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Exact controllability for the semilinear string equation in non cylindrical domains

by

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Abstract: In this paper, we investigate the exact controllability for a mixed problem for the equation

$$u'' - \left[\frac{\tau_0}{m} + \frac{k}{m}\frac{\gamma(t) - \gamma_0}{\gamma_0}\right] u_{xx} + f(u) = 0$$

in a non cylindrical domain. This model, without the resistance represented for $f(u)$, is a linearization of Kirchhoff's equation for small vibrations of a stretched elastic string when the ends are variables, see Medeiros, Limaco, Menezes (2002). We employ a variant, due to Zuazua (1990b), of the Hilbert Uniqueness Method (HUM), idealized by Lions (1988a, b).

Keywords: exact controllability, string equation, semilinear, non cylindrical.

1. Introduction

Let us consider a stretched elastic string with ends $\alpha_0 < \beta_0$ on the x axis, with $a < \alpha_0 < \beta_0 < b$, and fixed a, b. Suppose that the ends α_0 and β_0 move continuously to the position $\alpha(t) < \alpha_0$ and $\beta_0 < \beta(t)$, where $a \leq \alpha(t) < \beta(t) \leq b$, and we consider the transversal vibrations of the string at the position $]\alpha(t), \beta(t)[$. In Medeiros, Limaco, Meneses (2002), they obtained a nonlinear model which describes this type of vibrations with moving ends, which contains the Kirchhoff's model as a particular case, see Medeiros, Limaco, Meneses (2002) part 2. When we linearized it, we obtained the equation

$$u'' - \left[\frac{\tau_0}{m} + \frac{k}{m}\frac{\gamma(t) - \gamma_0}{\gamma_0}\right] u_{xx} = 0,$$

where τ_0 is the initial tension, m the mass of the string, k is the constant which depends on the material of the string, $\gamma(t) = \beta(t) - \alpha(t)$, $\alpha_0 = \alpha(0)$, $\beta_0 = \beta(0)$ and $\gamma_0 = \gamma(0)$. By $u' = u'(x,t)$ we represent the derivative $\dfrac{\partial u}{\partial t}$ and by $u_{xx} = u_{xx}(x,t)$ the one dimensional Laplace operator $\dfrac{\partial^2 u}{\partial x^2}$.

We consider in our investigation the above model perturbed by a nonlinear term of the type $f(u)$. For $T > 0$, we denote by $\widehat{Q}$ the non cylindrical domain of the plane $\mathbb{R}^2$ defined by

$$\widehat{Q} = \left\{(x,t) \in \mathbb{R}^2;\ \alpha(t) < x < \beta(t),\ \forall t \in]0,T[\right\},$$

with lateral boundary $\widehat{\Sigma}$ defined by

$$\widehat{\Sigma} = \widehat{\Sigma}_0 \cup \widehat{\Sigma}_1,$$

where

$$\widehat{\Sigma}_0 = \{(t,\alpha(t));\ \forall t \in]0,T[\} \quad \text{and} \quad \widehat{\Sigma}_1 = \{(t,\beta(t));\ \forall t \in]0,T[\}.$$

In this work, we shall consider the mixed problem

$$\left| \begin{array}{l} u'' - \left[\dfrac{\tau_0}{m} + \dfrac{k}{m}\dfrac{\gamma(t)-\gamma_0}{\gamma_0}\right] u_{xx} + f(u) = 0 \quad \text{in} \quad \widehat{Q}, \\ u = \left| \begin{array}{lll} \varphi & \text{on} & \widehat{\Sigma}_0 \\ 0 & \text{on} & \widehat{\Sigma}_1 \end{array} \right., \\ u(0) = u_0, \quad u'(0) = u_1 \qquad \text{in} \quad]\alpha_0,\beta_0[. \end{array} \right. \tag{1}$$

The exact controllability problem for (1) is formulated as follows: given $T > 0$ large enough, find a Hilbert space H such that for each pair of initial and final data $\{u_0,u_1\}$, $\{z_0,z_1\}$ belonging to H, there exists a control φ in $L^2\left(\widehat{\Sigma}_0\right)$, such that a solution $u = u(x,t)$ of (1) satisfies the condition

$$u(T) = z_0, \quad u'(T) = z_1. \tag{2}$$

Considering in (1), $f(s) = \sigma s$, we investigate this problem by mean of Hilbert Uniqueness Method (HUM) idealized by Lions (1988a, b). See also Zuazua (1990a), Komornik (1994) and Milla Miranda (1995) among others. In Milla Miranda (1995), the author studied the exact controllability of (1), without the linear non term in a particular domain. When we employ HUM, we need certain inequality called "inverse inequality". In this one dimensional case, there is an argument used by Zuazua (1990a) which we consider here, see 3.2. The argument was used by Medeiros (1993), when he investigated exact controllability for Timoshenko's system for beams.

In the general case, with f non linear, we employ a method idealized by Zuazua (1990b) that consists in the fix point argument.

Concerning the simultaneous controllability of a pair of linear hyperbolic

2. Notations, assumptions and results

As it was done in Medeiros, Limaco, Menezes (2002), when (x,t) varies in $\widehat{Q}$ the point (y,t), with $y = \dfrac{x-\alpha(t)}{\gamma(t)}$, varies in $Q =]0,1[\times]0,T[$. Then the application

$$\mathcal{T} : \widehat{Q} \to Q, \quad \mathcal{T}(x,t) = (y,t)$$

is of class C^2 and the inverse $\mathcal{T}^{-1}$ is also of class C^2. Therefore the change of variable $u(x,t) = v(y,t)$, with $y = \dfrac{x-\alpha(t)}{\gamma(t)}$, transforms the equation $(1)_1$ into the equivalent equation

$$v'' - [a(y,t)\, v_y]_y + b(y,t)\, v'_y + c(y,t)\, v_y + f(v) = 0,$$

where

$$a(y,t) = \frac{1}{\gamma^2(t)} \left\{ \frac{\tau_0}{m} + \frac{k}{m} \frac{\gamma(t)-\gamma_0}{\gamma_0} - [\alpha'(t) + \gamma'(t)\, y]^2 \right\},$$

$$b(y,t) = -2 \left[\frac{\alpha'(t) + \gamma'(t)\, y}{\gamma(t)} \right]$$

and

$$c(y,t) = -\left[\frac{\alpha''(t) + \gamma''(t)\, y}{\gamma(t)} \right].$$

In this way, it is enough to investigate the exact controllability for the equivalent problem

$$\left| \begin{array}{lll} v'' - [a(y,t)\, v_y]_y + b(y,t)\, v'_y + c(y,t)\, v_y + f(v) = 0 & \text{in} & Q, \\ v(0,t) = w(t), \quad v(1,t) = 0 & \text{on} &]0,T[, \\ v(x,0) = v_0(x), \quad v'(x,0) = v_1(x) & \text{in} &]0,1[. \end{array} \right. \tag{3}$$

To study (3), we need the hypotheses

$$\begin{array}{l} \alpha, \beta \in W^{3,1}(]0,\infty[) \cap W^{3,\infty}_{\text{loc}}(]0,\infty[), \\ \qquad \alpha(t) < \beta(t),\ \alpha'(t) < 0 < \beta'(t),\ \forall t \in [0,T], \end{array} \tag{4}$$

$$|\alpha'(t) + \gamma'(t)\, y| \le \left(\frac{\tau_0}{2m} \right)^{\frac{1}{2}}, \quad \forall (y,t) \in Q, \tag{5}$$

$$|\alpha''(t) + \gamma''(t)\, y| < \frac{[\alpha'(t) + \gamma'(t)\, y]^2}{\gamma(t)}, \quad \forall (y,t) \in Q \tag{6}$$

and f satisfying

$$f' \in L^\infty(\mathbb{R}); \quad \exists \lim \frac{f(s)}{s} = \sigma, \tag{7}$$

this is, f is asymptotically linear since it behaves like σs as $|s| \longrightarrow \infty$.

