Control and Cybernetics

vol. **34** (2004) No. 2

$C^{1,1}$ vector optimization problems and Riemann derivatives

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Abstract: In this paper we introduce a generalized second-order Riemann-type derivative for $C^{1,1}$ vector functions and use it to establish necessary and sufficient optimality conditions for vector optimization problems. We show that these conditions are stronger than those obtained by means of the second-order subdifferential in Clarke sense considered in Guerraggio, Luc (2001) and also to some extent than those obtained in Guerraggio, Luc, Minh (2001).

Keywords: vector optimization, Riemann-type directional derivatives, second-order optimality conditions.

1. Introduction

The class of $C^{1,1}$ functions, that is differentiable scalar functions whose derivatives are locally Lipschitz was first brought into attention by Hiriart-Urruty, Strodiot and Hien Nguyen (1984). The need for investigating such functions, as pointed out in Hiriart-Urrury, Strodiot and Hien Nguyen (1984) and Klatte, Tammer (1988), comes from the fact that several problems of applied mathematics including variational inequalities, semi-infinite programming, iterated local minimization by decomposition etc., involve differentiable functions with no hope of being twice differentiable. By introducing generalized Hessian matrices with the help of Clarke's generalized Jacobians, Hiriart-Urruty, Strodiot and Hien Nguyen (1984) succeeded in extending Taylor's expansion and exploited it to derive the second-order optimality conditions for scalar problems with data from this class of functions. Further applications were developed in Klatte, Tammer (1988), Luc (1995), Luc, Schaible (1996), Yang, Jeyakumar (1992), Yang (1993, 1994).

The analysis has been generalized to vector functions by Guerraggio and Luc (2001), where by means of Clarke's second-order subdifferential second order necessary and sufficient optimality conditions for unconstrained vector optimization problems are established. In Guerraggio, Luc (2003) the same authors also give second-order optimality conditions for constrained vector problems.

In this paper a generalized Riemann derivative for $C^{1,1}$ vector functions is introduced. By means of this derivative we give necessary and sufficient second-order optimality conditions for unconstrained vector optimization problems. We prove that these conditions are stronger than those given in Guerraggio, Luc (2001) and give some comparison with the results from Guerraggio, Luc, Minh (2001). When f is a scalar $C^{1,1}$ function the obtained optimality conditions reduce to those proved in Ginchev, Guerraggio (1998).

2. Preliminary concepts

A function f from $\mathbf{R}^m$ to $\mathbf{R}^n$ is said to be of class $C^{0,1}$ at $x^0 \in \mathbf{R}^m$ when it is locally Lipschitz at x^0. We say that f is of class $C^{0,1}$ when f is locally Lipschitz at any point of $\mathbf{R}^m$. If f is locally Lipschitz at x^0, then, according to the Rademacher Theorem, it is almost everywhere differentiable in a neighborhood of x^0. Hence the Clarke's generalized Jacobian of f at $x^0 \in \mathbf{R}^m$, denoted by $\partial f(x^0)$ can be defined as the set:

$$\partial f(x^0) = cl\ conv\{\lim f'(x_i) : x_i \to x^0,\ f'(x_i)\ \ exists\},$$

where f' denotes the Jacobian of f and $cl\ conv\{\ldots\}$ stands for the closed convex hull of the set under the parentheses.Now assume that f is a differentiable vector functions from $\mathbf{R}^m$ to $\mathbf{R}^n$ whose derivative is of class $C^{0,1}$ at x^0. In this case we also say that f is of class $C^{1,1}$ at x^0. We say that f is of class $C^{1,1}$ when it is of class $C^{1,1}$ at any point of $\mathbf{R}^m$. Denote by f'' the Jacobian of the function $f' : \mathbf{R}^m \to \mathbf{R}^{m\times n}$. The Clarke's generalized Jacobian of f' at x^0 is then denoted by $\partial^2 f(x^0)$ and called the second-order subdifferential of f at x^0, more precisely:

$$\partial^2 f(x^0) = cl\ conv\{\lim f''(x_i) : x_i \to x^0,\ \ f''(x_i)\ exists\}.$$

Thus, $\partial^2 f(x^0)$ is a subset of the finite dimensional space $L(\mathbf{R}^m, L(\mathbf{R}^m, \mathbf{R}^n))$

to $\mathbf{R}^n$. The elements of $\partial^2 f(x^0)$ can therefore be viewed as bilinear functions on $\mathbf{R}^m \times \mathbf{R}^m$ with values in $\mathbf{R}^n$. For the case $n = 1$, the term "generalized Hessian matrix" was used in Hiriart-Urrury, Storiot, Hien Nguyen (1984) to denote the set $\partial^2 f(x^0)$. By the previous construction the second-order subdifferential enjoys the properties of the generalized Jacobian. For instance $\partial^2 f(x^0)$ is a nonempty convex compact set of the space $L(\mathbf{R}^m, L(\mathbf{R}^m, \mathbf{R}^n))$ and the set-valued map $x \to \partial^2 f(x)$ is upper semicontinuous (u.s.c.). Let $u \in \mathbf{R}^m$; in the following we will denote by Lu the value of a linear operator $L : \mathbf{R}^m \to \mathbf{R}^n$ at the point $u \in \mathbf{R}^m$ and by $H(u, v)$ the value of a bilinear operator $H : \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^n$ at the point $(u, v) \in \mathbf{R}^m \times \mathbf{R}^m$. So we will set:

$$\partial f(x)(u) = \{Lu : L \in \partial f(x)\}$$

and:

$$\partial^2 f(x)(u, u) = \{H(u, u) : H \in \partial^2 f(x)\}.$$

We recall some important properties from Guerraggio, Luc (2001, 2003):

