right). \tag{3.11}$$

In addition, we have the equality

$$\begin{aligned} & J\left(t, \hat{X}(t), u^{\varepsilon}(\cdot)\right) - J\left(t, \hat{X}(t), \hat{u}(\cdot)\right) \\ & = -\int_t^{t+\varepsilon} \left\{ \left\langle \mathbb{E}^t \left[H(s;t) \right], v \right\rangle + \frac{1}{2} \left\langle L(s)\, v, v \right\rangle \right\} ds + o(\varepsilon). \end{aligned} \tag{3.12}$$

Proof. See Appendix A.1. ■

Lemma 2 *The following two statements are equivalent*

1) $\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}^t \left[H(s;t) \right] ds = 0,\ d\mathbb{P} - a.s,\ \forall t \in [0,T].$

2) $H(t;t) = 0,\ d\mathbb{P} - a.s,\ dt - a.e.$

Proof. See Appendix A.2. ■

Proof of the Theorem 1 Given an admissible strategy $\hat{u}(\cdot) \in \mathcal{L}^2_{\mathcal{F},p}\left(0,T;\mathbb{R}^{n+1}\right)$, for which (3.7) holds, according to Lemma 2 we have for any $t \in [0,T]$

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}^t \left[H\left(s;t\right)\right] ds = 0.$$

Then by the representation (3.12) for any $t \in [0,T]$ and for any $v \in \mathbb{L}^2\left(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n+1}\right)$, this yields

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\{ J\left(t, \hat{X}\left(t\right), u^{\varepsilon}\left(\cdot\right)\right) - J\left(t, \hat{X}\left(t\right), \hat{u}\left(\cdot\right)\right) \right\} \\
&= -\lim_{\varepsilon \downarrow 0} \int_t^{t+\varepsilon} \left\{ \left\langle \mathbb{E}^t \left[H\left(s;t\right)\right], v \right\rangle + \frac{1}{2} \left\langle L\left(s\right) v, v \right\rangle \right\} ds, \\
&= -\frac{1}{2} \left\langle L\left(t\right) v, v \right\rangle, \\
&\geq 0.
\end{aligned}$$

Hence, $\hat{u}(\cdot)$ is an equilibrium strategy.

Conversely, assume that $\hat{u}(\cdot)$ is an equilibrium strategy. Then, by (3.2) together with (2.13), for any $(t,u) \in [0,T] \times \mathbb{R}^{n+1}$ the following inequality holds

$$\lim_{\varepsilon \downarrow 0} \left\langle \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}^t \left[H\left(s;t\right)\right] ds, u \right\rangle + \frac{1}{2} \left\langle L\left(t\right) u, u \right\rangle \leq 0. \tag{3.13}$$

Now, we define

$$\Psi\left(t,u\right) = \left\langle \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}^t \left[H\left(s;t\right)\right] ds, u \right\rangle + \frac{1}{2} \left\langle L\left(t\right) u, u \right\rangle, \ \forall \left(t,u\right) \in [0,T] \times \mathbb{R}^{n+1}. \tag{3.14}$$

Easy manipulations show that the inequality (3.13) is equivalent to

$$\Psi\left(t,0\right) = \max_{u \in \mathbb{R}^{n+1}} \Psi\left(t,u\right), \ d\mathbb{P} - a.s, \forall t \in [0,T]. \tag{3.15}$$

It is easy to prove that the maximum condition (3.15) leads to the following condition, $\forall t \in [0,T]$

$$\Psi_u\left(t,0\right) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}^t \left[H\left(s;t\right)\right] ds = 0, \ d\mathbb{P} - a.s. \tag{3.16}$$

According to Lemma 1, the equality (3.7) follows immediately. This completes the proof. ■

3.3. An explicit representation of the equilibrium control

In this section, we look at the (equilibrium) efficient frontier of the mean-variance problem, but with the risk aversion parameter being state-dependent. We face an ODE-type system of two equations instead of solving some very complicated highly nonlinear integro-differential equations as in Björk, Murgoci and Zhou (2014) and Li and Li (2013). The key point in the explicit resolution of the problem is that the adjoint process may be separated into functions of time and state variables. Then, one needs only to solve some linear ODEs in order to completely determine the equilibrium control. By standard arguments, we will prove the following result, first letting $\forall t \in [0,T]$

$$\alpha(t) = \frac{e^{-\int_t^T r_0(s)ds}}{\mu\left(1 - \frac{2}{\mu}\int_t^T e^{-\int_\tau^T 3r_0(\iota)d\iota}\Phi(\tau)\,d\tau\right)^{\frac{1}{2}}},$$

and $\Phi(t) = B(t)^\top \mathbf{\Theta}(t) B(t)$ with

$$\mathbf{\Theta}(t) = \left(D(t)D(t)^\top + \int_{\mathbb{R}^*} F(t,z)\,\mathrm{diag}(\nu^\star(dz))\,F(t,z)^\top\right)^{-1}.$$

Proposition 1 *Let **(H1)-(H2)** hold. The stochastic mean-variance control problem* (2.9) *subject to the SDE* (2.7), *has an open-loop Nash equilibrium solution having the following feedback representation*

$$\hat{u}_R(s) = \alpha(s)\,\lambda\mu_Z\theta_0\left(\sigma_0^2 + \int_0^{+\infty} z^2\nu_0(dz)\right)^{-1}\hat{X}(s),\ \ s \in [0,T], \tag{3.17}$$

$$\hat{u}_I(s) = \alpha(s)\left(\sigma(s)\sigma(s)^\top + \int_{\mathbb{R}^*}\phi(s,z)\,diag(\nu(dz))\,\phi(s,z)^\top\right)^{-1} r(s)\hat{X}(s),$$
$$s \in [0,T]. \tag{3.18}$$

Moreover, the associated expected terminal wealth is

$$\mathbb{E}\left[\hat{X}(T)\right] = e^{\int_0^T (r_0(l)+\alpha(l)\Phi(l))dl}\left(x_0 + \kappa\int_0^T e^{-\int_0^\tau (r_0(l)+\alpha(l)\Phi(l))dl}d\tau\right), \tag{3.19}$$

and the corresponding variance of the terminal wealth is

$$\mathrm{Var}\left[\hat{X}(T)\right] = \int_0^T e^{-2\int_\tau^T 2(r_0(l)+\alpha(l)\Phi(l))dl}\alpha(\tau)^2\,\Phi(\tau)\,\mathbb{E}\left[\hat{X}(\tau)^2\right]d\tau, \tag{3.20}$$

where, for any $s \in [0,T]$, the associated expected square wealth is given by

$$\mathbb{E}\left[\hat{X}(s)^2\right]$$
$$= e^{\int_0^s \left(2r_0(l)+\left(2\alpha(l)+\alpha(l)^2\right)\Phi(l)\right)dl}$$
$$\times\left\{x_0^2+\kappa\int_0^s e^{\int_0^\tau -\left(r_0(l)+\left(\alpha(l)+\alpha(l)^2\right)\Phi(l)\right)dl}\left(x_0+\kappa\int_0^\tau e^{\int_0^\varsigma -(r_0(l)+\alpha(l)\Phi(l))dl}d\varsigma\right)d\tau\right\}.$$