The problem (3) has a unique global solution. So, there exists a unique global solution for the problem (1).

Setting

$$k_0 = \frac{\tau_0 \gamma_0^2}{32m\,(b-a)^4}, \tag{8}$$

we obtain

$$a\,(y,t) - \sqrt{k_0}\,|b\,(y,t)| \geq k_0 > 0, \quad \forall\,(y,t) \in Q. \tag{9}$$

In fact, from (5), we get

$$a\,(y,t) \geq \frac{\tau_0}{2m\,(b-a)^2} \quad \text{and} \quad |b\,(y,t)| \leq \frac{2}{\gamma_0}\sqrt{\frac{\tau_0}{2m}}.$$

Therefore,

$$a\,(y,t) - \sqrt{k_0}\,|b\,(y,t)| \geq \frac{\tau_0}{2m\,(b-a)^2} \geq k_0 > 0, \quad \forall\,(y,t) \in Q.$$

We have also by (6) that

$$b_y'\,(y,t) - c_y\,(y,t) \geq \mu_0 > 0, \quad \forall\,(y,t) \in Q, \tag{10}$$

see Medeiros, Limaco, Menezes (2002), Remark 3.2.

3. Exact controllability

We are interested in obtaining the exact controllability of the following problem

$$\left|\begin{array}{lll} v'' - [a\,(y,t)\,v_y]_y + b\,(y,t)\,v_y' + c\,(y,t)\,v_y + f\,(v) = 0 & \text{in} & Q, \\ v\,(0,t) = w\,(t), \quad v\,(1,t) = 0 & \text{on} &]0,T[, \\ v\,(y,0) = v_0, \quad v'\,(y,0) = v_1 & \text{in} &]0,1[. \end{array}\right. \tag{11}$$

We announce the main result.

THEOREM 3.1 *We assume the hypotheses (4)-(7) are satisfied. Let* $T > \dfrac{2}{\sqrt{k_0}}$, *$k_0$ given by (8), then for every initial data* $\{v_0, v_1\} \in L^2\,(0,1) \times H^{-1}\,(0,1)$, *there exists a control* $w \in L^2\,(0,T)$ *such that the solution* $v = v\,(y,t)$ *of (11) satisfies*

$$v\,(y,T) = z_0, \quad v'\,(y,T) = z_1 \quad in \quad]0,1[\tag{12}$$

Proof. In the proof of this theorem we will use HUM. We will divide the proof into cases.

Case 1. The linear case $f(s) = \sigma s$, $\forall s \in \mathbb{R}$ with σ fixed in $\mathbb{R}$.

Let us consider the operator

$$Lv = v'' - \left[a(y,t)\, v_y\right]_y + b(y,t)\, v'_y + c(y,t)\, v_y + \sigma v \tag{13}$$

whose formal adjoint is

$$\begin{aligned} L^* z \;=\;& z'' - \left[a(y,t)\, z_y\right]_y + b(y,t)\, z'_y + \left[b'(y,t) - c(y,t)\right] z_y + \\ &+ b_y(y,t)\, z' + \left[b'_y(y,t) - c_y(y,t)\right] z + \sigma z. \end{aligned} \tag{14}$$

By the linearity and reversibility of this case, we consider the null final data.

Step 1. Given $\{\phi_0, \phi_1\} \in \mathcal{D}(0,1) \times \mathcal{D}(0,1)$, we consider the adjoint problem

$$\left| \begin{array}{lll} L^*\phi = 0 & \text{in} & Q, \\ \phi(0,t) = \phi(1,t) = 0 & \text{on} &]0,T[, \\ \phi(x,0) = \phi_0(x), \quad \phi'(x,0) = \phi_1(x) & \text{in} &]0,1[. \end{array} \right. \tag{15}$$

This problem has only one solution. Furthermore

$$\phi_y(0,t) \in L^2(0,T).$$

Step 2. Using the solution ϕ of the problem (15), we consider the backward problem

$$\left| \begin{array}{lll} L\psi = 0 & \text{in} & Q, \\ \psi(0,t) = -\phi_y(t), \quad \psi(1,t) = 0 & \text{on} &]0,T[, \\ \psi(x,T) = \psi'(x,T) = 0 & \text{in} &]0,1[. \end{array} \right. \tag{16}$$

The problem (16) is well set.

The operator Λ. Starting from the solution of (16), we define the operator

$$\{\phi_0, \phi_1\} \mapsto \Lambda\{\phi_0, \phi_1\} = \{\psi'(0) + b(0)\, \psi_y(0), -\psi(0)\}. \tag{17}$$

Step 3. Multiplying both sides of $(16)_1$ by ϕ and integrating in Q, we obtain

$$\begin{aligned} \langle L\psi, \phi\rangle =& -\langle \psi'(0) + b(0)\, \psi_y(0), \phi_0\rangle + (\psi(0), \phi_1) + \\ &+ \int_0^T a(0,t)\, |\phi_y(0,t)|^2\, dt + \langle \psi, L^*\phi\rangle. \end{aligned}$$

Observing $(15)_1$, $(16)_1$ and (17), then

$$\langle \Lambda\{\phi_0, \phi_1\}, \{\phi_0, \phi_1\}\rangle = \int_0^T a(0,t)\, |\phi_y(0,t)|^2\, dt \tag{18}$$

Let us define in $\mathcal{D}(0,1) \times \mathcal{D}(0,1)$ the quadratic form

$$\|\{\phi_0, \phi_1\}\|_F^2 = \int_0^T a(0,t) |\phi_y(0,t)|^2 \, dt. \tag{19}$$

It follows from Holmgren's theorem that the quadratic form (19) defines a norm in $\mathcal{D}(0,1) \times \mathcal{D}(0,1)$. The operator Λ defined by (17) is linear and continuous with the norm $\|\cdot\|_F$. Then it has a unique continuous extension to the closure of $\mathcal{D}(0,1) \times \mathcal{D}(0,1)$,with respect to $\|\cdot\|_F$, which we will denote by F. Thus, the bilinear form defined by (18) is continuous and coercive in $F \times F$. Hence, as consequence of the Lax-Milgram's theorem, the operator

$$\Lambda : F \to F'$$

is an isomorphism. Therefore, for $\{v_1, v_0\} \in F'$, there exists a unique $\{\phi_0, \phi_1\} \in F$ such that

$$\Lambda \{\phi_0, \phi_1\} = \left\{ v_1 + b(0) (v_0)_y, -v_0 \right\} \quad \text{in} \quad F'. \tag{20}$$

By (17) and (20) we conclude that the unique solution of (16)satisfies $(11)_1$. Then the unique solution of (11), with control

$$w(0,t) = -\phi_y(0,t) \tag{21}$$

satisfies (12), with $z_0 = z_1 = 0$. To complete the controllabibility of the problem (11), we characterize the spaces F and F' as being $H_0^1(0,1) \times L^2(0,1)$ and $H^{-1}(0,1) \times L^2(0,1)$, respectively. We will do it by the following lemmas:

LEMMA 3.1 *There exists a constant $C^* > 0$ such that*

$$C^* \int_0^T a(0,t) |\phi_y(0,t)|^2 \, dt \le \|\phi_0\|_{H_0^1(0,1)}^2 + \|\phi_1\|_{L^2(0,1)}^2 . \tag{22}$$

LEMMA 3.2 *We assume the hypothesis of Theorem 3.1. Then, there exists a constant $C^{**} > 0$ such that*

$$\|\phi_0\|_{H_0^1(0,1)}^2 + \|\phi_1\|_{L^2(0,1)}^2 \le C^{**} \int_0^T a(0,t) |\phi_y(0,t)|^2 \, dt. \tag{23}$$

Assuming the previous lemmas are true, we have that

$$\frac{1}{C^{**}} \|\{\phi_0, \phi_1\}\|_{H_0^1(0,1) \times L^2(0,1)}^2 \le$$
$$\int_0^T a(0,t) |\phi_y(0,t)|^2 \, dt \le \frac{1}{C^*} \|\{\phi_0, \phi_1\}\|_{H_0^1(0,1) \times L^2(0,1)}^2 .$$

Therefore $F = H_0^1(0,1) \times L^2(0,1)$ and its dual $F' = H^{-1}(0,1) \times L^2(0,1)$

REMARK 3.1 *Multiplying the equation* $L^*\phi(t) = 0$ *by* $\phi'(t)$ *and integrating on* $]0,1[$, *we obtain*

$$\begin{aligned} E'(t) = & \frac{1}{2}\int_0^1 a'(y,t)\,|\phi_y(y,t)|^2\,dy - \frac{1}{2}\int_0^1 b_y(y,t)\,|\phi'(y,t)|^2\,dy + \\ & -\sigma\int_0^1 \phi(y,t)\,\phi'(y,t)\,dy + \frac{1}{2}\int_0^1 \left[b_y''(y,t) - c_y'(y,t)\right]|\phi(y,t)|^2\,dy + \\ & -\int_0^1 \left[b'(y,t) - c(y,t)\right]\phi_y(y,t)\,\phi'(y,t)\,dy, \end{aligned}$$

where

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^1 \Big(& |\phi'(y,t)|^2 + a(y,t)\,|\phi_y(y,t)|^2 + \\ & + \left[b_y'(y,t) - c_y(y,t)\right]|\phi(y,t)|^2\Big)\,dy. \end{aligned} \tag{24}$$

Note that by (9) and (10) the quadratic form $E(t)$ is positive.

Therefore,

$$|E'(t)| \le C(t)\,E(t), \tag{25}$$

with

$$\begin{aligned} C(t) = & \frac{\|a'(t)\|_{L^\infty(0,1)}}{k_0} + \|b_y(t)\|_{L^\infty(0,1)} + \frac{\|b'(t) - c(t)\|_{L^\infty(0,1)}}{\sqrt{k_0}} + \\ & + \frac{\left\|b_y''(t) - c_y'(t)\right\|_{L^\infty(0,1)}}{\mu_0} + \left(1 + \frac{1}{\mu_0}\right)|\sigma|. \end{aligned}$$

From (25) we obtain

$$-C(t)\,E(t) \le E'(t) \le C(t)\,E(t). \tag{26}$$

From the hypothesis (4), we have

$$\int_0^\infty C(t)\,dt \le C_0. \tag{27}$$

Combining (26) and (27), we conclude that

$$C_1 E(0) \le E(t) \le C_2 E(0), \quad \forall t \in [0,T], \tag{28}$$

with $C_1 = e^{-C_0}$ and $C_2 = e^{C_0}$.

Proof of Lemma 3.1 In this proof, we use the multiplier method as in Lions obtain, after some calculations, that

$$\begin{aligned}
\frac{1}{2}\int_0^T a(0,t)\left|\phi_y(0,t)\right|^2 dt &= \frac{1}{2}\int_Q a(y,t)\left|\phi_y(y,t)\right|^2 dydt+ \\
&+\frac{1}{2}\int_Q a_y(y,t)(1-y)\left|\phi_y(y,t)\right|^2 dydt- \\
&-\left(\phi'(t)+\frac{1}{2}b(t)\phi_y(t),(1-y)\phi_y(t)\right)\Big|_0^T+ \\
&+\frac{1}{2}\int_Q \left|\phi'(y,t)\right|^2 dydt+\frac{1}{2}\int_Q c'(y,t)(1-y)\left|\phi_y(y,t)\right|^2 dydt- \\
&-\frac{1}{2}\int_Q \left[b'(y,t)-c(y,t)\right](1-y)\left|\phi_y(y,t)\right|^2 dydt- \\
&-\int_Q \sigma\phi(y,t)(1-y)\phi_y dydt-\frac{1}{2}\int_Q \left[b_y'(y,t)-c_y(y,t)\right]\left|\phi(y,t)\right|^2 dydt+ \\
&+\frac{1}{2}\int_Q \left[b_y'(y,t)-c_y(y,t)\right]_y(1-y)\left|\phi(y,t)\right|^2 dydt+ \\
&+\frac{1}{2}\int_Q b_y(y,t)(1-y)\phi'(y,t)\phi_y(y,t)\,dydt.
\end{aligned} \tag{29}$$

From (26), we have by (29) that

$$\begin{aligned}
\frac{1}{2}\int_0^T a(0,t)\left|\phi_y(0,t)\right|^2 dt &\leq TC_2E(0)+ \\
&+\frac{1}{2}\int_Q a_y(y,t)(1-y)\left|\phi_y(y,t)\right|^2 dydt- \\
&-\left(\phi'(t)+\frac{1}{2}b(t)\phi_y(t),(1-y)\phi_y(t)\right)\Big|_0^T+ \\
&+\frac{1}{2}\int_Q c'(y,t)(1-y)\left|\phi_y(y,t)\right|^2 dydt- \\
&-\frac{1}{2}\int_Q \left[b'(y,t)-c(y,t)\right](1-y)\left|\phi_y(y,t)\right|^2 dydt- \\
&-\int_Q \sigma\phi(y,t)(1-y)\phi_y dydt+ \\
&+\frac{1}{2}\int_Q \left[b_y'(y,t)-c_y(y,t)\right]_y(1-y)\left|\phi(y,t)\right|^2 dydt+ \\
&+\int_Q b_y(y,t)(1-y)\phi'(y,t)\phi_y(y,t)\,dydt.
\end{aligned} \tag{30}$$

We will analyze each term that appears on the right hand side of (30).

- Analysis of $\frac{1}{2}\int_Q a_y(y,t)(1-y)|\phi_y(y,t)|^2\,dydt$:

$$\begin{aligned}&\frac{1}{2}\int_Q a_y(y,t)(1-y)|\phi_y(y,t)|^2\,dydt \leq\\ &\leq \frac{\|a_y\|_{L^\infty(Q)}}{2k_0}\int_Q a(y,t)|\phi_y(y,t)|^2\,dydt.\end{aligned}\tag{31}$$

- Analysis of $\frac{1}{2}\int_Q c'(y,t)(1-y)|\phi_y(y,t)|^2\,dydt$:

$$\begin{aligned}&\frac{1}{2}\int_Q c'(y,t)(1-y)|\phi_y(y,t)|^2\,dydt \leq\\ &\leq \frac{\|c'\|_{L^\infty(Q)}}{2k_0}\int_Q a(y,t)|\phi_y(y,t)|^2\,dydt.\end{aligned}\tag{32}$$

