**Control and Cybernetics**

vol. 26 (1997) No. 1

# On openness and regularity of --y-paraconv̄ex multifunctions[1]

by

K. Allali* and T. Amahroq**

*Faculte des Sciences et Techniques,
B.P 577 Settat, Morocco

**Faculte des Sciences et Techniques Gueliz,
B.P 618 Marrakech, Morocco

Abstract: In this paper, openness and metric regularity results for multifunctions whose inverses are 1-paraconvex are derived from a general theorem by Borwein and Zhuang.

Keywords: Openness, metric regularity, 7-paraconvex multifunction.

## 1. Introduction

S. Rolewicz (1979;1981) introduced the notion of paraconvexity for multifunctions. A multifunction $M$ defined between Banach spaces $X$ and $Y$ is said to be 1-paraconvex (with 1 > 0) if there exists $c > 0$ such that for all $t$ E]O, 1[ and x1,x2 E X

$$tM(x1) + (1 - t)M(x2) \text{ C } M(t.'E1 + (1 - t)x2) + c\|x1 - x2\|^{1} B y \quad (1)$$

whe e By is the closed unit ball. When 1 > 1, Rolewicz proved that (1) is equivalent to

$$tM(x_1)+(1-t)M(x_2) \subset M(tx_1+(1-t)x_2)+c\min(t,1-t)\|x_1-x_2\|^{\gamma}\mathbf{B}_Y. \quad (2)$$

Many important results have been proved by Rolewicz for this class of m4ltifunctions. However an important question was left unsolved. Is a multifunction $M$ (between Banach spaces) whose inverse is 1-paraconvex (with 1 >, 1) open at any point (x, fi) of its graph satisfying

$$R_+ [M(X) - fi] = Y?$$

1This work was carried out while the second author was visiting the Universite de Montpellier in the setting of "Action Integree" n° 95849 between Universities of Cadi-Ayyad (Marrakech), Mohammed V (Rabat) and Montpellier.

Note that one knows from Robinson (1976) that the result holds for convex multifunctions. Recently, Jourani (1996) has provided an affirmative answer to the above question. His method consisted in adapting in an elegant way the method and technics by Robinson (1976) to multifunctions whose inverses are ')'-paraconvex. The aim of the present note is to show that the result can be derived from a general strong result by Borwein and Zhuang (1988) concerning the openness of general multifunctions.

## 2. Preliminaries

Throughout the paper $X$ and $Y$ will be real Banach spaces. Recall that a multifunction $M$ : $X$ -+ $Y$ is *open at linear rate* around a point (x, *y)* E *GrM* if there exist a real number /5 > 0 and two neighbourhoods V and W of x and y respectively such that for all *x* E V, *y* E W n *M(x)* and *t* E [O, 1]

$y$ + t/5By c *M(x* + tBx).

Here *GrM* denotes the *graph* of *M,* that is

*GrM* := *{(x,y)* EX x Y: *y* E *M(x)}.*

Recall also that M is *approximately open at linear rate* around *(x, y)* E *GrM* if there exist two real numbers /5 and p with O < p < 15 and neighbourhoods V and W of x and *y* respectively such that for all *x* E V, *y* E W n *M(x)* and t E [O, 1]

$y$ + t/5By c *M(x* + tBx) + tpBy.

The following general theorem about openness of general multifunctions was established by Borwein and Zhuang (1988). (In fact they proved something more general.) This theorem will be used in the next section in order to derive the openness result about paraconvex multifunctions.

Theorem 1 *Borwein and Zhuang (1988) Let M* : *X* -+ *Y be a multifunction with closed graph and let* (x, *y)* E *GrM. Then the following assertions are equivalent:*

- *M is open at linear rate around (x, y)*
- *M is approximately open at linear rate around* (x, *y)*
- *M is metrically regv,lar around (x, y), that is, there exist a constant k* > 0, *neighbov,rhoods V and W of* x *and* y *such that for ally* E *W and x* E *V with M(x)* n *W* =I=0

  *d(x, M*$^{-1}$(y)) ;= *kd(y, M(x)).*

3. Regularity of paraconvex multifunctions

Let us prove first th following proposition.

**Proposition 1** *Let $M : X \rightrightarrows Y$ be a multifunction such that $M^{-1}$ is $\gamma$-paraconvex and let $(\bar{x},\bar{y}) \in Gr M$. Assume that $\mathbb{R}_+[M(X) - \bar{y}]$ is symmetric. Then*

$$\mathbb{R}_+[M(X) - \bar{y}] = \bigcap_{n \in \mathbb{N}, m \in \mathbb{N}} m[B_Y \cap (M(\bar{x} + nB_X) - \bar{y}) \cap (\bar{y} - M(\bar{x} + nB_X))].$$