(i) Mean-value Theorem: *Let f be of class $C^{0,1}$ and $a, b \in \mathbf{R}^m$. Then:*
$$f(b) - f(a) \in cl\ conv\{\partial f(x)(b - a) : x \in [a, b]\},$$
where $[a, b] = conv\{a, b\}$;

(ii) Taylor's expansion: *Let f be of class $C^{1,1}$ and $a, b \in \mathbf{R}^m$. Then:*
$$\begin{aligned} f(b) - f(a) \in\ & f'(a)(b - a) + \\ & \frac{1}{2} cl\ conv\{\partial^2 f(x)(b - a, b - a) : x \in [a, b]\}. \end{aligned}$$

Guerraggio and Luc (2001, 2003) have given necessary and sufficient optimality conditions for vector optimization problems, expressed by means of $\partial^2 f(x)$.

In the following f will always denote a function of class $C^{1,1}$ at the considered point x^0.

Now we set:

$$\Delta_R^2 f(x^0, t, u) = \frac{f(x^0 + 2tu) - 2f(x^0 + tu) + f(x^0)}{t^2}.$$

The following theorem can be easily deduced from Theorem 2.1 in La Torre, Rocca (2001/02) and characterizes functions of class $C^{1,1}$ in terms of $\Delta_R^2 f(x^0, t, u)$.

Theorem 2.1 *Assume that the function $f : \mathbf{R}^m \to \mathbf{R}^n$ is bounded on a neighborhood of the point $x^0 \in \mathbf{R}^m$. Then f is of class $C^{1,1}$ at x^0 if and only if there exist neighborhoods U of x^0 and V of $0 \in \mathbf{R}$ and a constant $M \geq 0$ such that $\|\Delta_R^2 f(x, t, u)\| \leq M$ for every $x \in U$, $t \in V \setminus \{0\}$ and $u \in S^1 = \{u \in \mathbf{R}^m : \|u\| = 1\}$.*

DEFINITION 2.1 *The second upper Riemann derivative of the function f at the point $x^0 \in \mathbf{R}^m$ in the direction $u \in \mathbf{R}^m$ is defined as:*

$$f''_R(x^0, u) = Limsup_{t \to 0^+} \Delta^2_R f(x^0, t, u),$$

where Limsup denotes the upper limit of sets in the sense of Kuratowski, that is, the set of all cluster points of sequences $\Delta^2_R f(x^0, t_k, u)$, taken as $t_k \to 0^+$.

REMARK 2.1 *Riemann introduced (for scalar functions) the homonymous notion of second-order derivative while he was studying the convergence of trigonometric series, Riemann (1892). If g is a function from $\mathbf{R}$ to $\mathbf{R}$, the second-order Riemann derivative of g at the point $x \in \mathbf{R}$ is given by:*

$$\lim_{t \to 0^+} \frac{g(x+2t) - 2g(x+t) + g(x)}{t^2},$$

if this limit exists. Taking lim sup *or* lim inf *instead of* lim *one obtains upper and lower Riemann derivatives. For properties and applications of Riemann derivatives one can see Ash (1967, 1985), Marcinkiewicz, Zygmund (1936), Zygmund (1959). Hence the previous definition generalizes the notion of Riemann derivative to functions from $\mathbf{R}^m$ to $\mathbf{R}^n$.*

The following theorem states basic properties of $f''_R(x^0, u)$ for a $C^{1,1}$ function f.

THEOREM 2.2 *$f''_R(x^0, u)$ is a nonempty and compact subset of $\mathbf{R}^n$.*

Proof. The thesis is an obvious consequence of Theorem 2.1. ■

The next result links the second upper Riemann derivative to $\partial^2 f(x)$.

THEOREM 2.3 $f''_R(x^0, u) \subseteq \partial^2 f(x^0)(u, u)$.

