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# A new method for analytic determination of extremum of the transients in linear systems

by

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**Abstract:** The relation between extremal values of the error and the coefficients of its differential equations is one of the central problems of control systems in chemical industry, because extremal values of the error sometimes cause serious damages to the environment or to the system itself. Analytical formulae for the determination of these values are known only for the second-order systems. In this paper a method which permits to determine extremal values of the error in higher-order systems is proposed.

**Keywords:** transcendental equations, extremal dynamic error, linear stationary system, parametric optimization, analytic formulae, discriminants of exponential functions, Vandermonde's determinant, Viéte's formulae, process control.

## 1. Introduction

In the process of design of the dynamic control systems we encounter the problem of determining the maximal transient error $x_e$ and the moment of time $t_e$ when it appears. The maximal error $x_e$ characteries the attainable accuracy, and time $t_e$ - the velocity of the rise of the transients. Let us consider the differential equation determining the transient error in a linear control system of the $n$-th order with lumped and constant parameters:

$$a_0\frac{d^n x(t)}{dt^n} + a_1\frac{d^{n-1}x(t)}{dt^{n-1}} + \ldots + a_{n-1}\frac{dx(t)}{dt} + a_n x(t) = 0 \ . \tag{1}$$

The initial conditions are determined by the force function and the system parameteres $a_0, a_1, a_2, \ldots, a_n$.

Let us assume, in general that

$$x^{(i-1)}(0) = c_i \neq 0 \qquad \text{for} \quad i = 1, 2, \ldots, n$$

The solution of equation (1) takes the form

$$x(t) = \sum_{k=1}^{n} A_k e^{s_k t} \ , \tag{2}$$

where $s_k$ are the real, different roots of the characteristic equation

$$a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0 \ . \tag{3}$$

The necessary condition for the transient error $x(t)$ to attain an extremal value at $t = t_e$ is given by the relation

$$\frac{dx(t)}{dt} = \sum_{k=1}^{n} s_k A_k e^{s_k t} \ . \tag{4}$$

We will also need higher derivatives and use the relations

$$\frac{d^p x(t)}{dt} = \sum_{k=1}^{n} s_k^p A_k e^{s_k t}, \qquad p = 1, 2, \ldots, n-1 \ . \tag{5}$$

The equations (2) and (5) represent a system of $n$ linear equations with respect to unknown terms $A_k e^{s_k t}$. Its matrix of coefficients is the Vandermonde's matrix:

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ s_1 & s_2 & \ldots & s_n \\ \vdots & \vdots & & \vdots \\ s_1^{n-1} & s_2^{n-1} & \ldots & s_n^{n-1} \end{pmatrix} . \tag{6}$$

Without loss of generality we assume for the sake of simplicity that equation (3) has only single roots: $s_i \neq s_j$ for $i \neq j$. With this assumption the matrix (6) has an inverse and the system (2) and (5) can be solved.

For this purpose we denote by $V$ the Vandermonde's determinant of the matrix (6) and by $V_j$ the Vandermonde's determinant of order $(n-1)$ of the variables $s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n$.

We denote also by $\Phi_r^{(j)}$ the fundamental symmetric function of the $r$-th order of $(n-1)$ variables $s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n$; $r = 0, 1, \ldots, n-1$:

$$\left.\begin{aligned} \Phi_0^{(j)} &= \ 1 \\ \Phi_r^{(j)} &= \ \sum_{i=0}^{r} (-1)^r a_{r+i} \, s_j^i, \quad j = 1, 2, \ldots, n-1 \end{aligned}\right\} .$$

It is possible to show that the elements of the inverse matrix to the matrix (6) have the form:

$$(-1)^{i+j} \quad \Phi^{(i)} \quad V \tag{7}$$

The solution of the system (2) and (5) is as follows

$$A_k e^{s_k t} = \sum_{j=1}^{n} \alpha_{kj} x^{(j-1)} = \sum_{j=1}^{n} \frac{(-1)^{k+j}}{V} \cdot \Phi_{n-j}^{(k)} \; V_k x^{(j-1)}(t)$$

or

$$A_k e^{s_k t} = \frac{(-1)^k V_k}{V} \sum_{j=1}^{n} (-1)^j \Phi_{n-j}^{(k)} x^{(j-1)}(t), \quad k = 1, 2, \ldots, n \ . \tag{8}$$

It is evident that for $t = 0$ we know $x^{(j-1)}(0)$, and the substitution of $t = 0$ into the equations (8) gives:

$$A_k = \frac{(-1)^k V_k}{V} \sum_{j=1}^{n} (-1)^j \Phi_{n-j}^{(k)} x^{(j-1)}(0) \ , \tag{9}$$

or in the explicit form

$$A_k = \sum_{j=1}^{n} (-1)^j \Phi_{n-j}^{(k)} x^{(j-1)}(0) \prod_{\substack{v=1 \\ v \neq k}}^{n} (s_v - s_k)^{-1}, \quad k = 1, 2, \ldots, n \ .$$

After the substitution of (9) into (8) we obtain

$$e^{s_k t} \frac{(-1)^k V_k}{V} \sum_{j=1}^{n} (-1)^j \Phi_{n-j}^{(k)} x^{(j-1)}(0) = \frac{(-1)^k V_k}{V} \sum_{j=1}^{n} (-1)^j \Phi_{n-j}^{(k)} x^{(j-1)}(t) \ ,$$

and finally for $k = 1, 2, \ldots, n$ we have:

$$e^{s_k t} \sum_{j=1}^{n} (-1)^j \Phi_{n-j}^{(k)} x^{(j-1)}(0) = \sum_{j=1}^{n} (-1)^j \Phi_{n-j}^{(k)} x^{(j-1)}(t) \ . \tag{10}$$

