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Symmetry and sufficient condition of optimality in a domain optimization problem

by

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Abstract: The method of deriving the sufficient condition in a domain optimization problem is given. This method is based on the consideration of symmetry, namely, we take only the shape of domains into account and do not take the place of the domains on the plane into consideration. We apply this method to the two parameter family of domain optimization problems where the objective functions have quadratic and linear parts. For these problems we give the sufficient conditions and show when the unit circle is the solution of the optimization problem. In particular, we obtain that every unit circle is the solution of maximum torsional rigidity problem. Then we consider the general case with an arbitrary function in the integral and derive the sufficient condition which provides that the unit circle is a solution of the optimization problem.

1. Introduction

In this paper we suggest the method of deriving the sufficient condition of optimality in some class of domain optimization problems. Domain optimization problems are problems in which the objective function depends on the domain through the solution of a boundary-value problem defined on the domain. There are many such problems in different branches of science and high-technology industry. When we try to find the best shape of some physical system we come to a domain optimization problem. Various examples can be found in Cea (1981). There exist quite a number of papers devoted to various aspects of domain optimization problem including existence of solution, continuity and differentiability of the objective functional with respect to a domain and necessary conditions of optimality. Appropriate references can be found in Sokolowski and Zolesio (1992). However, effective sufficient conditions in these problems have not been obtained yet. vVe mean that although some particular shape optimization problems are solved the methods of proving optimality in these problems cannot be applied in sufficiently general case. For example, Polya (1948) proved that a circle is an optimal cross section in the problem of maximum torsional rigidity of an isotropic uniform elastic bar. The geometric method of Steiner's symmetrization was applied, which is very specific and, as authors suppose, can not be used widely in domain optimization.

We think that there arc some reasons explaining the difficulties which arise when someone tries to obtain sufficient conditions. If we consider domain optimization problems from the point of view of multidimensional calculus of variations, we see that these problems are Lagrange problems with not fixed domain. The theory of sufficient conditions for such Lagrange problems does not exist yet. If we try to obtain sufficient condition of proper local minimum (maximum) by usual way, we have to establish uniform positive (negative) definiteness of second variation (with respect to appropriate norm). However, almost all domain optimization problems have symmetry because objective functional depends on the shape of domains and not on the place of the domains on a plane. This means that, in any neighborhood of optimal domain, there are other domains having the same shape and providing the same functional value. Therefore, minimmn (maximum) is not proper for these optimization problems and the second variation is not uniformly positiv(' (1wg-ativc) definite and even not positive (negative). The second variation is milY semi-definite. This article is about proving optimality in this case. We haw Iu note here that the sufficient conditions in general case were obtained by Fujii (1994). However, these conditions require positiveness of the second variation and could not be applied to the problems having symmetry. Therefore, at least, the illustrative example in Fujii (1994) was not appropriate. That is why the attempt to apply this condition to the problem of maximum torsional rigidity was unsuccessful. We have managed to find the mistake in the illustrative example. As a result of discussion of this situation the present article appeared.

First we consider following domain optimization problem: the functional

$$J(\Omega; u) \equiv \int_{\Omega} (c_1 u^2(x) + c_2 u(x))\, dx \tag{1}$$

should be maximized (or minimized) with respect to domain D , where c_1, c_2 are given constants and v, is the solution of the boundary-value problem

$$Lv,(x) = -1 \quad (x \text{ E D}), \tag{2}$$

$$v,(x) = 0 \quad (x \text{ E f}). \tag{3}$$

Here D is a smooth simply connected domain in R2 with boundary r of $C^{2,a}$ class and with prescribed area

$$J(D) = \int_{n} dx = 1r \tag{4}$$

When $c_1 = 0$, $c_2 = 1$ we have well-known problem of maximum torsional rigidity of elastic bar (Banichuk, 1976)[1]. In this paper we shall prove that unit circle provides maximum functional value not only for this particular case. If we suppose that $c_1 \neq 0$ we can put $c_1 = 1$ and in this case we will show that the unit circle gives the maximum functional value if $c_1 = 1$ and $c_2 \geq -1/4$. If $c_1 = 1$ and $c_2 < -1/2$ a unit circle yields minimum in this optimization problem. We will show that unit circle is not a solution of the optimization problem provided that $c_1 = 1$ and $-1/2 < c_2 < -1/4$), and our method can help to find the solution of optimization problem in this case.

