

# Sensitivity of optimal solutions to control problems for systems described by hemivariational inequalities<sup>1</sup>

by

Zdzisław Denkowski and Stanisław Migórski

Jagiellonian University

Faculty of Mathematics and Computer Science

Institute of Computer Science

ul. Nawojki 11, 30-072 Cracow, Poland

**Abstract:** In this paper the sensitivity of optimal solutions to control problems for the systems described by stationary and evolution hemivariational inequalities (HVIs) under perturbations of state relations and of cost functionals is investigated. First, basing on the theory of sequential  $\Gamma$ -convergence we recall the abstract scheme concerning convergence of minimal values and minimizers. The abstract scheme works provided we can establish two properties: the Kuratowski convergence of solution sets for HVIs (state relations) and some complementary  $\Gamma$ -convergence of the cost functionals. Then these two properties are implemented in each considered case.

**Keywords:** hemivariational inequality, control problem, sensitivity, the Clarke subdifferential, multifunction, pseudomonotone and maximal monotone operators,  $G$ ,  $PG$  and  $\Gamma$  convergences.

## 1. Introduction

It is well known (Panagiotopoulos, 1985a, 1985b, 1993; Naniewicz and Panagiotopoulos, 1995;) that many problems from mechanics (elasticity theory, semipermeability, electrostatics, hydraulics, fluid flow), economics and so on can be modeled by hemivariational inequalities (HVIs for short). The latter are generalizations of partial differential equations (PDEs) and variational inequalities (Duvaut and Lions, 1976) in the sense that besides the physical phenomena leading to classical PDEs one has to take into consideration some nonlinear, nonmonotone and possibly multivalued laws (e.g. stress-strain, reaction-displacement, generalized forces-velocities, etc.) which can be expressed by means of the Clarke subdifferential.

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In this paper, which in a sense is the continuation of Denkowski (2002), where some of the results below were conjectured (see Theorem 4.2 and Remark 4.2 in Denkowski, 2002), we deal with control problems for systems governed by the stationary (elliptic) as well as by the evolution first order (parabolic) HVIs. More precisely, we consider

$$(CP)_e \quad \text{minimize} \quad \left\{ \mathcal{F}_e(u, y) := \mathcal{F}^{(1)}(y) + \mathcal{F}^{(2)}(u) \right\}$$

subject to

$$(HVI_e) \quad \begin{cases} Ay + \partial J(y) \ni f + Cu \\ y \in V, \quad u \in \mathcal{U} \end{cases}$$

or

$$(CP)_p \quad \text{minimize} \quad \left\{ \mathcal{F}_p(u, y) := \mathcal{F}^{(1)}(y) + \mathcal{F}^{(2)}(u, y_0) \right\}$$

subject to

$$(HVI_p) \quad \begin{cases} y' + Ay + \partial J(y) \ni f + Cu \\ y(0) = y_0, \quad y \in \mathcal{W}_{pq} = \mathcal{Y}_p, \quad u \in \mathcal{U}, \quad 1/p + 1/q = 1, \quad p \geq 2, \end{cases}$$

where  $A$  is a pseudomonotone operator (possibly multivalued in the existence theorems),  $\mathcal{A}$  is the Nemitsky operator corresponding to  $A$ ,  $J$  is a locally Lipschitz superpotential ( $\partial J$  denotes its Clarke subdifferential),  $C$  is a controller operator acting on the space of controls  $\mathcal{U}$  and the cost functionals  $\mathcal{F}^{(i)}$  are in integral form (for details and definitions of spaces  $V$  and  $\mathcal{W}_{pq}$ , see Sections 4 and 5 below).

Our goal is to investigate the sensitivity of optimal solutions to these control problems; i.e. we are interested in the behavior of optimal solutions under perturbations of systems (state relations; e.g. coefficients in equations or parameters in superpotentials are perturbed,...) as well as of perturbations of cost functionals (e.g. integrands depend on parameters).

Our approach is based on the sequential  $\Gamma$ -convergence (epi-convergence in terms of Attouch, 1984) theory (see De Giorgi and Franzoni, 1975; De Giorgi and Spagnolo, 1973; Spagnolo, 1975; Buttazzo and Dal Maso, 1982; Denkowski and Mortola, 1993) in the sensitivity part, while for existence of optimal solutions we use the direct method. The nonemptiness of the solution set for HVIs follows from the theory of pseudomonotone operators (see Zeidler, 1990; Denkowski et al., 2003b).

The basic properties assuring the convergence of minimal values and minimizers of perturbed control problems to the minimal value and to a minimizer, respectively, of unperturbed problem are: on one hand the Kuratowski convergence of solution sets (which can be expressed as  $\Gamma$ -convergence of their indicator functions) and on the other hand some "complementary  $\Gamma$ -convergence" of cost

We underline that nonemptiness of solution sets for HVIs can be obtained (by surjectivity theorems for pseudo-monotone multivalued operators) for much more general classes of operators, while for sensitivity results we have to restrict ourselves to special classes of maximal monotone operators for which the notion of G-convergence can be applied.

The sensitivity of control problems was largely considered in the literature in papers on optimal control for systems governed by ordinary differential equations (Buttazzo and Dal Maso, 1982; Buttazzo and Freddi, 1993, 1995; Freddi, 2000), partial differential equations (Denkowski and Migórski, 1987; Migórski, 1992a, 1992b, 1995, 1999; Chapter 4.2 of Denkowski et al., 2003b), partial differential equations and differential inclusions (Denkowski and Mortola, 1993; Briani, 2000; Arada and Raymond, 1999; Acquistapace and Briani, 2002). We mention that the related control problems for systems described by HVIs were studied by Haslinger and Panagiotopoulos, 1995; Migórski and Ochal, 2000b; Denkowski, 2002; Migórski, 2003, the shape optimization problems for HVIs were considered by Denkowski and Migórski (1998a, 1998b), Gasiński (1998), Ochal (2000), Denkowski (2000, 2001) and the corresponding inverse and identification problems were treated by Migórski and Ochal (2000a).

The paper is organized as follows. In Section 2 we present an abstract setting for the sensitivity analysis, which is based on the  $\Gamma$ -convergence theory. In Section 3 we recall some material on the sequential  $\Gamma$ -convergence, the Clarke subdifferential and the multivalued operators. Section 4 is devoted to control problems for stationary hemivariational inequalities and contains the results on the sensitivity of the solution sets to hemivariational inequalities and on the stability of the control problems. In the last section the analogous sensitivity results are provided for control problems for systems governed by parabolic hemivariational inequalities.

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## 2. General setting

In this section we recall the abstract scheme based on the  $\Gamma$ -convergence theory, which we use to study the stability of optimal control problems.

We consider a control system governed by a relation  $\mathcal{R}$  which links the state  $y \in \mathcal{Y}_{\mathcal{R}}$  to the control variable  $u \in \mathcal{U}$ ,  $\mathcal{Y}_{\mathcal{R}}$  and  $\mathcal{U}$  being the spaces of states and controls, respectively. Generally, the relation  $\mathcal{R}$  can be chosen as an ordinary differential equation (ODE), a partial differential equation (PDE), a differential inclusion (DI), a variational inequality (VI) and a hemivariational inequality (hVI).

The optimal control problem under consideration reads as follows: find  $(u^*, y^*) \in \Lambda$  which minimizes the cost functional  $\mathcal{F}$ :

where the set  $\Lambda$  of admissible control-state pairs is defined by:

$$\Lambda = \text{graph } \mathcal{S}_{\mathcal{R}} = \{(u, y) : y \in \mathcal{S}_{\mathcal{R}}(u), u \in \mathcal{U}\}$$

and the solution map is given by

$$\mathcal{S}_{\mathcal{R}} : \mathcal{U} \ni u \longrightarrow \mathcal{S}_{\mathcal{R}}(u) = \{y \in \mathcal{Y}_{\mathcal{R}} : (u, y) \in \mathcal{R}\} \subset \mathcal{Y}_{\mathcal{R}}.$$

The set of optimal solutions to  $(CP)_{\mathcal{R}}$  is denoted by  $\mathcal{S}_{\mathcal{R}}^*$ , i.e.

$$\mathcal{S}_{\mathcal{R}}^* = \{(u^*, y^*) \in \Lambda : \mathcal{F}(u^*, y^*) = m\}.$$

The sensitivity (stability) is understood as a "nice-continuous" asymptotic behavior of optimal solutions to the perturbed problems, i.e. perturbed state relations  $\mathcal{R}_k$  and perturbed cost functionals  $\mathcal{F}_k$ . So we consider the sequence of optimal control problems indexed by  $k \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ , where the index  $k \in \mathbb{N}$  indicates "a perturbation" and  $k = \infty$  corresponds to the unperturbed original problem:

$$(CP)_{\mathcal{R}_k} \quad \text{minimize} \quad \{\mathcal{F}_k(u, y) : (u, y) \in \Lambda_k\} \quad (= \mathcal{F}_k(u_k^*, y_k^*) =: m_k)$$

and  $\Lambda_k = \text{graph } \mathcal{S}_{\mathcal{R}_k}$ . We are looking for conditions which assure the following stability results:

- (i)  $m_k \rightarrow m_\infty$  as  $k \rightarrow \infty$ ,
- (ii)  $K\text{-}\limsup \mathcal{S}_{\mathcal{R}_k}^* \subset \mathcal{S}_{\mathcal{R}_\infty}^*$ ,

where  $K\text{-}\limsup$  stands for the Kuratowski upper limit of sets. It is worth to recall (see e.g. Proposition 4.3 of Denkowski and Mortola, 1993) that (ii) is equivalent to the following condition: if  $\{k_n\}$  is an increasing sequence in  $\mathbb{N}$ ,  $(u_{k_n}^*, y_{k_n}^*) \in \mathcal{S}_{\mathcal{R}_{k_n}}^*$ ,  $u_{k_n}^*$  converges to  $u_\infty^*$  in  $\mathcal{U}$  and  $y_{k_n}^*$  converges to  $y_\infty^*$  in  $\mathcal{Y}_{\mathcal{R}}$ , then  $(u_\infty^*, y_\infty^*) \in \mathcal{S}_{\mathcal{R}_\infty}^*$ .