Multiplying both sides of all the equations (10) and using Viéte's relation between the roots $s_n$ and the coefficient $a_i$ of the characteristic equation,

$$\sum_{k=1}^{n} s_k = -a_1 \tag{11}$$

we obtain the main result:

$$e^{-a_1 t} \prod_{k=1}^{n} \sum_{j=1}^{n} (-1)^j \Phi_{n-j}^{(k)} x^{(j-1)}(0) = \prod_{k=1}^{n} \sum_{j=1}^{n} (-1)^j \Phi_{n-j}^{(k)} x^{(j-1)}(t) \ . \tag{12}$$

Both sides of equation (12) are composed of the symmetric polynomials of variables $s_1, \ldots, s_n$. Due to this it is possible to present these terms as the polynomials of the coefficients $a_1, \ldots, a_n$. Using Viéte's relations it is possible to replace the solution of algebraic equations (3). It is now possible to find the relation between the extremal error $x_e$ and the time $t_e$ using the necessary condition $x^{(1)}(t_e) = 0$, in the relation (12). We have the basic result

$$\prod_{k=1}^{n}\sum_{\substack{j=1\\j\neq 2}}^{n}(-1)^j\Phi_{n-j}^{(k)}x_e^{(j-1)}(t_e) = e^{-a_1 t_e}\prod_{k=1}^{n}\sum_{j=1}^{n}(-1)^j\Phi_{n-j}^{(k)}x^{(j-1)}(0)\ . \tag{13}$$

From the relation (13) it is evident that the values of the sequence of extremums $x_{e_i}$, diminish with the rate of $e^{-a_1 t_{e_i}}$, $i = 1, 2, \ldots$.

From Viéte's relation we have:

$$\left.\begin{array}{rl}\Phi_0^k = & 1\\ \Phi_1^k = & s_1 + s_2 + \ldots + s_{k-1} + \ldots + s_n = -a_1 - s_k\\ \Phi_2^k = & s_1s_2 + s_1s_3 + \ldots + s_1s_{k-1} + s_1s_{k+1} + \ldots + s_1s_n +\\ & +s_2s_3 + \ldots + s_2s_{k-1} + s_2s_{k+1} + \ldots + s_2s_n + s_{n-1}s_n +\\ & + \ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots =\\ = & a_2 - s_1s_k - s_2s_k - \ldots - s_ns_k\end{array}\right\}. \tag{14}$$

Taking into account the relations (14) in the equation (13) we obtain the final relation between $x_e, x_e^{(2)}, \ldots, x_e^{(n-1)}; x(0), x^{(1)}(0), \ldots, x^{(n-1)}(0); t_e, a_1, a_2, \ldots, a_n$. The explicit relations for $n = 2$ and 3 are as follows:

$$n = 2 \qquad (x_e)^2 \cdot e^{a_1 t_e} = \big(x(0)\big)^2 + \frac{a_1}{a_2}x(0)x^{(1)}(0) + \frac{1}{a_2}\big(x^{(1)}(0)\big)^2\ .$$

The value of the time $t_e$ can be calculated from the equation (4).

If the characteristic equation (3) has real single roots then we have

$$t_e = \frac{1}{\sqrt{a_1^2 - 4a_2}}\ln\frac{2a_2x(0) + \big(a_1 + \sqrt{a_1^2 - 4a_2}\big)x^{(1)}(0)}{2a_2x(0) + \big(a_1 - \sqrt{a_1^2 - 4a_2}\big)x^{(1)}(0)}, \quad a_1^2 \geq 4a_2\ .$$

In the case when equation (3) has complex conjugate roots we have for $k = 0, 1, 2, \ldots$ and $a_1^2 < 4a_2$

$$t_e = \frac{1}{\sqrt{4a_2 - a_1^2}}\ \cdot$$
$$\cdot\left[\arctan\frac{2\big(2a_2x(0) + (a_1 + \sqrt{4a_2 - a_1^2})\ x^{(1)}(0)\big)}{\big(2a_2x(0) + a_1x^{(1)}(0)\big)^2 - \big((4a_2 - a_1^2)\ x^{(1)}(0)\big)} + k\pi\right].$$

For $n = 3$

$$
\begin{aligned}
&\left[a_3^2 x_e^3 + a_1 a_3 x_e^{(2)} x_e^2 + a_2 \left(x_e^{(2)}\right)^2 x_e + \left(x_e^{(2)}\right)^3\right] e^{a_1 t_e} = \\
&= a_3^2 x^3(0) + 2a_2 a_3 x^{(1)}(0) x^{(2)}(0) + \left(a_1 a_3 + a_2^2\right)^2 \left(x^{(1)}(0)\right)^2 x(0) + \\
&+ \left(a_1 a_2 - a_3\right)\left(x^{(1)}(0)\right)^3 + (a_1 a_2 + 3a_3) x^{(2)}(0) x^{(1)}(0) x(0) + \\
&+ a_1 a_3 x^{(2)}(0) x^2(0) + a_2 \left(x^{(2)}(0)\right)^2 x(0) + \\
&+ (a_1^2 + a_2) x^{(2)}(0) \left(x^{(1)}(0)\right)^2 + 2a_1 \left(x^{(2)}(0)\right)^2 x^{(1)}(0) + \left(x^{(2)}(0)\right)^3 \ .
\end{aligned}
\tag{15}
$$

The determination of the times $t_{e_i}$ from the equation (4) is difficult and will be done later.

Before doing this we make the following remark. The plot of a solution of equation of the $3^{rd}$ order is shown in Fig. 1.

It is evident that the times $t_{e_i}$ are invariant with respect to the perpendicular displacement of the curve (see Figs. 1, 2).