Proof. The result of the previous subsections leads to the following flow of forward and backward stochastic differential system with jumps, parameterized by t:

$$\begin{cases} d\hat{X}(s)=\left(r_0(s)\hat{X}(s)+\hat{u}(s)^\top B(s)+\kappa\right)ds+\hat{u}(s)^\top D(s)\,dW^\star(s) \\ \qquad\qquad +\int_{\mathbb{R}^*}\hat{u}(s)^\top F(s,z)\,\widetilde{\gamma}^\star(ds,dz),\ s\in[0,T], \\ dp(s;t)=-r_0(s)\,p(s;t)\,ds+q(s;t)^\top dW^\star(s)+\int_{\mathbb{R}^*}r(s,z;t)^\top\,\widetilde{\gamma}^\star(ds,dz), \\ \qquad\qquad 0\le t\le s\le T, \\ \hat{X}(0)=x_0,\ p(T;t)=-\left(\hat{X}(T)-\mathbb{E}^t\left[\hat{X}(T)\right]\right)+\dfrac{\hat{X}(t)}{\mu},\ \text{for } t\in[0,T], \end{cases} \tag{3.21}$$

with the condition

$$B(t)\,p(t;t)+D(t)\,q(t;t)+\int_{\mathbb{R}^*}F(t,z)\,\mathrm{diag}(\nu^\star(dz))\,r(t,z;t)=0,\ d\mathbb{P}-a.s.,\ dt-a.e. \tag{3.22}$$

From the terminal condition of (3.21), we consider the following Ansatz

$$p(s;t)=-M(s)\left(\hat{X}(s)-\mathbb{E}^t\left[\hat{X}(s)\right]\right)+\Upsilon(s)\,\hat{X}(t),\ \forall 0\le t\le s\le T \tag{3.23}$$

for some deterministic functions $M(\cdot),\Upsilon(\cdot)\in C^1([0,T],\mathbb{R})$ such that, $M(T)=1$ and $\Upsilon(T)=\dfrac{1}{\mu}$. We would like to determine the equations that $M(\cdot)$ and $\Upsilon(\cdot)$ should satisfy. To this end we differentiate (3.23) and we get

$$\begin{aligned} &dp(s;t)= \\ &-\left(\frac{dM}{ds}(s)\left(\hat{X}(s)-\mathbb{E}^t\left[\hat{X}(s)\right]\right)+\frac{d\Upsilon}{ds}(s)\,\hat{X}(t)\right)ds \\ &-M(s)\,d\left(\hat{X}(s)-\mathbb{E}^t\left[\hat{X}(s)\right]\right). \end{aligned} \tag{3.24}$$

We remark that

$$d\mathbb{E}^t\left[\hat{X}(s)\right]=\left(r_0(s)\,\mathbb{E}^t\left[\hat{X}(s)\right]+\mathbb{E}^t\left[\hat{u}(s)^\top\right]B(s)+\kappa\right)ds,$$

then

$$
\begin{aligned}
& d\left(\hat{X}(s)-\mathbb{E}^t\left[\hat{X}(s)\right]\right)= \\
& \left(r_0(s)\left(\hat{X}(s)-\mathbb{E}^t\left[\hat{X}(s)\right]\right)+\left(\hat{u}(s)-\mathbb{E}^t[\hat{u}(s)]\right)^{\top} B(s)\right) ds \\
& +\hat{u}(s)^{\top} D(s)\, dW(s)+\int_{\mathbb{R}^*} \hat{u}(s)^{\top} F(s,z)\, \widetilde{\gamma}^{\star}(ds,dz). \qquad (3.25)
\end{aligned}
$$

Now, by invoking (3.24) and (3.25), and then by comparing with the BSDE in (3.21), we easily check that

$$
\begin{aligned}
& -r_0(s)\left(-M(s)\left(\hat{X}(s)-\mathbb{E}^t\left[\hat{X}(s)\right]\right)+\Upsilon(s)\hat{X}(t)\right) \\
& =-\frac{dM}{ds}(s)\left(\hat{X}(s)-\mathbb{E}^t\left[\hat{X}(s)\right]\right)+\frac{d\Upsilon}{ds}(s)\hat{X}(t) \\
& \quad -M(s)\left\{r_0(s)\left(\hat{X}(s)-\mathbb{E}^t\left[\hat{X}(s)\right]\right)+\left(\hat{u}(s)-\mathbb{E}^t[\hat{u}(s)]\right)^{\top} B(s)\right\}, \qquad (3.26)
\end{aligned}
$$

and also we get

$$
(q(s;t), r(s,z;t))=\left(-M(s)D(s)^{\top}\hat{u}(s), -M(s)F(s,z)^{\top}\hat{u}(s)\right). \qquad (3.27)
$$

Moreover, by taking (3.23) and (3.27) in (3.22), we obtain

$$
\begin{aligned}
& B(t)\Upsilon(t)\hat{X}(t)-M(t)\left(D(t)D(t)^{\top}+\int_{\mathbb{R}^*} F(t,z)\operatorname{diag}(\nu^{\star}(dz))F(t,z)^{\top}\right)\hat{u}(t) \\
& \quad =0.
\end{aligned}
$$

Subsequently, we obtain that $\hat{u}(t)$ admits the following representation

$$
\hat{u}(t)=M(t)^{-1}\Theta(t)B(t)\Upsilon(t)\hat{X}(t). \qquad (3.28)
$$

Next, from (3.26) and (3.28) we obtain

$$
\begin{aligned}
0= & \left(-\frac{dM}{ds}(s)-2r_0(s)M(s)+M(s)^{-1}\Phi(s)\Upsilon(s)\right)\left(\hat{X}(s)-\mathbb{E}^t\left[\hat{X}(s)\right]\right) \\
& +\left(\frac{d\Upsilon}{ds}(s)+r_0(s)\Upsilon(s)\right)\hat{X}(t). \qquad (3.29)
\end{aligned}
$$

This suggests that the functions $M(\cdot)$ and $\Upsilon(\cdot)$ solve the following system of equations

$$
\begin{cases}
\dfrac{dM}{ds}(s)+2r_0(s)M(s)-\dfrac{1}{M(s)}\Phi(s)\Upsilon(s)=0,\ s\in[0,T], \\
\dfrac{d\Upsilon}{ds}(s)+r_0(s)\Upsilon(s)=0,\ s\in[0,T], \\
M(T)=1,\ \Upsilon(T)=\dfrac{1}{\mu}.
\end{cases} \qquad (3.30)
$$