In order to establish the conditions (i) and (ii), first we reformulate the problem  $(CP)_{\mathcal{R}_k}$  as the unconstrained optimization one:

$$(CP)_{\mathcal{R}_k} \quad \text{minimize} \quad \{\mathcal{F}_k(u, y) + \chi_{\Lambda_k}(u, y) : (u, y) \in \mathcal{U} \times \mathcal{Y}_{\mathcal{R}_k}\},$$

where  $\chi_\Lambda$  denotes the indicator function of the set  $\Lambda$ , i.e.

$$\chi_\Lambda(x) = \begin{cases} 0 & x \in \Lambda \\ +\infty & x \notin \Lambda \end{cases}$$

and then we apply an approach based on the theory of  $\Gamma$ -convergence (epi-convergence), see De Giorgi and Spagnolo (1973), Spagnolo (1975), Buttazzo and Dal Maso (1982), and the references therein.

Another possible approach can be based on "discrete convergence", see Grig-

### 3. Preliminaries

For the convenience of the reader in this section we recall some material from the  $\Gamma$ -convergence theory, the generalized Clarke subdifferential and the theory of multivalued operators of monotone type.

#### 3.1. Sequential $\Gamma$ -convergence

We quote here the definition of  $\Gamma_{seq}$ -convergence for functions of two variables. The case of one variable follows easily by omitting the other. For the case of functions of many variables we refer to Buttazzo and Dal Maso (1982).

Let  $\mathcal{U}$  and  $\mathcal{Y}$  be two topological spaces. For  $u \in \mathcal{U}$  and  $y \in \mathcal{Y}$  we put  $\sigma_u := \{\{u_k\} \subset \mathcal{U} : u_k \rightarrow u\}$  and  $\sigma_y := \{\{y_k\} \subset \mathcal{Y} : y_k \rightarrow y\}$ . Given  $\mathcal{F}_k: \mathcal{U} \times \mathcal{Y} \rightarrow \mathbb{R} = \mathbb{R} \cup \{\pm\infty\}$ ,  $k \in \mathbb{N}$ , we define

$$\Gamma_{seq}(\mathcal{U}^-, \mathcal{Y}^+) \liminf_{k \rightarrow \infty} \mathcal{F}_k(u, y) = \inf_{\sigma_u} \sup_{\sigma_y} \liminf_{k \rightarrow \infty} \mathcal{F}_k(u_k, y_k),$$

$$\Gamma_{seq}(\mathcal{U}^-, \mathcal{Y}^+) \limsup_{k \rightarrow \infty} \mathcal{F}_k(u, y) = \inf_{\sigma_u} \sup_{\sigma_y} \limsup_{k \rightarrow \infty} \mathcal{F}_k(u_k, y_k),$$

and if both these extended numbers are equal, we say that there exists

$$(j) \quad \Gamma_{seq}(\mathcal{U}^-, \mathcal{Y}^+) \lim_{k \rightarrow \infty} \mathcal{F}_k(u, y).$$

Similarly, for other combinations of signs (+ and - denote sup and inf, respectively) we have

$$\Gamma_{seq}(\mathcal{U}^-, \mathcal{Y}^-) \liminf_{k \rightarrow \infty} \mathcal{F}_k(u, y) = \inf_{\sigma_u} \inf_{\sigma_y} \liminf_{k \rightarrow \infty} \mathcal{F}_k(u_k, y_k),$$

$$\Gamma_{seq}(\mathcal{U}^-, \mathcal{Y}^-) \limsup_{k \rightarrow \infty} \mathcal{F}_k(u, y) = \inf_{\sigma_u} \inf_{\sigma_y} \limsup_{k \rightarrow \infty} \mathcal{F}_k(u_k, y_k),$$

and if they are equal there exists

$$(jj) \quad \Gamma_{seq}(\mathcal{U}^-, \mathcal{Y}^-) \lim_{k \rightarrow \infty} \mathcal{F}_k(u, y).$$

In turn, if the numbers in (j) and (jj) are equal, we say that there exists

$$\Gamma_{seq}(\mathcal{U}^-, \mathcal{Y}^\pm) \lim_{k \rightarrow \infty} \mathcal{F}_k(u, y)$$

and then we write simply

$$\Gamma_{seq}(\mathcal{U}^-, \mathcal{Y}) \lim_{k \rightarrow \infty} \mathcal{F}_k(u, y) = (j) = (jj).$$



Now we are in a position to formulate

**THEOREM 3.1** (*De Giorgi and Franzoni, 1975*) *Let  $X$  be a topological space and let  $f_k: X \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ ,  $k \in \mathbb{N}$  be such that  $f_\infty = \Gamma(X^-) \lim_{k \rightarrow \infty} f_k$ . If*

$$\liminf_{k \rightarrow \infty} f_k(\hat{x}_k) = \liminf_{k \rightarrow \infty} \left( \inf_X f_k(x) \right)$$

(in this case  $\hat{x}_k$  is called "quasioptimal") and

$$\hat{x}_{k_n} \rightarrow \hat{x}_\infty \text{ as } n \rightarrow \infty,$$

then  $f_\infty(\hat{x}_\infty) = \inf_X f_\infty(x) = \lim_{k \rightarrow \infty} f_k(\hat{x}_k)$ .

In the sequel we put

$$X = \mathcal{U} \times \mathcal{Y}_{\mathcal{R}}, \quad \mathcal{R} = (HVI_e) \text{ or } (HVI_p),$$

$$\mathcal{S}_k = \mathcal{S}_{\mathcal{R}_k} \text{ and } f_k(x) = \mathcal{F}_k(u, y) + \chi_{\Lambda_k}(u, y).$$

**REMARK 3.1** *If the topological space  $X$  satisfies the first axiom of countability, then the sequential  $\Gamma_{seq}(X^-)$ -convergence coincides (see Proposition 8.1 of Dal Maso, 1993) with the topological  $\Gamma(X^-)$ -convergence introduced by De Giorgi and Franzoni (1975). Moreover, the sequential  $\Gamma$ -limit operation is not additive, i.e. it is not enough to know  $\Gamma\text{-}\lim \mathcal{F}_k$  and  $\Gamma\text{-}\lim \chi_{\Lambda_k}$  in order to calculate  $\Gamma\text{-}\lim(\mathcal{F}_k + \chi_{\Lambda_k})$ , see Example 6.18 in Dal Maso (1993).*

In order to calculate the  $\Gamma$ -limit of the sum of two functions we use the following

**THEOREM 3.2** (*Buttazzo and Dal Maso, 1982*) *If*

$$\mathcal{F}(u, y) = \Gamma_{seq}(\mathcal{U}^-, \mathcal{Y}) \lim_{k \rightarrow \infty} \mathcal{F}_k(u, y),$$

$$\mathcal{G}(u, y) = \Gamma_{seq}(\mathcal{U}, \mathcal{Y}^-) \lim_{k \rightarrow \infty} \mathcal{G}_k(u, y),$$

then

$$\mathcal{F}(u, y) + \mathcal{G}(u, y) = \Gamma_{seq}(\mathcal{U}^-, \mathcal{Y}^-) \lim_{k \rightarrow \infty} (\mathcal{F}_k(u, y) + \mathcal{G}_k(u, y)).$$

Thus, due to the above theorem, the convergences

- (i)  $m_k \rightarrow m_\infty$  (of minimal values) and
- (ii)  $K\text{-}\limsup \mathcal{S}_{\mathcal{R}_k}^* \subset \mathcal{S}_{\mathcal{R}_{c,j}}^*$ ,

follow from the following result (see also Propositions 4.1 and 4.5 in Denkowski

PROPOSITION 3.1 *Suppose*

$$\mathcal{F}_\infty = \Gamma_{seq}(\mathcal{U}^-, \mathcal{Y}_{\mathcal{R}}) \lim \mathcal{F}_k, \quad (1)$$

$$\chi_{\Lambda_\infty} = \Gamma_{seq}(\mathcal{U}, \mathcal{Y}_{\mathcal{R}}^-) \lim \chi_{\Lambda_k}. \quad (2)$$

Let  $(\hat{u}_k, \hat{y}_k)$  be optimal or "quasioptimal" solutions to the problems  $(CP)_{\mathcal{R}_k}$  such that

$$\liminf_{k \rightarrow \infty} \mathcal{F}_k(\hat{u}_k, \hat{y}_k) = \liminf_{k \rightarrow \infty} \left( \inf_{\mathcal{U} \times \mathcal{Y}_{\mathcal{R}_k}} \mathcal{F}_k \right) \quad (3)$$

and

$$(\hat{u}_{k_n}, \hat{y}_{k_n}) \rightarrow (\hat{u}_\infty, \hat{y}_\infty) \text{ as } n \rightarrow \infty. \quad (4)$$

Then  $\mathcal{F}_\infty(\hat{u}_\infty, \hat{y}_\infty) = \inf_{\Lambda_\infty} \mathcal{F}_\infty(u, y) = \lim_{k \rightarrow \infty} \left( \inf_{\Lambda_k} \mathcal{F}_k(u, y) \right)$ .

