Control and Cybernetics

vol. **47** (2018) No. 1

Optimal heat distributions by a gradient-based shape optimization method*

by

Zakaria Belhachmi[1], **Belhassen Meftahi**[2] **and Houcine Meftahi**[2*]

[1]Mathematics, Information Technology and Applications Laboratory (LMIA, EA 3993), 4 rue des Frères Lumière, 68093 Mulhouse cedex, France,
[2]Department of Mathematics ENIT of Tunisia, Tunisia
zakaria.belhachmi@uha.fr, belhassenmeftahi@hotmail.fr,
*corresponding author: houcine.meftahi@enit.utm.tn

Abstract: In this paper, we consider the problem of locating coated inclusions in a 2D dimensional conductor material in order to obtain a suitable thermal environment. The mathematical model is described by elliptic partial differential equation with linear boundary condition, including heat transfer coefficient. A shape optimization problem is formulated by introducing a cost functional to solve the problem under consideration. The shape sensitivity analysis is rigorously performed for the problem by means of a Lagrangian formulation. The optimization problem is solved by means of gradient-based strategy and numerical experiments are carried out to demonstrate the feasibility of the approach.

Keywords: heat conduction, shape optimization, sensitivity analysis, minimax differentiability

1. Introduction

Advanced shape optimization techniques have become a very powerful tool in the design and construction of industrial structures. The shape optimization problem for such structure is formulated as the minimization of a given shape functional, where the variable is the geometry of the subsets of $\mathbb{R}^n$. In general, the cost functional takes the form of an integral over the domain or its boundary, where the integrand depend smoothly on the solution of a boundary value problem. In such problems the sensitivity analysis plays a central role and was intensively studied by many authors (see, for instance, Céa, 1986; Delfour and Zolésio, 1988; Pantz, 2005; Sturm, 2013; Meftahi and Zolésio, 2015; Gangl et al., 2015; Meftahi, 2017 and the references therein).

In this work, we consider the problem of locating coated inclusions inside a thermal conductor domain in order to improve the thermal environment. This

*Submitted: December 2017; Accepted: August 2018

problem can be encountered in the design of electrical multicables, where the optimization of heat transfer to avoid overheating and irreparable damages of the machines is of great interest (see Harbrecht and Loos, 2016). The latter has been considered in Belhachmi et al. (2018) from the point of view of the topological sensitivity. The derivation of the so-called topological derivative is rigorously performed by means of an adjoint method.

In this paper, we focus on the shape optimization by means of shape derivatives, which, contrary to the topological derivatives, proceeds by smooth deformations of the boundary of the design domain. There exists a certain number of methods available to prove the differentiability of shape functions depending on the solution of a partial differential equation (PDE). The established methods comprise the material/shape derivative method (Sokołowski and Zolésio, 1992; Afraites et al., 2007; Dambrine et al., 2015; Dambrine and Laurain, 2016), the min approach for energy cost functions (Delfour, 2012) and the rearrangement methods (Ito et al., 2008; Kasumba and Kunisch, 2011; Kasumba and Kunisch, 2014). Lagrangian methods are also commonly used in shape optimization and have the advantage of providing the shape derivative without the need to compute the material derivative of the partial differential equations.

In Harbrecht and Loos (2016) the shape derivative has been computed for the similar problem by the finite element method and finite differences have been used to verify the implementation.

In the present paper we compute rigorously the shape derivative following a Lagrangian approach in the spirit of Delfour and Zolésio (1988). The method is based on the minimax approach from Delfour and Zolésio (1988), which is applicable to a large class for shape functionals. This allows us to obtain a smooth deformation field used as a descend direction in a gradient method. For the computation times, the adjoint method works much faster than the finite difference method.

Our approach works very well in the case of small numbers of inclusions. For the case of large number of inclusions, the gradient based approach algorithm runs into several local minima. We intend in the future work to tackle this situation using the topological derivation strategy (Giusti et al., 2017; Amigo et al., 2016; Novotny and Sokołowski, 2013; Giusti et al., 2010), coupled with a gradient based shape optimization approach.

The content of the paper is summarized as follows: In Section 2, we formulate the direct and the optimization problem. In Section 3, we give a brief review of the velocity method, which allows to build a family of parameterized domains and to define the appropriate tools for the differentiability of the cost functional, evaluated at these domains. In Section 4, we introduce the saddle point formulation of the shape optimization problem, and the Lagrange functions, associated with the cost functional. Then, we can perform the shape sensitivity analysis for the Lagrangian functional. In Section 5, the min-max formulation, coupled with the function space parameterization technique is described. In Section 6, the numerical algorithm is described and the results are given.

2. Problem formulation

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and $\omega \in \mathcal{O}_{ad}$ with

$$\mathcal{O}_{ad} := \{\omega \text{ of classe } C^2 : \quad \omega \subset \Omega, \operatorname{dist}(\partial\omega; \partial\Omega) > \eta\},$$

for some $\eta > 0$. Let $\Omega_0, \Omega_1, \Omega_2$ be subsets of Ω such that Ω_2 is surrounded by Ω_1 and the latter is surrounded by $\Omega_0 (\Omega_0 = \Omega \setminus \overline{\Omega_1 \cup \Omega_2})$. We denote by $\Gamma_2 = \partial\Omega_2$ and Γ_1 the external boundary of Ω_1; see Fig. 1 for the illustration of this geometry. In the sequel we will denote the coated inclusion ω by: $\omega = \overline{\Omega_2} \cup \Omega_1$ (ω is composed of two materials with different conductivities). We note that in the practical/physical situation, e.g. the design of electrical multi-cables, the inclusion is made of at least two materials (conducting and insulating and even air bubbles) different from the background material. This makes our problem non-standard.

Assume that thermal conductivity in Ω is

$$\sigma = \sigma_0 \chi_{\Omega_0} + \sigma_1 \chi_{\Omega_1} + \sigma_2 \chi_{\Omega_2},$$

where $\sigma_0, \sigma_1, \sigma_2 > 0$, are the positive constants and χ denotes the indicator function. For a given piecewise C^1 source term $f = f_0 \chi_{\Omega_0} + f_1 \chi_{\Omega_1} + f_2 \chi_{\Omega_2}$, the

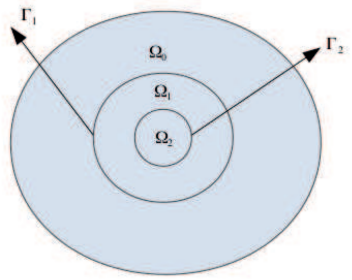


Figure 1. The domain $\Omega = \Omega_2 \cup \Gamma_2 \cup \Omega_1 \cup \Gamma_1 \cup \Omega_0$

temperature u satisfies the following problem:

$$\begin{cases} -\operatorname{div}(\sigma \nabla u) = f & \text{in } \Omega \\ [\![u]\!]_{\Gamma_i} = 0 & \text{on } \Gamma_i, i = 1, 2, \\ [\![\sigma \partial_n u]\!]_{\Gamma_i} = 0 & \text{on } \Gamma_i, i = 1, 2, \\ \sigma_0 \partial_n u + \alpha(u - u_a) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where n is the unit normal vector to the interface $\Gamma_i, i = 1, 2$ or $\partial\Omega$ pointing outward of Ω_i or Ω, α is the heat transfer coefficient (assumed to be positive constant), and u_a is the ambient temperature. Defining by ϕ_i the restriction of some function ϕ to Ω_i, we denote by $[\![\phi]\!]_{\Gamma_i}$ the jump of ϕ across Γ_i, i.e.,

$$[\![\phi]\!]_{\Gamma_i} = \phi_{i-1|\Gamma_i} - \phi_{i|\Gamma_i}.$$

The weak form of (2.1) reads

$$\text{find } u \in H^1(\Omega) \quad \text{such that} \quad b(u,v) = l(v), \tag{2.2}$$

where

$$b(u,v) = \int_\Omega \sigma \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \alpha u v \, dx \quad \text{ and } l(v) = \int_\Omega f v \, dx + \int_{\partial\Omega} \alpha u_a v \, dx,$$

for all $v \in H^1(\Omega)$. The existence and uniqueness of the weak solution follows from the Riesz representation theorem and the Poincaré-Wirtinger inequality (with the usual modification when $\alpha = 0$, where $H^1(\Omega)$ has to be replaced with $H^1(\Omega) \setminus \mathbb{R}$). Then, the optimization problem under consideration is:

Find the location of the inclusion $\omega \subset \Omega$ to get an efficient cooling temperature. (2.3)

To solve numerically problem (2.3), we consider the objective function

$$J_p(\omega, u) = \frac{1}{p} \int_\Omega |u|^p \, dx, \quad p \geq 2, \quad u \text{ solution of (2.1).}$$

Then, the optimization problem reads:

$$\begin{cases} \text{minimize } J_p(\omega, u) := \dfrac{1}{p} \displaystyle\int_\Omega |u|^p \, dx \\ \text{subject to } \omega \in \mathcal{O}_{ad} \text{ and } u \text{ the solution of (2.1),} \end{cases} \tag{2.4}$$

Remark 1 *The adequate shape functional to minimize the maximum temperature is*

$$J_\infty(\omega, u_\omega) = \|u\|_{L^\infty(\Omega)}.$$

The functional J_∞ is not differentiable but admits a subgradient (see Habbal, 1998). In order to use a gradient method, we replace J_∞ by the functional J_p, for $p \geq 2$.

The numerical resolution of (2.4) requires the sensitivity analysis of J_p with respect to ω. The differentiation with respect to the shape ω is the main purpose of the following sections.

3. Preliminaries

In this section we recall some basic facts about the velocity method from shape optimization, used to calculate the shape derivatives of the functional J_p; see Delfour and Zolésio (2011); Sokołowski and Zolésio (1992). In the velocity (or speed) method a domain Ω is deformed by the action of a velocity field V. The evolution of the domain is described by the following dynamical system:

$$\begin{cases} \dfrac{d}{dt}x(t) = V(x(t)), t \in [0, \varepsilon) \\ \quad x(0) = X \end{cases} \tag{3.1}$$

for some real number $\varepsilon > 0$. Assume $V \in \mathcal{D}^1(\Omega; \mathbb{R}^2)$, where $\mathcal{D}^1(\Omega; \mathbb{R}^2)$ denotes the space of continuously differentiable functions with compact support in Ω, then the ordinary differential equation (3.1) has a unique solution. This allows us to define the diffeomorphism

$$T_t : \mathbb{R}^2 \to \mathbb{R}^2 : X \mapsto T_t(X) := x(t). \tag{3.2}$$

For $t \in [0, \varepsilon)$, T_t is invertible. Furthermore, the Jacobian $\xi(t)$ is strictly positive

$$\forall\, t \in [0, \varepsilon), \quad \xi(t) = |\mathrm{det} DT_t(X)| > 0, \tag{3.3}$$

where $DT_t(X)$ is the Jacobian matrix of the transformation T_t, associated with the velocity field V. In the sequel, we use the following notation : M^{-1} for the inverse of M and M^{-*} for the transpose of its inverse. We also denote by

$$w(t) = \xi(t)|(DT_t)^{-*}n| \tag{3.4}$$

the tangential Jacobian of T_t on $\partial\Omega$.

Remark 2 *As the sets ω are made of two pieces (Ω_1 and Ω_2), we will restrict the fields V to those preserving such structure. This is easily done, for example, if we concatenate local smooth fields $V_i = V_{|\Omega_i}, i = 1, 2$.*

Proposition 1 *(Delfour and Zolésio, 2011; Sokołowski and Zolésio, 1992) For a function $\varphi \in W^{1,1}_{loc}(\mathbb{R}^2)$ and $V \in \mathcal{D}^1(\mathbb{R}^2)$, we have the following formulae*

$$\nabla(\varphi \circ T_t) = DT_t^*(\nabla\varphi) \circ T_t, \tag{3.5}$$

$$\frac{d}{dt}(\varphi \circ T_t) = (\nabla\varphi \cdot V) \circ T_t, \tag{3.6}$$

$$\frac{d\xi(t)}{dt} = [\operatorname{div} V] \circ T_t \xi(t), \tag{3.7}$$

$$w'(0) = \operatorname{div}(V) - DVn \cdot n. \tag{3.8}$$

Let J be a real valued function $J : \Omega \longrightarrow \mathbb{R}$. We say that J has a *Eulerian semiderivative* at Ω in the direction V if the following limit exists and is finite:

$$dJ(\Omega; V) = \lim_{t \searrow 0} \frac{J(T_t(\Omega)) - J(\Omega)}{t}.$$

If the map $V \longrightarrow dJ(\Omega; V))$ is linear and continuous, we say that J is shape differentiable at Ω.

Definition 1 *(Delfour and Zolésio, 2011; Sokołowski and Zolésio, 1992) Let Ω be an open domain of class C^2 with compact boundary $\partial\Omega$. We denote by $U(\partial\Omega)$ a neighborhood of $\partial\Omega$.*

(i) Let $f \in C^1(\partial\Omega)$ and $\widetilde{f}$ be an extension of f in a neighborhood of $\partial\Omega$. The tangential gradient of f at a point of $\partial\Omega$ is defined as

$$\nabla_\tau f = \nabla\widetilde{f} - \partial_n \widetilde{f} n,$$

where n is the outward unit normal vector to $\partial\Omega$.

(ii) For a vector function $v \in C^1(\partial\Omega)^d, d \geq 1$ and its extension $\widetilde{v}$, the tangential divergence is defined as

$$\operatorname{div}_\tau v = \operatorname{div}(\widetilde{v}) - D\widetilde{v}n \cdot n$$

where $D\widetilde{v}$ denotes the Jacobian matrix of $\widetilde{v}$.

Note that the tangential divergence and gradient are independent of the extension.

4. Min-sup formulation

In what follows we focus on the computation of the shape derivative of J_p. We introduce the Lagrangian functional

$$G(\omega, \varphi, \psi) = J_p(\omega, \varphi) + b(\omega, \varphi, \psi) - l(\psi), \quad \forall \varphi, \psi \in H^1(\Omega).$$

Then, it is easy to check that

$$J_p(\omega, u(\omega)) = \min_{\varphi \in H^1(\Omega)} \sup_{\psi \in H^1(\Omega)} G(\omega, \varphi, \psi),$$

since

$$\sup_{\psi \in H^1(\Omega)} G(\omega, \varphi, \psi) = \begin{cases} J_0(\omega, u(\omega)) & \text{if } \varphi = u(\omega), \\ +\infty & \text{otherwise.} \end{cases}$$