Note that the second equation in (3.30) is a linear ordinary differential equation with unique solution given by

$$\Upsilon(t) = \frac{1}{\mu} e^{\int_t^T r_0(\tau) d\tau}, s \in [0, T].$$

Now, by multipling by $M(s)$, the first equation in (3.30) can be rewritten as

$$\begin{cases} \frac{dM}{ds}(s) M(s) + 2r_0(s) M^2(s) = \Phi(s) \Upsilon(s), \ s \in [0, T], \\ M(T) = 1. \end{cases}$$

By the change of variable $z(s) = M^2(s)$ we get

$$\begin{cases} \frac{dz}{ds}(s) + 4r_0(s) z(s) = 2\Phi(s) \Upsilon(s), \ s \in [0, T], \\ z(T) = 1, \end{cases}$$

which is a linear ordinary differential equation that is explicitly solved by

$$z(t) = e^{4\int_t^T r_0(\iota) d\iota} \left(1 - \int_t^T e^{-\int_\tau^T 4r_0(\iota) d\iota} 2\Phi(\tau) \Upsilon(\tau) d\tau \right), \ t \in [0, T].$$

Then

$$M(t) = e^{2\int_t^T r_0(\iota) d\iota} \left(1 - \frac{2}{\mu} \int_t^T e^{-\int_\tau^T 3r_0(\iota) d\iota} \Phi(\tau) d\tau \right)^{\frac{1}{2}}, \ t \in [0, T]. \quad (3.31)$$

By (3.28) we get

$$(\hat{u}_R(t), \hat{u}_I(t))^T = \alpha(t) \Theta(t) (\lambda \mu_Z \theta_0, r(t))^T \hat{X}(t). \quad (3.32)$$

Note that $\forall t \in [0, T]$, $\Theta(t)$ can be represented as follows

$$\Theta(t) =$$

$$\begin{pmatrix} \left(\sigma_0^2 + \int_0^{+\infty} z^2 \nu_0(dz) \right)^{-1} & \mathbf{0}_{\mathbb{R}^n}^\top \\ \mathbf{0}_{\mathbb{R}^n} & \left(\sigma(t) \sigma(t)^\top + \int_{\mathbb{R}^*} \phi(t, z) \operatorname{diag}(\nu(dz)) \phi(t, z)^\top \right)^{-1} \end{pmatrix}.$$

It follows immediately that the open loop Nash equilibrium solution is given by (3.17) and (3.18).

Next, we derive the equilibrium efficient frontier of the mean-variance problem. By substituting the equilibrium solution (3.17) and (3.18) into the wealth process we obtain

$$\begin{cases} d\hat{X}(s) = \Big\{ \left(r_0(s) + \alpha(t)\Phi(s) \right) \hat{X}(s) + \kappa \Big\} ds \\ \qquad\qquad + \hat{X}(s)\alpha(s) B(s)^\top \Theta(s) D(s) dW^\star(s) \\ \qquad\qquad + \hat{X}(s)\alpha(s) \int_{\mathbb{R}^\star} B(s)^\top \Theta(s) F(s,z) \widetilde{\gamma}^\star(ds,dz), \text{ for } s \in [t,T], \\ \hat{X}(0) = x_0. \end{cases} \tag{3.33}$$

By taking expectations on both sides of (3.33), we represent $\mathbb{E}\left[\hat{X}(s)\right]$ as follows

$$\begin{cases} d\mathbb{E}\left[\hat{X}(s)\right] = \Big\{ \left(r_0(s) + \alpha(s)\Phi(s) \right) \mathbb{E}\left[\hat{X}(s)\right] + \kappa \Big\} ds, \\ \mathbb{E}\left[\hat{X}(0)\right] = x_0, \end{cases} \tag{3.34}$$

and then, the unique solution to the linear ordinary differential equation (3.34) is explicitly given by

$$\mathbb{E}\left[\hat{X}(s)\right] = e^{\int_0^s (r_0(l)+\alpha(l)\Phi(l))dl} \left(x_0 + \kappa \int_0^s e^{-\int_0^\tau (r_0(l)+\alpha(l)\Phi(l))dl} d\tau \right),$$

which implies (3.19).

Now, applying Itô's formula to $s \to \hat{X}(s)^2$ we get

$$\begin{aligned} d\hat{X}(s)^2 &= \Big\{ \Big(2r_0(s) + \Big(2\alpha(t) + \alpha(s)^2 \Big) \Phi(s) \Big) \hat{X}(s)^2 + 2\kappa \hat{X}(s) \Big\} ds \\ &+ 2\hat{X}(s)^2 \alpha(s) B(s)^\top \Theta(s) D(s) dW^\star(s) \\ &+ \int_{\mathbb{R}^*} \Big\{ \Big(B(s)^\top \Theta(s) F(s,z) \Big)^2 + 2\hat{X}(s) B(s)^\top \Theta(s) F(s,z) \Big\} \widetilde{\gamma}^\star(ds,dz). \end{aligned}$$

Taking the expectation, we conclude that $\mathbb{E}\left[\hat{X}(s)^2\right]$ satisfies the following linear ordinary differential equation

$$\begin{cases} d\mathbb{E}\left[\hat{X}(s)^2\right] = \\ \qquad \Big\{ \Big(2r_0(s) + \Big(2\alpha(s) + \alpha(s)^2 \Big) \Phi(s) \Big) \mathbb{E}\left[\hat{X}(s)^2\right] + 2\kappa \mathbb{E}\left[\hat{X}(s)\right] \Big\} ds, \\ \mathbb{E}\left[\hat{X}(0)^2\right] = x_0^2. \end{cases}$$

A variation of constant formula leads to the following representation, for $s \in [0,T]$,

$$\begin{aligned}
&\mathbb{E}\left[\hat{X}(s)^2\right] \\
&= e^{\int_0^s \left(2r_0(l)+\left(2\alpha(l)+\alpha(l)^2\right)\Phi(l)\right)dl} \\
&\times\left(x_0^2+\kappa\int_0^s e^{\int_0^\tau -\left(r_0(l)+\left\{\alpha(l)+\alpha(l)^2\right\}\Phi(l)\right)dl}\left\{x_0+\kappa\int_0^\tau e^{\int_0^\varsigma -(r_0(l)+\alpha(l)\Phi(l))dl}d\varsigma\right\}d\tau\right).
\end{aligned}$$