- Analysis of $\int_Q [b'(y,t)-c(y,t)](1-y)|\phi_y(y,t)|^2\,dydt$:

$$\begin{aligned}&\int_Q [b'(y,t)-c(y,t)](1-y)|\phi_y(y,t)|^2\,dydt \leq\\ &\leq \frac{\|b'-c\|_{L^\infty(Q)}}{2k_0}\int_Q a(y,t)|\phi_y(y,t)|^2\,dydt.\end{aligned}\tag{33}$$

- Analysis of $\int_Q \sigma\phi(y,t)(1-y)\phi_y(y,t)\,dydt$:

$$\begin{aligned}&\int_Q \sigma\phi(y,t)(1-y)\phi_y(y,t)\,dydt \leq\\ &\leq \frac{|\sigma|}{2\sqrt{k_0\mu_0}}\int_Q \left\{\left[b'_y(y,t)-c_y(y,t)\right]|\phi(y,t)|^2 + a(y,t)|\phi_y(y,t)|^2\right\}dydt.\end{aligned}\tag{34}$$

- Analysis of $\frac{1}{2}\int_Q \left[b'_y(y,t)-c_y(y,t)\right]_y(1-y)|\phi(y,t)|^2\,dydt$:

$$\begin{aligned}&\frac{1}{2}\int_Q \left[b'_y(y,t)-c_y(y,t)\right]_y(1-y)|\phi(y,t)|^2\,dydt \leq\\ &\leq \frac{\left\|b'_y-c_y\right\|_{L^\infty(Q)}}{2\mu_0}\int_Q \left[b'_y(y,t)-c_y(y,t)\right]|\phi(y,t)|^2\,dydt:\end{aligned}\tag{35}$$

- Analysis of $\int_Q b_y(y,t)(1-y)\phi'(y,t)\phi_y(y,t)\,dydt$:

$$\begin{aligned}&\int_Q b_y(y,t)(1-y)\phi'(y,t)\phi_y(y,t)\,dydt \leq\\ &\leq \frac{\|b_y\|_{L^\infty(Q)}}{2\sqrt{k_0}}\int_Q \left\{|\phi'(y,t)|^2 + a(y,t)|\phi_y(y,t)|^2\right\}.\end{aligned}\tag{36}$$

Now, set

$$X(t)=\left(\phi'(t)+\frac{1}{2}b(t)\,\phi_y(t),(1-y)\,\phi_y(t)\right),$$

then

$$|X(t)|\leq\frac{C_3}{2}\left\{\int_0^1|\phi'(y,t)|^2\,dydt+\int_0^1 a(y,t)\,|\phi_y(y,t)|^2\,dydt\right\},$$

with $C_3=\dfrac{1}{\sqrt{k_0}}+\dfrac{\|b\|_{L^\infty(Q)}}{k_0}$.

Therefore

$$|X(t)|\leq C_3E(t)$$

and

$$|X(0)-X(T)|\leq 2\,\|X(\cdot)\|_{L^\infty(0,T)}\leq 2C_3E(t)\,. \tag{37}$$

Substituting (31)-(36) and (37) in (30) and using (26), we obtain that

$$\frac{1}{2}\int_0^T a(0,t)\,|\phi_y(0,t)|^2\,dt\leq C_4E(0)\,,$$

where

$$C_4=C_2T\left(\frac{\|a_y\|_{L^\infty(Q)}+\|c'\|_{L^\infty(Q)}+\|b'-c\|_{L^\infty(Q)}}{k_0}+\frac{\|b_y\|_{L^\infty(Q)}}{\sqrt{k_0}}+\right.$$
$$\left.+\frac{\|b_y'-c_y\|_{L^\infty(Q)}}{2\mu_0}++\frac{|\sigma|}{\sqrt{k_0\mu_0}}+\frac{2C_3}{T}\right),$$

concluding the proof of the lemma. ∎

Proof of Lemma 3.2 Here we employ the argument of Zuazua (1990a). Let us define the functional

$$G(y)=\frac{1}{2}\int_{\delta y}^{T-\delta y}\left\{|\phi'(y,t)|^2+\right.$$
$$\left.+a(y,t)\,|\phi_y(y,t)|^2+\left[b_y'(y,t)-c_y(y,t)\right]|\phi(y,t)|^2\right\}dt, \tag{38}$$

with $\delta=\dfrac{1}{\sqrt{k_0}}$ and k_0 given by (8).

Note that

$$G(0)=\frac{1}{2}\int_0^T a(0,t)\,|\phi_y(0,t)|^2\,dt. \tag{39}$$

The derivative of the functional G is

$$
\begin{aligned}
G'(y) = \int_{\delta y}^{T-\delta y} & \{\phi'(y,t)\,\phi_y'(y,t) + \frac{a_y(y,t)}{2}\,|\phi_y(y,t)|^2 + \\
& + a(y,t)\,\phi_y(y,t)\,\phi_{yy}(y,t) + \frac{\left[b_{yy}'(y,t) - c_{yy}(y,t)\right]}{2}\,|\phi(y,t)|^2 + \\
& + \left[b_y'(y,t) - c_y(y,t)\right]\phi(y,t)\,\phi_y(y,t)\}dt - \\
-\frac{\delta}{2}\sum_{t=T-\delta y,\delta y} & \left\{|\phi'(y,t)|^2 + a(y,t)\,|\phi_y(y,t)|^2 + \right. \\
& \left. + \left[b_y'(y,t) - c_y(y,t)\right]|\phi(y,t)|^2\right\}.
\end{aligned} \tag{40}
$$

Integrating by parts, we get

$$
\begin{aligned}
\int_{\delta y}^{T-\delta y} \phi'(y,t)\,\phi_y'(y,t)\,dt = & -\int_{\delta y}^{T-\delta y} \phi''(y,t)\,\phi_y(y,t)\,dt + \\
& [\phi'(y,t)\,\phi_y(y,t)]|_{\delta y}^{T-\delta y}.
\end{aligned} \tag{41}
$$

Since

$$
\begin{aligned}
\phi'' \; = \; & a_y(y,t)\,\phi_y + a(y,t)\,\phi_{yy} - b(y,t)\,\phi_y' - [b'(y,t) - c(y,t)]\,\phi_y - \\
& - b_y(y,t)\,\phi' - \left[b_y'(y,t) - c_y(y,t)\right]\phi - \sigma\phi \quad \text{in} \quad Q,
\end{aligned}
$$

we conclude, using (41), that

$$
\begin{aligned}
G'(y) = & -\frac{1}{2}\int_{\delta y}^{T-\delta y} a_y(y,t)\,|\phi_y(y,t)|^2\,dt + \int_{\delta y}^{T-\delta y} b(y,t)\,\phi_y'(y,t)\,\phi_y(y,t)\,dt + \\
& + \int_{\delta y}^{T-\delta y} [b'(y,t) - c(y,t)]\,|\phi_y(y,t)|^2\,dt + \int_{\delta y}^{T-\delta y} b_y(y,t)\,\phi'(y,t)\,\phi_y(y,t)\,dt + \\
& + 2\int_{\delta y}^{T-\delta y} \left[b_y'(y,t) - c_y(y,t)\right]\phi(y,t)\,\phi_y(y,t)\}dt + \sigma\int_{\delta y}^{T-\delta y} \phi(y,t)\,\phi_y(y,t)\,dt + \\
& + \frac{1}{2}\int_{\delta y}^{T-\delta y} \left[b_{yy}'(y,t) - c_{yy}(y,t)\right]|\phi(y,t)|^2\,dt + [\phi'(y,t)\,\phi_y(y,t)]|_{\delta y}^{T-\delta y} - \\
& - \frac{\delta}{2}\sum_{t=T-\delta y,\delta y}\left\{|\phi'(y,t)|^2 + a(y,t)\,|\phi_y(y,t)|^2 + \left[b_y'(y,t) - c_y(y,t)\right]|\phi(y,t)|^2\right\}.
\end{aligned}
$$