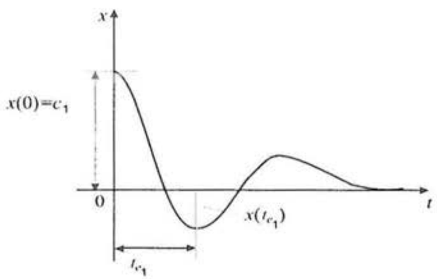


Figure 1. Plot of the third order function

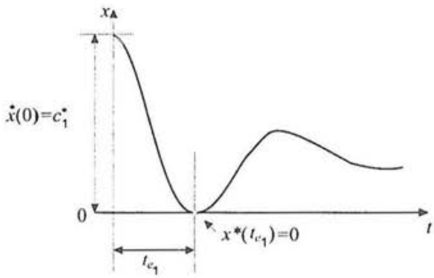


Figure 2. Plot of third order functions after prependicular displacement

The perpendicular displacement leads to changes of values of extremums and the initial value $x(0)$, but the values of moments of time $t_{e_i}$, and the form of the curve remain the same. We make use of this remark by performing such perpendicular displacement of the whole curve that one of the extremums assumes the value $x(t_{e_i}) = 0$, and of course the initial value $x(0)$ changes its value to $x^*(0)$.

It is evident that these values may be zero, one, or two, when we have three real different roots $s_1, s_2, s_3$, and infinite when we have a case with one real root $s_i$, and one pair of the complex, conjugate roots $s_{2,3} = \alpha \pm j\omega$.

After the displacement of the whole curve we have such a situation that at the extremal point the following relations are true:

$$
\left.\begin{aligned}
\dot{x}[x(0), t_{e_i}] &= \ 0 \\
\dot{x}[x^*(0), t_{e_i}] &= \ 0
\end{aligned}\right\} \ . \tag{16}
$$

These three homogeneous equations determine two independent values of time $t_{e_i}, t_{e_2}$.

The main determinant of these three equations is equal to zero, and from this it is possible to find the unknown value $x^*(0)$. As the illustration of this method we take the third order equation.

## 2. Solution of the third order equation

Let us consider the equation

$$\frac{d^3x}{dt^3} + a_1\frac{d^2x}{dt^2} + a_2\frac{dx}{dt} + a_3x = 0 \tag{17}$$

with the following initial conditions

$$\left.\begin{array}{rcl} x(0) & = & c_1 \\ \dot{x}(0) & = & c_2 \\ \ddot{x}(0) & = & c_3 \end{array}\right\} , \tag{18}$$

where $a_1, a_2, a_3$ are constant parameters.

The solution of the equation (17) is as follows:

$$x(t) = \sum_{k=1}^{3} A_k e^{s_k t} \tag{19}$$

where $s_k$ are the nonzero different roots of the characteristic equation $s^3 + a_1 s^2 + a_2 s + a_3 = 0$ and $A_k$, $k = 1, 2, 3$ are determined by the relations

$$\left.\begin{array}{rcl} A_1 & = & \dfrac{c_3 - (s_2 + s_3)\, c_2 + s_2 s_3 c_1}{(s_1 - s_2)(s_1 - s_3)} \\ A_2 & = & \dfrac{c_3 - (s_1 + s_3)\, c_2 + s_1 s_3 c_1}{(s_2 - s_1)(s_2 - s_3)} \\ A_3 & = & \dfrac{c_3 - (s_1 + s_2)\, c_2 + s_1 s_2 c_1}{(s_3 - s_1)(s_3 - s_2)} \end{array}\right\} . \tag{20}$$

For determination of the extremal values of the solution (19) we use the necessary condition

$$\left.\frac{dx(t)}{dt}\right|_{t_{e_i}} = 0 \ . \tag{21}$$

The differentiation of the relation (19) gives $\frac{dx(t)}{dt} = \sum_{k=1}^{3} s_k A_k e^{s_k t}$.

We apply now the method proposed in Section 1, and use the set of equations (16) with the notation (18).

$$\left.\begin{aligned} s_1 A_1 e^{s_1 t_{e_i}} + s_2 A_2 e^{s_2 t_{e_i}} + s_3 A_3 e^{s_3 t_{e_i}} &= 0 \\ s_1 A_1^* e^{s_1 t_{e_i}} + s_2 A_2^* e^{s_2 t_{e_i}} + s_3 A_3^* e^{s_3 t_{e_i}} &= 0 \\ A_1^* e^{s_1 t_{e_i}} + A_2^* e^{s_2 t_{e_i}} + A_3^* e^{s_3 t_{e_i}} &= 0 \end{aligned}\right\} \tag{22}$$

where $A_1^*$, $A_2^*$, $A_3^*$ denote the coefficients (20) but with the initial condition $c_1^*$ and $c_2, c_3$ unchanged.

The main determinant of equations (22) with respect to $e^{s_1 t_{e_i}}$, $e^{s_2 t_{e_i}}$, $e^{s_3 t_{e_i}}$ is equal to zero:

$$\Delta = \begin{vmatrix} s_1 A_1 & s_2 A_2 & s_3 A_3 \\ s_1 A_1^* & s_2 A_2^* & s_3 A_3^* \\ A_1^* & A_2^* & A_3^* \end{vmatrix} = 0 \ . \tag{23}$$

After the substitution of $A_1, A_2, A_3$ from (20) to the determinant (23) we obtain

$$\Delta = \frac{s_1 s_2 s_3 \left[-c_3 (c_1^*)^2 + (c_2^2 + c_1 c_3) c_1^* - c_1 c_2^2\right]}{(s_1 - s_2)(s_2 - s_3)(s_3 - s_1)} = 0 \tag{24}$$

or in an equivalent form (see Górecki, 1966c)

$$\Delta = \frac{a_3 \left[(c_1^*)^2 c_3 - c_1^* (c_2^2 + c_1 c_3) + c_1 c_2^2\right]}{\sqrt{27 a_3^3 + (4a_1^3 - 18 a_1 a_2) a_3^2 + (4a_2^3 - a_1^2 a_2^2) a_3}} = 0 \ . \tag{25}$$

From this relation it results that for the existence of a real $c_1^*$ there has to be

$$(c_2^2 + c_1 c_3)^2 - 4 c_1 c_2^2 c_3 \geq 0 \tag{26}$$

or equivalently $(c_2^2 - c_1 c_3)^2 \geq 0$, but this is always true.

