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A formulation for optimal continuum structures with a decomposition of material properties into specified and designable parts[1]

by

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Abstract: A formulation is presented for optimal design in the setting of linear elastostatics for continuum structures, where the modulus tensor field is expressed as a decomposition into a set of independent tensors. For the linear case at hand, net material properties simply equal the sum of the constituent tensor fields. The design variables are comprised of one or more of the these fields, while the remaining ones in the 'mix' are taken to be specified and fixed. The optimal continuum structure is designated to be such combination of designed and specified constituent fields that minimizes structural compliance. A set of unit energies is introduced to serve as a basis for the general expression of both unit cost in the isoperimetric (cost) constraint, and unit strain energy. The design problem is characterized in the form of a max-min problem, where the max applies to design and the (inner) minis w.r.t. the state variables. Partitioning of energies among constituents in the optimal mixture is identified directly from the governing equations.

Keywords: material properties, optimal continuum

1. Introduction

The present model for optimal design of continuum structures follows the sort of formulation wherein the design variable is comprised of the modulus tensor of the structural material itself (see e.g. Bends!ile et al., 1994, 1995). The objective in the studies cited was to predict the optimal modulus field from

[1]This paper is dedicated to Zenon Mroz, creative and prolific in his research and development work, renown contributor to progress in the field of structural optimization.

among unrestricted tensor fields within the set that meets the requirements for qualification as a linearly elastic continuum material. This concept is generalized here to a form that allows the net material properties to be interpreted in terms of a decomposition, i.e the net elasticity tensor is expressed simply as the sum of constituent tensors, each of which is required to qualify independently as a 'material properties tensor field' within classical linear elasticity theory. This is, of course, meaningful for the design problem only where there is purpose to represent distinct features in such individual constituents that combine to create the design. Specifically, the model serves to predict the optimal combination over the structure of two or more constituent material tensors, where one or more constituents are to be designed. Again, both fixed (specified) and design constituents are represented by pointwise varying material properties tensor fields.

The present model for 'design using the concept of a decomposition' is elaborated via the introduction of a generalized form for the isoperimetric or 'cost' constraint. In the original treatments for design of the material tensor this constraint was stated in terms of a single designated invariant of the tensor, e.g. its trace, and accordingly the solution identifies a specific modulus tensor field. In contrast, for the generalized cost model a basis for the expression of the argument of the constraint is established in the form of a linear combination of 'reference strain energies'. The optimal design result predicted from this model depends on the choice of coefficients designated in the linear expression. The dimension of the basis for the linear expression corresponds to the number of independent elements in the arbitrary modulus tensor for linear elastic material. It may be so that the model for optimal design comprises a complete map from 'the statement of unit cost' to 'predicted optimal material tensor'. In the present problem setting, the interpretation just given for the generalized cost applies to *each designable constituent tensor* in the decomposition.

This paper includes a brief description of the construction of the set of reference strain energies, and of its application in a model for linear elastostatics analysis of continuum structures. The model for analysis is incorporated within an expression in maxmin form for the above described design problem. An interpretation of necessary conditions for the design optimization indicates that for the optimal structure, a measure of *unit energy per unit cost* should be rendered (as nearly as possible) uniform over the designable constituent tensors.

2 Establishment of a basis

Our purpose here is to describe the means for the expression in general form for the argument of the isoperimetric or cost constraint, where the constraint is to reflect unit cost associated with a material modulus tensor. This measure is necessarily invariant w.r.t. reference frame, and so one might expect to exploit results from the area of analysis dealing with general tensor invariants to express cost (see e.g. the treatises by Zheng, 1994, and Jemiolo & Telega, 1997). In fact, the present needs are served by a much simpler approach, one that refers to a set of 'unit strain energies' associated with the mechanics of linear elastostatics as a basis. The construction described below for the determination of such a basis follows in slightly modified form the presentations in Taylor & Washabaugh (1995a,b, 1997).