On the other hand, simple computations show that $\operatorname{Var}\left[\hat{X}(\cdot)\right] = \mathbb{E}\left[\hat{X}(\cdot)^2\right] - \mathbb{E}\left[\hat{X}(\cdot)\right]^2$ satisfies the following ODE

$$\begin{cases}
d\operatorname{Var}\left[\hat{X}(s)\right] = \left\{2\left(r_0(s)+\alpha(s)\Phi(s)\right)\operatorname{Var}\left[\hat{X}(s)\right]+\alpha(s)^2\Phi(s)\mathbb{E}\left[\hat{X}(s)^2\right]\right\}ds,\\
\operatorname{Var}\left[\hat{X}(0)\right]=0,
\end{cases}$$

which is explicitly solved by

$$\operatorname{Var}\left[\hat{X}(s)\right] = \int_0^s e^{-2\int_\tau^s 2(r_0(l)+\alpha(l)\Phi(l))dl}\alpha(\tau)^2\Phi(\tau)\mathbb{E}\left[\hat{X}(\tau)^2\right]d\tau,$$

which implies (3.20). This completes the proof. ■

Remark 2 *A simple computation shows that the objective function value of the equilibrium strategy* $\hat{u}(\cdot)$ *is given by*

$$\begin{aligned}
&J\left(0,x_0,\hat{u}(\cdot)\right)\\
&=\frac{1}{2}\int_0^T e^{-2\int_\tau^T 2(r_0(l)+\alpha(l)\Phi(l))dl}\alpha(\tau)^2\Phi(\tau)\mathbb{E}\left[\hat{X}(\tau)^2\right]d\tau\\
&\qquad -\frac{x_0}{\mu}e^{\int_0^T(r_0(l)+\alpha(l)\Phi(l))dl}\left(x_0+\kappa\int_0^T e^{-\int_0^\tau(r_0(l)+\alpha(l)\Phi(l))dl}d\tau\right).
\end{aligned}$$

4. Special cases

Equilibrium strategies for mean–variance models under state dependent risk aversion have been studied, in particular, in Björk, Murgoci and Zhou (2014), Hu, Jin and Zhou (2012), and Li and Li (2013), among others, in different frameworks. In this section, we will compare our results with some of the existing ones in the literature.

4.1. Cramér–Lundberg model

Suppose that the surplus of the insurer is modelled with the classical Cramér–Lundberg (CL) model (i.e. the model (2.2), in which $\sigma_0 = 0$). In addition, we assume that the financial market consists of one risk-free asset, whose price process is given by (2.3) and n risky assets, whose price processes do not have jumps (i.e. the model (2.4) with $\phi_{ij}(s,z)=0$, $ds-a.e.$). Then, the dynamics of the wealth process $X(\cdot)=X^{t,x_t}(\cdot)$, which corresponds to an admissible strategy $u(\cdot)=(u_R(\cdot),u_I(\cdot))$ and the initial pair $(t,x_t)\in[0,T]\times\mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R})$ can be described by

$$\begin{cases} dX(s)=\{r_0(s)X(s)+(\delta+\theta_0 u_R(s))\lambda\mu_Z+u_I(s)r(s)\}ds \\ \qquad\qquad +u_I(s)^\top\sigma(s)dW(s)-u_R(s)\int_{\mathbb{R}^\star}z\widetilde{\gamma}_0(ds,dz), \text{ for } s\in[t,T], \\ X(t)=x_t. \end{cases} \tag{4.1}$$

In this case, the equilibrium strategy, given by the expressions (3.17) and (3.18), changes to

$$\hat{u}_R(s)=\frac{\lambda\mu_Z\theta_0 e^{-\int_s^T r_0(\tilde{v})d\tilde{v}}}{\mu\left(1-\frac{2}{\mu}\int_s^T e^{-\int_v^T 3r_0(\tilde{u})d\tilde{u}}\Phi(\tilde{v})d\tilde{v}\right)^{\frac{1}{2}}\left(\int_0^{+\infty}z^2\nu_0(dz)\right)}\hat{X}(s), \quad s\in[0,T], \tag{4.2}$$

$$\hat{u}_I(s)=\frac{e^{-\int_s^T r_0(\tilde{v})d\tilde{v}}}{\mu\left(1-\frac{2}{\mu}\int_s^T e^{-\int_v^T 3r_0(\tilde{u})d\tilde{u}}\Phi(\tilde{v})d\tilde{v}\right)^{\frac{1}{2}}}\left(\sigma(s)\sigma(s)^\top\right)^{-1}r(s)\hat{X}(s), \quad s\in[0,T], \tag{4.3}$$

where

$$\Phi(t)=\left(\frac{(\lambda\mu_Z\theta_0)^2}{\left(\sigma_0^2+\int_0^{+\infty}z^2\nu_0(dz)\right)}+r(s)^\top\left(\sigma(s)\sigma(s)^\top\right)^{-1}r(s)\right).$$

The above equilibrium reinsurance-investment solution is comparable with the one obtained in Li and Li (2013) in which the equilibrium is, however, defined within the class of feedback controls. Note that in Li and Li (2013) the authors have used the approach developed by Björk, Murgoci and Zhou (2014) and they have obtained feedback equilibrium solutions via some well posed integral equations, for which they had not obtained explicit solutions.

4.2. The investment only

We conclude this section with the case where the insurer is not allowed to purchase reinsurance or acquire new business, i.e. $u_R(s) \equiv 1$, and the financial market consists of one risk-free asset, whose price process is given by (2.3) and n risky assets whose price processes do not have jumps. In this case a trading strategy $u(\cdot)$ reduces to an n-dimensional stochastic process $u_R(s) = (u_1(s), ..., u_n(s))^\top$, where $u_i(s)$ represents the amount invested in the i-th risky stock at time s. The dynamics of the wealth process $X(\cdot)$, which corresponds to an admissible investment strategy $u_R(\cdot)$ and initial pair $(t, \xi) \in [0, T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ can be described by

$$\begin{cases} dX(s) = \left\{ r_0(s) X(s) + \eta\lambda\mu_Z + u_R(s)^\top r(s) \right\} ds + u_R(s)^\top \sigma(s) dW(s) \\ \qquad\qquad - \int_0^{+\infty} z\widetilde{\gamma}_0(ds, dz), \text{ for } s \in [t, T], \\ X(0) = x_0. \end{cases} \tag{4.4}$$

Similarly as in the previous section, for the investment only case, we can derive the open loop equilibrium strategy, which is described as

$$\hat{u}_I(s) = \frac{e^{-\int_s^T r_0(\tilde{v}) d\tilde{v}}}{\mu \left(1 - \frac{2}{\mu} \int_s^T e^{-\int_v^T 3r_0(\tilde{u}) d\tilde{u}} \Phi(\tilde{v}) d\tilde{v} \right)^{\frac{1}{2}}} \left(\sigma(s) \sigma(s)^\top \right)^{-1} r(s) \hat{X}(s), \tag{4.5}$$

where $\Phi(t) = r(s)^\top \left(\sigma(t) \sigma(t)^\top \right)^{-1} r(s)$. This essentially covers the solution obtained by Hu, Jin and Zhou (2012).