Proof. Applying Taylor's expansion we can write for $t > 0$ "small enough":

$$\begin{aligned} f(x^0 + 2tu) - f(x^0 + tu) \in\ & t f'(x^0 + tu)u + \\ & \frac{t^2}{2} cl\ conv\{\partial^2 f(x)(u, u) : x \in [x^0 + tu, x^0 + 2tu]\} \end{aligned}$$

and:

$$\begin{aligned} f(x^0) - f(x^0 + tu) \in\ & -t f'(x^0 + tu)u + \\ & \frac{t^2}{2} cl\ conv\{\partial^2 f(x)(u, u) : x \in [x^0 + tu, x^0]\}. \end{aligned}$$

Hence, by addition:

$$\begin{aligned} \Delta^2_R f(x^0, t, u) \in\ & \frac{1}{2} cl\ conv\{\partial^2 f(x)(u, u) : x \in [x^0 + tu, x^0 + 2tu]\} + \\ & \frac{1}{2} cl\ conv\{\partial^2 f(x)(u, u) : [x^0 + tu, x^0]\} \end{aligned}$$

Since the map $x \to \partial^2 f(x)$ is u.s.c., then for any $\varepsilon > 0$, there exists a neighborhood U of x^0 such that whenever $x \in U$ there holds:

$$\partial^2 f(x)(u,u) \subseteq \partial^2 f(x^0)(u,u) + \varepsilon B,$$

where B is the closed unit ball in $\mathbf{R}^n$. So, for t "small enough" we have:

(i) $cl\ conv\{\partial^2 f(x)(u,u) : x \in [x^0 + tu, x^0 + 2tu]\} \subseteq cl\ conv\left[\partial^2 f(x^0)(u,u) + \varepsilon B\right] = \partial^2 f(x^0)(u,u) + \varepsilon B$;

(ii) $cl\ conv\{\partial^2 f(x)(u,u) : x \in [x^0 + tu, x^0]\} \subseteq cl\ conv\left[\partial^2 f(x^0)(u,u) + \varepsilon B\right] = \partial^2 f(x^0)(u,u) + \varepsilon B$.

Hence we have, for t "small enough":

$$\Delta^2_R f(x^0, t, u) \in \partial^2 f(x^0)(u,u) + 2\varepsilon B.$$

If $t_k \to 0^+$ is a sequence such that $\Delta^2_R f(x^0, t_k, u) \to L \in \mathbf{R}^n$, then L is an element of $f''_R(x^0, u)$ and $L \in \partial^2 f(x^0)(u,u) + 2\varepsilon B$, since this set is compact. Since ε is arbitrary and $\partial^2 f(x^0)$ is closed we obtain:

$$L \in \partial^2 f(x^0)(u,u)$$

and the theorem in proved. ∎

REMARK 2.2 *The set $f''_R(x^0, u)$ is not necessarily convex, but since $\partial^2 f(x^0)(u,u)$ is convex, it follows also that $conv f''_R(x^0, u) \subseteq \partial^2 f(x^0)(u,u)$.*

REMARK 2.3 *The forthcoming Example 3.1 shows that the inclusions in Theorem 2.3 and remark 2.2 can be strict.*

The search for "second order subdifferentials" smaller than $\partial^2 f(x)$ has a recent development in Guerraggio, Luc, Minh (2001), Jeyakumar, Luc (1998). In these papers, the authors introduce the notion of approximate Hessian for C^1 functions and by means of this concept give second order optimality conditions for a vector optimization problem. We recall below the notion of approximate Jacobian and the related notion of approximate Hessian.

DEFINITION 2.2 *Let f be a continuous function from $\mathbf{R}^m$ to $\mathbf{R}^n$. An approximate Jacobian $\partial_A f(x)$ of f at x is defined as a closed set of $M \in L(\mathbf{R}^m, \mathbf{R}^n)$ such that for every $u \in \mathbf{R}^m$ and $v \in \mathbf{R}^n$ it holds:*

$$(vf)^+(x,u) \leq \sup_{M \in \partial_A f(x)} \langle v, Mu \rangle,$$

where $vf(x) = \sum_{j=1}^n v_j f_j(x)$ and $(vf)^+(x,u)$ is the upper Dini directional derivative of the function vf at x in the direction u, that is:

$$(vf)^+(x,u) = \limsup \frac{(vf)(x+tu) - (vf)(x)}{t}.$$

Now let $f : \mathbf{R}^m \to \mathbf{R}^n$ be of class C^1. The Jacobian matrix map ∇f is a continuous vector function from $\mathbf{R}^m$ to the space $L(\mathbf{R}^m, \mathbf{R}^n)$. An approximate Hessian $\partial_A^2 f(x)$ of f at x is defined as a closed subset of the space $L(\mathbf{R}^m, L(\mathbf{R}^m, \mathbf{R}^n))$ being an approximate Jacobian of ∇f at x.

Second order optimality conditions in terms of approximate Hessians for vector optimization problems are given by Guerraggio, Luc, Minh (2001). Their approach is not fully comparable with the one based on Riemann derivatives. Nevertheless, Example 5.1 in the last section shows that the approach based on Riemann derivatives can give better results. Let us underline also that the application of Riemann derivatives gives a computational advantage in comparison to the conditions based on approximate Hessians.

3. Necessary optimality conditions for weakly efficient and ideal solutions

In this section we prove second-order necessary optimality conditions for unconstrained vector optimization problems, which are stronger than those provided by Guerraggio and Luc (2001).

Assume that the space $\mathbf{R}^n$ is partially ordered by a closed, convex, pointed cone C, with a nonempty interior and denote by A^c the complement of the set A.