It is easily shown that the functional G is convex, continuously differentiable with respect to φ, and concave continuously differentiable with respect to ψ. Therefore, according to Ekeland and Temam (1974), the functional G has a saddle point (u, v) if and only if (u, v) solves the following system:

$$\partial_\psi G(\omega, u, v; \hat{\psi}) = \partial_\psi b(\omega, u, v; \hat{\psi}) - \partial_\psi l(v; \hat{\psi}) = 0,$$
$$\partial_\varphi G(\omega, u, v; \hat{\varphi}) = \partial_\varphi J_p(\omega, u; \hat{\varphi}) + \partial_\varphi b(\omega, u, v; \hat{\varphi}) = 0,$$

for all $\hat{\psi} \in H^1(\Omega)$ and $\hat{\varphi} \in H^1(\Omega)$. This yields that G has a saddle point (u, v), where the state u is the unique solution of (2.1) and the adjoint state $v = v(\omega)$ is the solution of the following adjoint problem:

$$\begin{cases} \operatorname{div}(\sigma\nabla v) = u|u|^{p-2}, p \geq 2 & \text{in } \Omega \\ \sigma_0 \partial_n v + \alpha v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Since u is Hölder continuous (from elliptic regularity results), $u|u|^{p-2}$ is at least in $L^2(\Omega)$. Therefore, the problem (4.1) has unique solution $v \in H^1(\Omega)$. Summarizing the above, we have obtained

THEOREM 1 *The functional $J_p(\omega, u(\omega))$ is given by*

$$J_p(\omega, u(\omega)) = \min_{\varphi \in H^1(\Omega)} \sup_{\psi \in H^1(\Omega)} G(\omega, \varphi, \psi). \tag{4.2}$$

The unique saddle point for G is given by (u, v), where u solves the direct problem (2.1) *and v solves the adjoint problem* (4.1).

Similarly, the previous analysis holds for the functional depending on the transformed subdomain $\omega_t = T_t(\omega)$. Thus, we have

$$J_p(\omega_t, u(\omega_t) = \min_{\varphi \in H^1(\Omega)} \sup_{\psi \in H^1(\Omega)} G(\omega_t, \varphi, \psi). \tag{4.3}$$

The corresponding saddle point $(u(\omega_t), v(\omega_t))$ is characterized by

$$\partial_\psi G(\omega_t, u(\omega_t), v(\omega_t); \hat{\psi}) = 0, \quad \forall\, \hat{\psi} \in H^1(\Omega),$$

$$\partial_\varphi G(\omega_t, u(\omega_t), v(\omega_t); \hat{\varphi}) = 0, \quad \forall\, \hat{\varphi} \in H^1(\Omega).$$

5. Shape derivative by the min-sup differentiability

In this section we apply Theorem 5 to compute the shape derivative of J_p. Let us consider the transformation T_t, defined by (3.2), with $V \in \mathcal{D}^1(\Omega, \mathbb{R}^2)$. In this case, $T_t(\Omega) = \Omega$, but in general $T_t(\omega) \neq \omega$. Our aim is to compute the derivative of $J_p(\omega_t, u(\omega_t))$ with respect to the parameter $t \geq 0$.

In order to differentiate $G(\omega_t, \varphi, \psi)$ with respect to t, the integrals appearing in $G(\omega_t, \varphi, \psi)$ need to be transported back on the reference interface ω using the transformation T_t. However, composing by T_t inside the integrals creates terms $\varphi \circ T_t$ and $\psi \circ T_t$, which might be non-differentiable. To avoid this problem, we need to parameterize the space $H^1(\Omega)$ by composing the elements of $H^1(\Omega)$ with T_t^{-1}. Following this argument, we rewrite (4.3) as

$$J_p(\omega_t, u(\omega_t)) = \min_{\varphi \in H^1(\Omega)} \sup_{\psi \in H^1(\Omega)} \tilde{G}(t, \varphi, \psi), \tag{5.1}$$

where

$$\tilde{G}(t, \varphi, \psi) := G(\omega_t, \varphi \circ T_t^{-1}, \psi \circ T_t^{-1}). \tag{5.2}$$

Note that since $T_t(\Omega) = \Omega$, we have $H^1(T_t(\Omega)) = H^1(\Omega)$, and the sets, over which the minimum and supremum are taken in (5.1) stay unchanged. Furthermore, $(u^t, v^t) = (u(\omega_t) \circ T_t, v(\omega_t) \circ T_t)$ is the saddle point of $\tilde{G}$.

We can rewrite expression (5.2) on the fixed domain Ω by using the transformation T_t. This yields

$$\begin{aligned}\tilde{G}(t,\varphi,\psi) =&\frac{1}{p}\int_\Omega |\varphi|^p \xi(t)\,dx + \int_\Omega \sigma A(t)\nabla\varphi\cdot\nabla\psi\,dx + \int_{\partial\Omega} w(t)\alpha\varphi\psi\,ds\\ &-\int_\Omega f\circ T_t\psi\xi(t)\,dx - \int_{\partial\Omega} w(t)\alpha u_a\circ T_t\psi\,ds,\end{aligned} \tag{5.3}$$

where

$$A(t) := (DT_t)^{-*}(DT_t)^{-1}\xi(t),$$

and $\xi(t)$, $\omega(t)$ are defined in (3.3),(3.4), respectively. The saddle point (u^t, v^t) is characterized by

$$\int_\Omega \sigma A(t)\nabla u^t\cdot\nabla\psi\,dx + \int_{\partial\Omega} w(t)\alpha u^t\psi\,ds = \int_\Omega \xi(t) f\circ T_t\psi\,ds + \int_{\partial\Omega} w(t)\alpha u_a\circ T_t\psi\,ds, \tag{5.4}$$

$$\int_\Omega \sigma A(t)\nabla v^t\cdot\nabla\varphi\,dx + \int_{\partial\Omega} w(t)\alpha v^t\psi\,ds = -\int_\Omega \xi(t)u^t|u^t|^{p-2}\varphi\,dx, \tag{5.5}$$

for all $\psi\in H^1(\Omega)$ and $\varphi\in H^1(\Omega)$.

Now we are ready to compute the limit

$$dJ_p(\omega;V) := \lim_{t\to 0}\frac{\mathcal{J}_p(\omega_t)-\mathcal{J}_p(\omega)}{t}$$

where

$$\mathcal{J}_p(\omega_t) := J_p(\omega_t, u(\omega_t)).$$

Theorem 2 (Volume expression) *The functional $\mathcal{J}_p$ is shape differentiable and its shape derivative in the direction V is given by*

$$\begin{aligned}d\mathcal{J}_p(\omega,V) =& \frac{1}{p}\int_\Omega |u|^p\operatorname{div}(V)\,dx + \int_\Omega \sigma A'(0)\nabla u\cdot\nabla v\,dx\\ &-\sum_{i=0}^{2}\int_{\Omega_i}(f_i\operatorname{div}(V)+\nabla f_i\cdot V)\,v\,dx.\end{aligned} \tag{5.6}$$

Proof We apply Theorem 5, given in the Appendix, to compute the shape derivative of $\mathcal{J}_p$; see Section 7. To this end, we should verify the following four respective assumptions $(H_1)-(H_4)$.

Assumption (H_1)*:* Given (β,γ) satisfying $0<\beta<\gamma$, we can find $\varepsilon>0$ such that

$$\forall\eta\in\mathbb{R}^2,\quad \beta|\eta|^2 \le \sigma A(t)\eta\cdot\eta \le \gamma|\eta|^2,\ \text{for } t\in[0,\varepsilon]. \tag{5.7}$$

As in Theorem 5, introduce the sets

$$X(t) := \left\{ x^t \in H^1(\Omega) : \sup_{y \in H^1(\Omega)} \tilde{G}(t, x^t, y) = \inf_{x \in H^1(\Omega)} \sup_{y \in H^1(\Omega)} \tilde{G}(t, x, y) \right\},$$

$$Y(t) := \left\{ y^t \in H^1(\Omega) : \inf_{x \in H^1(\Omega)} \tilde{G}(t, x, y^t) = \sup_{y \in H^1(\Omega)} \inf_{x \in H^1(\Omega)} \tilde{G}(t, x, y) \right\}.$$

We obtain

$$\forall\, t \in [0, \varepsilon] \quad S(t) = X(t) \times Y(t) = \{u^t, v^t\} \neq \varnothing,$$

and assumption (H_1) is satisfied.

