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Book review:

STOCHASTIC AND DIFFERENTIAL GAMES: THEORY AND NUMERICAL METHODS

by

M. Bardi, T. Parthasarathy and T.E.S. Raghavan (editors)

The book is a collection of papers organised into Volume 4 of the *Annals of the International Society of Dynamic Games* (1999). Bardi, Raghavan and Parthasarathy, the invited editors, have done a fine job in collecting ten papers on topics of interest to the dynamic games community and to mathematicians interested in optimisation. Bardi is the first four papers' editor, while Raghavan and Parthasarathy have put together the remaining six. The volume is dedicated to the memory of the late Russian mathematician Andrei I. Subbotin, an eminent contributor to the theory of partial differential equations, optimal control and differential games.

Dynamic games is an emerging discipline of science at an intersection of economics, control engineering and mathematics. However, this volume was written by mathematicians, and mainly for mathematicians. Still, some papers contain a potential for the economic and engineering applications and I will hint to them in the review.

Part I, edited by Bardi, is on (two-person) zero-sum differential games. Such games were first studied by Isaacs. One of his main contributions was the (heuristic) derivation of the fact that the *value* function of a zero-sum differential game must satisfy, wherever smooth, a Hamilton–Jacobi, also called Isaacs, equation. However, the value does not have to be a smooth function. For such a situation, a *viscosity* solution of the equation needs to be considered. In general, viscosity solutions are the correct class of generalised solutions to first and second order elliptic and parabolic partial differential equations. The Hamilton–Jacobi equation is such an equation, i.e., it is partial differential and "fully" non-linear. It is henceforth easy to understand the interest of the differential games' community in viscosity solutions.

The notion of viscosity solution is due to Lions (the 1980s). Interestingly, also in the eighties, Subbotin obtained a *minimax* solution to the Hamilton–Jacobi equation. It was later recognised that viscosity and minimax solutions are equivalent in most cases. A third method of solving differential games, which is based on differential inclusions, is also provided in Part I.

In summary, all four chapters of Part I are concerned with the solution of the Isaacs equation. They are largely self-contained and cover a wide range of the mathematical literature on the subject.

The first paper is by the late Subbotin and is about the theory of feedback (or positional) minimax solutions to time-optimal and fixed-horizon zero-sum differential games. Constructive solution methods and numerical procedures are developed and tested on a few examples. An extensive bibliography of more than 70 papers on partial differential equations and zero-sum differential games is provided.

The second chapter, written by Souganidis, is probably the most accessible to a non-specialist interested in zero-sum differential games. First, the author explains the theory of viscosity solutions of parabolic and elliptic partial differential equations. Then, he uses this theory for the solution of differential games. The exposure is complete in that the stochastic differential games are also treated (albeit briefly).

The third paper is by Bardi himself, co-authored by Falcone and Soravia. It is on numerical methods for pursuit-evasion games solved through viscosity solutions. Pursuit-evasion games are a class of zero-sum differential games especially important for military and engineering applications. The authors thoroughly review these games and then use the viscosity solution approach to solve such games with state constraints. A discretisation scheme useful for dynamic programming is developed, with a proof of convergence and estimates of the error. Numerical tests on several examples including the classical "Homicidal Chauffeur Game" are provided.

The fourth and final chapter in this part, devoted to zero-sum differential games is by Cardaliaguet, Quincampoix and Saint-Pierre. They use a set valued concept for the solution of an optimal control problem, which they subsequently extend to a differential game. The concept is to represent an optimal control problem (or a differential game problem) through a differential inclusion. The computational solution methods are based on numerical approximations of the so-called viability kernels for control problems, or discriminating kernels for games. For a control problem, a *viability* kernel is a key notion borrowed from *viability* theory. In broad terms, a viability kernel is a collection of the state space points, from which certain targets are always achieved. For a differential game, a discriminating kernel is introduced with a similar interpretation. The authors solve *qualitative* problems, such as finding the victory domain for each player, as well as *quantitative* problems, such as computing the minimal hitting time in a pursuit-evasion game. The qualitative problems, both control- and game-theoretic seem of much relevance to a central bank problem of controlling inflation within a prescribed band. Included are a number of interesting numerical examples that illustrate the solution to a range of typical control and game problems.

Part II of the volume is entitled "Stochastic and Nonzero-Sum Games and Applications". Perhaps, the title is a bit misleading. The games considered in this part are mainly *finite* (the paper by Nowak and Szajowski is an exception). A general remark about the coming chapters is that they are much less homogeneous in topic selection than the papers in Part I.

The first paper in this part (Chapter 5) is on gambling theory and its connections with stochastic games. The authors, Maitra and Sudderth, solve a few typical gambling problems (e.g., red-and-black). Then, they demonstrate how the gambling theory solution techniques can be applied to two-person zero-sum stochastic finite games. A brief note on the extensions of results to infinite games is also provided.