In Section 4 we consider the general problem with an arbitrary function $g(u)$ in the integral (1). Applying our symmetry method and doing almost the same calculations we derive the sufficient condition which provides that the every unit circle is the solution of the optimization problem. This sufficient condition is very simple for use in applications.

2. Calculus of functional variations

In this section, we shall obtain formula for the first and second Frechet derivative of objective function. Similar calculation has already been carried out in some other papers (see, for example, Fujii, 1990, or Sokolowski and Zolesio, 1992). However, for our purpose we need specific estimate of the rest in the final formula (29) so we give the outline of our calculations. First of all we can apply necessary conditions of optimality (Fujii, 1990) for optimization problem (1)-(4) and see that unit circle satisfies necessary condition for every c_1, c_2. The first order necessary condition is that there exists a constant λ such that

$$g(u) - \frac{\partial p}{\partial n}\frac{\partial u}{\partial n} - \lambda h(x) = 0 \quad (x \in \Gamma), \tag{5}$$

must hold for an optimal domain D with boundary Γ, where $u(x)$ is the corresponding solution of the boundary-value problem (2)-(3) on D, n outward normal, $g(u) = c_1u^2 + c_2u$, $h = 1$ and $p(x)$, the solution of the adjoint problem:

$$\Delta p(x) - k(x)p(x) = \frac{dg}{du}(u(x)) \quad (x \in D), \tag{6}$$
$$p(x) = 0 \quad (x \in \Gamma). \tag{7}$$

Note that the solution of boundary-value problem (2)-(3) for unit circle with center D at the origin is

$$u = \frac{1}{4}(1 - r^2) \tag{8}$$

[1] Banichuk has considered more general problem for anisotropic elastic bar and shown that necessary condition is satisfied for elliptic cross-section. The method of obtaining sufficient condition that we suggest in this paper can be applied for this problem without any serious correction.

in polar coordinate (r, <b>) Note also that this function is defined not only on D but everywhere. By solving the adjoint problem we obtain

$$p = 2c_1\left(\frac{r^2}{16} - \frac{r^4}{64} - \frac{3}{64}\right) - \frac{c_2}{4}(1 - r^2), \tag{9}$$

$$\frac{\partial p}{\partial n}\Big|_{\Gamma} = \frac{c_1}{8} + \frac{c_2}{2}. \tag{10}$$

So we have that necessary condition holds with A= ½(· ¼ + cl) and the unit circle is the critical point for all c1, c2.

Let us introduce a polar coordinate system with its origin at the center of the circle D. At every point x on the boundary Γ we plot segment $h(x)$ in the direction of outward normal. We will consider $C^{2,\alpha}$ functions $h(x)$. If $h(x)$ is sufficiently small the end points of segment form $C^{2,\alpha}$ smooth closed curve Γ_h which encloses domain D_h. Let v_h be the solution of the boundary-value problem:

$$L_1 v_h = -1 \quad (x \in D_h),$$
$$v_h = 0 \quad (x \in \Gamma_h).$$

Let us consider the difference

$$J(D_h, u_h) - J(D, v) = \int_{\Omega_h} g(u_h)\, d\Gamma - \int_{\Omega} g(u(x))\, dx$$

$$= \left(\int_{\Omega_h} g(u_h)\, dx - \int_{\Omega_h} g(v_h)\, dx\right) + \left(\int_{\Omega_h} g(v_h)\, dx - \int_{\Omega} g(v_h)\, dx\right)$$

$$= A_1 + A_2, \tag{11}$$

Now we obtain estimate for A_2.