*Proof* Let $z \in \mathbb{R}_+[M(X) - \bar{y}]$. Then there exist $\lambda, \lambda' \in \mathbb{R}_+$ and $(x,y)$, $(x',y') \in Gr M$ such that $z = \lambda(y - \bar{y})$ and $-z = \lambda'(y' - \bar{y})$. Let $m, n_1$ and $n \in \mathbb{N} \setminus \{0\}$ such that

$$m \geq \max(\lambda, \lambda', \|y - \bar{y}\|, \|y' - \bar{y}\|),$$

$$n_1 \geq (\|x - \bar{x}\|, \|x' - \bar{x}\|)$$

and

$$n \geq \max\left(\frac{\lambda}{m}, \frac{\lambda'}{m}\right) n_1 + c(m)t$$

where $c$ is the constant of the paraconvexity of $M^{-1}$. One can easily verify that

$$\bar{x} + \frac{\lambda}{m}(x - \bar{x}) \in M^{-1}\left[\bar{y} + \frac{\lambda}{m}(y - \bar{y})\right] + c(m)tB_X.$$

Therefore there exists $b \in B_X$ such that by (3)

$$\bar{y} + \frac{\lambda}{m}(y - \bar{y}) \in M\left[\bar{x} + \frac{\lambda}{m}(x - \bar{x}) + c(m)tu\right] \subset M(\bar{x} + nB_X)$$

and hence

$$\lambda(y - \bar{y}) \in m[M(\bar{x} + nB_X) - \bar{y}].$$

Similarly we get that $\lambda'(y' - \bar{y}) \in m[M(\bar{x} + nB_X) - \bar{y}]$. ∎

When a multifunction $M : X \rightrightarrows Y$ is $\gamma$-paraconvex with $\gamma > 1$, it is easy to see (Jourani, 1996, proof of Theorem 2.2, step 2) that for all $p > 0$, $p \in \mathbb{N} \setminus \{0\}$, $t_i \geq 0$ and $x_i \in pB_X$, $i = 1, \ldots, P$ with $\sum_{i=1} t_i = 1$ one has

$$\sum t_i M(x_i) \subset M\left(\sum t_i x_i\right) + c(2p)^{\gamma} B_Y. \quad (3)$$

We will need the following lemma which is contained in the proof of Theorem 2.2. in Jourani (1996), step 3:

**Lemma 1** *Let $M : X \rightrightarrows Y$ be a multifunction such that $M^{-1}$ is $\gamma$-paraconvex with $\gamma > 1$. Then for all $\lambda, p > 0$ there exists $r > 0$ such that*

$$co[(M(\bar{x} + pB_X) - \bar{y}) \cap \lambda B_Y] \subset M(\bar{x} + rB_X) - \bar{y},$$

*where co denotes the convex hull.* ∎

Recall that a point $y$ is in the *core* of a subset $S$ of $Y$ (denoted by *core* $S$) if for any $z \in Y$ there exists some $r > 0$ such that $\{(1-t)y + tz : t \in [0, r]\} \subset S$

Proposition 2 *Let* $M : X \to Y$ *be a multifunction whose inverse* $M^{-1}$ *is* 1-paraconvex *with* $\gamma > 1$ *and* $(\bar{x}, \bar{y}) \in GrM$. *Suppose that* $\bar{y} \in$ *core* $M(X)$. *Then there exist* $r > 0$ *and* $\delta > 0$ *such that for all* $\rho > 0$ *one has*

$$\bar{y} + \delta B_Y \subset M(\bar{x} + rB_X) + \rho B_Y.$$

Proof: Without loss of generality we may suppose $\bar{x} = 0$ and $\bar{y} = 0$. Since $M^{-1}$ is 1-paraconvex, the set $M(X) = \operatorname{dom} M^{-1}$ is convex, then $0 \in$ core $M(X)$ is equivalent to $Y = \mathbb{R}_+ M(X)$. Therefore by Proposition 1 we have (for $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$)

$$Y = \bigcup_{n,m \in \mathbb{N}^*} cl\{m[coM(nB_X) \cap co(-M(nB_X)) \cap B_Y]\}$$

(where $cl$ denotes the closure) and hence by Baire theorem there exist $n$ and $m \in \mathbb{N}^*$ such that

$$\operatorname{int} cl\{m[coM(nB_X) \cap co(-M(nB_X)) \cap B_Y]\} \neq \emptyset.$$

Thus there exists $\delta > 0$ such that

$$\delta B_Y \subset cl[coM(nB_X) \cap co(-M(nB_X)) \cap B_Y] \subset clcoM(nB_X) \cap B_Y,$$

that is for all $\rho > 0$

$$\delta B_Y \subset B_Y \cap coM(nB_X) + \rho B_Y.$$

Hence by Lemma 1 we have for all $\rho > 0$

$$\delta B_Y \subset M(rB_X) + \rho B_Y$$

where $r := n + c2^{\gamma}$. ∎

Using some ideas of Borwein and Zhuang (1988) we can now prove the following result.