We have that

$$\begin{aligned}&\int_{\delta y}^{T-\delta y} b(y,t)\,\phi_y'(y,t)\,\phi_y(y,t)\,dt = -\frac{1}{2}\int_{\delta y}^{T-\delta y} b'(y,t)\,|\phi_y(y,t)|^2\,dt+\\&+\frac{1}{2}\left[b(y,t)\,|\phi_y(y,t)|^2\right]\Big|_{\delta y}^{T-\delta y} \le -\frac{1}{2}\int_{\delta y}^{T-\delta y} b'(y,t)\,|\phi_y(y,t)|^2\,dt+\\&+\frac{1}{2}\sum_{t=T-\delta y,\delta y}|b(y,t)|\,|\phi_y(y,t)|^2\end{aligned} \tag{43}$$

and

$$\begin{aligned}&|\phi'(y,t)\,\phi_y(y,t)| \le \frac{1}{2\sqrt{k_0}}\left\{|\phi'(y,t)|^2 + k_0\,|\phi_y(y,t)|^2\right\} \le \frac{\delta}{2}\left\{|\phi'(y,t)|^2+\right.\\&\left.+\left[a(y,t)-\frac{|b(y,t)|}{\delta}\right]|\phi_y(y,t)|^2 + \left[b_y'(y,t)-c_y(y,t)\right]|\phi(y,t)|^2\right\}.\end{aligned} \tag{44}$$

Thus, substituting (43) and (44) in (42), we deduce that

$$\begin{aligned}&G'(y) \le -\frac{1}{2}\int_{\delta y}^{T-\delta y}\left[a_y(y,t)+b'(y,t)+2c(y,t)\right]|\phi_y(y,t)|^2\,dt+\\&+\int_{\delta y}^{T-\delta y} b_y(y,t)\,\phi'(y,t)\,\phi_y(y,t)\,dt + \sigma\int_{\delta y}^{T-\delta y}\phi(y,t)\,\phi_y(y,t)\,dt+\\&+2\int_{\delta y}^{T-\delta y}\left[b_y'(y,t)-c_y(y,t)\right]\phi(y,t)\,\phi_y(y,t)\}dt+\\&+\frac{1}{2}\int_{\delta y}^{T-\delta y}\left[b_{yy}'(y,t)-c_{yy}(y,t)\right]|\phi(y,t)|^2\,dt.\end{aligned} \tag{45}$$

We will analyze each term of the second member of (45).

- Analysis of $-\frac{1}{2}\int_{\delta y}^{T-\delta y}\left[a_y(y,t)+b'(y,t)+2c(y,t)\right]|\phi_y(y,t)|^2\,dt$:

$$\begin{aligned}&\int_{\delta y}^{T-\delta y}\left[a_y(y,t)+b'(y,t)+2c(y,t)\right]|\phi_y(y,t)|^2\,dt \le\\&\le \frac{\|a_y+b'+2c\|_{L^\infty(Q)}}{2k_0}\int_{\delta y}^{T-\delta y} a(y,t)\,|\phi_y(y,t)|^2\,dt.\end{aligned} \tag{46}$$

- Analysis of $\int^{T-\delta y} b_y(y,t)\,\phi'(y,t)\,\phi_y(y,t)\,dt$:

$$
\begin{aligned}
&\int_{\delta y}^{T-\delta y} b_y(y,t)\,\phi'(y,t)\,\phi_y(y,t)\,dt \leq \\
&\leq \frac{\|b_y\|_{L^\infty(Q)}}{2\sqrt{k_0}} \int_{\delta y}^{T-\delta y} \left\{ |\phi'(y,t)|^2 + a(y,t)\,|\phi_y(y,t)|^2 \right\} dt.
\end{aligned} \tag{47}
$$

- Analysis of $\sigma \int_{\delta y}^{T-\delta y} \phi(y,t)\,\phi_y(y,t)\,dt$:

$$
\begin{aligned}
&\sigma \int_{\delta y}^{T-\delta y} \phi(y,t)\,\phi_y(y,t)\,dt \leq \\
&\leq \frac{|\sigma|}{2\sqrt{k_0\mu_0}} \int_{\delta y}^{T-\delta y} \left\{ a(y,t)\,|\phi_y(y,t)|^2\,dt + \left[b'_y(y,t) - c_y(y,t)\right] |\phi(y,t)|^2 \right\} dt.
\end{aligned} \tag{48}
$$

- Analysis of $2\int_{\delta y}^{T-\delta y} \left[b'_y(y,t) - c_y(y,t)\right] \phi(y,t)\,\phi_y(y,t)\}dt$:

$$
\begin{aligned}
&2\int_{\delta y}^{T-\delta y} \left[b'_y(y,t) - c_y(y,t)\right] \phi(y,t)\,\phi_y(y,t)\}dt \leq \\
&\leq \frac{\left\|\sqrt{b'_y - c_y}\right\|_{L^\infty(Q)}}{\sqrt{k_0}} \int_{\delta y}^{T-\delta y} \left\{ a(y,t)\,|\phi_y(y,t)|^2\,dt + \right. \\
&\left. + \left[b'_y(y,t) - c_y(y,t)\right] |\phi(y,t)|^2 \right\} dt.
\end{aligned} \tag{49}
$$

- Analysis of $\frac{1}{2}\int_{\delta y}^{T-\delta y} \left[b'_{yy}(y,t) - c_{yy}(y,t)\right] |\phi(y,t)|^2\,dt$:

$$
\begin{aligned}
&\frac{1}{2}\int_{\delta y}^{T-\delta y} \left[b'_{yy}(y,t) - c_{yy}(y,t)\right] |\phi(y,t)|^2\,dt \leq \\
&\leq \frac{\left\|b'_{yy} - c_{yy}\right\|_{L^\infty(Q)}}{2\mu_0} \int_{\delta y}^{T-\delta y} \left[b'_y(y,t) - c_y(y,t)\right] |\phi(y,t)|^2\,dt.
\end{aligned} \tag{50}
$$

From (46)-(50), we get, by (45), that

$$
G'(y) \leq C_5 G(y),
$$

where

$$
\begin{aligned}
C_5 = &\frac{\|a_y + b' + 2c\|_{L^\infty(Q)}}{k_0} + \frac{\|b_y\|_{L^\infty(Q)} + \left\|\sqrt{b'_y - c_y}\right\|_{L^\infty(Q)}}{\sqrt{k_0}} + \\
&+ \frac{\left\|b'_{yy} - c_{yy}\right\|_{L^\infty(Q)}}{} + \frac{|\sigma|}{}.
\end{aligned}
$$

Hence

$$G(y) \leq C_6 G(0), \quad \forall y \in]0,1[, \tag{51}$$

with $C_6 = e^{C_5}$.