$$A_2 = \int_0^{2\pi} d\phi \int_1^{1+h(\phi)} rg\left(\frac{1}{4}(1-r^2)\right) dr$$

$$= -\frac{c_2}{4}\int_0^{2\pi} h^2(\phi)\, d\phi + \alpha(h)\int_0^{2\pi} h^2(\phi)\, d\phi, \tag{12}$$

Remark. Here and below $\alpha(h) \to 0$, if $\|h\|_{C^{2,\alpha}} \to 0$. This estimate of the rest is important for our purpose.
In order to estimate the expression A_1 we consider function v_h defined by

$$v_h(x) = u_h(x) - u(x)(x \in \Omega_h). \tag{13}$$

This function is the solution of the following boundary-value problem:

$$L_1 v_h = 0 \quad (x \in D_h), \tag{14}$$

$$v_h = -\frac{1}{4}(1 - (1+h)^2) = \frac{1}{2}h + \frac{1}{4}h^2. \quad (x \in \Gamma_h) \tag{15}$$

Then, from Green's formula we have:

$$
\begin{aligned}
A_1 &= \int_{\Omega_h} \frac{\partial g}{\partial u}(v_1) v_h \, dx + \frac{1}{2} \int_{\Omega_h} \frac{\partial^2 g}{\partial u^2}(v_1) v_h^2 \, dx \\
&= \int_{\Gamma_h} v_h \frac{\partial p_h}{\partial n_h} \, ds + \int_{\Omega_h} c_1 v^2 \, dx, \qquad (16)
\end{aligned}
$$

where p_h is the solution of the boundary-value problem:

$$
\Delta p_h = \frac{\partial g}{\partial u}(u) = 2c_1 u + c_2 \quad (x \in \Omega_h), \qquad (17)
$$

$$
p_h = 0 \quad (x \in \Gamma_h). \qquad (18)
$$

Let us introduce the function F_h as solution of the boundary-value problem:

$$
\Delta F_h = 0 \quad (x \in \Omega_h), \qquad (19)
$$

$$
F_h = \tfrac{1}{2} h \quad (x \in \Gamma_h). \qquad (20)
$$

Then we can rewrite A_1 in the form

$$
\begin{aligned}
A_1 &= \int_{\Gamma_h} F_h \left(\frac{\partial p_h}{\partial n_h} - \frac{\partial p}{\partial n_h}\right) ds + \int_{\Gamma_h} F_h \frac{\partial p}{\partial n_h} \, ds + \frac{1}{4} \int_{\Gamma} h^2 \frac{\partial p}{\partial n} \, ds \\
&+ c_1 \int_{\Omega_h} F_h^2 \, dx + \alpha(h) \int_{\Gamma_h} h^2 \, ds. \qquad (21)
\end{aligned}
$$

We used here the global Schauder's estimate (Gilbarg and Trudinger, 1983) for differences $v_h - F_h$ and $p_h - p$, which satisfies the boundary-value problem:

$$
\Delta(p_h - p) = 0 \quad (x \in \Omega_h), \qquad (22)
$$

$$
p_h - p = -\frac{\partial p}{\partial n} h - \frac{1}{2} \frac{\partial^2 p}{\partial n^2}(\theta) h^2 \quad (x \in \Gamma_h), \qquad (23)
$$

(we used in (23) the mean-value theorem).
Hence, after using interior Schauder's estimate for derivatives $p_h - p + \frac{1}{2}(\ldots + 1)F_h$, global Schauder's estimate and the maximum principle, we have

$$
\int_{\Gamma_h} F_h \left(\frac{\partial p_h}{\partial n_h} - \frac{\partial p}{\partial n_h}\right) ds = -\left(\frac{c_1}{4} + c_2\right) \int_{\Gamma_h} F_h \frac{\partial F_h}{\partial n_h} \, ds + \alpha(h) \int_{\Gamma} h^2 \, ds. \qquad (24)
$$