Let M be any of the cones C^c, $C\backslash\{0\}$, and $int\ C$. The unconstrained vector optimization problem corresponding to the pair (f, M) is written as:

$$min_M f(x), \qquad x \in \mathbf{R}^m,$$

which amounts to finding a point $x^0 \in \mathbf{R}^m$ (called the optimal solution) such that there is no $x \in \mathbf{R}^m$ with $f(x) \in f(x^0) - M$. If this is true for all x in some neighborhood of x^0, then we call x^0 a local optimal solution. The optimal solutions of the vector problem corresponding to $(f, C$ (respectively $(f, C\backslash\{0\})$ and $(f, int\ C)$ are called ideal solutions (respectively efficient solutions and weakly efficient solutions). It follows directly from the definition that x^0 is a local ideal solution if and only if there is a neighborhood $U \subset \mathbf{R}^m$ of x^0 such that:

$$f(x) - f(x^0) \in C, \quad \forall x \in U.$$

Guerraggio and Luc (2001) have proved necessary and sufficicient optimality conditions for vector problems, that we summarize in the following theorems.

THEOREM 3.1 *(i) Let $x^0 \in \mathbf{R}^m$ be a local weakly efficient solution. Then the following conditions hold:*
a) $f'(x^0)u \in (-int\ C)^c$, $\forall u \in \mathbf{R}^m$;

(ii) *Let $x^0 \in \mathbf{R}^m$ be a local ideal solution. Then the following conditions hold:*
a) $f'(x^0) = 0$;
b) $\partial^2 f(x^0)(u,u) \cap C \neq \emptyset$, $\forall u \in \mathbf{R}^m$.

THEOREM 3.2 (i) *Assume that one of the following conditions holds at a point $x^0 \in \mathbf{R}^m$:*
a) $f'(x^0)u \in (-C)^c$, $\forall u \in \mathbf{R}^m$;
b) $\partial^2 f(x^0)(u,u) \subseteq int\ C$, *for $u \in \mathbf{R}^m$ such that $f'(x^0)u = 0$.*
Then x^0 is a local efficient solution.
(ii) *Assume that the following conditions hold at a point $x^0 \in \mathbf{R}^m$:*
a) $f'(x^0) = 0$;
b) $\partial^2 f(x^0)(u,u) \subseteq int\ C$, $\forall u \in \mathbf{R}^m \backslash \{0\}$.
Then x^0 is a local ideal solution.

Now we prove second-order necessary optimality conditions for unconstrained vector optimization problems, expressed by means of Riemann derivatives.

THEOREM 3.3 *Let $x^0 \in \mathbf{R}^m$ be a local weakly efficient solution. Then the following conditions hold:*
(i) $f'(x^0)u \in (-int\ C)^c$, *for every $u \in \mathbf{R}^m$;*
(ii) $f''_R(x^0,u) \cap (-int\ C)^c \neq \emptyset$, *for $u \in \mathbf{R}^m$, with $f'(x^0)u \in -(C \backslash int\ C)$.*

Proof. Condition (i) has been given in Theorem 3.1 and so we prove only condition (ii).

We begin observing that for any $t > 0$, $u \in \mathbf{R}^m$ and $j = 1, 2, 3, \ldots$, the following identity holds (see also Ginchev, Guerraggio, 1998):

$$f(x^0 + tu) - f(x^0) = t\frac{f(x^0 + \frac{t}{2^j}u) - f(x^0)}{\frac{t}{2^j}} + \frac{t^2}{2}\sum_{i=1}^{j}\frac{1}{2^i}\Delta_R^2 f(x^0, \frac{t}{2^i}, u). \tag{1}$$

Let $t > 0$ and $u \in \mathbf{R}^m$ be fixed. Observe that as $j \to +\infty$, we have:

$$\frac{f(x^0 + \frac{t}{2^j}u) - f(x^0)}{\frac{t}{2^j}} \to f'(x^0)u$$

and therefore $\sum_{i=1}^{+\infty} \frac{1}{2^i}\Delta_R^2 f(x^0, \frac{t}{2^i}, u)$ converges. Furthermore, for $\bar{t}$ "small enough", from Theorem 2.1, we get that the set $\{\Delta_R^2 f(x^0, t, u) : t \in (0, \bar{t})\}$ is bounded. Hence, it readily follows that $\forall \gamma > 0$, $\exists \delta = \delta(\gamma) > 0$, such that:

2) $\Delta_R^2 f(x^0, t, u) \in f''_R(x^0, u) + \gamma B$,

for $t \in (0, \delta(\gamma))$

Now, for $\beta \in (0, \delta(\gamma))$, consider the sequence:

$$a_k = \sum_{i=1}^{k} \frac{1}{2^i} \Delta_R^2 f(x^0, \frac{t}{2^i}, u) + \left(1 - \sum_{i=1}^{k} \frac{1}{2^i}\right) \Delta_R^2 f(x^0, \beta, u) .$$