Integrating (51) in $]0,1[$, we have

$$\int_0^1 G(y)\, dy \leq C_6 G(0). \tag{52}$$

Since $T > \dfrac{2}{\sqrt{k_0}} = 2\delta$, we obtain, by (28), that

$$\begin{aligned}(T-2\delta) E(0) &= \int_\delta^{T-\delta} E(0)\, dt \leq \frac{1}{C_1}\int_\delta^{T-\delta} E(t)\, dt = \\ &= \frac{1}{2C_1}\int_\delta^{T-\delta}\int_0^1 \Big\{ |\phi'(y,t)|^2 + a(y,t)|\phi_y(y,t)|^2 + \\ &\quad + \left[b_y'(y,t) - c_y(y,t)\right]|\phi(y,t)|^2 \Big\}\, dy dt. \end{aligned} \tag{53}$$

From (38) and (52) we modify (53) to obtain

$$(T-2\delta) E(0) \leq \frac{1}{C_1}\int_0^1 G(y)\, dy \leq \frac{C_6}{C_1} G(0),$$

which implies in the inequality (23). ■

Case 2. The general case with f nonlinear.

Using the solution ϕ of the problem (15), we consider the backward problem

$$\left| \begin{array}{lll} \xi'' - \left[a(y,t)\, v_y\right]_y + b(y,t)\,\xi_y' + c(y,t)\,\xi_y + f(\xi) = 0 & \text{in} & Q, \\ \xi(0,t) = -\phi_y(t), \quad \xi(1,t) = 0 & \text{on} &]0,T[, \\ \xi(T) = z_0, \quad \xi'(T) = z_1 & \text{in} &]0,1[. \end{array} \right. \tag{54}$$

The solution ξ of the (54) can be written as

$$\xi = z + \theta + \eta,$$

where z, θ e η are solutions of the following problems:

$$\left| \begin{array}{lll} z'' - \left[a(y,t)\, z_y\right]_y + b(y,t)\, z_y' + c(y,t)\, z_y + \sigma z = 0 & \text{in} & Q, \\ z(0,t) = z(1,t) = 0 & \text{on} &]0,T[, \\ z(T) = z_0, \quad z'(T) = z_1 & \text{in} &]0,1[, \end{array} \right. \tag{55}$$

$$\left| \begin{array}{lll} \theta'' - \left[a(y,t)\,\theta_y\right]_y + b(y,t)\,\theta_y' + c(y,t)\,\theta_y + \sigma\theta = 0 & \text{in} & Q, \\ \theta(0,t) = -\phi_y(t), \quad \theta(1,t) = 0 & \text{on} &]0,T[, \end{array} \right. \tag{56}$$

and

$$\left| \begin{array}{lll} \eta'' - [a\eta_y]_y + b\eta'_y + c\eta_y + f(z+\theta+\eta) = \sigma(z+\theta) & \text{in} & Q, \\ \eta(0,t) = \eta(1,t) = 0 & \text{on} &]0,T[, \\ \eta(T) = \eta'(T) = 0 & \text{in} &]0,1[. \end{array} \right. \tag{57}$$

We can see in Milla, Miranda (1995) that the solutions z e θ of (55) and (56) belong to the class

$$z,\ \theta \in C\left([0,T]; L^2(0,1)\right) \cap C^1\left([0,T]; H^{-1}(0,1)\right), \tag{58}$$

and that for $T > 0$, there exists $C = C(T) > 0$, such that

$$||z||_{L^\infty(0,T;L^2(0,1))} + ||z'||_{L^\infty(0,T;H^{-1}(0,1))} \leq C\,||\{z_0,z_1\}||_{L^2(0,1)\times H^{-1}(0,1)} \tag{59}$$

and

$$||\theta||_{L^\infty(0,T;L^2(0,1))} + ||\theta'||_{L^\infty(0,T;H^{-1}(0,1))} \leq C\,||\{\phi_0,\phi_1\}||_{H_0^1(0,1)\times L^2(0,1)}. \tag{60}$$

For the solution η of the (57) we have the following result:

LEMMA 3.3 *For every $\{\phi_0,\phi_1\} \in H_0^1(0,1) \times L^2(0,1)$, $\{z_0,z_1\} \in L^2(0,1) \times H^{-1}(0,1)$ there exists a unique solution $\eta = \eta(y,t)$ of (57) in the class*

$$\eta \in C\left([0,T]; L^2(0,1)\right) \cap C^1\left([0,T]; H^{-1}(0,1)\right). \tag{61}$$

Moreover, for any $\varepsilon > 0$, there exists a positive constant $C(\varepsilon) > 0$ such that

$$\begin{aligned} &||\eta||_{L^\infty(0,T;H_0^1(0,1))} + ||\eta'||_{L^\infty(0,T;L^2(0,1))} \leq \\ &\leq \varepsilon\left\{||\{\phi_0,\phi_1\}||_{H_0^1(0,1)\times L^2(0,1)} + ||\{z_0,z_1\}||_{L^2(0,1)\times H^{-1}(0,1)}\right\} + C(\varepsilon). \end{aligned} \tag{62}$$

Proof. We remark that by (7), the function $\eta \longmapsto f(z+\theta+\eta)$ is Lipschitz. In this way, we can see in Milla, Miranda (1995) that there exists a unique solution η in the class (61).

Consider the energy

$$E(\eta,t) = \frac{1}{2}\int_0^1 \left\{|\eta'(y,t)|^2 + a(y,t)\,|\eta_y(y,t)|^2\right\} dy,\ \forall t \in [0,T]. \tag{63}$$

Multiplying the equation in (57) for η' and integrating on $(0,1)$ we obtain

$$\begin{aligned} E'(\eta,t) \leq\ & \frac{||a'||_{L^\infty(Q)}}{2}\int_0^1 |\eta_y(y,t)|\,dy + \frac{||b_y||_{L^\infty(Q)}}{2}\int_0^1 |\eta'(y,t)|\,dy + \\ &+ \frac{||c||_{L^\infty(Q)}}{2}\int_0^1 |\eta_y(y,t)|\,dy + \frac{||c||_{L^\infty(Q)}}{2}\int_0^1 |\eta'(y,t)|\,dy - \\ &- \int_0^1 f(z+\theta+\eta)\,\eta'\,dy + \int_0^1 \sigma(z+\theta)\,\eta'\,dy. \end{aligned} \tag{64}$$

Using (9), we have

$$
\begin{aligned}
E'(\eta,t) \leq & \left[\frac{\|a'\|_{L^\infty(Q)} + \|c\|_{L^\infty(Q)}}{k_0} + \|b_y\|_{L^\infty(Q)} + \|c\|_{L^\infty(Q)}\right] E(\eta,t) - \\
& -\int_0^1 f(z+\theta+\eta)\,\eta' dy + \int_0^1 \sigma(z+\theta)\,\eta' dy.
\end{aligned} \tag{65}
$$

Now,

$$
\begin{aligned}
&\int_0^1 f(z+\theta+\eta)\,\eta' dy + \int_0^1 \sigma(z+\theta)\,\eta' dy \leq \\
&\leq \|f'\|_{L^\infty(\mathbb{R})} \int_0^1 \{|z+\theta+\eta| + |f(0)| + |\sigma|\,|z+\theta|\}\,|\eta'|\,dy \leq \\
&\leq \|f'\|_{L^\infty(\mathbb{R})}(1+|\sigma|)\left(\|z\|^2_{L^\infty(0,T;L^2)} + \|\theta\|^2_{L^\infty(0,T;L^2)}\right) + \\
&\qquad + \frac{\|f'\|_{L^\infty(\mathbb{R})}}{2}(3+|\sigma|)\int_0^1 |\eta'|^2\,dy + \\
&\qquad + \frac{\|f'\|_{L^\infty(\mathbb{R})}}{2}\int_0^1 |\eta|^2\,dy + \frac{1}{2}\|f'\|_{L^\infty(\mathbb{R})}|f(0)|^2,
\end{aligned}
$$

by (59), (60) we get

$$
\begin{aligned}
&-\int_0^1 f(z+\theta+\eta)\,\eta' dy + \int_0^1 \sigma(z+\theta)\,\eta' dy \leq \\
&\leq \|f'\|_{L^\infty(\mathbb{R})}(1+|\sigma|)\left\{\|\{z_0,z_1\}\|_{L^2(0,1)\times H^{-1}(0,1)} + \|\{\phi_0,\phi_1\}\|_{H_0^1(0,1)\times L^2(0,1)}\right\} + \\
&\qquad + \|f'\|_{L^\infty(\mathbb{R})}(|\sigma|+k_0+3)\,E(\eta,t) + \frac{1}{2}\|f'\|_{L^\infty(\mathbb{R})}|f(0)|^2.
\end{aligned} \tag{66}
$$