This means that if there exists $c_1$ for which an extremum $x_e$ exists, then there exists also $c_1^*$ for which the condition (11) is fulfilled. Taking into account (24) or (25) we obtain the relation between given initial conditions $c_1, c_2, c_3$ and an unknown initial condition $c_1^*$:

$$c_3 (c_1^*)^2 - (c_2^2 + c_1 c_3) c_1^* + c_1 c_2^2 = 0 \ . \tag{27}$$

From equation (27) we obtain the main result that either $c_1^* = c_1$ and the displacement of the function $x(t)$ is not needed, or

$$c_1^* = \frac{c_2^2}{c_3} \ . \tag{28}$$

Depending on whether it is a minimum ($c_3 > 0$) or maximum ($c_3 < 0$) of the function $x(t)$ the value of $c_1$ must be appropriately modified. If the roots ...

$s_3$ are real we obtain the following solutions for $t_{e_i}$, $i = 1, 2, 3$ from relations (22):

$$\left.\begin{aligned} t_{e_1} &= \frac{1}{s_2 - s_1} \ln \left| \frac{c_3 - (s_2 + s_3)c_2 + s_2 s_3 c_1^*}{c_3 - (s_1 + s_3)c_2 + s_1 s_3 c_1^*} \right| \\ t_{e_2} &= \frac{1}{s_3 - s_2} \ln \left| \frac{c_3 - (s_1 + s_3)c_2 + s_1 s_3 c_1^*}{c_3 - (s_1 + s_2)c_2 + s_1 s_2 c_1^*} \right| \\ t_{e_3} &= \frac{1}{s_1 - s_3} \ln \left| \frac{c_3 - (s_2 + s_1)c_2 + s_2 s_1 c_1^*}{c_3 - (s_2 + s_3)c_2 + s_2 s_3 c_1^*} \right| \end{aligned}\right\} . \tag{29}$$

Two of equations (29) are independent.

In the case of one real root $s_1$ and two complex conjugate roots $s_{2,3} = \alpha \pm j\omega$ we obtain an infinite number of solutions for $t_{e_k}$, $k = 0, 1, 2, \ldots$

$$t_{e_k} = \frac{1}{2\omega} \left[ -\arctan \frac{2\omega(c_2 - s_1 c_1^*)[c_3 - (\alpha + s_1)c_2 + \alpha s_1 c_1^*]}{[c_3 - (\alpha + s_1)c_2 + \alpha s_1 c_1^*]^2 - \omega^2 (c_2 - s_1 c_1^*)^2} + k\pi \right] . \tag{30}$$

The substitution of the relations (28), (29) or (30) into (19) gives the extremum values of $x(t)$.

## 3. Limitations of the method and their overcoming

In the case when the initial value of the first derivative $c_2 = 0$, we have an extremum for $t = 0$, but we cannot apply relation (28). In this case we calculate the value of the determinant (26), using approximate solutions of $x(t)$ and $\dot{x}(t)$. Developing in the solution (2) and in the relation (4) the exponential functions in the Taylor series around $t = 0$ we obtain

$$e^{s_k t} = 1 + \frac{s_k \Delta t}{1!} + 0(s_k \Delta t)^2, \qquad k = 1, 2, \ldots, n \ . \tag{31}$$

Substituting the relation (31) into the relations (2) and (4) we get

$$x(t) \approx \sum_{k=1}^{n} (1 + s_k \Delta t) A_k^{**} = 0 \tag{32}$$

$$\frac{dx(t)}{dt} \approx \sum_{k=1}^{n} s_k (1 + s_k \Delta t) A_k^{**} = 0 \ . \tag{33}$$

Elimination of the variable $\Delta t$ from equations (32) and (33) gives the discriminant $\Delta$ in the form

$$\Delta^* = \left( \sum_{k=1}^{n} A_k^{**} \right) \left( \sum_{k=1}^{n} s_k^2 A_k^{**} \right) - \left( \sum_{k=1}^{n} s_k A_k^{**} \right)^2 = 0$$

We take into account that

$$\left.\begin{aligned} \sum_{k=1}^{n} A_k^{**} &= c_1^{**} \\ \sum_{k=1}^{n} s_k A_k^{**} &= c_2^{**} \\ \sum_{k=1}^{n} s_k^2 A_k^{**} &= c_3^{**} \end{aligned}\right\} .$$

By eliminating on this basis we obtain after $\Delta t$ the new values of initial conditions and the relation $c_1^{**} = \dfrac{(c_2^{**})^2}{c_3^{**}}$, which is valid for an arbitrary degree $n$ and the increment of time $\Delta t$, as well as $c_1^{**}$, $c_2^{**}$, $c_3^{**}$ which are different from zero.

In this way the obstacle connected with $c_2 = 0$ is removed. Up to these results it was necessary to know the values of roots of the characteristic equation.

Now we show that it is possible to overcome also this difficulty and to obtain formulae which depend only on the coefficients $a_i$ of the characteristic equation.