Assumption (H_2)*:* Upon defining $B(t) = DT_t^{-*}$, we may compute

$$\begin{aligned}
&B'(t) = -B(t) DV^* B(t), \qquad \xi'(t) = \operatorname{tr}(DV B(t)^*) \xi(t),\\
&A'(t) = -A(t) \operatorname{tr}(DV B(t)^*) + A(t) B(t)^* DV + DV^* B(t) A(t),\\
&w'(t) = \xi'(t) |B(t) n| + \xi(t) |B(t) n|^{-1} B'(t) n.
\end{aligned}$$

Consequently, we obtain the derivatives

$$\begin{aligned}
\partial_t \tilde{G}(t, \varphi, \psi) = &\frac{1}{p} \int_\Omega |\varphi|^p \xi'(t)\, dx + \int_\Omega \sigma A'(t) \nabla\varphi \cdot \nabla\psi\, dx + \int_{\partial\Omega} w'(t) \alpha \varphi \psi\, ds\\
&- \sum_{i=0}^{2} \int_{\Omega_i} (\nabla f_i \cdot V) \circ T_t \xi(t)\, dx - \int_\Omega f \circ T_t \xi'(t) dx\\
&- \int_{\partial\Omega} w'(t) \alpha u_a \circ T_t \psi\, ds - \int_{\partial\Omega} w(t) \alpha \nabla u_a \cdot V \psi\, ds.
\end{aligned} \tag{5.8}$$

At $t = 0$ we also have

$$A'(0) = \operatorname{div}(V) I_2 - DV^* - DV, \quad w'(0) = \operatorname{div}_\tau V,$$

where I_2 is the 2×2 identity matrix. Since $V \in \mathcal{D}^1(\Omega)$, $t \to DT_t$ is continuous in $[0, \varepsilon]$, and consequently also $t \mapsto (B(t), A(t), A'(t), w(t), w'(t))$. Therefore, the partial derivatives $\partial_t \tilde{G}(t, \varphi, \psi)$ exist everywhere in $[0, \varepsilon]$ and the condition (H_2) is satisfied.

Assumptions (H_3) *and* (H_4)*:* We show first the boundedness of (u^t, v^t). Let $\psi = u^t$ in the variational equation (5.4). By the choice, of ε satisfying the condition (5.7), and noting that

$$T_t(x) = x, \quad \xi(t) = 1 \quad \text{and} \quad w(t) = 1 \text{ on } \partial\Omega,$$

we get from (5.4)

$$\beta\|\nabla u^t\|^2_{L^2(\Omega)}+\alpha\|u^t\|^2_{L^2(\partial\Omega)} \leq \|\xi(t)f\circ T_t\|_{L^2(\Omega)}\|u^t\|_{L^2(\Omega)}+\|u_a\|_{L^2(\partial\Omega)}\|u^t\|_{L^2(\partial\Omega)}.$$

Using Young's inequality, we obtain

$$\begin{aligned}\beta\|\nabla u^t\|^2_{L^2(\Omega)} + \alpha\|u^t\|^2_{L^2(\partial\Omega)} \leq{}& \frac{1}{2r}\|\xi(t)f \circ T_t\|^2_{L^2(\Omega)} + \frac{r}{2}\|u^t\|^2_{L^2(\Omega)} + \frac{1}{2s}\|u_a\|^2_{L^2(\partial\Omega)}\\ &+ \frac{s}{2}\|u^t\|^2_{L^2(\partial\Omega)}\end{aligned}$$

for some $r, s > 0$. This implies that

$$\begin{aligned}&\beta\|\nabla u^t\|^2_{L^2(\Omega)} - \frac{r}{2}\|u^t\|^2_{L^2(\Omega)} + \left(\alpha - \frac{s}{2}\right)\|u^t\|^2_{L^2(\partial\Omega)}\\ &\leq \frac{1}{2r}\|\xi(t)f \circ T_t\|^2_{L^2(\Omega)} + \frac{1}{2s}\|u_a\|^2_{L^2(\partial\Omega)}.\end{aligned}$$

We choose s such that $\alpha > s/2$, and using the fact that

$$\|u\|^2 := \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\Omega)}$$

is a norm on $H^1(\Omega)$, equivalent to the natural norm (Meftahi, 2017), we obtain

$$\min\left(\beta, \alpha - \frac{s}{2}\right)\|u^t\|^2_{H^1(\Omega)} - \frac{r}{2}\|u^t\|^2_{L^2(\Omega)} \leq \frac{1}{2r}\|\xi(t)f\circ T_t\|^2_{L^2(\Omega)} + \frac{1}{2s}\|u_a\|^2_{L^2(\partial\Omega)}.$$

Choosing r such that $\min(\beta, \alpha - s/2) - r/2 > 0$, we deduce that

$$\|u^t\|^2_{H^1(\Omega)} \leq C\left(\frac{1}{2r}\|\xi(t)f \circ T_t\|^2_{L^2(\Omega)} + \frac{1}{2s}\|u_a\|^2_{L^2(\partial\Omega)}\right),$$

where C is a positive constant. Since $\xi(t) \to 1$ as $t \to 0$ and $f \circ T_t \to f$ in $L^2(\Omega)$ (see Delfour and Zolésio, Lemma 2.1, p. 397), then u^t is bounded:

$$\text{there exists } c > 0 \text{ such that } \sup_{t\in[0,\epsilon]} \|u^t\|_{H^1(\Omega)} \leq c.$$

We apply the same technique to the variational equation (5.5) and we are able to show that the function v^t is bounded. The next step is to show the continuity with respect to t of the vector (u^t, v^t). Subtracting (5.4) at $t > 0$ and $t = 0$ and choosing $\psi = u - u^t$ yields

$$\begin{aligned}&\int_\Omega \sigma|\nabla(u - u^t)|^2\, dx + \int_{\partial\Omega} \alpha|u - u^t|^2\, ds\\ &= \int_\Omega (\sigma A(t) - \sigma I_2)\, \nabla u^t \cdot \nabla(u - u^t)\, dx + \int_\Omega (\xi(t)f \circ T_t - f)\,(u - u^t)\, dx\\ &\leq \|\sigma A(t) - \sigma I_2\|_{L^\infty(\Omega)}\|\nabla u^t\|_{L^2(\Omega)}\|\nabla(u - u^t)\|_{L^2(\Omega)}\\ &+ \|\xi(t)f \circ T_t - f\|_{L^2(\Omega)}\|u - u^t\|_{L^2(\Omega)}.\end{aligned}$$

Furthermore, due to the boundedness of u^t and the fact that

$$\|u\|^2 := \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\Omega)}$$

is a norm on $H^1(\Omega)$, equivalent to the natural norm, we obtain

$$\|u^t - u\|_{H^1(\Omega)} \leq c\left(\|\sigma A(t) - \sigma I_2\|_{L^\infty(\Omega)} + \|\xi(t) f \circ T_t - f\|_{L^2(\Omega)}\right).$$

Due to the strong continuity of $A(t)$ (as a function of t) and $\xi(t) f \circ T_t \to f$ in $L^2(\Omega)$ as $t \to 0$, one deduces that $u^t \to u$ in $H^1(\Omega)$ as $t \to 0$. Concerning the continuity of v^t, one may show from (5.5) that $v^t \to v$ in $H^1(\Omega)$. Finally, in view of the strong continuity of

$$(t, \varphi) \to \partial_t \tilde{G}(t, \varphi, \psi) \text{ and } (t, \psi) \to \partial_t \tilde{G}(t, \varphi, \psi),$$

assumptions (H_3) and (H_4) are verified. All assumptions of Theorem 5 are satisfied and therefore, we obtain

$$d\mathcal{J}_p(\omega; V) = \partial_t \tilde{G}(t, u, v)\big|_{t=0},$$

where

$$\partial_t \tilde{G}(t, u, v)\big|_{t=0} = \frac{1}{p}\int_\Omega \operatorname{div}(V)|u|^p\, dx + \int_\Omega \sigma A'(0)\nabla u \cdot \nabla v\, dx - \sum_{i=0}^{2}\int_{\Omega_i} (f_i \operatorname{div}(V) + \nabla f_i \cdot V)\, v\, dx.$$

To obtain the boundary expression of the shape derivative, presented in (5.6), one may differentiate (5.2) directly, using the following result for the differentiation of domain integrals; see for instance Delfour and Zolésio (2011).