If we consider function F which satisfies equations:

$$
\Delta F = 0 \quad (x \in \Omega), \qquad (25)
$$

$$
F = \tfrac{1}{2} h \quad (x \in \Gamma), \qquad (26)
$$

we can obtain (using again Schauder's estimates)

$$
\int_{\Gamma_h} F_h \frac{\partial A}{\partial n_h} \, ds = \int_{\Gamma} F \frac{\partial F}{\partial n} \, ds + \alpha(h) \int_{\Gamma} h^2 \, ds. \qquad (27)
$$

Note that p is the fixed function defined by ( ). After simple calculations we obtain:

$$\int_{\Gamma_h} F \frac{\partial p}{\partial n_h} ds = \frac{1}{2}\int_\Gamma \left(\frac{c_1}{8} + \frac{c_2}{2}\right) h\, ds + \frac{1}{2}\int_\Gamma c_2 h^2\, ds + \alpha(h)\int_\Gamma h^2\, ds, \tag{28}$$

where $\alpha(h) \to 0$, as $\|h\|_{C^{2,\alpha}} \to 0$.
Now using (12), (16), (21), (24), (27), (28) we can rewrite the right hand side of formula (11) as

$$J(\Omega_h, u_h) - J(\Omega, u) = \frac{1}{2}\int_\Gamma \left(\frac{c_1}{8} + \frac{c_2}{2}\right) h\, ds + \frac{1}{4}\int_\Gamma c_2 h^2\, ds + c_1 \int_\Omega F^2\, dx$$
$$+ \frac{1}{4}\int_\Gamma \left(\frac{c_1}{8} + \frac{c_2}{2}\right) h^2\, ds - \left(\frac{c_1}{4} + c_2\right)\int_\Gamma F \frac{\partial F}{\partial n}\, ds + \alpha(h)\int_\Gamma h^2\, ds, \tag{29}$$

where $\alpha(h) \to 0$, if $\|h\|_{C^{2,\alpha}} \to 0$.
For functional $I(\Omega)$ the appropriate formula is

$$I(\Omega_h) - I(\Omega) = \int_\Gamma h\, ds + \frac{1}{2}\int_\Gamma h^2\, ds \tag{30}$$

3. Proof of optimality

3.1. New representation of the second derivative

Now we consider again the difference of the objective functions on Ω_h and Ω, where Ω_h satisfies the constraint (4). Then

$$\begin{aligned} J(\Omega_h, u_h) - J(\Omega, u) &= (J(\Omega_h, u_h) - \lambda I(\Omega_h)) - (J(\Omega, u) - \lambda I(\Omega)) \\ &= L(\Omega_h, u_h, \lambda) - L(\Omega, u, \lambda), \end{aligned} \tag{31}$$

where L denotes Lagrange function and λ is an undefined multiplier. Comparing formulas (29),(30) we obtain

$$\lambda = \frac{1}{2}\left(\frac{c_1}{8} + \frac{c_2}{2}\right) \tag{32}$$

$$\begin{aligned} L(\Omega_h, v_h, \lambda) - L(\Omega, v, \lambda) &= c_2 \int_\Gamma F^2\, ds - \left(\frac{c_1}{4} + c_2\right)\int_\Gamma F \frac{\partial F}{\partial n}\, ds \\ &+ c_1 \int_\Omega F^2\, dx + \alpha(h)\int_\Gamma h^2\, ds, \end{aligned} \tag{33}$$

where $\alpha(h) \to 0$, if $\|h\|_{C^{2,\alpha}} \to 0$.

Let us consider the function F as a function of polar coordinates r and φ. Using Fourier series we can rewrite the boundary conditions (25) as follows

$$F(1, \varphi) = \frac{1}{2}h(\varphi) = \sum_0^\infty (a_n \cos n\varphi + b_n \sin n\varphi) \tag{34}$$

The solution of the boundary-value problem (24)-(25) in this case can be represented in the form:

∞
F(r,cp) = I:rn(ancosncp+bnsinncp) (35)
0

and

$$\frac{\partial F}{\partial r}(r,\phi) = \sum_{0}^{\infty} nr^{n-1}(a_n \cos n\phi + b_n \sin n\phi). \quad (36)$$