5. Numerical results

In this section we provide a numerical example to illustrate our results. Without loss of generality, we only consider the situation when all the parameters of the financial market are assumed be constants. For the case, in which the surplus of the insurers is modelled by (2.2), we suppose that the financial market consists of one risk-free asset, whose price process is given by (2.3) and only one risky asset whose price process is modelled by the geometric Lévy process

$$\begin{cases} dP(s) = P(s-) \left(r_1 ds + \sigma dW(s) + d\left(\sum_{i=1}^{\widetilde{N}(s)} Y_i \right) \right), \\ P(0) = p > 0. \end{cases} \tag{5.1}$$

Here r and σ are assumed to be constants, such that $r_1 \geq r_0$. The process $W(\cdot)$ is a one-dimensional standard Brownian motion, Y_i for $i = 1, 2, ...$ is a sequence of independent and identically distributed nonnegative random variables with a common distribution $\mathbb{P}_Y$, with finite first and finite second moments $\mu_Y = \int_0^{+\infty} z\mathbb{P}_Y(dz)$ and $\sigma_Y = \int_0^{+\infty} z^2\mathbb{P}_Y(dz)$, respectively. Note that Y_i denotes the amount of the i-th claim and $\widetilde{N}(s)$ represents the number of claims occurring within the time horizon $[0, s]$. We assume that $\widetilde{N}(\cdot)$ is a time homogeneous Poisson process with intensity $\lambda_Y > 0$. We assume also that $(Y_i)_{i\geq 1}$, $\widetilde{N}(\cdot)$ and $W(\cdot)$ are mutually independent. A trading strategy $\pi(.)$ is described by two-dimensional stochastic processes $(u_R(.), u_I(.))$, where $u_R(s)$ represents the retention level of reinsurance or new business acquired at time $s \in [0, T]$ and $u_I(s)$ represents the amount invested in the risky stock at time s. The dynamics of the wealth process $X(.) = X^{t,x_t}(\cdot)$, which corresponds to an admissible strategy $u(.) = (u_R(.), u_I(.))$ and initial pair $(t, x_t) \in [0, T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ can be described by

$$\begin{cases} dX(s) = \{r_0 X(s) + (\delta + (1+\theta_0) u_R(s)) \lambda \mu_Z + (r_1 - r_0) u_I(s)\} ds \\ \qquad\qquad + u_R(s) \sigma_0 dW_0(s) \\ \qquad\qquad + u_I(s) \sigma dW(s) - u_R(s) d\left(\sum_{i=1}^{N(s)} Z_i\right) + u_I(s) d\left(\sum_{i=1}^{\widetilde{N}(s)} Y_i\right), \\ \qquad\qquad \text{for } s \in [0, T], \\ X(t) = x_t. \end{cases}$$

In this case, the equilibrium strategies, given by the expressions (3.17) and (3.18), have the following representations

$$\hat{u}_R(s) = \alpha(t) \left(\frac{\lambda_Z \mu_Z \theta_0}{\sigma_0^2 + \lambda_Z \sigma_Z^2}\right) \hat{X}(t), \tag{5.2}$$

$$\hat{u}_I(s) = \alpha(t) \left(\frac{r_1 - r_0 + \lambda_Y m_Y}{\sigma^2 + \lambda_Y \sigma_Y^2}\right) \hat{X}(t), \tag{5.3}$$

where

$$\alpha(t) = e^{-r_0(T-t)} \left(\mu - \frac{2}{3r_0}\left(1 - e^{-3r_0(T-t)}\right)\left(\frac{(\lambda_Z \mu_Z \theta_0)^2}{\sigma_0^2 + \lambda_Z \sigma_Z^2} + \frac{(r_1 - r_0 + \lambda_Y m_Y)^2}{\sigma^2 + \lambda_Y \sigma_Y^2}\right)\right)^{-\frac{1}{2}}.$$

More specifically, in the framework of classical Cramér–Lundberg modelling, i.e. $\sigma_0 = 0$ and $\lambda_Y = 0$, the expressions (5.2) and (5.3) reduce to the following representations

$$\hat{u}_R(t) = \tilde{\alpha}(t)\left(\frac{\lambda_Z \mu_Z \theta_0}{\lambda_Z \sigma_Z^2}\right)\hat{X}(t),$$

$$\hat{u}_I(t) = \tilde{\alpha}(t)\left(\frac{r_1 - r_0}{\sigma^2}\right)\hat{X}(t),$$

with

$$\tilde{\alpha}(t) = e^{-r_0(T-t)}\left(\mu - \frac{2}{3r_0}\left(1 - e^{-3r_0(T-t)}\right)\left(\frac{(\lambda_Z \mu_Z \theta_0)^2}{\lambda_Z \sigma_Z^2} + \frac{(r_1 - r_0)^2}{\sigma^2}\right)\right)^{-1}.$$

Let us denote by $c_1(.)$, and $c_2(.)$ the propensities to equilibrium reinsurance strategy and investment strategy, respectively, i.e.

$$c_1(t) \equiv \frac{\hat{u}_R(t)}{\hat{X}(t)} \equiv \alpha(t)\left(\frac{\lambda_Z \mu_Z \theta_0}{\sigma_0^2 + \lambda_Z \sigma_Z^2}\right)$$

and

$$c_2(t) \equiv \frac{\hat{u}_I(t)}{\hat{X}(t)} \equiv \alpha(t)\left(\frac{r_1 - r_0 + \lambda_Y m_Y}{\sigma^2 + \lambda_Y \sigma_Y^2}\right).$$

We also denote by $c_3(.)$ and $c_4(.)$ the propensities to reinsurance and investment strategies in the case of Cramér–Lundberg model, i.e.

$$c_3(t) \equiv \tilde{\alpha}(t)\left(\frac{\lambda_Z \mu_Z \theta_0}{\lambda_Z \sigma_Z^2}\right)$$

and

$$c_4(t) \equiv \tilde{\alpha}(t)\frac{r_1 - r_0}{\sigma^2}.$$

We illustrate the propensities $c_1(.)$, $c_2(.)$, $c_3(.)$ and $c_4(.)$ with various choices of μ. We have used the parameter values: $T = 5$; $\lambda = 0.4$; $\mu_Y = 0.6$; $\theta_0 = 1.5$; $r_0 = 0.35$; $\sigma_0 = 0.5$; $\sigma_Y = 1$; $r_1 = 0.7$; $\lambda_Z = 0.5$; $\mu_Z = 0.3$; $\sigma = 0.3$; $\sigma_Z = 0.5$.