Theorem 3 *Let $\Phi : [0, T] \to W^{1,\infty}(\mathbb{R}^d)$ differentiable at $t = 0$ with $\Phi(0) = I$ and $\Phi'(0) = V$ and let $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary. Assume $[0, T] \ni t \to F(t, .) \in L^1(\mathbb{R}^d)$ is differentiable at 0 and $F'(0, .) \in W^{1,1}(\mathbb{R}^d)$. Then*

$$\int_{\Phi(t)(\Omega)} F(t, x)\, dx$$

is differentiable at 0 and we have

$$\frac{d}{dt}\int_{\Phi(t)(\Omega)} F(t, x)\, dx\Big|_{t=0} = \int_{\partial\Omega} F(0, x) V \cdot n\, ds + \int_\Omega \partial_t F(0, x)\, dx. \tag{5.9}$$

In order to apply formula (5.9), we split integrals in (5.2), and we obtain

$$\begin{aligned}\partial_t \tilde{G}(0, u, v) = &-\frac{1}{p}\int_{\Gamma_1} [\![|u|^p]\!]_{\Gamma_1} V \cdot n\, ds - \frac{1}{p}\int_{\Gamma_2} [\![|u|^p]\!]_{\Gamma_2} V \cdot n\, ds + \int_\Omega \dot{u}u|u|^{p-2}\, dx \\ &- \int_{\Gamma_1} [\![\sigma\nabla u \cdot \nabla v]\!]_{\Gamma_1} V \cdot n\, ds - \int_{\Gamma_2} [\![\sigma\nabla u \cdot \nabla v]\!]_{\Gamma_2} V \cdot n\, ds \\ &+ \int_\Omega \sigma(\nabla\dot{u} \cdot \nabla v + \nabla u \cdot \nabla\dot{v})\, dx - \int_\Omega f\dot{v}\, dx \\ &+ \int_{\Gamma_1} [\![f]\!]_{\Gamma_1} vV \cdot n\, ds + \int_{\Gamma_2} [\![f]\!]_{\Gamma_2} vV \cdot n\, ds\end{aligned}$$

(5.10)

where

$$\dot{\phi} = \frac{d}{dt}(\phi \circ T_t^{-1})\Big|_{t=0} = -\nabla\phi \cdot V.$$

Using Green's formula and the fact that $\dot{v} = -\nabla v \cdot V$ has compact support in Ω, we obtain

$$\begin{aligned}\int_\Omega \sigma\nabla u \cdot \nabla\dot{v}\,dx - \int_\Omega f\dot{v}\,dx &= \int_{\Omega_0} \sigma\nabla u \cdot \nabla\dot{v}\,dx + \int_{\Omega_1} \sigma\nabla u \cdot \nabla\dot{v}\,dx \\ &+ \int_{\Omega_2} \sigma\nabla u \cdot \nabla\dot{v}\,dx - \int_\Omega f\dot{v}\,dx \\ &= -\int_{\Omega_0} \operatorname{div}(\sigma\nabla u)\dot{v}\,dx - \int_{\Omega_1} \operatorname{div}(\sigma\nabla u)\dot{v}\,dx \\ &- \int_{\Omega_2} \operatorname{div}(\sigma\nabla u)\dot{v}\,dx - \int_\Omega f\dot{v}\,dx \\ &+ \int_{\Gamma_1} [\![\sigma\partial_n u \cdot \partial_n v]\!] V \cdot n\,ds + \int_{\Gamma_2} [\![\sigma\partial_n u \cdot \partial_n v]\!] V \cdot n\,ds \\ &= \int_{\Gamma_1} [\![\sigma\partial_n u \cdot \partial_n v]\!] V \cdot n\,ds + \int_{\Gamma_2} [\![\sigma\partial_n u \cdot \partial_n v]\!] V \cdot n\,ds.\end{aligned}$$

Similarly, we obtain

$$\int_\Omega \sigma\nabla v\cdot\nabla\dot{u}\,dx + \int_\Omega \dot{u}u|u|^{p-2}\,dx = \int_{\Gamma_1} [\![\sigma\partial_n u\cdot\partial_n v]\!] V\cdot n\,ds + \int_{\Gamma_2} [\![\sigma\partial_n u\cdot\partial_n v]\!] V\cdot n\,ds.$$

Due to the fact that $u \in H^1(\Omega)$, expression (5.10) becomes

$$\begin{aligned}\partial_t\tilde{G}(0,u,v) = &-\int_{\Gamma_1} [\![\sigma\nabla u \cdot \nabla v]\!]_{\Gamma_1} V \cdot n\,ds - \int_{\Gamma_2} [\![\sigma\nabla u \cdot \nabla v]\!]_{\Gamma_2} V \cdot n\,ds \\ &+ \int_{\Gamma_1} [\![f]\!]_{\Gamma_1} vV \cdot n\,ds + \int_{\Gamma_2} [\![f]\!]_{\Gamma_2} vV \cdot n\,ds \\ &+ 2\int_{\Gamma_1} [\![\sigma\partial_n u \cdot \partial_n v]\!] V \cdot n\,ds + 2\int_{\Gamma_2} [\![\sigma\partial_n u \cdot \partial_n v]\!] V \cdot n\,ds.\end{aligned} \tag{5.11}$$

Using the decomposition

$$[\![\sigma\nabla u\cdot\nabla v]\!]_{\Gamma_i} = [\![\sigma\partial_n u\cdot\partial_n v + \sigma\nabla_{\Gamma_i} u\cdot\nabla_{\Gamma_i} v]\!] = [\![\sigma\partial_n u\cdot\partial_n v]\!]_{\Gamma_i} + [\![\sigma]\!]_{\Gamma_i} \nabla_{\Gamma_i} u\cdot\nabla_{\Gamma_i} v,$$

we have arrived at the following result.

THEOREM 4 (BOUNDARY EXPRESSION) *The shape derivative of the functional J in the direction $V \in \mathcal{D}^1(\Omega)$ is given by*

$$\begin{aligned} D\mathcal{J}_p(\omega; V) = &\int_{\Gamma_1} [\![\sigma \partial_n u \cdot \partial_n v]\!]_{\Gamma_1} V \cdot n \, ds + \int_{\Gamma_2} [\![\sigma \partial_n u \cdot \partial_n v]\!]_{\Gamma_2} V \cdot n \, ds \\ &- \int_{\Gamma_1} [\![\sigma]\!]_{\Gamma_1} \nabla_{\Gamma_1} u \cdot \nabla_{\Gamma_1} v V \cdot n \, ds - \int_{\Gamma_2} [\![\sigma]\!]_{\Gamma_2} \nabla_{\Gamma_1} u \cdot \nabla_{\Gamma_1} v V \cdot n \, ds \\ &+ \int_{\Gamma_1} [\![f]\!]_{\Gamma_1} v V \cdot n \, ds + \int_{\Gamma_2} [\![f]\!]_{\Gamma_2} v V \cdot n \, ds. \end{aligned} \tag{5.12}$$

REMARK 3 *Note that the volume expression of the shape derivative, presented in* (5.6)*, can be rewritten in canonical form as*

$$d\mathcal{J}_p J(\omega; V) = \int_\Omega \mathbb{S} : DV \, dx + \sum_{i=0}^{2} \int_{\Omega_i} \mathbb{S}_i : DV + \mathbb{G}_i \cdot V \, dx$$

where

$$\mathbb{S} = -\sigma(\nabla u \otimes \nabla v + \nabla v \otimes \nabla u) + \sigma(\nabla u \cdot \nabla v) I + \frac{1}{p}|u|^p I, \quad \mathbb{S}_i = f_i v I \text{ and } \mathbb{G}_i = \nabla f_i v.$$

Using standard tensor relation $(\nabla u \otimes \nabla)n = (\nabla v \cdot n)\nabla u$, one may obtain directly the boundary expression of the shape derivative, presented in Theorem 4 (see Laurain and Sturm, 2016).