After simple calculations we can rewrite expression (32) in the form:

$$\begin{aligned} &L(\Omega_h, u_h, \lambda) - L(\Omega, u, \lambda) = a_0^2\pi(c_1 + 2c_2) \\ &+\pi \sum_{1}^{\infty}(\frac{c_1}{2n+2} + c_2 - (\frac{c_1}{4} + c_2)n)(a_n^2 + b_n^2) \\ &+\alpha(h) \int_\Gamma h^2\, ds = a_0^2\pi(c_1 + 2c_2) \\ &+\pi \sum_{2}^{\infty}(\frac{c_1}{2n+2} + c_2 - (\frac{c_1}{4} + c_2)n)(a_n^2 + b_n^2) \\ &+\alpha(h) \int_\Gamma h^2\, ds. \end{aligned} \quad (37)$$

3.2. ·The nearest domain

Let us consider again the unit disc D with the boundary r and introduce 8-neighborhood/of the unit circumference r:

$$B_\delta = \{x = (x_1, x_2) : 1 - \delta \leq (x_1^2 + x_2^2)^{\frac{1}{2}} \leq 1 + \delta\} \quad (38)$$

Let r h E B ,s and let D be an arbitrary unit disc with boundary I' which belongs to $B_{2\delta}$. Now we are able to introduce h which plays for r such a role as h for r. Namely, if we introduce new polar coordinates with the origin at the center of the disc n then h will be the distance between r and r h, Let now f*' h^* be such that

$$\int_{\Gamma^*} (h^*)^2\, ds = \min_{\hat\Gamma \in B_{2\delta}} \int_{\hat\Gamma} (\hat h)^2\, ds \quad (39)$$

Roughly speaking, f* is the boundary of the unit disc D* which is nearest to the domain D_h in some sense. It is easy to see that f*, h^* exist.

Let us introduce other polar coordinates,r*, φ* with the origin at the center of the disc D*. The function h^* has important property:

Lemma 1 *In the Fourier expansion*

$$h^*(\phi^*) = a_0 + \sum_{1}^{\infty}(a_n \cos n\phi^* + b_n \sin n\phi^*), \tag{40}$$

$a_1 = b_1 = 0$ *holds.*

Proof. Suppose the contrary. Then we can consider unit circle D with boundary Γ and with its center at (a_1^*, b_1^*). The boundary Γ is placed in $B_{2\delta}$; this follows from inequalities:

$$0 < a_1^{*2} + b_1^{*2} \le a_1^{*2} + \sum_{1}^{\infty}(a_n^{*2} + b_n^{*2})$$

$$= \frac{1}{\pi}\int_0^{2\pi} h^{*2}\, d\phi^* \le \frac{1}{\pi}\int_0^{2\pi} h^2\, d\phi \le 2\delta^2. \tag{41}$$

We used here property (39) ($\Gamma_h \in B_\delta$) and Parseval's equality.
If we represent the equation of the circumference Γ in the form:

$$r^* = 1 + f(\phi^*), \tag{42}$$

after simple calculations we will obtain:

$$f(\phi^*) = -1 + a_1^*\cos\phi^* + b_1^*\sin\phi^* + (1 - (a_1^*\sin\phi^* + b_1^*\cos\phi^*)^2)^{1/2}. \tag{43}$$

Indeed, the equation of the boundary Γ in the polar coordinates r^*, ϕ^* is given by

$$(r^*\cos\phi^* - a^*)^2 + (r^*\sin\phi^* - b^*)^2 = 1.$$

Solving this equation with respect to r^*, we obtain (43).
Let us rewrite $f(\phi^*)$ in the form

$$f(\phi^*) = g(\phi^*) + A(a_1^*, b_1^*), \tag{44}$$

where

$$g(\phi^*) = a_1^*\cos\phi^* + b_1^*\sin\phi^*, \quad |A| \le 2(a_1^{*2} + b_1^{*2}). \tag{45}$$