REMARK 3.2 The condition (2) of Proposition 3.1 is equivalent (see Proposition 4.3 of Denkowski and Mortola, 1993) to the Kuratowski convergence

$$(2') \quad \mathcal{S}_k(u_k) \xrightarrow{K(\mathcal{Y}_{\mathcal{R}})} \mathcal{S}_\infty(u) \text{ for all } u_k \xrightarrow{\mathcal{U}} u$$

i.e.

$$(2'')$$

$$K(\mathcal{Y}_{\mathcal{R}})\text{-}\limsup \mathcal{S}_k(u_k) \subset \mathcal{S}_\infty(u) \subset K(\mathcal{Y}_{\mathcal{R}})\text{-}\liminf \mathcal{S}_k(u_k) \text{ for all } u_k \xrightarrow{\mathcal{U}} u$$

while the condition (1) (the complementary  $\Gamma$ -convergence), roughly speaking, means a "continuous convergence" of cost functionals with respect to  $y$  and  $\Gamma(\mathcal{U}^-)$  convergence with respect to  $u$ . We recall that for a sequence of sets  $\{A_n\}_{n \in \mathbb{N}}$  in the topological space  $X$ , by  $K\text{-}\liminf A_n$  we mean the set of all limits of sequences  $\{x_n\}$  such that  $x_n \in A_n$ , while the set  $K\text{-}\limsup A_n$  consists of all limits of subsequences  $\{x_k\}$  such that  $x_k \in A_{n_k}$  for any increasing sequence  $\{n_k\} \subset \{n\}$ .

### 3.2. G-convergence of multivalued elliptic operators

Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^N$  with Lipschitz boundary. Following Chiado'Piat, Dal Maso and Defranceschi (1990), for fixed  $m_i \in L^1(\Omega)$ ,  $c_i > 0$ ,  $i = 1, 2$ , we introduce the following class of multivalued operators

$$\mathcal{M}_\Omega(\mathbb{R}^N) = \{a: \Omega \times \mathbb{R}^N \rightarrow 2^{\mathbb{R}^N} \text{ such that (i) - (iii) below hold}\}$$

(i)  $a(x, \xi)$  is maximal monotone with respect to  $\xi$  for all  $x \in \Omega$ ;

(ii)  $a$  is  $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{B}(\mathbb{R}^N)$  measurable

(iii) for every  $(\xi, \eta)$  with  $\eta \in a(x, \xi)$  we have

$$|\eta|^q \leq m_1(x) + c_1(\eta, \xi)_{\mathbb{R}^N} \quad (5)$$

$$|\xi|^p \leq m_2(x) + c_2(\eta, \xi)_{\mathbb{R}^N}. \quad (6)$$

REMARK 3.3 *The main examples of maps  $a \in \mathcal{M}_\Omega(\mathbb{R}^N)$  have the form*

$$a(x, \xi) = \partial_\xi \psi(x, \xi) \text{ for some } \psi: \Omega \times \mathbb{R}^N \rightarrow [0, +\infty),$$

*where  $\psi$  is measurable in both variables, convex in  $\xi$ , and satisfies*

$$c_1|\xi|^p \leq \psi(x, \xi) \leq c_2|\xi|^p \quad (7)$$

*with suitable constants  $0 < c_1 \leq c_2$ .*

DEFINITION 3.1 *For every function  $a \in \mathcal{M}_\Omega(\mathbb{R}^N)$  we define multivalued operators*

$$\overline{A}: W^{1,p}(\Omega) \ni y \rightarrow \overline{A}y := \{\eta \in L^q(\Omega; \mathbb{R}^N) : \eta(x) \in a(x, Dy(x)) \text{ a.e.}\},$$

$$A: W^{1,p}(\Omega) \ni y \rightarrow Ay := \{-\operatorname{div} \eta : \eta \in \overline{A}y\} \subset (W^{1,p}(\Omega))^*.$$

In the space  $L^q(\Omega; \mathbb{R}^N)$  we define topology  $\sigma$  according to:

DEFINITION 3.2

$$\eta_k \xrightarrow{\sigma} \eta \text{ if and only if } \begin{cases} \eta_k \rightarrow \eta \text{ in } w - L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} \eta_k \rightarrow \operatorname{div} \eta \text{ in } s - (W^{1,p}(\Omega))^*. \end{cases}$$

For  $1 < p < \infty$ , we admit the following definition of multivalued  $G$ -convergence.

DEFINITION 3.3 *We say that a sequence  $\{a_k\} \in \mathcal{M}_\Omega(\mathbb{R}^N)$   $G$ -converges to  $a \in \mathcal{M}_\Omega(\mathbb{R}^N)$  and we write  $a_k \xrightarrow{G} a$  if*

$$K(w, \sigma)\text{-}\limsup Gr\overline{A}_k \subset Gr\overline{A}.$$

We recall that the compactness of the class  $\mathcal{M}_\Omega(\mathbb{R}^N)$  with respect to the notion of  $G$ -convergence given in Definition 3.3 was proved by Chiado'Piat, Dal Maso and Defranceschi (1990). The definition of  $G$ -convergence and its properties for linear operators go back to De Giorgi and Spagnolo (1973), Spagnolo (1967, 1975), and Colombini and Spagnolo (1977).

PROPOSITION 3.2 (see Theorem 3.11 of Chiado'Piat, Dal Maso and Defranceschi, 1990) *If  $a_k, a \in \mathcal{M}_\Omega(\mathbb{R}^N)$  are such that  $a_k \xrightarrow{G} a$ , then*

$$K(w - V, s - V^*)\text{-}\lim_{k \rightarrow \infty} GrA_k = GrA.$$

(For the latter we also write  $GrA_k \xrightarrow{K(w-V, s-V^*)} GrA$ ).

The inverse of the Proposition 3.2 does not hold (see Remark 3.13 of Chia-



### 3.3. Clarke subdifferential

Given a locally Lipschitz function  $J: Z \rightarrow \mathbb{R}$ , where  $Z$  is a Banach space, we recall (see Clarke, 1983) the definitions of the generalized directional derivative and the generalized gradient of Clarke. The generalized directional derivative of  $J$  at a point  $u \in Z$  in the direction  $v \in Z$ , denoted by  $J^0(u; v)$ , is defined by

$$J^0(u; v) = \limsup_{y \rightarrow u, t \downarrow 0} \frac{J(y + tv) - J(y)}{t}.$$

The generalized gradient of  $J$  at  $u$ , denoted by  $\partial J(u)$ , is a subset of a dual space  $Z^*$  given by  $\partial J(u) = \{\zeta \in Z^* : J^0(u; v) \geq \langle \zeta, v \rangle_{Z^* \times Z} \text{ for all } v \in Z\}$ . The locally Lipschitz function  $J$  is called regular (in the sense of Clarke) at  $u \in Z$  if for all  $v \in Z$  the one-sided directional derivative  $J'(u; v)$  exists and satisfies  $J^0(u; v) = J'(u; v)$  for all  $v \in Z$ .

We recall a result concerning the Clarke subdifferential of the integral functional (see Theorem 2.7.5 of Clarke, 1983). Let  $\Omega$  be a bounded subset of  $\mathbb{R}^N$ ,  $1 \leq p < \infty$  and let  $f: \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ . We assume that:

- (i)  $f(\cdot, \xi)$  is measurable for all  $\xi \in \mathbb{R}^d$ ,  $f(\cdot, 0)$  is integrable;
- (ii)  $f(x, \cdot)$  is locally Lipschitz for each  $x \in \Omega$ ;
- (iii) there exists a constant  $c > 0$  such that for every  $\zeta \in \partial_v f(x, v)$ , we have

$$\|\zeta\|_{\mathbb{R}^d} \leq c \left(1 + \|v\|_{\mathbb{R}^d}^{p-1}\right).$$

**THEOREM 3.3** *Under the above hypotheses, the functional  $F: L^p(\Omega; \mathbb{R}^d) \rightarrow \mathbb{R}$  defined by  $F(v) = \int_{\Omega} f(x, v(x)) dx$  for  $v \in L^p(\Omega; \mathbb{R}^d)$  is well-defined and locally Lipschitz (in fact, Lipschitz continuous on bounded subsets of  $L^p(\Omega; \mathbb{R}^d)$ ) and we have*

$$\partial F(v) \subset \int_{\Omega} \partial_v f(x, v(x)) dx \text{ for } v \in L^p(\Omega; \mathbb{R}^d).$$

*The latter means that for any  $z \in \partial F(v)$ , there is a function  $\zeta \in L^q(\Omega; \mathbb{R}^d)$ ,  $1/p + 1/q = 1$  satisfying  $\zeta(x) \in \partial_v f(x, v(x))$  for a.e.  $x \in \Omega$  and such that for all  $y \in L^p(\Omega; \mathbb{R}^d)$  we have  $\langle \zeta, y \rangle_{L^q \times L^p} = \int_{\Omega} (\zeta(x), y(x))_{\mathbb{R}^d} dx$ .*

### 3.4. Multivalued operators

We give the basic definitions for multivalued operators and then we quote two main surjectivity results for the operator classes under consideration (see e.g. Denkowski et al., 2003b; Naniewicz and Panagiotopoulos, 1995; Papageorgiou et al., 1999).

Let  $Y$  be a real reflexive Banach space and  $Y^*$  be its dual space and let  $T: Y \rightarrow 2^{Y^*}$  be a multivalued operator. We say that  $T$  is:

- (1) upper semicontinuous if for any closed subset  $C \subseteq Y^*$  the set  $T^-(C) =$

(2) pseudomonotone if the following conditions hold:

- a) the set  $Ty$  is nonempty, bounded, closed and convex for each  $y \in Y$ ;
- b)  $T$  is upper semicontinuous from each finite-dimensional subspace of  $Y$  to  $Y^*$  furnished with the weak topology;
- c) if  $\{y_n\} \subseteq Y$ ,  $y_n \rightarrow y$  weakly in  $Y$ ,  $y_n^* \in Ty_n$ , and  $\limsup_{n \rightarrow +\infty} \langle y_n^*, y_n - y \rangle \leq 0$ , then for each element  $v \in Y$  there exists  $y^*(v) \in Ty$  such that  $\liminf_{n \rightarrow +\infty} \langle y_n^*, y_n - y \rangle \geq \langle y^*(v), y - v \rangle$ ;

Let  $L: D(L) \subset Y \rightarrow Y^*$  be a linear, densely defined and maximal monotone operator.