We denote the right-hand side of the equation (15) by $w(c_1, c_2, c_3)$. If we put the value $c_1^*$ instead of $c_1$ in equation (15) we must put $x_e = 0$. We obtain then from equation (15) that

$$\left(x_e^{(2)}\right)^3 e^{a_1 t_e} = w(c_1^*, c_2, c_3) \ . \tag{34}$$

Substracting both sides of equation (34) from equation (15) we obtain

$$\left[a_3^2 x_e^3 + a_1 a_3 x_e^{(2)} x_e^2 + a_2 \left(x_e^{(2)}\right)^2 x_e\right] e^{a_1 t_e} = w(c_1, c_2, c_3) - w(c_1^*, c_2, c_3) \tag{35}$$

but

$$\begin{aligned} & w(c_1, c_2, c_3) - w(c_1^*, c_2, c_3) = \\ & \quad = a_3^2 (c_1^3 - c_1^{*\,3}) + (2a_2 a_3 c_2 + a_1 a_3 c_3)(c_1^2 - c_1^{*\,2}) + \\ & \quad + \left[(a_1 a_3 + a_2^2) c_2^2 + (a_1 a_2 + 3a_3) c_2 c_3 + a_2 c_3^2\right](c_1 - c_1^*) = \\ & \quad = \left\{a_3^2 \left[c_1^2 + c_1^* c_1 + c_1^{*\,2}\right] + \left[2a_2 a_3 c_2 + a_1 a_3 c_3\right](c_1 + c_1^*) + \right. \\ & \quad \left. + \left[(a_1 a_3 + a_2^2) c_2^2 + (a_1 a_2 + 3a_3) c_2 c_3 + a_2 c_3^2\right]\right\}(c_1 - c_1^*) \end{aligned} \tag{36}$$

and

$$c_1 - c_1^* = x_e \ . \tag{37}$$

Taking into account relations (36) and (37) in the dependence (35) we have

$$\left[a_3^2 x_e^2 + a_1 a_3 x_e^{(2)} x_e + a_2 \left(x_e^{(2)}\right)^2\right] e^{a_1 t_e} = R(c_1, c_1^*, c_2, c_3) \tag{38}$$

where

$$R = a_3^2 \left[c_1^2 + c_1^* c_1 + c_1^{*\,2}\right] + \left[2a_2 a_3 c_2 + a_1 a_3 c_3\right](c_1 + c_1^*) +$$

Dividing both sides of the equality (38) by the formula (34) yields

$$a_3^2\left(\frac{x_e}{x_e^{(2)}}\right)^2 + a_1 a_3 \frac{x_e}{x_e^{(2)}} + a_2 = \frac{R(c_1, c_1^*, c_2, c_3)}{w(c_1^*, c_2, c_3)}\, x_e^{(2)} \ .$$

Finally, taking into account formula (34) we have

$$x_{e_{1,2}} = \frac{e^{-\frac{1}{3}a_1 t_e} w^{\frac{1}{3}}(c_1^*, c_2, c_3)}{2a_3} \cdot \left[a_1 \pm \sqrt{a_1^2 - 4\left[a_2 - R(c_1, c_1^*, c_2, c_3) w^{\frac{2}{3}}(c_1^*, c_2, c_3) e^{-\frac{1}{3}a_1 t_e}\right]}\,\right] \ . \tag{39}$$

Relations (29), (30) and (39) can be used for parametric optimization, when the coefficients $a_i$ are functions of parameters of the controller.

## 4. Example

Let us consider the problem of synthesis of the proportional controller for the third order object. The transfer function of the object is

$$G(s) = \frac{b_1 s^2 + b_2 s + b_3}{a_0 s^3 + a_1 s^2 + a_2 s + a_3} \ .$$

The input transform is $U(s) = \dfrac{1}{s}$ and the transfer function of the ideal controller has the form $G_R(s) = K$.

The transform function of the whole closed-loop system is given by

$$\begin{aligned} G_c(s) = &\ \frac{KG(s)}{1 + KG(s)} = \\ = &\ K \frac{b_1 s^2 + b_2 s + b_3}{a_0 s^3 + (a_1 + Kb_1)s^2 + (a_2 + Kb_2)s + a_3 + Kb_3} \ . \end{aligned} \tag{40}$$

The transient error is equal to

$$\varepsilon(t) = x(\infty) - x(t) \ . \tag{41}$$

The output signal in the steady state is

$$x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} G_c(s) = \frac{Kb_3}{a_3 + Kb_3} \ . \tag{42}$$

The transform of the transient error can be calculated from (41) and (42):

$$E(s) = \frac{1}{s}\left[\frac{Kb_3}{a_3 + Kb_3} - G_c(s)\right] \ .$$

The initial values of the transient error and its derivatives are

$$\alpha_0 = \varepsilon(0) = \lim sE(s) = \frac{Kb_3}{\ldots}$$

From relation (41) we deduce that $\varepsilon^{(1)}(t) = -x^{(1)}(t)$ and the initial value of the derivative of error

$$c_2 = \varepsilon^{(1)}(0) = -x^{(1)}(0) = \lim_{s\to\infty} s\big[-sX(s) + x(0)\big] \tag{43}$$

$$x(0) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty}\big[-G_c(s)\big] = 0 \ . \tag{44}$$

From (43), (44) and (40) we have

$$c_2 = \varepsilon^{(1)}(0) = \lim_{s\to\infty} s\big[-sX(s)\big] = \lim_{s\to\infty} s\big[-G_c(s)\big] = -\frac{Kb_1}{a_0}$$

and similarly

$$\begin{aligned} c_3 = \varepsilon^{(2)}(0) &= \lim_{s\to\infty} s\big[-s^2X(s) - sx(0) - x^{(1)}(0)\big] = \\ &= -\frac{Kb_2}{a_0} + \frac{Kb_1}{a_0}\frac{a_1 + Kb_1}{a_0} \end{aligned}$$

and in general for the transfer function

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{a_1 s^n + a_2 s^{n-1} + \ldots + a_n}$$

for $i = 1, 2, \ldots, n$

$$\begin{aligned} \varepsilon^{(i)}(0) &= -\frac{b_0}{a_0} + \left[\frac{a_1}{a_0}\varepsilon^{(i-1)}(0) + \frac{a_2}{a_0}\varepsilon^{(i-2)}(0) + \ldots + \frac{a_{i-1}}{a_0}\varepsilon^{(1)}(0)\right], \\ \varepsilon^{(0)}(0) &= -\frac{b_n}{a_n} \ . \end{aligned}$$