Here we used

$$(1+x)^{\frac{1}{2}} \le 1 + \frac{1}{2}|x| \quad (x \ge -1).$$

Note now that $|h^*(\phi^*) - f(\phi^*)|$ is the distance between the points on Γ_h and Γ which is placed on the polar ray. Let us consider the difference

$$\int_0^{2\pi} h^{*2}\, d\phi^* - \int_0^{2\pi}(h^* - f)^2\, d\phi^*$$

$$= \int_0^{2\pi} h^{*2}\, d\phi^* - \int_0^{2\pi}(h^* - g)^2\, d\phi^* - \int_0^{2\pi}(2(h^* - g)A + A^2)\, d\phi^*$$

$$= \pi(a_1^{*2} + b_1^{*2}) + K(a_1^*, b_1^*), \tag{46}$$

where

$$|K| \leq \text{const}(a_1^2 + b_1^2)\delta.$$

We used the formulas (40),(41),(44),(45), Parseval's equality and Holder inequality. Now if we put δ sufficiently small we have that left side of the formula (46) positive, which contradicts optimality. Lemma 1 is thereby proved. ∎

3.3. The sufficient conditions

For any admissible domain D_h the following equality holds

$$\int_{\partial D_h} 1\,ds - \int_{\partial D^*} 1\,ds = \int_0^{2\pi} (h^* + \frac{1}{2}h^{*2})\,d\varphi^* = 0. \tag{47}$$

Then,

$$a_1^2 = \left(\frac{2}{2\pi}\int_0^{2\pi} h^*\,d\varphi^*\right)^2 = \left(\frac{2}{2\pi}\int_0^{2\pi}\frac{1}{2}h^{*2}\,d\varphi^*\right)^2 = a(h)\int_0^{2\pi} h^{*2}\,d\varphi^* \tag{48}$$

Let us consider now $c_1 = 0$, $c_2 = 1$. Note that this case is the classical problem of maximum torsional rigidity. If we take formula (37) and the last formula into account we have

$$\begin{aligned} J(D_h, u_h) - J(D_*, u_*) &= J(D_h, u_h) - J(D_*, u_*) \\ &= L(D_h, u_h, \lambda) - L(D_*, u_*, \lambda) \\ &\leq -\frac{1}{2}(a_1^2 + b_1^2) + a(h)\int_0^{2\pi} h^{*2}\,d\varphi^* \\ &= -\int_0^{2\pi} h^{*2}\,d\varphi^* + a(h)\int_0^{2\pi} h^{*2}\,d\varphi^* < 0 \end{aligned} \tag{49}$$

if δ is sufficiently small, because $a(h) \to 0$. So we have proved that a unit circle is the solution of maximum torsional rigidity problem.

Let now $c_1 \neq 0$, then we can put $c_1 = 1$ without loss of generality. Then from (37),(48) we easily obtain that if $c_2 \geq -\frac{1}{4}$, then

$$\frac{c_1}{2n+2} + c_2 - (\frac{c_1}{4} + c_2)n \leq \frac{1}{2n+2} + c_2 - \frac{1}{4} - c_2 \leq -\frac{1}{12}, \ (n \geq 2) \tag{50}$$

Hence

$$J(D_h, v_h) - J(D_*, v_*) \leq -\frac{1}{12}\sum_{n=2}^{\infty}(a_n^{*2} + b_n^{*2}) + a(h)\int_0^{2\pi} h^{*2}\,d(\varphi) < 0, \tag{51}$$

if h is sufficiently small, and we see that a unit circle provides maximum functional value.

If $c_2 < -\frac{1}{2}$ after similar considerations we have that a unit circle provides minimum functional value. Thus, we have the following theorem

Theorem 1 *Every unit circle attains the local maximum for the functional (1) provided that $c_2 > -\frac{1}{4}$ $(c_1 = 1)$. It attains the local minimum provided that $c_2 < -\frac{1}{2}$ $(c_1 = 1)$. Note that we can put $c_1 = 1$ without loss of generality.*

If $-\frac{1}{2} < c_2 < -\frac{1}{4}$, a unit circle is not a solution of the optimization problem (1)-(4) $(c_1 = 1)$, because members of the sum in (37) have different signs and, in this case, formula (37) will be useful for finding the solution of the optimization problem.