6. Algorithm and numerical results

6.1. Descent direction

DEFINITION 2 *Let $V \in \mathcal{D}^1(\Omega, \mathbb{R}^d)$ and denote by T_t the associated transformation, defined in* (3.2)*. We say that V is a descent direction for a functional $J : \Omega \to \mathbb{R}$ if there exists $\varepsilon > 0$ such that*

$$J(T_t(\Omega)) < J(\Omega) \quad \forall\, t \in (0, \varepsilon).$$

If the Eulerian semiderivative of J at Ω in the direction V exists and if it is a descent direction, then by definition

$$dJ(\Omega; V) < 0.$$

We use descent descent directions in iterative methods to find a possible local minimizers of the functional J. The strategy is to start with the initial shape Ω and compute descent direction V, then we proceed along this direction as long as the cost functional J reduces sufficiently, using a line search strategy.

In our problem the shape ω is known and invariant under a rotation, but its location is unknown. It is meaningful to use translations to move the shape ω.

One may choose a velocity V as a piecewise linear function so that V is a translation on Γ_1, Γ_2 and vanishes on $\partial\Omega$. In order to obtain transformation, which is locally a translation, one may choose the class of vector fields $V = (b_1\zeta, b_2\zeta)^T := (V_1, V_2)^T$, where $b_1, b_2 \in \mathbb{R}$ and ζ is a smooth function equal to one in a neighborhood ω^* of ω and equal to zero on $\partial\Omega$.

The volume expression of the shape derivative can be written as

$$dJ(\omega; V) = \int_\Omega G_1(V_1) + G_2(V2)\, dx = b_1 \int_\Omega G_1(\zeta)\, dx + b_2 \int_\Omega G_2(\zeta)\, dx.$$

Therefore, the descent direction is easily found as

$$b_1 = -\int_\Omega G_1(\zeta)\, dx, \quad b_2 = -\int_\Omega G_2(\zeta)\, dx. \tag{6.1}$$

The reconstruction of the function ζ depends on the geometry of ω and Ω.

The boundary expression of the shape derivative can be written as

$$dJ(\Omega; V) = \sum_{i=1}^{2} \int_{\Gamma_i} g_i V \cdot n\, ds.$$

On the boundary Γ_i, $V = (b_1, b_2)^T$. Then, plugging V with $n = (n_1, n_2)^T$ in the above expression, one obtains

$$dJ(\Omega; V) = b_1 \sum_{i=1}^{2} \int_{\Gamma_i} g_i n_1\, ds + b_2 \sum_{i=1}^{2} \int_{\Gamma_i} g_i n_2\, ds.$$

To get a descent direction, one may choose

$$b_1 = -\sum_{i=1}^{2} \int_{\Gamma_i} g_i n_1\, ds \text{ and } b_2 = -\sum_{i=1}^{2} \int_{\Gamma_i} g_i n_2\, ds. \tag{6.2}$$

For our numerical results, we use the boundary expression of the shape derivative to get the descent directions and the gradient algorithm with backtracking line search to solve the optimization problem.

To avoid the overlap of the inclusions during the optimization process, we follow the routine presented in Loos et al. (2014). To prevent the exit to the external boundary, we introduce a constraint on the distance between the center of the inclusions and the center of the domain Ω. Since we use a gradient-based method, we implement a line search to adjust the time-stepping. The algorithm is stopped when the decrease of the functional becomes insignificant.

6.2. Numerical results

In this section we provide some preliminary numerical results, meant to assess the shape gradient approach. We take Ω to be the unit disk, Ω_2 the disk with radius $r_2 = 0.1$ and Ω_1 the annulus with internal radius r_2 and external radius $r_1 = 0.2$. The conductivity values are set as $\sigma_0 = 1$, $\sigma_1 = 0.5$ and $\sigma_2 = 30$. The term source values are set as $f_0 = 0$, $f_1 = 0$ and $f_2 = 20$. We take the ambient temperature $u_a = 32$ and the heat transfer coefficient $\alpha = 10$. We use pdetoolbox of MATLAB for the implementation. The domain Ω is meshed using 6016 elements.

6.2.1. Example 1

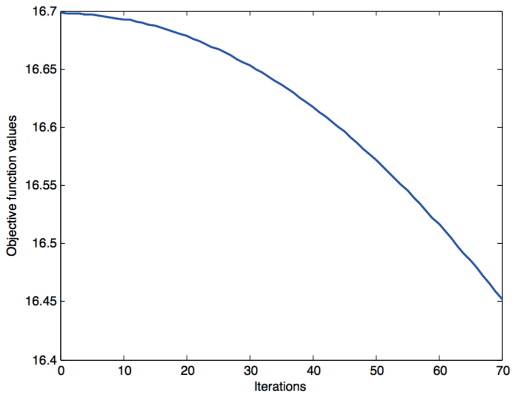
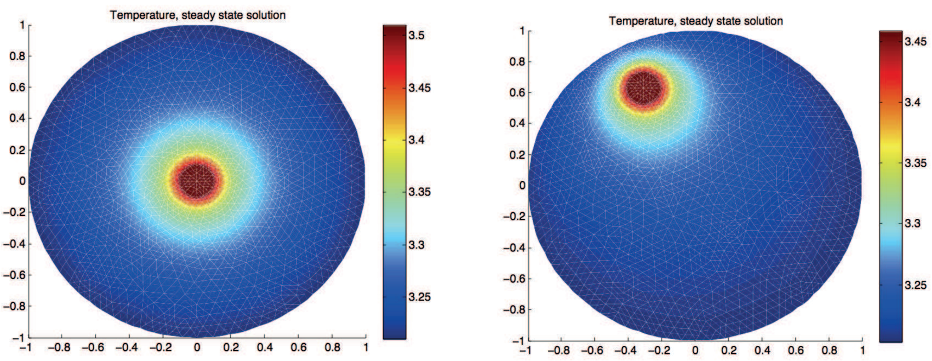


Figure 2. History of the objective function for $p = 2$.

We first optimize the position of only a single inclusion. For the shape functional, we set two different values of p ($p = 2$ and $p = 10$). We start with the origin of the coordinate system as the first position of the inclusion. The objective function is monotonically decreasing with respect to the iterations (see Figs. 2, 4). Obviously, the optimization works very well. During the optimization process, the single inclusion starts from the origin (see Figs. 3a, 5c) and runs out to the external boundary(see Figs. 3b, 5d). Figures 3 and 5 depict the temperature distributions for two different configurations.