(3)  $T$  is  $L$ -generalized pseudomonotone if the following conditions hold:

- a) for every  $y \in Y$ ,  $Ty$  is a nonempty, convex and weakly compact subset of  $Y^*$ ;
- b)  $T$  is upper semicontinuous from each finite-dimensional subspace of  $Y$  into  $Y^*$  equipped with the weak topology;
- c) if  $\{y_n\} \subseteq D(L)$ ,  $y_n \rightarrow y$  weakly in  $Y$ ,  $y \in D(L)$ ,  $Ly_n \rightarrow Ly$  weakly in  $Y^*$ ,  $y_n^* \in Ty_n$ ,  $y_n^* \rightarrow y^*$  weakly in  $Y^*$  and  $\limsup_{n \rightarrow +\infty} \langle y_n^*, y_n - y \rangle \leq \langle y^*, y \rangle$ , then  $y^* \in Ty$  and  $\langle y_n^*, y_n \rangle \rightarrow \langle y^*, y \rangle$ .

The crucial point in the proofs of the existence of a solution to the hemivariational inequalities considered below are the following surjectivity results.

**PROPOSITION 3.3** *If  $Y$  is a reflexive Banach space, and  $T: Y \rightarrow 2^{Y^*} \setminus \{\emptyset\}$  is a pseudomonotone and coercive operator, then  $T$  is surjective.*

**PROPOSITION 3.4** *If  $Y$  is a reflexive, strictly convex Banach space,  $L: D(L) \subset Y \rightarrow Y^*$  is a linear, densely defined, maximal monotone operator and  $T: Y \rightarrow 2^{Y^*} \setminus \{\emptyset\}$  is a bounded, coercive and  $L$ -generalized pseudomonotone operator, then  $L + T$  is surjective.*

The proof of Proposition 3.3 can be found in Denkowski et al. (2003b), Theorem 1.3.70, while the proof of Proposition 3.4 can be found in Papageorgiou et al. (1999), Theorem 2.1, p.345.

## 4. Control problem for elliptic hemivariational inequality

In this section we deliver a sensitivity result for optimal control problem for systems governed by stationary hemivariational inequality. First we give an existence theorem for elliptic HVI, then we provide results on the sensitivity of the solution set and on the convergence of the cost functionals.

Given an open bounded set  $\Omega \subset \mathbb{R}^N$  with Lipschitz boundary, we introduce the following spaces  $V = W_0^{1,p}(\Omega)$ ,  $Z = L^p(\Omega)$ ,  $H = L^2(\Omega)$ ,  $Z^* = L^q(\Omega)$ ,

evolution fivefold of spaces  $V \subset Z \subseteq H \subseteq Z^* \subset V^*$  with compact embedding  $V \subset Z$ .

We consider the following sequence of hemivariational inequalities

$$(HVI_e)_k \quad \langle A_k y, v - y \rangle + J_k^0(y; v - y) \geq \langle f_k + C_k u, v - y \rangle, \quad \forall v \in V.$$

The hypotheses on the data of  $(HVI_e)_k$  are the following:

$H(A)$ :  $A_k: V \rightarrow 2^{V^*}$  are multivalued pseudomonotone, bounded and coercive operators;

$H(J)$ :  $J_k: Z \rightarrow \mathbb{R}$  are locally Lipschitz functions such that

(i)  $\|\partial J_k(z)\|_{Z^*} \leq \tilde{c}_1(1 + \|z\|_Z^{2/q})$  for all  $z \in Z$  and for some  $\tilde{c}_1 > 0$ ;

(ii)  $J_k^0(z; -z) \leq \tilde{c}_2(1 + \|z\|^r)$  for all  $z \in Z$  with  $r < p$  and  $\tilde{c}_2 \geq 0$ ;

$(H_0)$ :  $f_k \in V^*$ .

$H(C)$ :  $C_k \in \mathcal{L}(\mathcal{U}, V^*)$ , where  $\mathcal{U}$  is a reflexive separable Banach space modeling the control space.

We remark that the problem  $(HVI_e)_k$  is equivalent to the following differential inclusion

$$\begin{cases} A_k y + \partial J_k(y) \ni f_k + C_k u \\ y \in V \end{cases}$$

where  $\partial J_k$  denotes the Clarke subdifferential of  $J_k$ . Given  $u \in \mathcal{U}$ , by a solution of  $(HVI_e)_k$  we mean an element  $y \in V$  such that  $A_k y + \eta_k = f_k + C_k u$  with some  $\eta_k \in \partial J_k(y)$  and  $\eta_k \in Z^*$ .

**PROPOSITION 4.1** *If hypotheses  $H(A)$ ,  $H(J)$ ,  $H(C)$  and  $(H_0)$  hold, then for a fixed  $k \in \mathbb{N}$  and for all  $u \in \mathcal{U}$ , we have  $S_k(u) := S_{(HVI_e)_k}(u) \neq \emptyset$ . Moreover, if  $A_k: V \rightarrow 2^{V^*}$  is strongly monotone and  $\partial J_k$  is monotone, then  $S_k(u) = \{y_k\}$  (i.e. we have the uniqueness of solution).*

*Proof.* The above existence result follows from Proposition 3.3 (see also Chapter 4.3 of Naniewicz and Panagiotopoulos, 1995). To this end, it is enough to remark that if the operator  $A_k$  is coercive, i.e.  $\langle A_k v, v \rangle \geq \alpha(\|v\|)\|v\|$  for all  $v \in V$  with  $\alpha: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ ,  $\alpha(t) \rightarrow +\infty$  as  $t \rightarrow +\infty$  and  $J_k^0(z; -z) \leq \tilde{c}_2(1 + \|z\|^r)$  for all  $z \in Z$ , then  $A_k + \partial J_k$  is a coercive operator. The uniqueness is a consequence of the strong monotonicity of  $A_k + \partial J_k$ . ■

**REMARK 4.1** *A simple example of a superpotential  $J_k$  which satisfies  $H(J)$  is an integral functional  $J_k: Z \rightarrow \mathbb{R}$ ,*

$$J_k(z) = \int \tilde{j}_k(x, z(x)) \, dx, \quad z \in Z = L^p(\Omega).$$

where the integrand  $j_k: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is given by  $j_k(x, \xi) = \min\{g_1(\xi), g_2(\xi)\}$ . We suppose that  $g_i: \mathbb{R} \rightarrow \mathbb{R}$ ,  $g_i(x) = \alpha_i x^2 + \beta_i$ ,  $\alpha_i > 0$  for  $i = 1, 2$ . Using Theorem 2.5.1 of Clarke (1983), we know that  $\partial j_k(x, \xi) \subset \text{co}\{g'_1(\xi), g'_2(\xi)\}$  and hence the subdifferential  $\partial j_k(x, \cdot)$  has at most a linear growth. So, by exploiting Theorem 3.3, we obtain  $H(J)(i)$ . Next, by Proposition 2.1.2 of Clarke (1983), we have  $j_k^0(x, \xi; \eta) = \max\{\xi^* \eta : \xi^* \in \partial j_k(x, \xi)\}$ . Therefore

$$j_k^0(x, \xi; -\xi) = \max\{\xi^*(-\xi) : \xi^* = \lambda g'_1(\xi) + (1-\lambda)g'_2(\xi), \lambda \in (0, 1)\} \leq 0,$$

because  $g'_i(\xi) \xi \geq 0$ ,  $i = 1, 2$ . Hence and from the inequality

$$J_k^0(z; v) \leq \int_{\Omega} j_k^0(x, z(x); v(x)) dx \quad \text{for all } z, v \in Z$$

(which is a consequence of the Fatou lemma), it follows that  $H(J)(ii)$  holds with  $\tilde{c}_2 = 0$ .

#### 4.1. Sensitivity of solution sets for $(HVI_e)_k$

We are now in a position to state the result on the Kuratowski convergence of the solution sets for elliptic hemivariational inequalities.

**PROPOSITION 4.2** *In addition to the hypotheses of Proposition 4.1, we suppose the operators  $A_k$  in  $(HVI_e)_k$  correspond (see Definition 3.1) to multifunctions  $a_k \in \mathcal{M}_{\Omega}(\mathbb{R}^N)$  and assume*

- (i)  $Gr A_k \xrightarrow{K(w-V, s-V^*)} Gr A_{\infty}$
- (ii)  $K(s-Z, w-Z^*)\text{-}\limsup_{k \rightarrow \infty} Gr \partial J_k \subset Gr \partial J_{\infty}$
- (iii)  $C_k, C_{\infty} \in \mathcal{L}(\mathcal{U}, V^*)$ ,  $C_k \xrightarrow{c} C_{\infty}$  continuously
- (iv)  $f_k \rightarrow f_{\infty}$  in  $s-V^*$ .

Then

- 1° for every  $k \in \overline{\mathbb{N}}$ ,  $S_k(u) = S_{(HVI_e)_k}(u) \neq \emptyset$  for all  $u \in \mathcal{U}$ ;
- 2°  $K(w-V)\text{-}\limsup_{k \rightarrow \infty} S_k(u_k) \subset S_{\infty}(u_{\infty})$ , for all  $u_k \xrightarrow{\mathcal{U}} u_{\infty}$ .

Moreover,

- (v) if  $S_{\infty}(u_{\infty}) = \{y_{\infty}\}$  and for every  $u_k \rightarrow u_{\infty}$  we can find weakly compact sequence of solutions  $y_k \in S_k(u_k)$ ,  $k \in \mathbb{N}$ ,

then

- 3°  $S_{\infty}(u_{\infty}) \subset K(w-V)\text{-}\liminf S_k(u_k)$ .