The establishment of an energy basis is described here in a form which has elements of the basis identified as values associated with a given modulus tensor, and specified and fixed reference strains. The reference strains, say η_{ij}^{β}, $\beta = 1, 2, \dots, N$, are comprised of a set of orthogonal, symmetric but otherwise arbitrary second order tensors. The range N of the set equals the number of independent elements in the conventional description of strain state for the elastic continuum, i.e. three or six in 2D or 3D respectively. For example, the system

$$\eta_{ij}^{1,2,3} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \tag{1}$$

comprises a suitable set of reference strains in 2D (the specific values (1) are chosen for convenience). For a given positive definite modulus tensor $E_{ijkl}(x)$, values λ_{ν} are to be evaluated according to:

$$\lambda_{\nu} = E_{ijkl}\eta_{ij}^{\nu}\eta_{kl}^{\nu} \qquad \nu = 1, 2, \dots, N \tag{2}$$

with no sum over ν.

The λ_{ν} of (2), which correspond in value to twice the strain energy associated with material E_{ijkl} at the respective strain η_{ij}^{ν}, are designated as the first N elements of the energy basis. For the 2D example with reference strains (1), these values are

$$\lambda^{1,2,3} = E_{1111};\ E_{1212};\ E_{2222}$$

The remainder of the basis may be established in a similar way. We first identify the symmetric second order tensors ('strains') ς_{ij}^{μ} comprised of all the possible linearly independent combinations of the original reference strains in the form

$$\varsigma_{ij}^{\mu} = a_{\mu}\left(\eta_{ij}^{\alpha} + \eta_{ij}^{\beta}\right) \qquad \alpha \neq \beta; \quad \mu = 1, 2, \dots, C. \tag{3}$$

Constants a_{μ} are chosen so that $\varsigma_{ij}^{\mu}\varsigma_{ij}^{\mu} = 1$, *no sum over* μ, again for convenience. Combinatorics provides that the number C of such elements is given by

$$C = {}^{N}C_{2} = \frac{N(N-1)}{2!}$$

Thus C = 3 for the 2D case, and for η_{ij}^{β} of (1) the three values of ς_{ij}^{μ} are

$$\varsigma_{ij}^{1,2,3} = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \sqrt{\frac{2}{3}}\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}; \sqrt{\frac{2}{3}}\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \right\} \tag{4}$$

The set of (twice the) strain energies associated with tensors (4), namely

$$\hat{\lambda}_{\mu} = E_{ijkl}\hat{\varepsilon}^{\mu}_{ij}\hat{\varepsilon}^{\mu}_{kl} \quad \mu = 1, 2, \ldots, C \quad \text{no sum over } \mu \tag{5}$$

complete the basis. The values $\hat{\lambda}_{\mu}$ for this example are evaluated as:

$$\hat{\lambda}_{1,2,3} = \left\{ \tfrac{1}{4}(E_{1111} + 2E_{2112} + \bar{E}_{2222}); \quad \tfrac{2}{3}(E_{1111} + 2E_{1112} + E_{1212}); \quad (\bar{E}_{2222} + 2E_{1222} + E_{1212}) \right\}$$

From this point on, the complete basis is symbolized simply by $\bar{R}_y$ defined as

$$B_{\gamma} := \left\{ \lambda_{\nu};\ \hat{\lambda}_{\mu} \right\}, \tag{6}$$

and material constitution is fully identified in terms of this basis.