6.2.2. Example 2

In the second example, we optimize the position of three inclusions. Starting with the positions $(-0.3, 0), (0, 0.3), (0.3, -0.1)$(see Figs. 7e, 7), the objective function decreases monotonically with respect to the iterations (see Figs. 6, 8) and the inclusions run out to the external boundary again (see Figs. 7f, 9h). They are forced to find a configuration, in which the inclusions have the largest distance from each other. Figures 5 and 9 depict the temperature distributions for two different configurations.



(a) (b)

Figure 3. (a) the temperature distribution of the initial configuration and (b) the temperature distribution for the final configuration ($p = 2$)

Remark 4 *Although with $p = 2$ or $p = 10$ the functional J_p does not approximate the L^∞-norm very accurately, the maximum temperature is lower than at the initial configuration for both cases presented in Examples 1 and 2. The reason is that if the inclusions are near the external boundary, more heat is emitted to the environment by radiation and convection than it is the case if they are located far form the boundary.*

7. Appendix

An abstract differentiability result

In this section, we give an abstract result for differentiating Lagrangian functionals with respect to a parameter. This result is used to prove Theorem 2. We first introduce some notations. Consider the functional

$$G : [0, \varepsilon] \times X \times Y \to \mathbb{R} \tag{7.1}$$

for some $\varepsilon > 0$ and the Banach spaces X and Y. For each $t \in [0, \varepsilon]$, define

$$g(t) = \inf_{x\in X} \sup_{y\in Y} G(t, x, y), \qquad h(t) = \sup_{y\in Y} \inf_{x\in X} G(t, x, y), \tag{7.2}$$

and the associated sets

$$X(t) = \left\{ x^t \in X : \sup_{y\in Y} G(t, x^t, y) = g(t) \right\}, \tag{7.3}$$

$$Y(t) = \left\{ y^t \in Y : \inf_{x\in X} G(t, x, y^t) = h(t) \right\}. \tag{7.4}$$

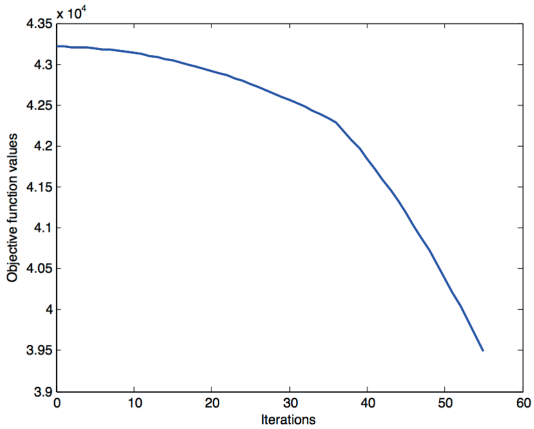


Figure 4. History of the objective function for $p = 10$

Note that inequality $h(t) \leq g(t)$ holds. If $h(t) = g(t)$, the set of saddle points is given by

$$S(t) := X(t) \times Y(t). \tag{7.5}$$

We state now a simplified version of a result from Delfour and Zolésio (1988), derived from Correa and Seeger (1985), which gives realistic conditions that allows for differentiating $g(t)$ at $t = 0$. The main difficulty is to obtain conditions, which make it possible to exchange the derivative with respect to t and the inf-sup in (7.2).

THEOREM 5 *(Correa and Seeger, 1985; Delfour and Zolésio, 2011) Let X, Y, G and ε be given as above. Assume that the following conditions hold :*

(H1) $S(t) \neq \emptyset$ *for* $0 \leq t \leq \varepsilon$.

(H2) The partial derivative $\partial_t G(t, x, y)$ *exists for all* $(t, x, y) \in [0, \varepsilon] \times X \times Y$.

(H3) For any sequence $\{t_n\}_{n \in \mathbf{N}}$, *with* $t_n \to 0$, *there exist a subsequence* $\{t_{n_k}\}_{k \in \mathbf{N}}$ *and* $x^0 \in X(0)$, $x_{n_k} \in X(t_{n_k})$ *such that for all* $y \in Y(0)$,

$$\lim_{t \searrow 0, k \to \infty} \partial_t G(t, x_{n_k}, y) = \partial_t G(0, x^0, y),$$

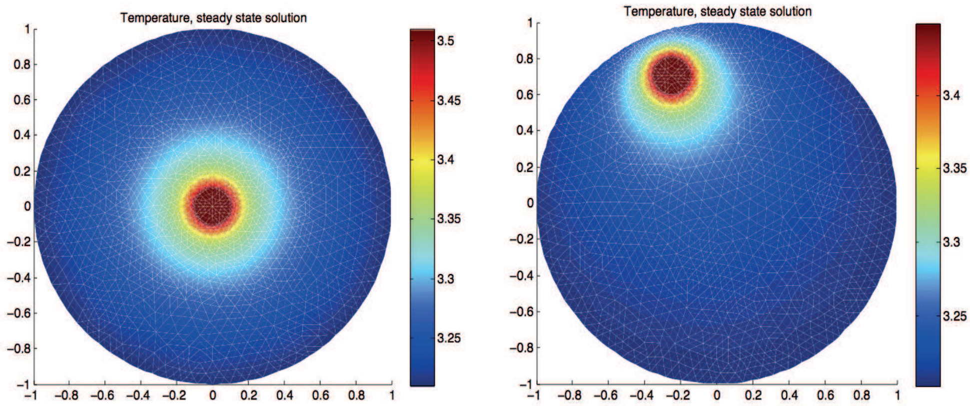
(H4) For any sequence $\{t_n\}_{n \in \mathbf{N}}$, *with* $t_n \to 0$, *there exist a subsequence* $\{t_{n_k}\}_{k \in \mathbf{N}}$ *and* $y^0 \in Y(0)$, $y_{n_k} \in Y(t_{n_k})$ *such that for all* $x \in X(0)$,

$$\lim_{t \searrow 0, k \to \infty} \partial_t G(t, x, y_{n_k}) = \partial_t G(0, x, y^0).$$

Then there exists $(x^0, y^0) \in X(0) \times Y(0)$ *such that*

$$\frac{dg}{dt}(0) = \partial_t G(0, x^0, y^0).$$

Acknowledgement: The authors thank the referees for their helpful remarks and comments, which contributed to the improved quality of the paper.



(c) (d)

Figure 5. (c) the temperature distribution of the initial configuration and (d) the temperature distribution for the final configuration ($p = 10$)

References

Afraites, L., Dambrine, M. and Kateb, D. (2007) Shape methods for the transmission problem with a single measurement. *Numer. Funct. Anal. Optim.*, **28**(5-6): 519–551.

Amigo, R., Giusti, S. M., Novotny, A. A., Silva, E. and Sokołowski, J. (2016) Optimum design of flextensional piezoelectric actuators into two spatial dimensions. *SIAM Journal on Control and Optimization*, **54**(2): 760–789.

Belhachmi, Z., Ben Abda, A., Meftahi, B., and Meftahi, H. (2018) Topology optimization method with respect to the insertion of small coated inclusion. *Asymptotic Analysis*, **106**(2): 99–119.

Céa, J. (1986) Conception optimale ou identification de formes, calcul rapide de la derivée directionelle de la fonction coût. *Math. Mod. and Num. Anal.*, 20: 371–402.

Correa, R. and Seeger, A. (1985) Directional derivative of a minimax function. *Nonlinear Anal.*, **9**(1): 13–22.

Dambrine, M., Harbrecht, H. and Puig, B. (2015) Computing quantities of interest for random domains with second order shape sensitivity analysis. *ESAIM Math. Model. Numer. Anal.*, **49**(5): 1285–1302.

Dambrine, M. and Laurain, A. (2016) A first order approach for worst-case shape optimization of the compliance for a mixture in the low contrast regime. *Struct. Multidiscip. Optim.*, **54**(2): 215–231.