Clearly, for $t \in (0, \gamma(\delta))$, a_k belongs to the convex compact set $conv\ \{f_R''(x^0, u) + \gamma B\}$. By letting k go to $+\infty$, we get the inclusion:

$$\sum_{i=1}^{+\infty} \frac{1}{2^i} \Delta_R^2 f(x^0, \frac{t}{2^i}, u) \in conv\ \{f_R''(x^0, u) + \gamma B\}.$$

Now, assume that $u \in \mathbf{R}^m$ is such that $f'(x^0)u \in -(C \backslash int\ C)$, and ab absurdo suppose that:

$$f_R''(x^0, u) \cap (-int\ C)^c = \emptyset,$$

that is $f_R''(x^0, u) \subseteq -int\ C$. Since $f_R''(x^0, u)$ is compact and $int\ C$ is open and convex, we obtain the existence of a number $\gamma > 0$ such that:

$$conv[f_R''(x^0, u) + \gamma B] \subseteq -int\ C.$$

Hence, for t "small enough", we obtain:

$$f(x^0 + tu) - f(x^0) = tf'(x^0)u + \frac{t^2}{2} \sum_{i=1}^{+\infty} \frac{1}{2^i} \Delta_R^2 f(x^0, \frac{t}{2^i}, u)$$

$$\in -(C \backslash int\ C) - int\ C \subseteq -int\ C,$$

which is absurd. So the proof is complete. ■

REMARK 3.1 *Since $f_R''(x^0, u) \subseteq \partial^2 f(x^0)(u, u)$, the necessary conditions of the previous theorem are stronger than those proved by Guerraggio and Luc (2001) in terms of $\partial^2 f(x^0)(u, u)$. The same remark holds for Theorem 3.4. Example 3.1 shows that the conditions expressed by means of Riemann derivatives can work when the conditions of Theorem 3.1 do not.*

EXAMPLE 3.1 *Consider the function $f : \mathbf{R}^n \to \mathbf{R}^n$ defined for $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ as $f(x) = \left(\int_0^{x_1} |z|\, dz, \dots, \int_0^{x_n} |z|\, dz\right)$ and let $C = \mathbf{R}_+^n$. It is easy to see that the point $x^0 = 0 \in \mathbf{R}^n$ is not a local weakly efficient solution. We have $f'(x)u = (|x_1|u_1, \dots, |x_n|u_n)$ for $u = (u_1, \dots, u_n) \in \mathbf{R}^n$ and in particular $f'(x^0) = 0 \in \mathbf{R}^{n \times n}$. For the second-order subdifferential we have $\partial^2 f(x^0)(u, u) = I\, u_1^2 \times \dots \times I\, u_n^2$, where $I = [-1, 1] \subset \mathbf{R}$. The second-order necessary condition from Theorem 3.1, (i), is satisfied, therefore we cannot conclude on this basis that x^0 is not a local weakly efficient point. For the second-order Riemann derivative we have $f_R''(x^0, u) = (\mathrm{sign}(u_1)\, u_1^2, \dots, \mathrm{sign}(u_n)\, u_n^2)$. In particular, if all the coordinates of u are negative we have $f_R''(x^0, u) = (-u_1^2, \dots, -u_n^2) \in -int\ C$. Therefore for such u the second-order necessary condition from Theorem 3.3 is not satisfied and on this basis we can conclude*

The following theorem states necessary conditions for local ideal solutions.

THEOREM 3.4 *Let $x^0 \in \mathbf{R}^m$ be a local ideal solution. Then the following conditions hold:*

(i) $f'(x^0) = 0$;

(ii) for every $u \in \mathbf{R}^m$ we have $conv\, f''_R(x^0, u) \cap C \neq \emptyset$.

Proof. Condition i) is stated in Theorem 3.1 and so we prove only condition ii).

Ab absurdo, assume that x^0 is a local ideal solution, but ii) does not hold, so that there exists a vector $u \in \mathbf{R}^m$ such that:

$$conv\, f''_R(x^0, u) \cap C = \emptyset,$$

that is $conv\, f''_R(x^0, u) \subseteq C^c$. Since $conv\, f''_R(x^0, u)$ is compact and C^c is open, there exists a number $\gamma > 0$ such that:

$$conv[f''_R(x^0, u) + \gamma B] = conv\, f''_R(x^0, u) + \gamma B \subseteq C^c.$$

From the proof of the previous theorem, we know that for every $\gamma > 0$, there exists $\delta > 0$ such that for all $t \in (0, \delta)$ there is:

$$\sum_{i=1}^{+\infty} \frac{1}{2^i} \Delta^2_R f(x^0, \frac{t}{2^i}, u) \in conv\{f''_R(x^0, u) + \gamma B\}.$$

Using identity (1), we find that for t "small enough", $f(x^0 + tu) - f(x^0) \in C^c$, which is a contradiction. ■

It is easy to see that when a function $f : \mathbf{R}^m \to \mathbf{R}$ is considered, then from the previous theorems we recover the necessary conditions for a local extremum, proved by Ginchev and Guerraggio (1998), as stated in the following result.