REMARK 4.2 *The hypothesis (i) of Proposition 4.2 follows, for instance, if  $a_k \xrightarrow{G} a_\infty$  (see Proposition 3.2). The assumption (ii) holds, for example, if  $J_k: Z \rightarrow \mathbb{R}$  are locally Lipschitz, equi-lower semidifferentiable, locally equi-bounded and  $J_k \xrightarrow{\Gamma_{seq}} J_\infty$  (see Theorem 1 of Zolezzi, 1994). The continuous convergence of  $C_k$  to  $C_\infty$  in (iii) means that for every  $u_k \xrightarrow{\mathcal{U}} u_\infty$  we have  $C_k u_k \xrightarrow{V^*} C_\infty u_\infty$ . The condition (v) is satisfied in the case  $\{S_k(u)\}$  are equicoercive for all  $u \in \mathcal{U}$  and  $k \in \mathbb{N}$  (which happens if  $A_k$  are equicoercive; e.g.  $A_k = -\operatorname{div} a_k$ ,  $a_k \in \mathcal{M}_\Omega(\mathbb{R}^N)$ ).*

COROLLARY 4.1 *Under the assumptions (i)–(v) of Proposition 4.2, we have*

$$S_k(u_k) \xrightarrow{K(w-V)} S_\infty(u_\infty) \text{ for all } u_k \xrightarrow{\mathcal{U}} u_\infty.$$

*Proof of Proposition 4.2.* The existence of solution to the problem  $(HVI_e)_k$  follows from Proposition 4.1. For the proof of  $2^\circ$ , let  $u_k \xrightarrow{\mathcal{U}} u_\infty$  and  $y_\infty \in K(w-V)\text{-}\limsup_{k \rightarrow \infty} S_k(u_k)$ . Thus we can find a sequence  $\{k_n\} \subset \mathbb{N}$  and  $\{y_{k_n}\} \subset V$  such that  $y_{k_n} \in S_{k_n}(u_{k_n})$  and  $y_{k_n} \rightarrow y_\infty$  weakly in  $V$ . Clearly  $A_{k_n} y_{k_n} + \eta_{k_n} = f_{k_n} + C_{k_n} u_{k_n}$  with  $\eta_{k_n} \in \partial J_{k_n}(y_{k_n})$ . From hypothesis  $H(J)(i)$ , we know that  $\{\eta_{k_n}\}$  lies in a bounded subset of  $Z^*$  and so we may assume that

$$\eta_{k_n} \rightarrow \eta_\infty \text{ weakly in } Z^* \quad (8)$$

for some  $\eta_\infty \in Z^*$ . Since  $(y_{k_n}, \eta_{k_n}) \in \operatorname{Gr} \partial J_{k_n}$  and  $(y_{k_n}, \eta_{k_n}) \rightarrow (y_\infty, \eta_\infty)$  in  $(s-Z) \times (w-Z^*)$  topology, by the assumption (ii) we deduce that

$$\eta_\infty \in \partial J_\infty(y_\infty). \quad (9)$$

Next, from hypotheses (iii) and (iv), (8) and the compactness of the embedding  $Z^* \subset V^*$ , it follows that

$$A_{k_n} y_{k_n} = f_{k_n} + C_{k_n} u_{k_n} - \eta_{k_n} \rightarrow f_\infty + C_\infty u_\infty - \eta_\infty \text{ in } V^*.$$

By the assumption (i), we obtain  $f_\infty + C_\infty u_\infty - \eta_\infty = A_\infty y_\infty$  which, together with (9), implies  $y_\infty \in S_\infty(u_\infty)$  and finishes the proof of  $2^\circ$ .

Finally, the conclusion in  $3^\circ$  follows from  $2^\circ$  and the following Urysohn property of the Kuratowski convergence:  $S_k(u_k) \xrightarrow{K(w-V)} S_\infty(u_\infty)$  for  $u_k \xrightarrow{\mathcal{U}} u_\infty$  if and only if every subsequence of  $S_k(u_k)$  contains a further subsequence which  $K(w-V)$ -converges to  $S_\infty(u_\infty) = \{y_\infty\}$ . The proof of the proposition is finished.  $\blacksquare$

We close this section by providing the sufficient conditions for the integral functionals under which the hypothesis (ii) of Proposition 4.2 holds. Consider the functionals  $J_k: Z \rightarrow \mathbb{R}$ ,  $k \in \mathbb{N} \cup \{\infty\}$  of the form

$$J_k(z) = \int i_k(x, z(x)) dx \text{ for } z \in Z = L^p(\Omega)$$

We admit the following assumption:

$H(j)$ :  $j_k: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $k \in \mathbb{N} \cup \{\infty\}$ , are such that

- (i)  $j_k(\cdot, \xi)$  are measurable for all  $\xi \in \mathbb{R}$  and  $j_k(\cdot, 0) \in L^1(\Omega)$ ;
- (ii)  $j_k(x, \cdot)$  are locally Lipschitz for all  $x \in \Omega$ ;
- (iii)  $|\eta| \leq c_3 (1 + |\xi|^{p-1})$  for all  $\eta \in \partial j_k(x, \xi)$  with  $c_3 \geq 0$ ;
- (iv)  $j_\infty(x, \cdot)$  is regular in the sense of Clarke;
- (v)  $K\text{-}\limsup_{k \rightarrow \infty} Gr \partial j_k(x, \cdot) \subset Gr \partial j_\infty(x, \cdot)$  for  $x \in \Omega$ .

PROPOSITION 4.3 *Under hypothesis  $H(j)$ , we have*

$$K(s\text{-}Z, w\text{-}Z^*)\text{-}\limsup_{k \rightarrow \infty} Gr \partial J_k \subset Gr \partial J_\infty.$$

*Proof.* Let  $(u, v) \in K(s\text{-}Z, w\text{-}Z^*)\text{-}\limsup_{k \rightarrow \infty} Gr \partial J_k$ . Then there is a sequence  $\{k_n\}$  of  $\mathbb{N}$  and  $(u_{k_n}, v_{k_n}) \in Z \times Z^*$  such that

$$v_{k_n} \in \partial J_{k_n}(u_{k_n}) \quad (10)$$

$$u_{k_n} \rightarrow u \text{ in } Z \quad (11)$$

$$v_{k_n} \rightarrow v \text{ weakly in } Z^*. \quad (12)$$

We will prove that  $(u, v) \in Gr J_\infty$ . Since the integrands  $j_k$  satisfy  $H(j)(i)$ ,  $(ii)$  and  $(iii)$ , we apply Theorem 3.3 to the functional  $J_k$  and we obtain that it is bounded on every bounded subset of  $Z$  and for every  $k \in \mathbb{N}$ , we have

$$\partial J_k(z) \subset \int_{\Omega} \partial j_k(x, z(x)) dx \text{ for all } z \in Z. \quad (13)$$

From (10) and (13), we have  $v_{k_n} \in \int_{\Omega} \partial j_{k_n}(x, u_{k_n}(x)) dx$ . The latter means (see Theorem 3.3) that there exists a sequence  $s_{k_n} \in Z^*$  satisfying

$$s_{k_n} \in \partial j_{k_n}(x, u_{k_n}(x)) \text{ a.e. } x \in \Omega \quad (14)$$

and such that, for every  $\varphi \in Z$ ,

$$\langle v_{k_n}, \varphi \rangle = \int_{\Omega} s_{k_n}(x) \varphi(x) dx. \quad (15)$$

Using  $H(j)(iii)$  from (14) we obtain

$$\|s_{k_n}\|_{Z^*} \leq c_4 \left(1 + \|u_{k_n}\|_Z^{p-1}\right) \text{ with } c_4 > 0. \quad (16)$$

From (16) and (11) we know that  $\{s_{k_n}\}$  remains in a bounded subset of  $Z^*$  and hence we may suppose



Again by (11) for a next subsequence, we have

$$u_{k_n}(x) \rightarrow u(x) \text{ for a.e. } x \in \Omega. \quad (18)$$

Combining (14), (17), (18) and applying Theorem 7.2.1 of Aubin and Frankowska (1990), we get

$$s(x) \in \overline{\text{conv}} K - \limsup_{z \rightarrow u(x), k \rightarrow \infty} \partial j_k(x, z) \subset \partial j_\infty(x, u(x)) \text{ a.e. } x \in \Omega. \quad (19)$$

The latter inclusion follows from  $H(j)(v)$  since  $\partial j_\infty(x, \cdot)$  has convex and closed values. Moreover, due to (12) and (17) we pass to the limit in (15) and we have

$$\langle v, \varphi \rangle = \int_{\Omega} s \varphi \, dx \text{ for every } \varphi \in Z. \quad (20)$$

From (19) and (20) we now infer  $v \in \int_{\Omega} \partial j_\infty(x, u(x)) \, dx$ . Applying again Theorem 3.3 to  $J_\infty$ , by exploiting the regularity assumption  $H(j)(iv)$ , we have  $\partial J_\infty(u) = \int_{\Omega} j_\infty(x, u(x)) \, dx$  which implies that  $v \in \partial J_\infty(u)$ . This means that  $(u, v) \in \text{Gr } \partial J_\infty$  and completes the proof. ■

## 4.2. Complementary $\Gamma$ -convergence of cost functionals

The goal of this section is to give conditions under which the cost functionals in the control problem  $(CP)_{(HVI)_k}$  for systems described by elliptic HVIs satisfy the convergence condition

$$\mathcal{F}_\infty = \Gamma_{seq}(\mathcal{U}^-, \mathcal{Y}_{\mathcal{R}}) \lim \mathcal{F}_k$$

of Proposition 3.1. The functionals  $\mathcal{F}_k: \mathcal{U} \times \mathcal{Y}_e \rightarrow \mathbb{R}$  have the form

$$\mathcal{F}_k(u, y) = \mathcal{F}_k^{(1)}(y) + \mathcal{F}_k^{(2)}(u), \quad u \in \mathcal{U}, \quad y \in \mathcal{Y}_e = V, \quad (21)$$

$$\mathcal{F}_k^{(1)}(y) = \int_{\Omega} F_k^{(1)}(x, y(x)) \, dx, \quad (22)$$

$$\mathcal{F}_k^{(2)}(u) = \int_{\Omega} F_k^{(2)}(x, (C_k u)(x)) \, dx. \quad (23)$$