Figures 1 and 2 show that the propensity to equilibrium amount invested in the risky asset and the propensity to equilibrium reinsurance strategy both increase with respect to time. Then, Figs. 3 and 4 plot the propensities to reinsurance and investment strategies for Cramér–Lundberg model for $\mu = 1$, 0.5, 0.2, 0.1.

In addition, Figs. 1 and 3 illustrate that the proportion of wealth invested in risky stock is decreasing with respect to the coefficient of risk aversion μ,

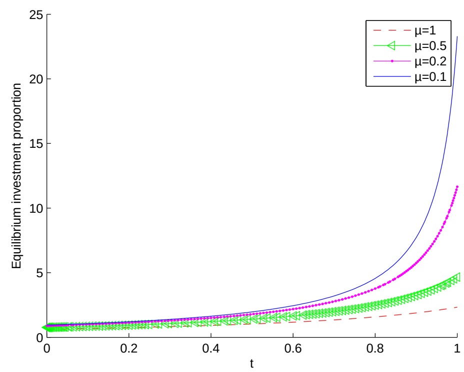

Figure 1. The equilibrium investment proportion in our model with different values of μ

i.e., the more the insurer dislikes risk, the lower investment proportion the insurer invests in the risky asset. Figures 2 and 4 tell us that the equilibrium reinsurance proportion decreases with respect to the coefficient of risk aversion, that is to say, the more risk averse the insurer is, the less insurance proportion the insurer keeps.

On comparing the results in our model (Figs. 1 and 2) with the ones obtained in the case of Cramér–Lundberg model (Figs. 3 and 4), we can see that in the framework of CL model both the equilibrium investment proportion and the equilibrium reinsurance proportion increase. Especially, the insurer should invest in the second case a higher proportion of wealth in the risky asset.

6. Conclusion and future work

In this paper, the open-loop time-consistent equilibrium control is investigated for a kind of investment and reinsurance problem under the assumption that the risk aversion is a function of current wealth level for insurer with mean–variance utility. We provide a necessary and sufficient condition for equilibrium and derive explicitly the time-consistent investment–reinsurance strategy and the equilibrium value function. Some comparison results are illustrated. Furthermore, we present some numerical illustrations to demonstrate the results we have derived. For future research, the closed-loop time-consistent solution should be studied.

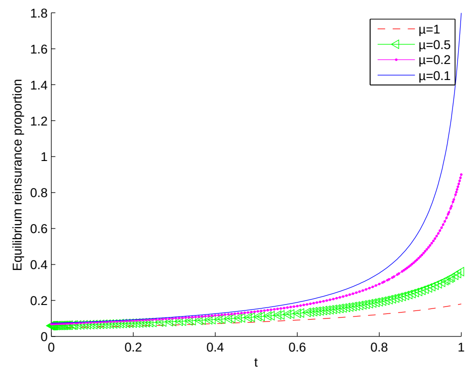

Figure 2. The equilibrium reinsurance proportion in our model with different values of μ

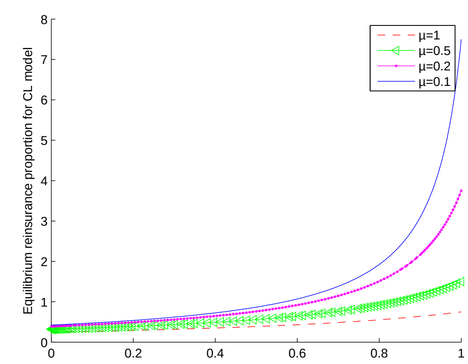

Figure 3. The equilibrium reinsurance proportion in CL model with different values of μ

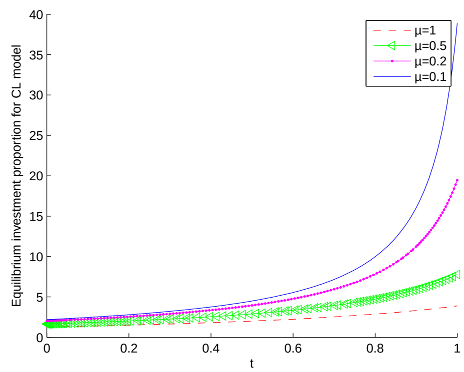

Figure 4. The equilibrium investment proportion in CL model with different values of μ

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7. Appendix: Additional proofs

A.1. Proof of Lemma 1

First let us prove (3.10). By taking conditional expectations into (3.9) , we find that

$$\begin{cases} d\mathbb{E}^t\left[y^\varepsilon(s)\right] = \left\{r_0(s)\,\mathbb{E}^t\left[y^\varepsilon(s)\right] + v^\top B(s)\,1_{[t,t+\varepsilon)}(s)\right\} ds\ , s \in [t,T]\,, \\ \mathbb{E}^t\left[y^\varepsilon(t)\right] = 0. \end{cases} \tag{A.1.1}$$

Accordingly, in the integral form we get for any $s \in [t,T]$

$$\mathbb{E}^t\left[y^\varepsilon(s)\right] = \int_t^s \left\{r_0(\tau)\,\mathbb{E}^t\left[y^\varepsilon(\tau)\right] + v^\top B(\tau)\,1_{[t,t+\varepsilon)}(\tau)\right\} d\tau. \tag{A.1.2}$$

Thus,

$$\left|\mathbb{E}^t\left[y^\varepsilon(s)\right]\right|^{2k} = \left|\int_t^s r_0(\tau)\,\mathbb{E}^t\left[y^\varepsilon(\tau)\right] d\tau + v^\top \int_t^s B(\tau)\,1_{[t,t+\varepsilon)}(\tau)\,d\tau\right|^{2k}. \tag{A.1.3}$$

By using the inequality $(a+b)^{2k} \le 2^{2k-1}\left(a^{2k}+b^{2k}\right)$, for $a, b \in \mathbb{R}$, we obtain

$$\begin{aligned} &\left|\mathbb{E}^t\left[y^\varepsilon(s)\right]\right|^{2k} \le \\ &2^{2k-1}\left(\left|\int_t^s r_0(\tau)\,\mathbb{E}^t\left[y^\varepsilon(\tau)\right] d\tau\right|^{2k} + \left|v^\top \int_t^s B(\tau)\,1_{[t,t+\varepsilon)}(\tau)\,d\tau\right|^{2k}\right). \end{aligned} \tag{A.1.4}$$