COROLLARY 3.1 *Let $f : \mathbf{R}^m \to \mathbf{R}$ be a function of class $C^{1,1}$ at a point $x^0 \in \mathbf{R}^m$. If x^0 is a local minimizer of the function f, then the following conditions hold:*

(i) $f'(x^0) = 0$;

(ii) $\limsup_{t \to 0^+} \Delta^2_R(x^0, t, u) \geq 0, \quad \forall u \in \mathbf{R}^m$.

4. Sufficient optimality conditions for efficient and ideal solutions

Before giving sufficient optimality conditions, we prove the following lemma.

LEMMA 4.1 $f''_R(x^0, u) = f''_r(x^0, u)$,
where $f''_r(x^0, u) = Limsup_{t \to 0^+,\ u' \to u} \Delta^2_R f(x^0, t, u')$, that is the set of all cluster

Proof. The inclusion $f''_R(x^0,u) \subseteq f''_r(x^0,u)$ is obvious, so that we have only to prove the reverse inclusion. Let $L \in f''_r(x^0,u)$; hence there exist sequences $t_k \to 0^+$ and $u_k \to u$ as $k \to +\infty$, such that:

$$L = \lim_{k\to+\infty} \frac{1}{t_k^2}\left(f(x^0+2t_ku_k) - 2f(x^0+t_ku_k) + f(x^0)\right).$$

We have:

$$\begin{aligned}
&\frac{1}{t_k^2}\left(f(x^0+2t_ku_k) - 2f(x^0+t_ku_k) + f(x^0)\right)\\
&= \frac{1}{t_k^2}\left[f(x^0+2t_ku_k) - 2f(x^0+t_ku_k) + f(x^0)\right.\\
&\left. - f(x^0+2t_ku) + f(x^0+2t_ku) - 2f(x^0+t_ku) + 2f(x^0+t_ku)\right].
\end{aligned}$$

Without loss of generality we can assume that:

$$\frac{1}{t_k^2}\left(f(x^0+2t_ku) - 2f(x^0+t_ku) + f(x^0)\right) \to L'.$$

Let $f = (f_1,\ldots,f_n)$, with $f_i : \mathbf{R}^m \to \mathbf{R}$. For every $i = 1,\ldots,n$, applying the mean value theorem we have:

$$\begin{aligned}
&\left[f_i(x^0+2t_ku_k) - f_i(x^0+2t_ku) - 2(f_i(x^0+t_ku_k) - f_i(x^0+t_ku))\right]\\
&= 2t_k\left[f_i'(x^0+2t_ku+2\theta_{i,k}t_k(u_k-u))(u_k-u)\right.\\
&\left. - f_i'(x^0+t_ku+\theta'_{i,k}t_k(u_k-u))\,(u_k-u)\right],
\end{aligned}$$

where $\theta_{i,k},\ \theta'_{i,k} \in (0,1)$. Since $f \in C^{1,1}$ we obtain:

$$\begin{aligned}
&\Big|2t_k\left[f_i'(x^0+2t_ku+2\theta_{i,k}t_k(u_k-u))(u_k-u)\right.\\
&\quad\left. - f_i'(x^0+t_ku+\theta'_{i,k}t_k(u_k-u))(u_k-u)\right]\Big|\\
&\le 2K_it_k\|t_ku+2\theta_{i,k}t_k(u_k-u) - \theta'_{i,k}t_k(u_k-u)\|\|u_k-u\|\\
&\le 2K_it_k^2\|u\|\|u_k-u\| + 2K_it_k^2|2\theta_{i,k}-\theta'_{i,k}|\|u_k-u\|^2,
\end{aligned}$$

where K_i is a Lipschitz constant for f_i'. Hence it is easily seen that:

$$\frac{1}{t_k^2}\left[f_i(x^0+2t_ku_k) - f_i(x^0+2t_ku) - 2(f_i(x^0+t_ku_k) - f_i(x^0+t_ku))\right] \to 0.$$

It follows that $L = L'$ and the lemma is proved. ∎

THEOREM 4.1 *Let f be a function of class $C^{1,1}$ and assume that at the point $x^0 \in \mathbf{R}^m$ for every $u \in S^1$ one of the following conditions holds:*

(i) $f'(x^0)u \in (-C)^c$;

Then x^0 is a local efficient solution.

Proof. Assume, ab absurdo, that for every $u \in S^1$ condition i) or ii) holds, but x^0 is not a local efficient solution. Then there exists a sequence $x^j \to x^0$ such that:

$$f(x^j) - f(x^0) \in -C \backslash \{0\}. \tag{2}$$

We can put $x^j = x^0 + t_j u_j$, where $t_j \to 0^+$, $u_j \in S^1$, and without loss of generality we can accept that $u_j \to u \in S^1$. For this u one of the following two possibilities holds:

(i) $f'(x^0)u \in (-C)^c$;

(ii) $f'(x^0)u \in -(C \backslash int\ C)$ and $f''_R(x^0, u) \subseteq int\ C$.