As a preparation to the treatment the combined analysis and design problem, we consider how the reference to a basis system applies in the case where the (net) constitutive tensor is expressd via a decomposition into a set of tensors, i.e. $E_{ijkl} = \sum_{1} E^{f}_{ijkl}$. The admissible constituent terms in this decomposition belong to the set of all representations of an elastic material, valid within the linear continuum model. The dccompositon is to provide the possibility in the design problem to design any one or more of the constituents. The material with net properties E_{ijkl} constituted in this way is a form of mixture, where contituents contribute as they would for a 'system in parallel'. In the case of a thin flat laminated sheet with perfectly bonded laminae, effective properties may be evaluated precisely according to this model of the mixture, where each constituent is simply the modulus tensor for the respective layer (see e.g. Pedersen, 1993). It may be of interest to note that in the setting of the design problems in general, the constituents themselves may be designated to reflect distinct *material form* corresponding to *any admissible constitutive tensor,* the net properties then amounting to a complex of the separate forms. As an example, one might seek to design the optimal combination of an orthotropic, zero shear stiffness relatively stiff material and an isotropic second material.

Given the unique representation of any material modulus tensor in terms of its basis determined as outlined above, the decomposition model can be expressed in terms of a set of such basis energies, each element in the set corresponding to one of the constituents. To demonstrate the expression of linear elastostatics for the mixture in a form that makes use of the bases (the purpose being to facilitate the unified modelling below for optimal design), we describe the calculation of total unit strain energy associated with (an arbitrary) strain field, s_{ay} Eij. Note that strain itself can be expressed in terms of the reference strains T/ta

$$\text{Eij } (x) = \sum_{fJ=1}^{N} \text{Cf } (x) \; T/t- \qquad (7)$$

in this representation coefficients Cf (x) serve to identify the response *strain field* in terms of the (constant) reference strains. Associated unit strain energy *U(x)* for the decomposition model is evaluated in turn as (this element of the modelling is described in Taylor, 1998):

$$\begin{aligned} U(x) &= \frac{1}{2} E_{ijkl}\epsilon_{ij}\epsilon_{kl} = \frac{1}{2}\sum_{s=1}^{N_s} E^s_{ijkl}\epsilon_{ij}\epsilon_{kl} \\ &= \frac{1}{2}\sum_{s=1}^{N_s} E^s_{ijkl}\left(\sum_{\alpha=1}^{N} c_\alpha \eta^\alpha_{ij}\right)\left(\sum_{\beta=1}^{N} c_\beta \eta^\beta_{kl}\right), \qquad (8) \\ &= \ldots := \sum_{s=1}^{N_s}\sum_{6=1}^{N+C} e_\delta B^s_\delta = \sum_{6=1}^{N+C} e_\delta \sum_{s=1}^{N_s} B^s_\delta \end{aligned}$$

Coefficients e6 (x) in this result for net strain energy, quadratic in Cf (x) of expression (7) for strain, now represent *response state* in terms of strain energy. Note from (8) that the description m a y be brought full circle with identification of a 'net basis', B6 = : 1 B f However, to cover meaningful problems in design, constituent material properties must be represented individually in the formulation.

It m a y be demonstrated that the following constrained min problem corresponds (up to a scale factor) to the minimum potential energy characterization for linear elastostatics (this aspect of the formulation is discussed in detail in T a y lor, 1997):

[E] c i!\ { k } ; e6 (ea) B8dV}

subject to

W – {L fiuidV + ht tiuidS } ::; 0

1/2(ui,j + Uj,i) - L CT/0 = 0 X E n

In this expression u symbolizes the admissible deformation displacement field; note that displacement boundary conditions exclude the possibility of rigid body displacement. Bound W on compliance, loads Ji and t_i, reference strains T/i, and bases (constitution) B6, are all data. Elements of the basis are positive valued functions within the region of the structure. We note that problem [E] has the favorable form 'convex quadratic in the objective with linear constraints'.

The equilibrium boundary value problem statement is identified with stationarity conditions for problem [El, where the state variables are *Ca* (x) and *Uk* (x). With the introduction of A and % as the multipliers associated respectively with the constraints of [E], the equilibrium system is stated:

N+C tJ Nâ
L a::: L bJ -. %'r/ij = 0 x E n; a = 1, 2, ... , N (9)
6=1 s=1

%,j + Afi = 0 x E n; i = 1, 2, 3 (10)

%nj + Ati = 0 x E ft; i = 1,2,3 (11)

with *qijnj* = 0 on the remainder of the boundary where displacement is not prescribed. Clearly, the symbol *qij* may be identified as the measure of stress. Accordingly, the first term of (9) is understood to identify the component of total energy associated with the a-th element among the prescribed reference strains 'f/{j; the equation can be recognized in this way as an implicit expression of the constitutive relations. Also from (9) *qij* # 0 almost everywhere, and so the constraint on deformation kinematics is enforced. It follows as well from (10) that A # 0, and so the compliance constraint of [E] is active at the solution.