The initial value after displacement is

$$c_1^* = \frac{c_2^2}{c_3} = \frac{Kb_1^2}{-a_0b_2 + b_1(a_1 + Kb_1)} \ .$$

The extremum value of the transient error is

$$\varepsilon_e = c_1 - c_1^* = K\left[\frac{b_3}{a_3 + Kb_3} - \frac{b_1^2}{b_1(a_1 + Kb_1) - a_0b_2}\right] \ .$$

Using the necessary condition for the calculation of the extremal value of $\varepsilon_e$ with respect to gain $K$ we obtain

$$\begin{aligned} \frac{d\varepsilon_e}{dK} = &\ \frac{b_3}{a_3 + Kb_3} - \frac{b_1^2}{b_1(a_1 + Kb_1) - a_0b_2} + \\ &+K\left[-\frac{b_3^2}{\text{[illegible]}} + \frac{b_1^4}{\text{[illegible]}}\right] = 0 \ . \end{aligned} \tag{45}$$

From (45) we find finally that the optimal gain for the minimum $\varepsilon_e$ is equal to

$$K^*_{1,2} = \eta \frac{1}{b_1} \sqrt{\frac{a_3}{b_3}(a_1 b_1 - a_0 b_2)}, \qquad \eta = \pm 1 \ . \tag{46}$$

The minimal value of the extremal transient error after using (46) is given by

$$x_{e_{1,2}} = \frac{2\eta \sqrt{\frac{b_3 a_3}{b_1^2}(a_1 b_1 - a_0 b_2)} - \left[a_3 + \frac{b_3}{b_1^2}(a_1 b_1 - a_0 b_2)\right]}{a_3 - \frac{b_3}{b_1^2}(a_1 b_1 - a_0 b_2)} \ .$$

## 5. Solution of the $n$-th order equation

Let us consider the relation (2), which represents the solution of equation (1), i.e. $x(t) = \sum_{k=1}^{n} A_k e^{s_k t}$.

For the elimination of $n$-exponential elements $e^{s_k t}$ it is necessary to find $(n-3)$ additional equations in comparison to the solution of the $3^{rd}$ order equation. To do this, let us observe that after the perpendicular displacement the curve representing $x(t)$ relation (1), all the derivatives of $x(t)$ are not changed. In comparison to the equation of the $3^{rd}$ order there is an essential difference, because the values of the higher derivatives of the extremal points, beginning from the second order, are not known. To overcome this difficulty we take into account the fact that the values of higher derivatives remain invariant during the perpendicular displacement. Let us find the differences between apropriate derivatives at the extremal points before and after the displacement

These differences are equal to zero, and in this way we obtain the additional $(n-3)$ homogeneous equations. The set of the $n$ linear independent equations takes the form

$$\left.\begin{aligned}
A_1^* e^{s_1 t_e} + A_2^* e^{s_2 t_e} + \ldots + A_n^* e^{s_n t_e} &= 0 \\
s_1 A_1^* e^{s_1 t_e} + s_2 A_2^* e^{s_2 t_e} + \ldots + s_n A_n^* e^{s_n t_e} &= 0 \\
s_1 A_1 e^{s_1 t_e} + s_2 A_2 e^{s_2 t_e} + \ldots + s_n A_n e^{s_n t_e} &= 0 \\
s_1^2 (A_1 - A_1^*) e^{s_1 t_e} + s_2^2 (A_2 - A_2^*) e^{s_2 t_e} + \ldots & \\
\ldots + s_n^2 (A_n - A_n^*) e^{s_n t_e} &= 0 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &= \ldots \\
s_1^{n-2} (A_1 - A_1^*) e^{s_1 t_e} + s_2^{n-2} (A_2 - A_2^*) e^{s_2 t_e} + \ldots & \\
\ldots + s_n^{n-2} (A_n - A_n^*) e^{s_n t_e} &= 0
\end{aligned}\right\} \tag{47}$$

where, similarly as in the case of the $3^{rd}$ order equation $A_1^*, \ldots, A_n^*$ denote the coefficients with the initial condition $x(0) = c_1^*$, and the remaining initial conditions are unchanged.

has the form

$$\Delta = \begin{vmatrix} A_1^* & A_2^* & \dots & A_n^* \\ s_1 A_1^* & s_2 A_2^* & \dots & s_n A_n^* \\ s_1 A_1 & s_2 A_2 & \dots & s_n A_n \\ s_1^2(A_1 - A_1^*) & s_2^2(A_2 - A_2^*) & \dots & s_n^2(A_n - A_n^*) \\ \dots\dots & \dots\dots & \dots & \dots\dots \\ s_1^{n-2}(A_1 - A_1^*) & s_2^{n-2}(A_2 - A_2^*) & \dots & s_n^{n-2}(A_n - A_n^*) \end{vmatrix} = 0 \ . \tag{48}$$

## 6. Main results

The determinant (48) is obtained in the form

$$\Delta = \frac{(-1)^{n-2} \prod_{i=1}^{n} s_i^{n-2}}{\prod_{\substack{i=1 \\ i \neq j}}^{n} (s_i - s_j)} \left[ (c_1 - c_1^*)^{n-2} (c_2^2 - c_1^* c_3) \right] \qquad n \geq 2 \tag{49}$$

or in the equivalent form

$$\Delta = \frac{a_n^{n-2}}{\sqrt{D}} \left[ (c_1 - c_1^*)^{n-2} (c_2^2 - c_1^* c_3) \right] \qquad n \geq 2 \ ,$$

where $D$ is determined by the formula (50)

$$D = \begin{vmatrix} na_0 & a_0 & 0 & 0 & \dots \\ (n-1)a_1 & a_1 & na_0 & a_0 & \dots \\ (n-2)a_2 & a_2 & (n-1)a_1 & a_1 & \dots \\ (n-3)a_3 & a_3 & (n-2)a_2 & a_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & a_n \end{vmatrix}_{2n \times 2n} \ . \tag{50}$$

Proof of this result is in Appendix to the paper.

We assume that the roots of equation (3) are single and different from zero, for that reason the determinant $\Delta$ can be equal zero only with the adjusted initial conditions $c_1 = c_1^*$ or $c_1^2 = c_1^* c_3$.