Delfour, M. (2012) *Introduction to Optimization and Semidifferential Calculus.* MOS-SIAM Series on Optimization. Society for Industrial and

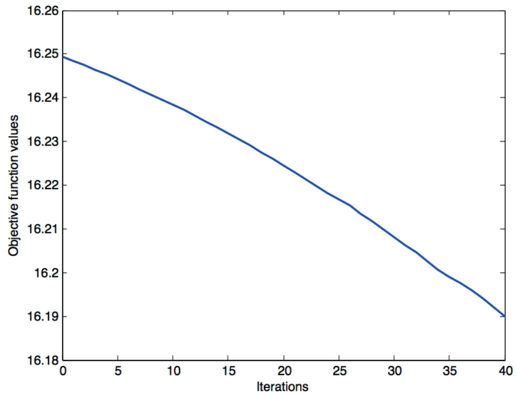


Figure 6. History of the objective function for $p = 2$

Applied Mathematics.

Delfour, M. C. and Zolésio, J.-P. (1988) Shape sensitivity analysis via min max differentiability. *SIAM J. Control Optim.*, **26**(4): 834–862.

Delfour, M. C. and Zolésio, J.-P. (2011) *Shapes and Geometries*. Volume 22 of *Advances in Design and Control*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition.

Ekeland, I. and Temam, R. (1974) *Analyse convexe et problèmes variationnels*. Dunod, Collection Études Mathématiques.

Gangl, P., Langer, U., Laurain, A., Meftahi, H. and Sturm, K. (2015) Shape optimization of an electric motor subject to nonlinear magnetostatics. *SIAM J. Sci. Comput.*, **37**(6): B1002–B1025.

Giusti, S., Mróz, Z., Novotny, A. and Sokołowski, J. (2017) Topology design of thermomechanical actuators. *Structural and Multidisciplinary Optimization*, **55**(5): 1575–1587.

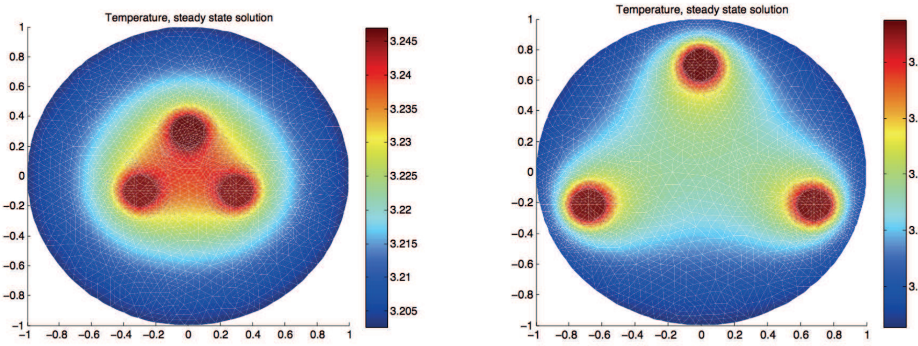
Giusti, S. M., Novotny, A. A. and Sokoł owski, J. (2010) Topological derivative for steady-state orthotropic heat diffusion problem. *Struct. Multidiscip. Optim.*, **40**(1-6): 53–64.

Habbal, A. (1998) Nonsmooth shape optimization applied to linear acoustics. *SIAM Journal on Optimization*, **8**(4): 989–1006.

Harbrecht, H. and Loos, F. (2016) Optimization of current carrying multicables. *Computational Optimization and Applications*, **63**(1): 237–271.

Ito, K., Kunisch, K. and Peichl, G. H. (2008) Variational approach to shape derivatives. *ESAIM Control Optim. Calc. Var.*, **14**(3): 517–539.

Kasumba, H. and Kunisch (2011) On shape sensitivity analysis of the cost functional without shape sensitivity of the state variable. *Control and Cybernet.*, **14**(4): 989–1017.



(e) (f)

Figure 7. (e) the temperature distribution of the initial configuration and (f) the temperature distribution for the final configuration ($p = 2$)

Kasumba, H. and Kunisch, K. (2014) On computation of the shape hessian of the cost functional without shape sensitivity of the state variable. *Journal of Optimization Theory and Applications*, 1–26, DOI 10.1007/S10957-013-0520-4.

Laurain, A. and Sturm, K. (2016) Distributed shape derivative via averaged adjoint method and applications. *ESAIM: Mathematical Modelling and Numerical Analysis*, **50**(4): 1241–1267.

Loos, F., Dvorsky, K. and Liess, H.-D. (2014) Two approaches for heat transfer simulation of current carrying multicables. *Mathematics and Computers in Simulation*, 101: 13–30.

Meftahi, H. (2017) Stability analysis in the inverse Robin transmission problem. *Math. Methods Appl. Sci.*, **40**(7): 2505–2521.

Meftahi, H. and Zolésio, J.-P. (2015) Sensitivity analysis for some inverse problems in linear elasticity via minimax differentiability. *Appl. Math. Model.*, **39**(5-6): 1554–1576.

Novotny, A. A. and Sokołowski, J. (2013) *Topological Derivatives in Shape Optimization. Interaction of Mechanics and Mathematics*. Springer, Heidelberg.

Pantz, O. (2005) Sensibilité de l'équation de la chaleur aux sauts de conductivité. *C. R. Math. Acad. Sci. Paris*, **341**(5): 333–337.

Sokołowski, J. and Zolésio, J.-P. (1992) *Introduction to Shape Optimization*, *Springer Series in Computational Mathematics* **16**. Springer-Verlag, Berlin.

Sturm, K. (2013) Lagrange method in shape optimization for non-linear partial differential equations: A material derivative free approach. Technical Report No. 1817. Weierstrass-Institut f. Angewandte Analysis u. Stochastik, Berlin.

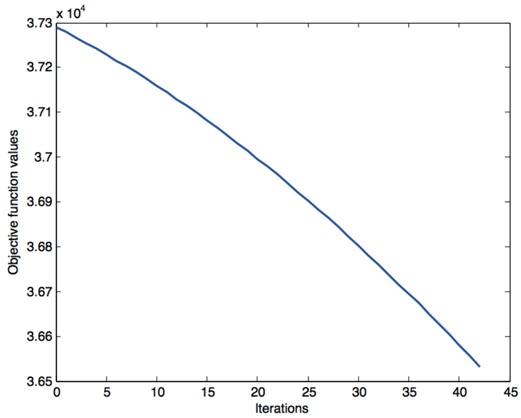
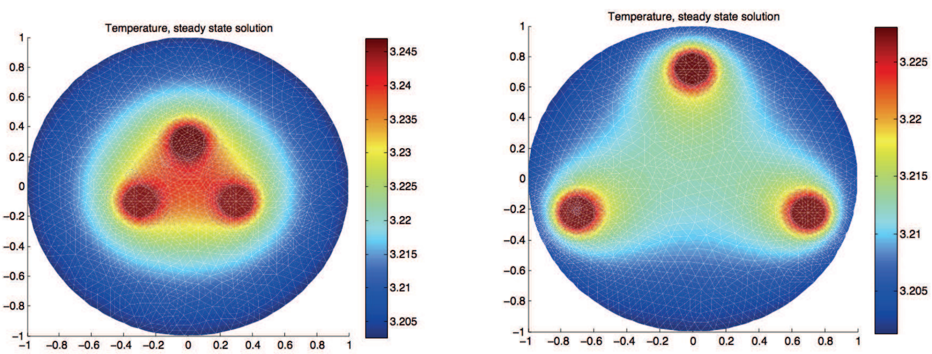


Figure 8. History of the objective function for $p = 10$



(g) (h)

Figure 9. (g) the temperature distribution of the initial configuration and (h) the temperature distribution for the final configuration ($p = 10$)