From (66) and taking in consideration that $E(\eta,T)=0$, we obtain from (65) that

$$
E(\eta,t) \leq C_7\left\{\|\{\phi_0,\phi_1\}\|_{H_0^1(0,1)\times L^2(0,1)} + \|\{z_0,z_1\}\|_{L^2(0,1)\times H^{-1}(0,1)} + 1\right\}, \tag{67}
$$

where

$$
C_7 = C_7\left(\|a'\|_{L^\infty(Q)}, \|b_y\|_{L^\infty(Q)}, \|c\|_{L^\infty(Q)}, k_0, \|f'\|_{L^\infty(\mathbb{R})}, |f(0)|, |\sigma|, T\right) > 0,
$$

for every $t \in [0,T]$, $\{\phi_0,\phi_1\} \in H_0^1(0,1) \times L^2(0,1)$ and $\{z_0,z_1\} \in L^2(0,1) \times H^{-1}(0,1)$.

From (59), (60) and (67) we have

$$
\|\xi\|^2_{L^\infty(0,T;L^2(0,1))} + \|\xi'\|^2_{L^\infty(0,T;H^{-1}(0,1))} \leq \tag{68}
$$

where $C_8 = C_8(C_7, T) > 0$, for every $\{\phi_0, \phi_1\} \in H_0^1(0,1) \times L^2(0,1)$ and $\{z_0, z_1\} \in L^2(0,1) \times H^{-1}(0,1)$.

To prove (62) we write the equation corresponding to η as follows

$$\begin{aligned} \eta'' - [a\eta_y]_y + b\eta_y' + c\eta_y + \sigma\eta &= \sigma(z+\theta+\eta) - f(z+\theta+\eta) \\ &= \sigma\xi + f(\xi) \quad \text{in} \quad Q. \end{aligned} \tag{69}$$

Multiplying (69) by η' and integrating from 0 to 1, we obtain

$$\begin{aligned} &E'(\eta,t) \leq \\ &\quad \left[\frac{\|a'\|_{L^\infty(Q)}}{k_0} + \|b_y\|_{L^\infty(Q)} + \|c\|_{L^\infty(Q)}\left(\frac{1}{k_0}+1\right) + |\sigma|\left(\frac{1}{k_0}+1\right)+1\right] \\ &\quad E(\eta,t) + +\frac{1}{2}\|\sigma\xi - f(\xi)\|^2_{L^\infty(0,T;L^2(0,1))}. \end{aligned} \tag{70}$$

By the change of variables $s = T - t$ and solving the equation in (70) it follows that

$$E(\eta,t) \leq \frac{1}{2C_9}e^{C_9 T}\|\sigma\xi - f(\xi)\|^2_{L^\infty(0,T;L^2(0,1))},\ \forall t \in [0,T],$$

where

$$C_9 = \frac{\|a'\|_{L^\infty(Q)}}{k_0} + \|b_y\|_{L^\infty(Q)} + \|c\|_{L^\infty(Q)}\left(\frac{1}{k_0}+1\right) + |\sigma|\left(\frac{1}{k_0}+1\right)+1.$$

From (7), it follows that for each $\varepsilon > 0$,

$$E(\eta,t) \leq \varepsilon\|\xi\|^2_{L^\infty(0,T;L^2(0,1))},\ \forall t \in [0,T]. \tag{71}$$

Using (68) and (71), we obtain the inequality (62). ■

To conclude the exact controllability of (11) it is enough to prove that the operator

$$\mu : H_0^1(0,1) \times L^2(0,1) \longrightarrow H^{-1}(0,1) \times L^2(0,1),$$

defined by

$$\mu\{\phi_0,\phi_1\} = \{\xi'(0) + b(0)\,\xi_y(0), -\xi(0)\}, \tag{72}$$

is surjective.

We can write the operator μ as follows

$$\begin{aligned} \mu\{\phi_0,\phi_1\} = \quad &\{\eta'(0) + b(0)\,\eta_y(0), -\eta(0)\} + \\ &+ \{\theta'(0) + b(0)\,\theta_y(0), -\theta(0)\} + \end{aligned} \tag{73}$$

We will denote by K the operator from $H_0^1(0,1) \times L^2(0,1)$ in $H^{-1}(0,1) \times L^2(0,1)$ defined by

$$K\{\phi_0, \phi_1\} = \{\eta'(0) + b(0)\eta_y(0), -\eta(0)\}.$$

By Lemma 3.3 and the compact immersions $H_0^1(0,1) \subset L^2(0,1)$, $L^2(0,1) \subset H^{-1}(0,1)$, the operator K is compact.

We can write (73) as

$$\mu\{\phi_0, \phi_1\} = K\{\phi_0, \phi_1\} + \Lambda\{\phi_0, \phi_1\} + \{z'(0) + b(0) z_y(0), -z(0)\},$$

where $\Lambda : H_0^1(0,1) \times L^2(0,1) \longrightarrow H^{-1}(0,1) \times L^2(0,1)$ is the isomorphism of the linear case.

Showing that the operator μ is surjective is equivalent to solving the following problem:

$$\begin{aligned}\Lambda^{-1}\{\xi'(0) + b(0)\xi_y(0), -\xi(0)\} = \;& \Lambda^{-1}K\{\phi_0, \phi_1\} + \{\phi_0, \phi_1\} + \\ & + \Lambda^{-1}\{z'(0) + b(0) z_y(0), -z(0)\},\end{aligned}$$

that is,

$$\begin{aligned}\{\phi_0, \phi_1\} &= \Lambda^{-1}\{\xi'(0) + b(0)\xi_y(0) - z'(0) - b(0) z_y(0), -\xi(0) + z(0)\} - \\ &\quad - \Lambda^{-1}K\{\phi_0, \phi_1\}.\end{aligned}$$

By setting

$$\begin{aligned}\Theta\{\phi_0, \phi_1\} &= \Lambda^{-1}\{\xi'(0) + b(0)\xi_y(0) - z'(0) - b(0) z_y(0), -\xi(0) + z(0)\} - \\ &\quad - \Lambda^{-1}K\{\phi_0, \phi_1\} \end{aligned} \tag{74}$$

we are looking for a fixed point of the operator

$$\Theta : H_0^1(0,1) \times L^2(0,1) \longrightarrow H_0^1(0,1) \times L^2(0,1).$$

We will apply the Shauder's fixed point theorem. As the operator Θ is compact, it is enough prove that the image of Θ is limited, that is, there exists a positive constant M, such that

$$\|\Theta\{\phi_0, \phi_1\}\|_{H_0^1(0,1)\times L^2(0,1)} \leq M, \tag{75}$$

for every $\{\phi_0, \phi_1\} \in H_0^1(0,1) \times L^2(0,1)$ such that $\|\{\phi_0, \phi_1\}\|_{H_0^1(0,1)\times L^2(0,1)} \leq N$, with N being constant.