In Chapter 6, a complex analytic perspective to discounted stochastic matrix games is presented by the team of Connell, Filar, Szczechla and Vrieze. The paper speaks of dynamic games in the context of some classical works and ideas of such famous mathematicians as Newton, Hadamard and Riemann (to name a few of them referred to in the paper). The teams' objective is to show that the proper tool for the analysis of the limit discount equations (Bellman-related) of stochastic games is complex analysis (in particular, complex analytic varieties). Indeed, in the real domain not every smooth function is real analytic. By moving the function argument to the complex domain one finds that such anomalies do not occur. It is in the complex domain where the authors prove the classical result that the solutions to the limit discount equations of stochastic games are given by the Puiseux series (i.e., fractional power series) in the discount factor, when it is close to 1.

Before I move on to the discussion of Chapters 7, 8 and 10, which, unlike the other papers, deal with non zero-sum stochastic games, I will comment on Chapter 9. Its author, Altman, considers a dynamic optimal routing problem with two queues. A controller has to decide to which queue should an arriving customer be sent. The controller seeks a minimax strategy that guarantees the minimum cost under the worst case conditions. Altman models this problem as a zero-sum Markov game between the "server" and the "router". A few new structural results concerning the optimal policies and the value function are obtained in the paper for the finite and infinite horizon cases. In broad terms, the monotonicity properties of the value function were proved and the monotone switching-curve character of pure Markov policies was established.

As said above, the remaining three chapters differ from the rest of the volume in that their interest is in non-zero-sum games. This is the area of dynamic games where the existence results are scarce. At the same time, a large number of economic and managerial conflict problems can only be modelled as non zero-sum dynamic games. So, there is a "demand" for the results reported in these papers.

In Chapter 7, Nowak and Szajowski define a general model for a stochastic Markov game where the state space is measurable. It is still an open problem whether such games have stationary equilibrium solutions. A positive answer for the discounted games is known for some special cases where transition rules and players' payoff satisfy certain conditions, like, e.g., separability and supermodularity, respectively. First, Nowak and Szajowski report on their work on the correlated equilibrium notion and apply it to the solution of discounted and non-discounted games. Next, stopping games, like in the two-firm extension to the *secretary problem* are extensively treated. However, it is the correlate equilibrium that in my view has more real life applications. It is a Nash kind of equilibrium in strategies that depend on an *extended* state. The extended state comprises the "natural" state of the game and a *public signal*, which is read by all the players. Nowak, who introduced this equilibrium concept elsewhere calls it a *correlated equilibrium with public signals.* The reason is that after the outcome at any period of the game, the players can coordinate their next choices by exploiting the next *public* signal and using some coordination mechanism to tell which action is to be played by them. Under quite mild conditions, every non zero-sum discounted stochastic game has a correlated equilibrium stationary solution; under more conditions, even non discounted games can be solved using this equilibrium concept. All these conditions and theorems are thoroughly documented by Nowak and Szajowski. The chapter finishes with a huge bibliography on many facets of dynamic games, containing more than 100 papers.

Thuijsman and Vrieze, in Chapter 8, write on the power of threats in stochastic games. They, too, are considering non zero-sum games. However, their results apply to finite games (in state and actions) rather than the infinite ones, as was the case of the previous chapter. The authors point to the fact that applying threats in stochastic games might be inefficient. This is because a stochastic disturbance may take the game to a state, from which punishment has no effect on players. Henceforth, an equilibrium strategy based on a threat might not exist in a stochastic game. However, Thuijsman and Vrieze provide sufficient conditions, under which limiting average ε-equilibria exist.

The final paper is by Mohan, Neogy, and Parthasarathy (one of the volume's editors) on linear complementarity and discounted polystochastic games. This is a short report on their results published and proved elsewhere. The main conclusion is that computing a stationary strategy for a player, in a game where the transition probabilities depend on the action of a single player, amounts to the solution of a linear complementarity problem using Lemke's algorithm.

Few criticisms, mainly about the volume editing, have to be listed. The title promises more than the volume delivers. In particular, zero-sum games are discussed much more extensively than the non zero-sum ones. The bibliographic items are quite diversely referenced: numbers, letters and many combinations of both are used. Nowak and Szajowski's bibliography is ordered according to a non-decipherable criterion. This means that the volume does not score very well as an easy introduction of the reader to the literature. Furthermore, the volume could have been a better reference book if it contained an index. As all papers are written in LaTeX (good!), it would be easy to construct an index by asking the authors to tag the index entries by the `\index` command. Finally, although all papers keep their focus sharp on the issues announced in their titles, the papers' very disparate lengths hampered my reading. The shortest paper counts 4 pages, the longest 72 pages. However, the book's value is in collecting and presenting the most important facts and features, discussed above, about dynamic games and is indisputable. It outweighs by far the criticisms so, I can recommend this volume of the *Annals of the International Society of Dynamic Games* as a reliable source of the state-of-the-art knowledge on (mostly zero-sum) stochastic and differential games.

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