Theorem 2 *Let* $M : X \to Y$ *be a multifunction with closed graph and such that* $M^{-1}$ *is* 1-paraconvex *with* $\gamma > 1$. *Let* $(\bar{x}, \bar{y}) \in GrM$ *such that* $\bar{y} \in$ *core* $M(X)$. *Then* $M$ *is open at linear rate around* $(\bar{x}, \bar{y})$ *and hence metrically regular at* $(\bar{x}, \bar{y})$.

Proof. By translation, we may suppose that $\bar{x} = 0$ and $\bar{y} = 0$. By Proposition 2 there exist $\delta > 0$, $r > 0$ such that for all $\rho > 0$ with $\rho < \delta$ one has $\delta B_Y \subset M(rB_X) + \rho B_Y$, that is, for each $p \in B_Y$ there exist $b \in B_X$ and $u \in B_Y$ such that $rb \in M^{-1}(\delta p - \rho u)$. Now let $\varepsilon > 0$ with $\varepsilon < \delta - \rho$, $y \in \varepsilon B_Y$, $x \in \varepsilon B_X \cap M^{-1}(y)$ and $s \in ]0, 1[$. We have

$$\begin{aligned}(1-s)x + srb &\in (1-s)M^{-1}(y) + sM^{-1}(\delta p - \rho u) \\ &\subset M^{-1}((1-s)y + s(\delta p - \rho u)) + cs\|y - (\delta p + \rho u)\|^{\gamma} B_X\end{aligned}$$

and hence

$$x + srb \in M^{-1}[y - sy + s(\delta p - \rho q)] + s(\epsilon + c(\epsilon + \delta + \rho)^{\gamma})B_X .$$

Putting $\lambda := c(\epsilon + \delta + \rho)^{\gamma}$ we get for all $t \in ]0,1[$

$$(1-t)x + t(x + srb) \in (1-t)M^{-1}(y) + tM^{-1}[y - sy + s(\delta p - \rho q)] + ts(\epsilon + \lambda)B_X$$

which ensures that

$$x + tsrb \in M^{-1}[y - sty + st(\delta p - \rho q)] + [st(\epsilon + \lambda) + t\lambda s^{\gamma}]B_X .$$

Then there exists $u \in B_X$ such that

$$y - ts(y + \rho q) + ts\delta p \in M[x + tsrb - (ts(\epsilon + \lambda) + t\lambda s^{\gamma})u]$$

and hence

$$y + ts\delta p \in M(x + s[tr + t(\epsilon + \lambda) + t\lambda s^{\gamma-1}]\mathbf{B}_X) + ts(\epsilon + \rho)\mathbf{B}_Y.$$

Hence choosing $t < \frac{1}{2}\min($[illegible]$)$ we get for all $s \in ]0,1[$

$$y + s\delta' B_Y \subset M(x + sB_X) + s\rho' B_Y$$

where $\delta' := t\delta$, $\rho' := t(\epsilon + \rho)$ and $\rho' < \delta'$. So $M$ is approximately open at linear rate around $(x, y)$ and hence Theorem 1 completes the proof. ∎

**Remark 1** *It is worth noting that the result can be obtained by using Dolecki's paper, Dolecki (1978), about 1-Hausdorff upper semi-continuous multifunction called later pseudo-Hausdorff upper semi-continuous. In fact Dolecki proved that a multifunction is uniformly pseudo-Hausdorff upper-semi continuous if and only if it is uniformly lower semi-continuous. Moreover, it is easy to see that a multifunction is uniformly lower semi continuous in a neighbourhood of a point $(x_0, y_0)$ if and only if the inverse multifunction is q-regular around $(x_0, y_0)$ in the sense of Borwein and Zhuang (1988).*

## Acknowledgement

The author would like to thank Professors S. Rolewicz and L. Thibault, and the referee for their suggestions and corrections.

## References

BORWEIN, J.M. and ZHUANG, D.M. (1988) Verifable necessary and sufficient conditions for openness and regularity of set-valued maps. *J. Math Appl.*, 134, 441-459.

DOLECKI, S. (1978A) Semicontinuity in constrained optimization I. Metric spaces. *Control and Cybernetics,* 7, 2, 5-lG.

DOLECKI, S. (1978B) Semicontinuity in constrained optimization I. Metric spaces. *Control and Cybernetics,* 7, 3, 17-25.

DOLECKI, S. (1978c) Semicontinuity in constrained optimization I. Metric spaces. *Control and Cyberneti:cs,* 7, 4, 51-68.

JOURANI, A. (1996) Open mapping theorem and inversion theorem for 1-paraconvex multivalued mappings and applications. *Stv,dia Mathematica,* 117, 123-136.

ROBINSON, S.M. (1976) Regularity and stability for convex multivalued functions. *Math. Oper. Res.,* 1, 130-143.

ROLEWICZ, S. (1979A) On paraconvcx multifunctions. *Oper. Res. Verfahren,* 31, 539-546.

ROLEWICZ, S. (1979B) On 1-paraconvcx multifunctions. *Math. Japonica,* 24, 293-300.

ROLEWICZ, S. (1981) On conditions warranting <fl2-subdifferentiability. *Math. Prag. Stv.dy,* 14, 215-224.