Our aim is to assure that

$$1^0 \quad \text{for every } k \in \overline{\mathbb{N}}, \quad \mathcal{F}_k^{(1)}(\cdot) \text{ is } (w-V)\text{-lsc}, \quad \mathcal{F}_k^{(2)}(\cdot) \text{ is } \tau_{\mathcal{U}}\text{-lsc}$$

$$2^0 \quad \mathcal{F}_k^{(1)} \xrightarrow{\Gamma_{seq}(w-V^\pm)} \mathcal{F}_\infty^{(1)}, \quad \mathcal{F}_k^{(2)} \xrightarrow{\Gamma_{seq}(\mathcal{U}^-)} \mathcal{F}_\infty^{(2)}$$

To this end we admit the hypotheses  $\mathcal{U} = L^q(\Omega)$  with  $2 \leq p < \infty$  so  $1 < q \leq 2$  and  $C_k$  is the embedding of  $L^q(\Omega)$  into  $V^* = W^{-1,q}(\Omega)$ .

$H(F^{(1)})$ :  $F_k^{(1)}: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is measurable in  $x \in \Omega$ ,  $F_k^{(1)}(x, 0) \in L^p(\Omega)$  and

$$|F_k^{(1)}(x, z_1) - F_k^{(1)}(x, z_2)| \leq c_5(1 + |z_1|)|z_1 - z_2| \text{ with } c_5 > 0;$$

$$F_k^{(1)}(\cdot, z) \longrightarrow F_\infty^{(1)}(\cdot, z) \quad w\text{-}L^1(\Omega) \text{ for all } z \in \mathbb{R};$$

$H(F^{(2)})$ :  $F_k^{(2)}: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is measurable in  $x \in \Omega$ , convex in  $z \in \mathbb{R}$  and

$$0 \leq F_k^{(2)}(x, z) \leq \lambda|z|^q \quad \text{for some } \lambda > 0;$$

$$F_k^{(2)}(\cdot, z) \longrightarrow F_\infty^{(2)}(\cdot, z) \quad w\text{-}L^1(\Omega) \text{ for all } z \in \mathbb{R}.$$

PROPOSITION 4.4 *If hypotheses  $H(F^{(1)})$  and  $H(F^{(2)})$  hold, then*

1<sup>0</sup> *for  $k \in \overline{\mathbb{N}}$ ,  $\mathcal{F}_k^{(1)}(\cdot)$  is  $(s\text{-}L^p(\Omega))$ -continuous and also sequentially  $(w\text{-}W^{1,p}(\Omega))$ -continuous, and  $\mathcal{F}_k^{(2)}(\cdot)$  is  $(s\text{-}L^q(\Omega))$ -continuous;*

2<sup>0</sup>  $\mathcal{F}_k^{(1)}(y_k) \rightarrow \mathcal{F}_\infty^{(1)}(y_\infty)$ ,  $\forall y_k \xrightarrow{w\text{-}W^{1,p}} y_\infty$  and  $\mathcal{F}_k^{(2)}(u) \xrightarrow{\Gamma_{seq}(s\text{-}L^q(\Omega)^-)} \mathcal{F}_\infty^{(2)}(u)$ .

*Proof.* Ad 1<sup>0</sup>. From the hypotheses it follows that the intergrands  $F_k^{(1)}$ ,  $F_k^{(2)}$  ( $k \in \overline{\mathbb{N}}$ ) are Carathéodory type functions, i.e. they are measurable in  $x$ , continuous in  $z$  and bounded by integrable functions on bounded sets. So the functionals  $\mathcal{F}_k^{(1)}$ ,  $\mathcal{F}_k^{(2)}$  ( $k \in \overline{\mathbb{N}}$ ) are continuous, respectively on  $L^p(\Omega)$  and  $L^q(\Omega)$  in the strong topologies (see the Carathéodory Continuity Theorem in Example 1.22 of Dal Maso, 1993). Then,  $\mathcal{F}_k^{(1)}$  is also sequentially  $(w\text{-}W^{1,p}(\Omega))$ -continuous owing to the compactness of the embedding  $W^{1,p}(\Omega)$  into  $L^p(\Omega)$ .

Ad 2<sup>0</sup>. For the first convergence (continuous convergence) of  $\mathcal{F}_k^{(1)}$  in 2<sup>0</sup>, assume  $y_k \rightarrow y_\infty$  in  $w\text{-}W^{1,p}(\Omega)$ , so also in  $s\text{-}L^p(\Omega)$  (and in  $s\text{-}L^2(\Omega)$  as we have  $p \geq 2$ ). By the direct calculation we have

$$|\mathcal{F}_k^{(1)}(y_k) - \mathcal{F}_\infty^{(1)}(y_\infty)| \leq |\mathcal{F}_k^{(1)}(y_k) - \mathcal{F}_k^{(1)}(y_\infty)| + |\mathcal{F}_k^{(1)}(y_\infty) - \mathcal{F}_\infty^{(1)}(y_\infty)|.$$

The first term of the right hand side can be estimated (see  $H(F^{(1)})$ ), by using the Hölder inequality, as follows

$$\begin{aligned} \int_\Omega |F_k^{(1)}(x, y_k) - F_k^{(1)}(x, y_\infty)| dx &\leq c_5 \int_\Omega (1 + |y_\infty(x)|) |y_k(x) - y_\infty(x)| dx \leq \\ &\leq c_5 \|1 + y_\infty\|_{L^2(\Omega)} \|y_k - y_\infty\|_{L^2(\Omega)} \end{aligned}$$

so it tends to zero, since  $y_k \rightarrow y_\infty$  in  $s\text{-}L^2(\Omega)$ . The convergence of the second term to zero can be proved in the similar way as the convergence (13) in Lemma 4.1 in Denkowski and Staicu (1994). For the second assertion in 2<sup>0</sup> observe that  $\mathcal{F}_k^{(2)}$  as convex and locally equibounded functions are locally equi-Lipschitz continuous. Hence, owing to Proposition 5.9 of Dal Maso (1993),

convergence  $\mathcal{F}_k^{(2)}(u) \rightarrow \mathcal{F}_\infty^{(2)}(u)$  for all  $u \in L^q(\Omega)$ , and the latter again can be proved as (13) in Lemma 4.1 in Denkowski and Staicu (1994) which completes the proof.  $\blacksquare$

Thus, according to Proposition 3.1, from Propositions 4.2 and 4.4, applying the direct method, we get

**THEOREM 4.1** *Under the assumptions of Proposition 4.2 concerning the operators  $A_k$  and superpotentials  $J_k$  appearing in  $(HVI_e)_k$  we set  $\mathcal{U} = L^q(\Omega)$  with the strong topology,  $C_k = \text{id}: L^q(\Omega) \rightarrow V^*$  and we admit hypotheses  $H(F^{(1)})$ ,  $H(F^{(2)})$  for the cost functionals given by (21)-(23). Then*

(i) *for every  $k \in \bar{\mathbb{N}}$  the control problem  $(CP)_{(HVI_e)_k}$  has at least one optimal solution  $(u_k^*, y_k^*)$  (so  $S_k^* = S_{(HVI_e)_k}^* \neq \emptyset$ ) with minimal value  $m_k := \mathcal{F}_k(u_k^*, y_k^*)$ ;*

(ii) *if the original problem  $(HVI_e)_\infty$  has the uniqueness property i.e.*

$$S_{(HVI_e)_\infty}(u) = \{y_\infty(u)\} \text{ for all } u \in \mathcal{U}$$

*and  $S_{(HVI_e)_\infty}(u)$  are equicoercive ( $u \in \mathcal{U}$ ,  $k \in \mathbb{N}$ ), then every accumulation point of the sequence  $(u_k^*, y_k^*)$  is an optimal solution to the problem  $(CP)_{(HVI_e)_\infty}$ , i.e.*

$$\{(u_k^*, y_k^*) \in S_k^*, (u_k^*, y_k^*) \rightarrow (u_\infty^*, y_\infty^*)\} \implies (u_\infty^*, y_\infty^*) \in S_\infty;$$

*Moreover, in this case, we have also*

$$(iii) \quad m_k \rightarrow m_\infty \text{ as } k \rightarrow \infty.$$

## 5. Control problem for parabolic hemivariational inequality

In this section we consider optimal control problem for systems described by evolution of first order hemivariational inequality. Similarly as in the previous section, we first recall the notion of parabolic  $G$ -convergence of operators, then we state a result on the sensitivity of the solution set and on the convergence of the cost functionals.