Moreover, by Hölder's inequality and the bondedness condition of $r_0(\cdot)$ and $B(\cdot)$ we deduce that

$$\begin{aligned} &\left|\int_t^s r_0(\tau)\,\mathbb{E}^t\left[y^\varepsilon(\tau)\right] d\tau\right|^{2k} \\ &\le \left\{\left(\int_t^s \left|\mathbb{E}^t\left[y^\varepsilon(\tau)\right]\right|^{2k} d\tau\right)^{\frac{1}{2k}} \left(\int_t^s \left|r_0(\tau)\right|^{\frac{2k}{2k-1}} d\tau\right)^{\frac{2k-1}{2k}}\right\}^{2k}, \\ &\le K_1 \int_t^s \left|\mathbb{E}^t\left[y^\varepsilon(\tau)\right]\right|^{2k} d\tau, \end{aligned} \tag{A.1.5}$$

and

$$\begin{aligned}&\left|v^{\top}\int_t^s B(\tau)\,1_{[t,t+\varepsilon)}(\tau)\,d\tau\right|^{2k}\\&\le |v|^{2k}\left\{\left(\int_{[t,t+\varepsilon]}|B(\tau)|^{2k}\,d\tau\right)^{\frac{1}{2k}}\left(\int_{[t,t+\varepsilon]}1d\tau\right)^{\frac{2k-1}{2k}}\right\}^{2k},\\&\le K_2\,|v|^{2k}\,\varepsilon^{2k},\end{aligned}\tag{A.1.6}$$

for some constants $K_1,\ K_2>0$. Invoking $(A.1.5)$ and $(A.1.6)$ in $(A.1.4)$ we obtain with $C_1=2^{2k-1}K_1$ and $C_2=2^{2k-1}K_2$ that

$$\left|\mathbb{E}^t\left[y^{\varepsilon}(s)\right]\right|^{2k}\le C_1\int_t^s\left|\mathbb{E}^t\left[y^{\varepsilon}(u)\right]\right|^{2k}d\tau+C_2\,|v|^{2k}\,\varepsilon^{2k}.$$

By Grönwall Inequality we deduce that

$$\left|\mathbb{E}^t\left[y^{\varepsilon}(s)\right]\right|^{2k}\le C_2\,|v|^{2k}\,\varepsilon^{2k}\exp\left(C_1(s-t)\right),$$

which implies that

$$\sup_{s\in[t,T]}\left|\mathbb{E}^t\left[y^{\varepsilon}(s)\right]\right|^{2k}\le C_2\,|v|^{2k}\,\varepsilon^{2k}\exp\left(C_1(T-t)\right).$$

Then,

$$\sup_{s\in[t,T]}\left|\mathbb{E}^t\left[y^{\varepsilon}(s)\right]\right|^{2k}=O\left(\varepsilon^{2k}\right).$$

The estimation (3.11) is a direct consequence of Lemma 2.1. in Tang and Li (1994).

To prove (3.12), we consider the difference

$$\begin{aligned}&J\left(t,\hat{X}(t),u^{\varepsilon}(\cdot)\right)-J\left(t,\hat{X}(t),\hat{u}(\cdot)\right)\\&=\frac{1}{2}\mathbb{E}^t\left[\left(\hat{X}^{\varepsilon}(T)^2-\hat{X}(T)^2\right)\right]-\frac{1}{2}\left(\mathbb{E}^t\left[\hat{X}^{\varepsilon}(T)\right]^2-\mathbb{E}^t\left[\hat{X}(T)\right]^2\right)\\&\quad-\frac{\hat{X}(t)}{\mu}\left(\mathbb{E}^t\left[\hat{X}^{\varepsilon}(T)\right]-\mathbb{E}^t\left[\hat{X}(T)\right]\right).\end{aligned}$$

Noting that, by the second order Taylor-Lagrange expanssion, see, e.g., Yong and Zhou (1999), and from (3.8) it results that

$$\begin{aligned}&J\left(t,\hat{X}(t),u^{\varepsilon}(\cdot)\right)-J\left(t,\hat{X}(t),\hat{u}(\cdot)\right)\\&=\mathbb{E}^t\left[\frac{1}{2}y^{\varepsilon}(T)^2+\left(\hat{X}(T)-\mathbb{E}^t\left[\hat{X}(T)\right]-\frac{\hat{X}(t)}{\mu}\right)y^{\varepsilon}(T)-\frac{1}{2}\mathbb{E}^t\left[y^{\varepsilon}(T)\right]^2\right].\end{aligned}\tag{A.1.7}$$

On the other hand, from (3.10) the following estimate is deduced

$$\frac{1}{2}\mathbb{E}^t\left[y^\varepsilon(T)\right]^2 = o(\varepsilon).$$

Then, from the terminal conditions in the adjoint equation (3.3), it follows that

$$J\left(t,\hat{X}(t),u^\varepsilon(\cdot)\right)-J\left(t,\hat{X}(t),\hat{u}(\cdot)\right) = -\mathbb{E}^t\left[p(T;t)\,y^\varepsilon(T)-\frac{1}{2}y^\varepsilon(T)^2\right]+o(\varepsilon). \tag{A.1.8}$$

Now, by applying Itô's formula to $s\mapsto p(s;t)\,y^\varepsilon(s)$ on $[t,T]$, we get, by taking conditional expectations,

$$\begin{aligned}\mathbb{E}^t\left[p(T;t)\,y^\varepsilon(T)\right] = \mathbb{E}^t\Bigg[\int_t^{t+\varepsilon}\Big\{&v^\top B(s)\,p(s;t)+v^\top D(s)\,q(s;t)\\ &+\int_{\mathbb{R}^*}v^\top F(s,z)\,\mathrm{diag}(\nu^\star(dz))\,r(s,z;t)\Big\}ds\Bigg].\end{aligned} \tag{A.1.9}$$

Again, by applying Itô's formula to $s\mapsto e^{\int_s^T 2r_0(\tau)d\tau}y^\varepsilon(s)^2$ on $[t,T]$, we get, by taking conditional expectations,

$$\begin{aligned}\mathbb{E}^t\left[y^\varepsilon(T)^2\right] = \mathbb{E}^t\Bigg[\int_t^{t+\varepsilon}e^{\int_s^T 2r_0(\tau)d\tau}\Big\{&2v^\top B(s)\,y^\varepsilon(s)\\ &+v^\top\left(D(s)\,D(s)^\top+\int_{\mathbb{R}^*}F(s,z)\,\mathrm{diag}(\nu^\star(dz))\,F(s,z)^\top\right)v\Big\}ds\Bigg].\end{aligned} \tag{A.1.10}$$

Moreover, we conclude from **(H1)** together with (3.10) − (3.11) that

$$\begin{aligned}\mathbb{E}^t\left[\int_t^{t+\varepsilon}e^{\int_s^T 2r_0(\tau)d\tau}v^\top B(s)\,y^\varepsilon(s)\,ds\right] &\leq K\varepsilon^{\frac{1}{2}}\left(\int_t^{t+\varepsilon}\mathbb{E}^t\left[y^\varepsilon(s)^2\right]ds\right)^{\frac{1}{2}},\\ &\leq K\varepsilon\left(\sup_{s\in[t,T]}\mathbb{E}^t\left[\left|y^\varepsilon(s)\right|^2\right]\right)^{\frac{1}{2}},\\ &= o(\varepsilon).\end{aligned} \tag{A.1.11}$$