In this way we have obtained the following basic result:

THEOREM 6.1 *If there exists an extremum of the solution of equation (1) with the initial condition* $c_1$ *then the value of the extremum is equal to* $x_e = c_1 - c_1^*$, *and the new initial condition* $c_1^*$ *is determined by the relation* $c_1^* = \frac{c_2^2}{\ }$

The moments of time at the extremum points can be calculated from the set of $(n-1)$ independent equations (47). Then the number of extremums depends on the initial conditions, and on the roots of the characteristic equation (3). In particular, if equation (3) has complex conjugate roots, then the number of extremums can be infinite.

If the characteristic equation has only real roots, then the number of extremums is between 0 and $(n-1)$ (see Górecki, 1993, pp. 121 and 135-136).

The necessary and sufficient conditions for the existence of only single real, and negative roots of equation (3) are determined by the following theorem (see Merrow, 1956).

THEOREM 6.2 *The necessary and sufficcient conditions for existence of only single real and negative roots of equation (3) are as follows*

$1^o$. *All the coefficients $a_0, a_1, \ldots, a_n$ must be positive.*

$2^o$. *All the main even (or odd) main determinants of matrix (51) must be positive*

$$\begin{bmatrix} na_0 & a_0 & 0 & 0 & \ldots \\ (n-1)a_1 & a_1 & na_0 & a_0 & \ldots \\ (n-2)a_2 & a_2 & (n-1)a_1 & a_1 & \ldots \\ (n-3)a_3 & a_3 & (n-2)a_2 & a_2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & a_n \end{bmatrix}_{2n\times 2n} \tag{51}$$

*so*

$$\Delta_2 = \begin{vmatrix} na_0 & a_0 \\ (n-1)a_1 & a_1 \end{vmatrix} > 0, \ \ \Delta_4 = \begin{vmatrix} na_0 & a_0 & 0 & 0 \\ (n-1)a_1 & a_1 & na_0 & a_0 \\ (n-2)a_2 & a_2 & (n-1)a_1 & a_1 \\ (n-3)a_3 & a_3 & (n-2)a_2 & a_2 \end{vmatrix} > 0, \ \ldots, \ \ \Delta_{2n} > 0 \ . \tag{52}$$

We can also establish conditions for non existence of the extremums of eq. (1).

THEOREM 6.3 *Let us consider the transfer function*

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{a_0 s^n + a_1 s^{n-1} + \ldots + a_n} = \frac{L(s)}{M(s)} \ . \tag{53}$$

*We assume that the coefficients $b_1, \ldots, b_n$ are real, and the coefficients $a_0, a_1, \ldots, a_n$ are real and positive, and fulfill the Hurwitz stability conditions.*

*The necessary and sufficient conditions for non existence of extremums of equation (1) are that the zeros $z_i$ of the numerator $L(s)$ and zeros $s_j$ of the denominator $M(s)$ of the transfer function $G(s)$ (53) fulfill the inequalities $s_1 > z_i > s_2 > z_2 \ldots s_n > z_n < 0$, Re $s_i < 0$, $i = 1, 2, \ldots, n$. All the zeros must be real and single.*

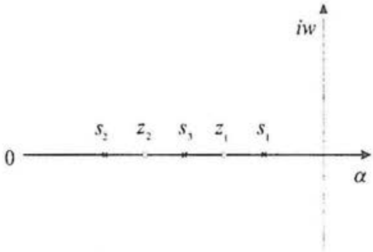


Figure 3. Illustration of the Theorem 6.3

## 7. Example 2: Equation of the $n$-th order

Let us consider the chain system which consists of $n$ equal elements of $R, L, C$, $G$ types as in Fig. 4.

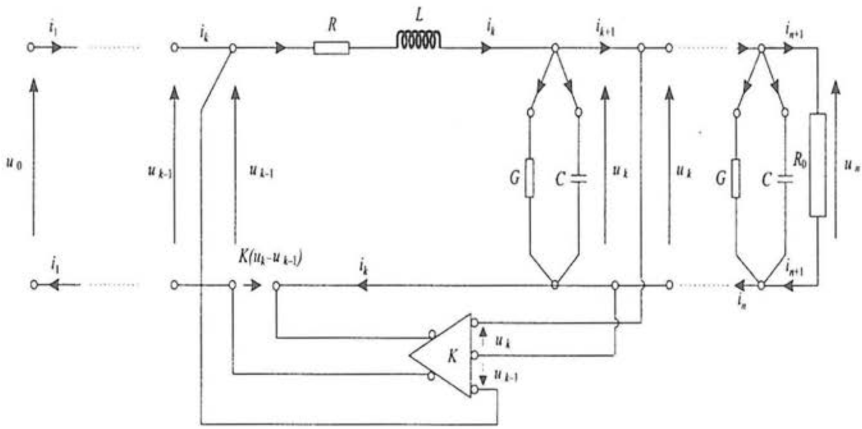


Figure 4. The chain $LRCG$ system

Each element of the chain is closed by feedback with a gain $K$ (see Górecki, 2000). The impedance of the load is equal to the impendance of every element of the chain. The transfer function of the whole system is equal to $G(s) = \dfrac{1}{M_n(s)}$, where

$$M_n(s) = \frac{\sin[(n+1)\phi(s)]}{\sin[\phi(s)]} \qquad (54)$$

and $\phi(s)$ can be determined from the relation

$$\frac{(G + sC)(R + sL)}{\phantom{x}} + 2 - 2\cos\phi \qquad (55)$$

From the equation $M_n(s) = 0$ and the relation (54) we obtain

$$\phi(s) = \frac{k\pi}{n+1}, \qquad k = 1, 2, \ldots, n \ . \tag{56}$$

Using the relations (55) and (56) we have that

$$LCs^2 + (GL + RC)s + GR + 4(K+1)\sin^2 \frac{k\pi}{2(n+1)} = 0, k = 1, \ldots, n.$$

The roots of this characteristic equation for $k = 1, 2, \ldots, n$ are

$$s_{1,2\ k} = \frac{-(GL+RC) \pm \sqrt{(GL+RC)^2 - 4LC[GR + 4(K+1)\sin^2 \frac{k\pi}{2(n+1)}]}}{2LC}$$

Knowing the roots $s_i$ we can calculate from equations (47) the moments of time $t_{e_i}$ of extremums, and after that from the relation (2) the values of extremums $x(t_{e_i})$.