Assume that the first case holds. Then, since f is of class $C^{1,1}$ and, accounting for (2), we have:

$$f'(x^0)u = \lim_{t \to 0^+} \frac{f(x^0 + tu) - f(x^0)}{t} = \lim_{j \to +\infty} \frac{f(x^j) - f(x^0)}{t_j} \in -C,$$

a contradiction.

Therefore the case ii) should be true. The inclusion $f''_R(x^0, u) \subseteq int\ C$ and $f''_R(x^0, u)$ compact implies that there exists $\gamma > 0$ such that $f''_R(x^0, u) + \gamma B \subseteq int\ C$, whence

$$conv\ (f''_R(x^0, u) + \gamma B) = conv\ f''_R(x^0, u) + \gamma B \subseteq int\ C.$$

Consequently, as in the proof of Theorem 3.3 and Lemma 4.1, there exists $\delta > 0$, such that for all $t \in (0, \delta)$ and $u' \in S^1$, $\|u' - u\| < \delta$, there holds:

$$\sum_{i=1}^{+\infty} \frac{1}{2^i} \Delta^2_R f(x^0, \frac{t}{2^i}, u') \in conv\ \{f''_R(x^0, u) + \gamma B\} \subseteq int\ C.$$

Applying the above reasonings, we see that for j "large enough" we have:

$$f(x^j) - f(x^0) = t_j f'(x^0) u_j + \frac{t_j^2}{2} \sum_{i=1}^{+\infty} \frac{1}{2^i} \Delta^2_R f(x^0, \frac{t_j}{2^i}, u_j)$$

and consequently

$$f(x^j) - f(x^0) \subseteq -(C \backslash int\ C) + int\ C,$$

which contradicts (2), since the set on the right-hand side does not intersect $-(C \backslash int\ C)$. Indeed, if the latter is not true, then we would have $-c_1 = -c_2 + c_0 \Leftrightarrow c_2 = c_0 + c_1$ for some $c_0 \in int\ C$, $c_1 = C \setminus \{0\}$, $c_2 \in C \setminus int\ C$. The last equality is contradictory, since the right-hand side belongs to $int\ C$, while

THEOREM 4.2 *Let f be a function of class $C^{1,1}$ and assume that the following conditions hold at a point $x^0 \in \mathbf{R}^m$:*

(i) $f'(x^0) = 0$;

(ii) $f''_R(x^0, u) \subseteq int\ C$, *for every* $u \in S^1$.

Then x^0 is a local ideal solution.

Proof. Ab absurdo, assume that x^0 is not a local ideal solution. Then there exists a sequence $x^j = x^0 + t_j u_j$, $u_j \in S^1$, $u_j \to u \in S^1$ such that $f(x^j) - f(x^0) \in C^c$. We have:

$$f(x^j) - f(x^0) = \sum_{i=1}^{+\infty} \frac{1}{2^i} \Delta_R^2 f(x^0, \frac{t_j}{2^i}, u_j).$$

By analogy to the previous theorem we can conclude that for j "large enough",

$$\sum_{i=1}^{+\infty} \frac{1}{2^i} \Delta_R^2 f(x^0, \frac{t_j}{2^i}, u_j) \in int\ C,$$

and this is a contradiction. ■

REMARK 4.1 *The sufficient optimality conditions proved in the previous theorems are stronger than those provided by Guerraggio and Luc (2001), since* $f''_R(x^0, u) \subseteq \partial^2 f(x^0)(u, u)$.

EXAMPLE 4.1 *Let $C = \mathbf{R}^2_+$ and consider the twice differentiable function $f : \mathbf{R} \to \mathbf{R}^2$ defined as:*

$$f(x) = (x^2, \int_0^x z^2 \sin \frac{1}{z} dz + cx^2),$$

where $c \in (0, 1/2)$. Then, at the local ideal point $x^0 = 0$ we have $f'(0) = 0$ and:

$$\partial^2 f(0)(u, u) = [(2u^2, (-1 + 2c)u^2), (2u^2, (1 + 2c)u^2)],$$

whenever $u \in \mathbf{R}$. Hence the sufficient condition on $\partial^2 f(0)(u, u)$ for 0 to be a local ideal solution is not satisfied. On the contrary, we have :

$$f''_R(0, u) = (2u^2, 2cu^2), \quad \forall u \in \mathbf{R}$$

and so the sufficient condition on $f''_R(0, u)$ for f to be a local ideal solution is satisfied.

When f is a function from $\mathbf{R}^m$ to $\mathbf{R}$, the previous theorems provide the following sufficient optimality conditions for f (see also Ginchev, Guerraggio, 1998).

COROLLARY 4.1 *Let $f : \mathbf{R}^m \to \mathbf{R}$ be a function of class $C^{1,1}$ at $x^0 \in \mathbf{R}^m$ and assume that at x^0 the following conditions hold:*

(i) $f'(x^0) = 0$;

(ii) $\liminf_{t \to 0^+} \Delta_R^2(x^0, t, u) > 0$, *for every* $u \in S^1$.