By taking $(A.1.8)-(A.1.11)$ in $(A.1.7)$, it can be deduced that

$$J\left(t,\hat{X}\left(t\right),u^{\varepsilon}\left(\cdot\right)\right)-J\left(t,\hat{X}\left(t\right),\hat{u}\left(\cdot\right)\right)=$$
$$-\mathbb{E}^{t}\left[\int_{t}^{t+\varepsilon}\left\{v^{\top}B\left(s\right)p\left(s;t\right)+v^{\top}D\left(s\right)q\left(s;t\right)\right.\right.$$
$$+\int_{\mathbb{R}^{*}}v^{\top}F\left(s,z\right)\operatorname{diag}\left(\nu^{\star}\left(dz\right)\right)r\left(s,z;t\right)$$
$$\left.\left.-\frac{1}{2}e^{\int_{s}^{T}2r_{0}(\tau)d\tau}v^{\top}\left(D\left(s\right)D\left(s\right)^{\top}+\int_{\mathbb{R}^{*}}F\left(s,z\right)\operatorname{diag}\left(\nu^{\star}\left(dz\right)\right)F\left(s,z\right)^{\top}\right)v\right\}ds\right]$$
$$+o\left(\varepsilon\right), \tag{A.1.12}$$

which is equivalent to

$$J\left(t,\hat{X}\left(t\right),u^{\varepsilon}\left(\cdot\right)\right)-J\left(t,\hat{X}\left(t\right),\hat{u}\left(\cdot\right)\right)$$
$$=-\int_{t}^{t+\varepsilon}\left\{\left\langle\mathbb{E}^{t}\left[H\left(s;t\right)\right],v\right\rangle+\frac{1}{2}\left\langle L\left(s\right)v,v\right\rangle\right\}ds+o\left(\varepsilon\right).$$

■

A.2. Proof of Lemma 2

First we put $\alpha\left(s\right)=-e^{\int_{s}^{T}r_{0}(\tau)d\tau}$ and define for $t\in[0,T]$ and $s\in[t,T]$ the process $\bar{p}\left(s;t\right)$ by

$$\bar{p}\left(s;t\right)=-\alpha\left(s\right)p\left(s;t\right)+\mathbb{E}^{t}\left[\hat{X}\left(T\right)\right]+\frac{\hat{X}\left(t\right)}{\mu}.$$

Then, by the integration by parts formula, on the time interval $[t,T]$, for any $t\in[0,T]$,

$$\begin{cases} d\bar{p}\left(s;t\right)=-p\left(s;t\right)d\alpha\left(s\right)-\alpha\left(s\right)dp\left(s;t\right), \; s\in[t,T], \\ \bar{p}\left(T;t\right)=\hat{X}\left(T\right), \end{cases}$$

the triple $\left(\bar{p}\left(\cdot,\cdot\right),\bar{q}\left(\cdot,\cdot\right),\bar{r}\left(\cdot,\cdot;\cdot\right)\right)$ satisfies the following equation

$$\begin{cases} d\bar{p}\left(s;t\right)=\bar{q}\left(s;t\right)dW^{\star}\left(s\right)+\int_{\mathbb{R}^{\star}}\bar{r}\left(s,z;t\right)\widetilde{\gamma}^{\star}\left(ds,dz\right), \; s\in[t,T], \\ \bar{p}\left(T;t\right)=\hat{X}\left(T\right), \end{cases} \tag{A.2.1}$$

where

$$\left(\bar{q}\left(s;t\right),\bar{r}\left(s,z;t\right)\right)\equiv-\alpha\left(s\right)\left(q\left(s;t\right),r\left(s,z;t\right)\right).$$

It is clear that neither the terminal condition nor the coefficients of the equation $(A.2.1)$ depend on t, so it can be taken as a BSDE on the time interval

$[0,T]$. This implies that the process $(\bar{p}(\cdot,\cdot), \bar{q}(\cdot,\cdot), \bar{r}(\cdot,\cdot;\cdot))$ does not depend on t. Thus, we denote the solution of $(A.2.1)$ by $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{r}(\cdot;\cdot))$. We have then, for any $t \in [0,T]$ and $s \in [t,T]$,

$$\begin{aligned}&(p(s;t), q(s;t), r(s,z;t)) = \\ &-\alpha(s)^{-1}\left(\left(\bar{p}(s) - \mathbb{E}^t\left[\hat{X}(T)\right] - \frac{\hat{X}(t)}{\mu}\right), \bar{q}(s), \bar{r}(s,z)\right).\end{aligned} \tag{A.2.2}$$

Now, using $(A.2.2)$ we have from the definition of $H(\cdot;\cdot)$ given by (3.5), for any $t \in [0,T]$ and $s \in [t,T]$

$$\begin{aligned}\mathbb{E}^t\left[H(s;t) - H(s;s)\right] &= \mathbb{E}^t\left[B(s)(p(s;t) - p(s;s))\right], \\ &= \alpha(s)^{-1}\frac{B(s)}{\mu}\mathbb{E}^t\left[\hat{X}(t) - \hat{X}(s)\right],\end{aligned} \tag{A.2.3}$$

where we have used the law of iterated expectations (i.e. $\mathbb{E}^t[\mathbb{E}^s[X]] = \mathbb{E}^t[X]$, for any $t \leq s$, and $X \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$). Moreover, since $B(s)$ and $\alpha(s)^{-1}$ are bounded and $\hat{X}(\cdot)$ is a right continuous with finite left limit, we have

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left[\int_t^{t+\varepsilon}\left|\mathbb{E}^t\left[H(s;t) - H(s;s)\right]\right|ds\right] \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\left|\mathbb{E}^t\left[B(s)(p(s;t) - p(s;s))\right]\right|ds, \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\left|\frac{B(s)}{\mu}\alpha(s)^{-1}\mathbb{E}^t\left[\hat{X}(t) - \hat{X}(s)\right]\right|, \\ &\leq K\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}^t\left|\hat{X}(t) - \hat{X}(s)\right|, \\ &= 0,\end{aligned}$$

and thus for any $t \in [0,T]$ we have

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}H(s;t)\,ds\right] = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}H(s;s)\,ds\right]. \tag{A.2.4}$$

From the above equality, it is clear that if 2) holds, then

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}H(s;t)\,ds\right] = 0.\ \mathbb{P}-a.s.$$

Conversely, according to Lemma 3.5 in Hu, Jin and Zhou (2017) if 1) holds, then $H(s;s) = 0$, $\mathbb{P}-a.s$, $a.e.\ s \in [0,T]$. This completes the proof. ∎