4. The general problem

Now we are ready to derive the sufficient conditions in the following general optimization problem

$$J(\Omega, u) = \int_\Omega g(u(x))\, dx, \tag{52}$$

where $u(x)$ is the solution of

$$\Delta u(x) = -1 \quad (x \in \Omega), \tag{53}$$

$$u(x) = 0 \quad (x \in \Gamma), \tag{54}$$

under the constraint

$$J(\Omega) = \int_\Omega dx = \pi. \tag{55}$$

Without loss of generality we can count that $g(0) = 0$. We will obtain here that if function g satisfies some condition, then a unit circle is the solution of optimization problem.

First of all we will show that a unit circle satisfies the necessary condition (5) for all functions g $(h = 1)$. Indeed, solving the adjoint problem (6)-(7) in the polar coordinates, we have

$$\frac{\partial p}{\partial n}\Big|_\Gamma = 2g\left(\frac{1}{4}\right) \tag{56}$$

Substituting $\frac{\partial p}{\partial n}$ to (5) we obtain that the unit circle satisfies the necessary condition with constant

$$\lambda = g\left(\frac{1}{4}\right). \tag{57}$$

Using the same calculation as in Section 2 we obtain the formulas for the difference of the Lagrange functions:

$$L(\Omega_h, u_h, \lambda) - L(\Omega, u, \lambda) = \int_\Gamma \frac{\partial g}{\partial u} F^2\, ds - 4\int_\Gamma g\left(\frac{1}{4}\right) F \frac{\partial F}{\partial n}\, ds + \frac{1}{2}\int_{\Omega} \frac{\partial^2 g}{\partial u^2} \nabla F^2\, dx + o(h) \int_\Gamma h^2\, ds, \tag{58}$$

where $\alpha(h) \to 0$, if $\|h\|_{C^{2,\alpha}} \to 0$. Using the representation (35) of the function F we can evaluate these integrals (using integration by parts in the third term). After simple calculations we obtain:

$$L(D_h, u_h, \lambda) - L(D, u, \lambda) = -4g(\tfrac{1}{4})\pi \sum_{n \geq 2} n(a_n^2 + b_n^2)$$

$$+2\pi \sum_{n \geq 2} \left(\int_0^1 \frac{\partial g}{\partial u}\Big|_{u=\frac{1-r^2}{4}} r^{2n-1}\, dr \right) n(a_n^2 + b_n^2) + \alpha(h) \int_\Gamma h^2\, ds. \quad (59)$$

Here we used the fact that the first terms (n=1) are equal and have different signs and also that a_0 is small. Using Lemma 1 we obtain the main result of this paper:

Theorem 2 *If for every $n \geq 2$ the following inequalities are satisfied*

$$8\left(\int_0^1 \frac{\partial g}{\partial u}\Big|_{u=\frac{1-r^2}{4}} r^{2n-1}\, dr \right) - 2g(\tfrac{1}{4}) > 0 \; (< 0) \quad (60)$$

then a unit circle is the local minimum (maximum) in the optimization problem (52)-(55).

It is very easy to use this sufficient condition in particular problems. As a matter of fact, if we consider the objective function (1) we immediately see that Theorem 1 is a corollary of Theorem 2.

From Theorem 2 we have a simple but interesting result:

Theorem 3 *Let* $\frac{\partial g}{\partial u} > 0 \; (< 0)$ *for all* $u \in [0, \frac{1}{4}]$. *Then if*

$$4\int_0^{\frac{1}{4}} g(u)\, du < g(\tfrac{1}{4}) \; (> g(\tfrac{1}{4})) \quad (61)$$

every unit circle is the local maximum (minimum) of the optimization problem (52)-(55).

Note that we can obtain from this theorem the result for maximum torsional rigidity mentioned in 3.3.

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