5. Final remarks

In the case of twice continuously differentiable function $f : \mathbf{R}^m \to \mathbf{R}^n$ the Riemann derivative $f''_R(x^0, u)$ is a singleton and coincides with the Hessian $f''(x^0)(u,u)$. In such a case the necessary conditions from Theorems 3.3 and 3.4 and the sufficient conditions from Theorems 4.1 and 4.2 can be reformulated in terms of the Hessian. The optimality conditions obtained in such a way can be referred to as the classical ones. We use the following simple example of a function $f : \mathbf{R} \to \mathbf{R}$ to underline that the conditions in Riemann derivatives work also when the classical conditions do not.

EXAMPLE 5.1 *Consider the function $f : \mathbf{R} \to \mathbf{R}$, defined as*

$$f(x) = \begin{cases} x^3 \sin \frac{1}{x} + \kappa x^2 , & x \neq 0, \\ 0 , & x = 0. \end{cases}$$

Obviously f is of class $C^{1,1}$ but not of class C^2. The point $x_0 = 0$ is a point of a local minimum for f in the case $\kappa > 0$ and is not a point of a local minimim in the case $\kappa < 0$.

The conclusion in this example follows directly by Theorems 3.3 and 4.1. We have to observe that $f'(0) = 0$ and $f''_R(0, \pm 1) = 2\kappa$. This result cannot be obtained by the classical second order conditions, since f is not twice differentiable at 0. Let us underline that also the classical first order conditions cannot be applied (we mean that $f'(x) \leq 0$, $x_0 - \varepsilon < x \leq x_0$ and $f'(x) \geq 0$, $x_0 \leq x < x_0 + \varepsilon$ for some $\varepsilon > 0$ implies that x_0 is a point of a local minimum), since the derivative $f'(x)$ is not monotonic near $x_0 = 0$.

Now we apply this example to give some comparison with the results obtained in Guerraggio, Luc, Minh (2001). The latter are supposed to improve the results from Guerraggio, Luc (2001, 2003). In the case of a function $f : \mathbf{R} \to \mathbf{R}$ the results of Guerraggio, Luc, Minh, 2001 simplify as follows:

THEOREM 5.1 *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuously differentiable function and denote by $\partial^2_A f(x_0)$ the approximate Hessian of f at x_0.*

a) Necessary Conditions *(compare with Guerraggio, Luc, Minh, 2001, Theorem 3.1)*
Let x_0 be a local minimizer of f. Then $f'(x_0) = 0$ and for each $u \in \mathbf{R}$ there exists $M \in cl\ conv\ \partial^2_A f(x_0)$ or $M \in \left(conv\ \partial^2_A f(x_0)\right)_\infty \setminus \{0\}$ such that $M(u,u) \geq 0$ (here A_∞ stands for the recession cone of a given set A)

b) Sufficient Conditions *(compare with Guerraggio, Luc, Minh, 2001, Theorem 4.1)*
Suppose that $f'(x_0) = 0$ and for each $u \neq 0$ and each $M \in cl\ conv\ \partial^2_A f(x_0) \cup \left(\left(conv\ \partial^2_A f(x_0)\right)_\infty \setminus \{0\}\right)$ it holds $M(u,u) > 0$. Then x_0 is a strong local mini-

The gradient $\nabla f(x)u = f'(x)u$ of the function from Example 5.1 is given by

$$f'(x) = \begin{cases} 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x} + 2\kappa x\,, & x \neq 0, \\ 0\,, & x = 0. \end{cases}$$

Applying the definitions from Section 2, we get $(v\,\nabla f)^+(0,u) = |uv| + 2\kappa\, uv$. Therefore, each approximate Hessian $\partial_A^2 f(0)$ contains points M_-, M_+, such that $M_- \le 2\kappa - 1 < 2\kappa + 1 \le M_+$ and consequently $[2\kappa - 1,\, 2\kappa + 1] \subseteq conv\ \partial_A^2 f(0)$.

Take now for κ one of the following:

1^0. $-1/2 < \kappa < 0$. Then $x_0 = 0$ is not a local minimizer, which can be established on the basis that the necessary conditions of Theorem 3.3 are not satisfied. At the same time the inequalities $2\kappa - 1 < 0 < 2\kappa + 1$ show that $0 \in conv\ \partial_A^2 f(0)$. Therefore, the necessary conditions of Theorem 5.1 are satisfied. Consequently, on the basis of the necessary conditions of Theorem 5.1 one cannot reject the suspicion that $x_0 = 0$ is a local minimizer.

2^0. $0 < \kappa < 1/2$. Then $x_0 = 0$ is a local minimizer, which can be established on the basis that the sufficient conditions of Theorem 4.1 are satisfied. At the same time the inequalities $2\kappa - 1 < 0 < 2\kappa + 1$ show that $0 \in conv\ \partial_A^2 f(0)$. Therefore, the sufficient conditions of Theorem 5.1 are not satisfied. Consequently, on the basis of the sufficient conditions of Theorem 5.1 one cannot establish that $x_0 = 0$ is a local minimizer.

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