3. A formulation for optimal design

As indicated in the introduction, in existing treatments for optimization of continuum structures having the form where design is represented by the free material modulus tensor (e.g., Bends0e, et al., 1994, 1995), the isoperimetric or 'cost' constraint is expressed in terms of one choice or another from among the selected invariants of the modulus tensor. The same assumption was made in the extension of such formulations to establish a procedure for the prediction of 'sharp image' (topology design) versions of optimal structures (Guedes and Taylor, 1997). Such formulations in fact amount to examples of the design problem, and clearly there is purpose to consider more general statements of the cost constraints. Where the measure of cost is to be linear in (elements of) the modulus tensor, Washabaugh and Taylor (1995a,b , 1997) describe means to express the cost constraint in general form. Generality here follows from the fact that the expression provides a unique measure for any such given material.

The 'basis of invariants' described in the latter papers has precisely the form of the basis described above, and so for present purposes we simply assert that the generalized cost is expressed as the integral over the structure of an arbitrary linear function within the set of functions described in terms of this basis. Thus for a specified single global bound R on all structural material resource, the cost constraint is stated:

$$\int_{\Omega}\left(\sum_{s=1}^{N_D}\sum_{\delta} b_{\delta}^{s}(x)\, B_{\delta}^{s}(x)\right) dV - R \leq 0 \quad b_{\delta}^{s}(x) > 0 \quad \forall\delta \tag{12}$$

The range $s = 1, 2, \ldots, ND$ covers all *designable constituents* from among the complete set of terms in the decomposition of the net material properties tensor. Variation of cost over the structure is reflected constituent by constituent in the (specified) coefficients *b8*(x) > 0 of the linear expression (12), and these coefficients are limited only to be positive-valued in position coordinates. They may be interpreted to represent componentwise relative unit cost of each constituent material. Given the local constraints on BJ, total *cost per unit volume* lies in the 'first quadrant'. Generality of this form of expression for total cost derives from completeness of the basis sets BJ. As a possible alternative to resource constraint (12), a constituent by constituent designation of resource is achieved with the set of constraints:

L (b;Hx)BJ(x)) dV-Rs ≤ 0 *(bs(x)* > 0\18); *s* = 1, 2, ... Nv(13)

This form is appropriate where the purpose in design is to predict distributions for the optimal compounding over a given set of material tensors, within limits on the 'amount available' for each tensor.

For the present statement of the design problem where the elastostatics and the cost constraint are represented via [E] and (12), the basis elements BJ themselves have the role of 'design variables'. Thus with upper and lower bounds $\overline{B}$; and B'/; on the local value of BJ and for the case of a single global resource constraint, the design problem is stated symbolically in the form:

[D] max B8 min c (x); uk(x) { ∫n L6 eo(c-y) L8 BJ dv}

BJ subject to

fn (:)iBt) dv - R:::; 0 0< B'/; ≤ BJ ≤ $\overline{B}$;

and c-y (x); *Uk* (x) subject to

W -{∫n *f*i*u*i dV + ∫r, *t*i*u*i *dS}* ≤ O *(Ui,j + Uj,i)* - LY *C-y Yij* = 0

The 'max' of total strain energy w.r.t. the basis energies does in fact correspond to the intended objective of minimizing compliance (this is confirmed in

Taylor, 1998). A sufficient condition that the system at the solution is globally stable (has positive strain energy) requires that bounds B8 and B; on design admit at least one positive definite material tensor.