Let  $\Omega$  be an open bounded subset of  $\mathbb{R}$  and let  $V = W_0^{1,p}(\Omega)$ ,  $Z = L^p(\Omega)$ ,  $H = L^2(\Omega)$ ,  $Z^* = L^q(\Omega)$ ,  $V^* = W^{-1,q}(\Omega)$ , where  $2 \leq p < \infty$  and  $1/p + 1/q = 1$ . Then  $V \subset Z \subseteq H \subseteq Z^* \subset V^*$  with compact embedding  $V \subset Z$ . Given  $0 < T < +\infty$ , let  $Q = (0, T) \times \Omega$ . We introduce the following spaces  $\mathcal{V} = L^p(0, T; V)$ ,  $\mathcal{Z} = L^p(0, T; Z)$ ,  $\mathcal{H} = L^2(0, T; H) \simeq L^2(Q)$ ,  $\mathcal{Z}^* = L^q(0, T; Z^*)$ ,  $\mathcal{V}^* = L^q(0, T; V^*)$ ,  $\mathcal{W}_{pq} = \{v \in \mathcal{V} : v' \in \mathcal{V}^*\}$ . It is well known (see, for instance, Zeidler, 1990) that

$$\mathcal{W}_{pq} \subset \mathcal{V} \subset \mathcal{Z} \subseteq \mathcal{H} \subseteq \mathcal{Z}^* \subset \mathcal{V}^*,$$

We consider the following sequence of parabolic hemivariational inequalities

$$(HVI_p)_k \quad \begin{cases} \langle y'(t) + A_k(t, y(t)), v - y(t) \rangle + J_k^0(y(t); v - y(t)) \geq \\ \geq \langle f_k(t) + C_k u, v - y(t) \rangle \text{ for all } v \in V \text{ and a.e. } t \in (0, T) \\ y(0) = y_k^0, \quad y \in \mathcal{Y}_p = \mathcal{W}_{pq}. \end{cases}$$

### 5.1. PG-convergence of parabolic operators

Following Svanstedt (1999) we start with the following definition:

**DEFINITION 5.1** *Given nonnegative constants  $m_0, m_1, m_2$  and  $0 < \alpha \leq 1$  we set*

$\mathcal{M} = \mathcal{M}(m_0, m_1, m_2, \alpha) := \{a: Q \times \mathbb{R}^N \rightarrow \mathbb{R}^N \text{ such that (i) – (iv) below hold}\}$

- (i)  $|a(t, x, 0)| \leq m_0$  a.e. in  $Q$ ;
- (ii)  $a(\cdot, \cdot, \xi)$  is Lebesgue measurable on  $Q$  for all  $\xi \in \mathbb{R}^N$ ;
- (iii)  $|a(t, x, \xi) - a(t, x, \eta)| \leq m_1(1 + |\xi| + |\eta|)^{p-1-\alpha} |\xi - \eta|^\alpha$  a.e., for all  $\xi, \eta$ ;
- (iv)  $(a(t, x, \xi) - a(t, x, \eta), \xi - \eta)_{\mathbb{R}^N} \geq m_2 |\xi - \eta|^\alpha$  a.e. in  $Q$  for all  $\xi, \eta \in \mathbb{R}^N$ .

**REMARK 5.1** *If  $a \in \mathcal{M}$ , then the following inequalities hold*

$$\begin{cases} |a(t, x, \xi)| \leq c_6(1 + |\xi|)^{p-1} \text{ a.e. in } Q, \text{ for all } \xi \in \mathbb{R}^N \\ |\xi|^p \leq c_7(1 + (a(t, x, \xi), \xi)_{\mathbb{R}^N}) \text{ for all } \xi \in \mathbb{R}^N \end{cases}$$

so the mappings from the class  $\mathcal{M}$  are uniformly bounded, coercive and monotone.

**DEFINITION 5.2** *A sequence of maps  $a_k \in \mathcal{M}$  is PG convergent to a map  $a_\infty \in \mathcal{M}$ , written as  $a_k \xrightarrow{PG} a_\infty$ , if for every  $g \in V^*$  we have*

$$\begin{cases} y_k \xrightarrow{w-\mathcal{W}_{pq}} y_\infty, \\ a_k(t, x, Dy_k) \xrightarrow{w-L^q(Q, \mathbb{R}^N)} a_\infty(t, x, Dy_\infty), \end{cases}$$

where  $y_k, k \in \bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ , is the unique solution to the problem

$$y' - \operatorname{div} a_k(t, x, Dy) = g, \quad y(0) = 0. \quad (24)$$

**REMARK 5.2** *Given  $a_k \in \mathcal{M}$ , it can be shown that the Nemitsky operators  $A_k: \mathcal{V} \rightarrow \mathcal{V}^*$  of the form*

$$(A_k y)(t) = A_k(t, y), \quad t \in (0, T)$$

corresponding to the family of operators  $A_k(t, y) = -\operatorname{div} a_k(t, x, Dy)$  are bounded, coercive, hemicontinuous and monotone. Therefore, for every  $k \in \bar{\mathbb{N}}$  and  $g \in V^*$ , there exists a unique solution  $y_k \in \mathcal{W}_{pq}$  to the problem (24). The compactness of the class  $\mathcal{M}$  with respect to the PG-convergence was established by Svanstedt (1999). The Definition 5.2 generalizes the one given for a class of



## 5.2. Sensitivity of solution sets for $(HVI_p)_k$

In this section we provide the result of the sensitivity of the solution set of the parabolic hemivariational inequality. First we observe that the problem  $(HVI_p)_k$  is equivalent to the following inclusion

$$(HVI_p)_k \quad \begin{cases} y'(t) + (\mathcal{A}_k y)(t) + \partial J_k(y(t)) \ni f_k(t) + C_k(t)u, & k \in \overline{\mathbb{N}} \\ y(0) = y_k^0, & y \in \mathcal{Y}_p = \mathcal{W}_{pq}, \end{cases}$$

where  $\mathcal{A}_k: \mathcal{V} \rightarrow 2^{\mathcal{V}^*}$  is the Nemitsky operator corresponding to  $A_k$ ,  $J_k: Z \rightarrow \mathbb{R}$  is a superpotential,  $f_k \in \mathcal{V}^*$ ,  $C_k: \mathcal{U} \rightarrow \mathcal{V}^*$  and  $y_0 \in H$ .

**REMARK 5.3** *The existence of solutions to  $(HVI_p)_k$ ,  $k \in \overline{\mathbb{N}}$  can be established under "mild" assumptions, e.g. that  $A_k + \partial J_k$  is bounded, coercive and pseudomonotone with respect to the domain of  $\frac{d}{dt}$  operator. These conditions imply that the operator  $\frac{d}{dt} + A_k + \partial J_k$  is surjective (see Proposition 3.4). In order to study the sensitivity of the system we consider a special class of operators  $A_k: \mathcal{V} \rightarrow \mathcal{V}^*$  of the form  $(A_k y)(t) = A_k(t, y)$  with  $A_k(t, y) = -\operatorname{div} a_k(t, x, Dy)$  for  $a_k \in \mathcal{M}$ .*

For the existence results for parabolic hemivariational inequalities we refer the Readers to Miettinen and Panagiotopoulos (1999), Migórski (2000, 2001, 2003), Migórski and Ochal (2000b) and Denkowski (2002).

The hypotheses on the data of  $(HVI_p)_k$  are the following:

$H(A)_p$ :  $A_k: (0, T) \times V \rightarrow V^*$  are the operators of the form

$$A_k(t, y) = -\operatorname{div} a_k(t, x, Dy) \text{ with } a_k \in \mathcal{M}, k \in \overline{\mathbb{N}} \text{ and } a_k \xrightarrow{PG} a_\infty;$$

$H(J)_p$ :  $J_k: Z \rightarrow \mathbb{R}$  are locally Lipschitz functions such that satisfy uniformly in  $k$  the conditions

- (i)  $\|\partial J_k(z)\|_{Z^*} \leq c_8(1 + \|z\|_Z^{2/q})$  for all  $z \in Z$  with some  $c_8 > 0$ ;
- (ii)  $J_k^0(z; -z) \leq c_9(1 + \|z\|^r)$  for all  $z \in Z$  with  $r < p$  and  $c_9 \geq 0$ ;
- (iii)  $K(s-Z, w-Z^*)\text{-}\limsup_{k \rightarrow \infty} Gr \partial J_k \subset Gr \partial J_\infty$ ;

$$\underline{H(H)_p}: \quad y_k^0 \in V, f_k \in \mathcal{V}^*, k \in \overline{\mathbb{N}}, y_k^0 \xrightarrow{w-V} y_\infty^0, f_k \xrightarrow{s-V^*} f_\infty;$$

$H(C)_p$ :  $C_k \in \mathcal{L}(\mathcal{U}, \mathcal{V}^*)$ ,  $k \in \overline{\mathbb{N}}$ , where  $\mathcal{U}$  is a reflexive separable Banach space modeling the control space and  $C_k \xrightarrow{c} C_\infty$  continuously.

**PROPOSITION 5.1** *Under the above hypotheses from any sequence*

*$\{y_k \in \mathcal{S}_{(HVI_p)_k}(u_k)\}_k$  with  $u_k \xrightarrow{\mathcal{U}} u_\infty$ , one can extract a convergent subsequence  $y_{k_n} \xrightarrow{w-\mathcal{W}_{pq}} y_\infty$  and  $y_\infty \in \mathcal{S}_{(HVI_p)_\infty}(u_\infty)$ , so we have*

Moreover, if we have the uniqueness of solution to the limit problem  $(HVI_p)_\infty$ , then we have also

$$\mathcal{S}_\infty(u_\infty) \subset K(w - \mathcal{W}_{pq})\text{-}\liminf \mathcal{S}_k(u_k),$$

so in this case  $\mathcal{S}_k(u_k) \xrightarrow{K(w - \mathcal{W}_{pq})} \mathcal{S}_\infty(u_\infty)$ .