From (74) we have

$$\begin{aligned}&\|\Theta\{\phi_0, \phi_1\}\|_{H_0^1(0,1)\times L^2(0,1)} \leq \left\|\Lambda^{-1}K\{\phi_0, \phi_1\}\right\|_{H_0^1(0,1)\times L^2(0,1)} + \\ &+ \left\|\Lambda^{-1}\{\xi'(0) + b(0)\xi_y(0) - z'(0) - b(0) z_y(0), -\xi(0) + z(0)\}\right\|_{H_0^1(0,1)\times L^2(0,1)}.\end{aligned}$$

Reminding that $\xi = z + \theta + \eta$ then

$$\left\|\Lambda^{-1}\left\{\xi'(0) + b(0)\,\xi_y(0) - z'(0) - b(0)\,z_y(0), -\xi(0) + z(0)\right\}\right\|_{H_0^1(0,1)\times L^2(0,1)} \le$$
$$\le \left\|\Lambda^{-1}\right\|\left\{\|\eta'(0) + b(0)\,\eta_y(0)\|_{H^{-1}(0,1)} + \|\theta'(0) + b(0)\,\theta_y(0)\|_{H^{-1}(0,1)} + \right.$$
$$\left. + \|\eta(0) + \theta(0)\|_{L^2(0,1)}\right\}, \tag{77}$$

where $\left\|\Lambda^{-1}\right\|$ denote $\left\|\Lambda^{-1}\right\|_{\mathcal{L}\left(H^{-1}(0,1)\times L^2(0,1);H_0^1(0,1)\times L^2(0,1)\right)}$.

By (77), we obtain

$$\left\|\Lambda^{-1}\left\{\xi'(0) + b(0)\,\xi_y(0) - z'(0) - b(0)\,z_y(0), -\xi(0) + z(0)\right\}\right\|_{H_0^1(0,1)\times L^2(0,1)} \le$$
$$\le C_{10}\left\|\Lambda^{-1}\right\|\left\{\|\eta'(0)\|_{L^2(0,1)} + \|\eta(0)\|_{H_0^1(0,1)} + \right.$$
$$\left. + \|\theta'(0) + b(0)\,\theta_y(0)\|_{H^{-1}(0,1)} + \|\theta(0)\|_{L^2(0,1)}\right\}, \tag{78}$$

where $C_{10} > 0$ depends on the constants of the continous immersions $H_0^1(0,1) \subset L^2(0,1)$ and $L^2(0,1) \subset H^{-1}(0,1)$. Using (62) in (78), we get

$$\left\|\Lambda^{-1}\left\{\xi'(0) + b(0)\,\xi_y(0) - z'(0) - b(0)\,z_y(0), -\xi(0) + z(0)\right\}\right\|_{H_0^1(0,1)\times L^2(0,1)} \le$$
$$\le C_{10}\left\|\Lambda^{-1}\right\|\left\{\varepsilon\left[\|\{\phi_0,\phi_1\}\|_{H_0^1(0,1)\times L^2(0,1)} + \|\{z_0,z_1\}\|_{L^2(0,1)\times H^{-1}(0,1)}\right] + C(\varepsilon) + \right.$$
$$\left. + \|\theta'(0) + b(0)\,\theta_y(0)\|_{H^{-1}(0,1)} + \|\theta(0)\|_{L^2(0,1)}\right\}.$$

Therefore, by (60), we have

$$\left\|\Lambda^{-1}\left\{\xi'(0) + b(0)\,\xi_y(0) - z'(0) - b(0)\,z_y(0), -\xi(0) + z(0)\right\}\right\|_{H_0^1(0,1)\times L^2(0,1)} \le$$
$$\le C_{11}\left\|\Lambda^{-1}\right\|\varepsilon\left\{\|\{\phi_0,\phi_1\}\|_{H_0^1(0,1)\times L^2(0,1)} + \|\{z_0,z_1\}\|_{L^2(0,1)\times H^{-1}(0,1)}\right\} +$$
$$+ \left\|\Lambda^{-1}\right\| C_{10}C(\varepsilon), \tag{79}$$

where $C_{11} = C_{11}\left(T, \|b(0)\|_{L^\infty(0,1)}\right)$.

Choosing $\varepsilon = \left[2C_{11}\left\|\Lambda^{-1}\right\|_{\mathcal{L}\left(H^{-1}(0,1)\times L^2(0,1);H_0^1(0,1)\times L^2(0,1)\right)}\right]^{-1}$, it follows from (79) that

$$\left\|\Lambda^{-1}\{\xi'(0) + b(0)\,\xi_y(0) - z'(0) - b(0)\,z_y(0), -\xi(0) + z(0)\}\right\|_{H_0^1(0,1)\times L^2(0,1)} \le$$
$$\le \frac{1}{2}\|\{\phi_0,\phi_1\}\|_{H_0^1(0,1)\times L^2(0,1)} + \|\{z_0,z_1\}\|_{L^2(0,1)\times H^{-1}(0,1)} + \left\|\Lambda^{-1}\right\| C_{10}C(\varepsilon). \tag{80}$$

Substituting (80) in (76) we obtain

$$\|\Theta\{\phi_0,\phi_1\}\|_{H_0^1(0,1)\times L^2(0,1)} \le \left\|\Lambda^{-1}K\{\phi_0,\phi_1\}\right\|_{H_0^1(0,1)\times L^2(0,1)} +$$

therefore,

$$\|\Theta\{\phi_0,\phi_1\}\|_{H_0^1(0,1)\times L^2(0,1)} \leq \left(\frac{1}{2}+\left\|\Lambda^{-1}K\right\|\right)N + C_{12},$$

where $C_{12} = C_{12}\left(\{z_0,z_1\},\left\|\Lambda^{-1}\right\|,C_{10}\right) > 0$, proving (75). [End of proof of Theorem 3.1] ■

REMARK 3.2 *By (21) and the change of variable* $u(x,t) = v(y,t)$, *with* $y = \dfrac{x-\alpha(t)}{\gamma(t)}$, *we obtain that the unique solution of (1), with control*

$$\varphi(\alpha(t),t) = -\frac{1}{\gamma(t)}\rho_x(\alpha(t),t)$$

satisfies (2), where ρ *is solution of the problem (15) after a transformation from* Q *to* $\widehat{Q}$, *by* $\rho(x,t) = \phi(y,t)$ *with* $x = \alpha(t) + \gamma(t)y$.

REMARK 3.3 *An example of non cylindrical domain is given by*

$$\alpha(t) = \alpha_0 - \sqrt{\frac{1}{t_0}} + \sqrt{\frac{1}{t+t_0}}, \quad t \geq 0,$$

$$\beta(t) = \beta_0 + \sqrt{\frac{1}{t_0}} - \sqrt{\frac{1}{t+t_0}}, \quad t \geq 0,$$

with

$$\frac{1}{t_0} = \sqrt[3]{\frac{\tau_0}{2m}}.$$

And

$$a = \alpha_0 - \sqrt{\frac{1}{t_0}} \quad \text{and} \quad b = \beta_0 + \sqrt{\frac{1}{t_0}}.$$

This example can be seen in Medeiros, Limaco, Menezes (2002).

REMARK 3.4 *In our paper we study exact controllability for vibrating strings when the ends are variable with the time, with mild nonlinearity. Our control act on the moving boundary. In Kangsheng, Yong (1999) the authors studied controllability, for the linear wave equation, when the control act on a domain in the interior that is variable with the time. These two seem to be different questions.*

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