Note that with absent constraints on the basis elements other than BJ 0, and with coefficients b_6 proportional among constituents, problem [D] has the case of the so-called *free material modulus tensor design* imbedded within it. In this case, the 'design constraints' may be stated in the form:

$$\int_{\Omega} \rho dV - R \leq 0$$

0 S Pmin S P S Pmax

where $p =$ 8 s $b_6 B$ $8''$ An analytical treatment for this case, where the argument of the isoperimetric constraint is simply the trace of the modulus tensor, appears in BendsOe et al. (1994). Analysis for the present formulation, with its decomposition of the modulus tensor and having cost expressed in general form, is covered by a slight extension of the earlier treatment. The analysis implies that, in the present setting of a coupled cost constraint, interpreted as indicated above in terms of *p,* an optimal design can be composed locally (pointwise) of one constituent, which will have a positive semidefinite elasticity tensor. Moreover, the associated response (displacement field) will be unique if Pmin > 0.

Maximization in [D] relates, in fact, directly to *local control* of the BJ. Introducing K, RJ and tsJ as multipliers on the isoperimetric constraint and the local upper and lower bounds, respectively, stationarity w.r.t. BJ requires (satisfaction of the 'optimality conditions'; note that where non-unique displacement might arise, the conditions should be interpreted in terms of generalized derivatives.):

$$\left.\begin{array}{l} -e_{\gamma} + \overline{\kappa}_{\delta}^{s} - \underline{\kappa}_{\delta}^{s} + b_{\delta}^{s} K = 0 \\ \overline{\kappa}_{\delta}^{s}\left(B_{\delta}^{s} - \overline{B}_{\delta}^{s}\right) = 0; \quad \overline{\kappa}_{\delta}^{s} \geq 0 \\ \underline{\kappa}_{\delta}^{s}\left(\underline{B}_{\delta}^{s} - B_{\delta}^{s}\right) = 0; \quad \underline{\kappa}_{\delta}^{s} \geq 0 \end{array}\right\} x \in \Omega;\ \gamma \in G_{\delta}^{s} \tag{14}$$

$$K\left[\int_{\Omega} \sum_{\delta} \sum_{s} b_{\delta}^{s} B_{\delta}^{s} dV - R\right] = 0 \quad K \geq 0 \tag{15}$$

G6 identifies the set of all values gamma. It is assumed that the data value R lies in the range such that at least one among the design variables BJ satisfies B8 < BJ < B;, i.e. lies off the local constraints at least somewhere in D. Identifying such intervals (design regions) and the set of associated indices, respectively, by DD$_8$ and GJJ$_8$, yields, from (14)

$$e_{\delta} = b_{\delta}^{s} K \quad x \in \Omega_{D\delta}^{s}; \quad \gamma \in G_{D\delta}^{s} \tag{16}$$

Since *b8* > 0 for all gamma and over the entire structure, if K = 0 then according to (16) e_6 = 0 for all *8* E GDo· This implies in turn that the optimal structure includes components *B8* > *B6* with positive cost and zero contribution to the objective of 'max', and this, of course, is a contradiction. Accordingly, the solution value of K satisfies K > 0, and so at the solution e_0 > 0 \:*k* E *DD*6; 8 E Gb, and also from (15):

$$\int_{\Omega} \sum_{\delta} \sum_{s} b_{\delta}^{s} B_{\delta}^{s} dV - R = 0 \qquad (17)$$

4. Summary

Problem formulation [D] can be interpreted for the design of continuum structures having separately identified, designable and fixed constituent material properties. The simple mixture-model used in the formulation does not generally admit meaningful explicit physical interpretation, beyond what is facilitated by the use of the concept of 'effective volume'. However, the results obtained as a solution to [D] may be subject to an additional procedure by which the individual constituents become spatially separated; this procedure has been applied in a narrower problem setting (Guedes & Taylor, 1997a,b) to predict optimal topology for single-material continuum structures.

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