*Proof.* The proof of the first part of the proposition is contained in Theorem 4.1 of Migórski (2000). Equicoercivity of  $\mathcal{S}_{(HVI_p)_k}(u)$ ,  $(u \in \mathcal{U}, k \in \overline{\mathbb{N}})$  follows from the a priori estimates which we establish below. Let  $u \in \mathcal{U}$ ,  $k \in \overline{\mathbb{N}}$  and  $y_k \in \mathcal{S}_{(HVI_p)_k}(u)$ . Then  $y_k \in \mathcal{Y}_p = \mathcal{W}_{pq}$  and

$$\begin{cases} y'_k(t) + (\mathcal{A}_k y_k)(t) + w_k(t) = f_k(t) + C_k(t)u & \text{a.e. } t \in (0, T) \\ w_k(t) \in \partial J_k(y_k(t)) & \text{a.e. } t \in (0, T) \\ y_k(0) = y_k^0. \end{cases}$$

Using the integration by parts formula (see Proposition 3.4.14 of Denkowski et al., 2003b), we have  $2 \int_0^T \langle y'_k(t), y_k(t) \rangle dt = |y_k(T)|_H^2 - |y_k^0|_H^2$ . From  $H(J)_p$ (ii) it follows that

$$-\langle w_k(t), y_k(t) \rangle \leq J_k^0(y_k(t); -y_k(t)) \leq c(1 + \|y_k(t)\|_Z^r) \leq c(1 + \|y_k(t)\|_V^r)$$

where  $c \geq 0$  denotes the generic constant and  $r < p$ . Hence

$$\langle w_k, y_k \rangle_Z \geq c(1 + \|y_k\|_V^r).$$

By exploiting the coercivity of  $\mathcal{A}_k$  (see Remark 5.1) from the equality

$$\langle y'_k, y_k \rangle_V + \langle \mathcal{A}_k y_k, y_k \rangle_V + \langle w_k, y_k \rangle_Z = \langle f_k + C_k u, y_k \rangle_V$$

we obtain

$$\frac{1}{2} |y_k(T)|_H^2 - \frac{1}{2} |y_k^0|_H^2 + c \|y_k\|_V^p - c(1 + \|y_k\|_V^r) \leq (\|f_k\|_{V^*} + \|C_k u_k\|_{V^*}) \|y_k\|_V.$$

Thus

$$c \|y_k\|_V^p \leq \frac{1}{2} |y_k^0|_H^2 + c(1 + \|f_k\|_{V^*}) \|y_k\|_V + c(1 + \|y_k\|_V^r)$$

which implies that  $\{y_k\}$  is bounded in  $V$  uniformly with respect to  $k$ . Next, since  $\mathcal{A}_k$  and  $\partial J_k$  are bounded operators, from  $y'_k = f_k + C_k u - \mathcal{A}_k y_k - w_k$  we deduce that  $\{y'_k\}$  is bounded in  $V^*$ . Therefore we infer that  $\{y_k\}$  is bounded in  $\mathcal{W}_{pq}$ . Finally, we remark that the second part of the proposition can be proved analogously as in the proof of Proposition 4.2 by using the Urysohn property of the Kuratowski convergence. Thus, the proof is completed. ■

In the parabolic case, besides control  $u \in \mathcal{U}$  (distributed control  $Cu \in V^*$ ), we can admit the initial value  $y_0 \in V$  as an additional control in  $(CP)_{(HVI)_p}$ . So in the next subsection we consider the cost functionals which depend on



### 5.3. $\Gamma$ -convergence of cost functionals

In this section we state conditions which guarantee the suitable  $\Gamma$ -convergence of the cost functionals in the control problem  $(CP)_{(HVI_p)_k}$ .

We consider the following costs

$$\mathcal{F}_k(y, u, y^0) := \mathcal{F}_k^{(1)}(y) + \mathcal{F}_k^{(2)}(u) + \mathcal{F}_k^{(3)}(y_0), \quad y \in \mathcal{Y}_p, \quad u \in \mathcal{U}, \quad y^0 \in V \quad (25)$$

where

$$\mathcal{F}_k^{(1)}(y) = \int_Q F_k^{(1)}(t, x, y(t, x)) \, dt dx, \quad (26)$$

$$\mathcal{F}_k^{(2)}(u) = \int_Q F_k^{(2)}(t, x, (C_k u)(t, x)) \, dt dx, \quad (27)$$

$$\mathcal{F}_k^{(3)}(y^0) = \int_\Omega F_k^{(3)}(x, y^0(x), Dy^0(x)) \, dx. \quad (28)$$

In the following hypothesis the conditions (i) and (ii) hold uniformly with respect to  $k \in \overline{\mathbb{N}}$ .

$$\underline{H(F^{(1)})_p} :$$

- (i)  $F_k^{(1)}: Q \times \mathbb{R} \rightarrow \mathbb{R}$  is measurable in  $(t, x) \in Q$ ,  $F_k^{(1)}(t, x, 0) \in L^p(Q)$ ;
- (ii)  $|F_k^{(1)}(t, x, z_1) - F_k^{(1)}(t, x, z_2)| \leq c_{10}(1 + |z_1|)|z_1 - z_2|$  in  $Q$  for some  $c_{10} > 0$ ;
- (iii)  $F_k^{(1)}(\cdot, \cdot, z) \xrightarrow{w-L^1(Q)} F_\infty^{(1)}(\cdot, \cdot, z)$  for all  $z \in \mathbb{R}$ ;

$$\underline{H(F^{(2)})_p} :$$

- (i)  $F_k^{(2)}: Q \times \mathbb{R} \rightarrow \mathbb{R}$  is measurable in  $(t, x) \in Q$ , convex in  $z \in \mathbb{R}$ ;
- (ii)  $0 \leq F_k^{(2)}(t, x, z) \leq \lambda|z|^q$  a.e. in  $Q$  with some  $\lambda > 0$ ;
- (iii)  $F_k^{(2)}(\cdot, \cdot, z) \xrightarrow{w-L^1(Q)} F_\infty^{(2)}(\cdot, \cdot, z)$  for all  $z \in \mathbb{R}$ ;

$$\underline{H(F^{(3)})_p} :$$

- (i)  $F_k^{(3)}: \Omega \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}$  is measurable in  $x \in \Omega$ , convex in  $z \in \mathbb{R}^{N+1}$ ;
- (ii)  $0 \leq F_k^{(3)}(x, z) \leq \lambda|z|^p$  a.e. in  $\Omega$  with some  $\lambda > 0$ ;
- (iii)  $F_k^{(3)}(\cdot, z) \xrightarrow{w-L^1(\Omega)} F_\infty^{(3)}(\cdot, z)$  for all  $z \in \mathbb{R}^{N+1}$ ;

**PROPOSITION 5.2** *For every fixed  $k \in \overline{\mathbb{N}}$  under regularity assumptions (i), (ii) of  $H(F^{(1)})_p$ ,  $H(F^{(2)})_p$  and  $H(F^{(3)})_p$ , respectively, we have*

(jj)  $\mathcal{F}_k^{(2)}(\cdot)$  is  $s$ - $L^q(Q)$  continuous,

(jjj)  $\mathcal{F}_k^{(3)}(\cdot)$  is  $w$ - $V$  continuous.

While, under convergence conditions (iii), we have

$$\begin{aligned}\mathcal{F}_k^{(1)}(y) &\xrightarrow{c} \mathcal{F}_\infty^{(1)}(y) \\ \mathcal{F}_k^{(2)}(u) &\xrightarrow{\Gamma_{seq}(s-L^q(Q)^-)} \mathcal{F}_\infty^{(2)}(u) \\ \mathcal{F}_k^{(3)}(y^0) &\xrightarrow{\Gamma_{seq}(w-W^{1,p}(\Omega)^-)} \mathcal{F}_\infty^{(3)}(y^0).\end{aligned}$$

So for functional  $\mathcal{F}_k(y, u, y^0)$  defined by (25) we obtain

$$\begin{aligned}(jv) \quad \mathcal{F}_\infty(y, u, y^0) &= \\ &= \Gamma_{seq}(w-\mathcal{W}_{pq}^\pm, s-L^q(Q)^-, w-V^-) \lim_{k \rightarrow \infty} \mathcal{F}_k(y, u, y^0).\end{aligned}$$

*Proof.* The proof goes along the same lines like that of Proposition 4.4 with  $\Omega$  replaced by  $Q$  in cases of  $\mathcal{F}_k^{(1)}$ ,  $\mathcal{F}_k^{(2)}$ , and with similar arguments for  $\mathcal{F}_k^{(3)}$ .

Now, the main result on the sensitivity of optimal control problems for parabolic hemivariational inequalities follows from Propositions 3.1, 5.1 and 5.2, and from the direct method for the existence part:

**THEOREM 5.1** *Under the assumptions of Proposition 5.1 for  $(HVI_p)_k$  with  $\mathcal{U} = L^q(0, T; L^q(\Omega)) \simeq L^q(Q)$ ,  $C_k = id: \mathcal{U} \rightarrow Z^* \simeq L^q(Q) \subset V^*$ , we admit the hypotheses  $H(F^{(j)})_p$ ,  $j = 1, 2, 3$  for cost functional  $\mathcal{F}_k(y, u, y^0)$  given by (25). Then*

(i) *For every  $k \in \overline{\mathbb{N}}$  the problem  $(CP)_{(HVI_p)_k}$  has at least one optimal solution  $(y_k^*, u_k^*, y_k^{0*}) \in S_k^*$ ,  $m_k := \mathcal{F}_k(y_k^*, u_k^*, y_k^{0*})$  being its minimal value.*

(ii) *If the limit (original) problem  $(CP)_{(HVI_p)_\infty}$  has the "uniqueness of solution property" i.e. for all  $u \in \mathcal{U}$ ,  $S_{(HVI_p)_\infty}(u) = \{y_\infty(u)\}$ , then every accumulation point of the sequence  $(y_k^*, u_k^*, y_k^{0*})$  is an optimal solution to the problem  $(CP)_{(HVI_p)_\infty}$ , i.e.*

$$(y_{k_n}^*, u_{k_n}^*, y_{k_n}^{0*}) \longrightarrow (y_\infty^*, u_\infty^*, y_\infty^{0*}), \text{ and } (y_\infty^*, u_\infty^*, y_\infty^{0*}) \in S_\infty^*.$$

(iii) *We also have*

$$m_k \rightarrow m_\infty \text{ as } k \rightarrow \infty.$$

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