## 8. Remark

The presentation of the new method gives the analytical solution of the transcendental equations. The method will be illustrated by an example of the $3^{rd}$ order transcendental equation. Let us consider the following equation

$$B_1 e^{s_1 t} + B_2 e^{s_2 t} + B_3 e^{s_3 t} = 0, \qquad s_1 \neq s_2 \neq s_3 \tag{57}$$

where $B_1, B_2, B_3$ are constant parameters. We interpret this equation as the derivative equal to zero of the system

$$\dot{x}(t_e) = 0 = A_1 s_1 e^{s_1 t_e} + A_2 s_2 e^{s_2 t_e} + A_3 s_3 e^{s_3 t_e} \ . \tag{58}$$

We look for such a solution $x(t)$ which besides the equation (58) fulfills the equations

$$x(t_e) = 0 = A_1^* e^{s_1 t_e} + A_2^* e^{s_2 t_e} + A_3^* e^{s_3 t_e} \tag{59}$$

and

$$\dot{x}(t_e) = 0 = A_1^* s_1 e^{s_1 t_e} + A_2^* s_2 e^{s_2 t_e} + A_3^* s_3 e^{s_3 t_e} \ . \tag{60}$$

The main determinant of the equations (58), (59) and (60) is equal to zero, if the initial condition satisfy the following relation

$$c_1^* = \frac{c_2^2}{c_3} \ . \tag{61}$$

Comparing the coefficients of the exponential terms in equations (57) and (58) we obtain

$$A_1 = \frac{B_1}{\ldots} \qquad A_2 = \frac{B_2}{\ldots} \qquad A_3 = \frac{B_3}{\ldots} \tag{62}$$

From the equation $x(t) = 0 = A_1 e^{s_1 t} + A_2 e^{s_2 t} + A_3 e^{s_3 t}$ we obtain, by putting $t = 0$ and using (62), the relations

$$A_1 + A_2 + A_3 = \frac{B_1}{s_1} + \frac{B_2}{s_2} + \frac{B_3}{s_3} = c_1$$

and from equation (57) we have for $t = 0$ that

$$B_1 + B_2 + B_3 = c_2 \ . \tag{63}$$

After differentiating equation (57) with respect to $t$ and putting $t = 0$ we have

$$B_1 s_1 + B_2 s_2 + B_3 s_3 = c_3 \ . \tag{64}$$

The unknown value of $c_1^*$ determined by (61) and using (63), (64) has the form

$$c_1^* = \frac{(B_1 + B_2 + B_3)^2}{B_1 s_1 + B_2 s_2 + B_3 s_3} \ .$$

The unknown values $t_e$ of the equation (57) can now be found according to the described general method.

## Appendix: Calculation of the determinant (48)

The determinant (48) was calculated for $n = 2, 3, 4, 5$. These results enable us to postulate the conjecture that for an arbitrary $n$ the formula (49) is valid. The general proof of the relation (49) for arbitrary $n$ was given by Professor S. Białas. This proof is as follows:

The elements of the inverse matrix to the matrix $V$ of Vandermonde have the form determined by the relation (7). $\alpha_{ij} = \dfrac{(-1)^{i+j}}{V} \varphi_{n-j}^{(i)} V_j$. Using these relations it is possible to write the explicit forms (9) for coefficient $A_k$

$$A_k = \frac{(-1)^k}{V} V_k \sum_{j=1}^{n} (-1)^j \varphi_{n-j}^{(k)} c_j \ , \tag{65}$$

$$A_k^* = \frac{(-1)^k}{V} V_k \Big[ \sum_{j=2}^{n} (-1)^j \ \varphi_{n-j}^{(k)} c_j - \varphi_{n-1}^{(k)} c_1^* \Big] \ . \tag{66}$$

By substracting (66) from (65) we obtain

$$A_k - A_k^* = \frac{(-1)^{k+1}}{V} V_k \varphi_{n-1}^{(k)} (c_1 - c_1^*) \qquad k = 1, 2, \ldots, n \tag{67}$$

The substitution of the relations (67) into the formula (49) gives

$$\Delta = \frac{(c_1 - c_1^*)^{n-2}}{V^{n-2}} \begin{vmatrix} A_1^* & A_2^* & \dots & A_n^* \\ s_1 A_1^* & s_2 A_2^* & \dots & s_n A_n^* \\ s_1 B_{11} & s_2 B_{12} & \dots & s_n B_{1n} \\ s_1^2 B_{11} & s_2^2 B_{12} & \dots & s_n^2 B_{1n} \\ \dots & \dots & \dots & \dots \\ s_1^{n-2} B_{11} & s_2^{n-2} B_{12} & \dots & s_n^{n-2} B_{1n} \end{vmatrix} \tag{68}$$

where

$$B_{ik} = (-1)^{k+1} V_k \; \varphi_{n-1}^{(k)}, \qquad k = 1, 2, \dots, n \ . \tag{69}$$

It is worth noting that in the formula (68) all the rows beginning from the third up to the $(n-2)$nd are independent of the initial conditions $c_1^*$, $c_1, c_2, \dots, c_n$.

Let us denote by $B$ the matrix whose determinant appears in the formula (68)

$$B = \begin{bmatrix} A_1^* & A_2^* & \dots & A_n^* \\ s_1 A_1^* & s_2 A_2^* & \dots & s_n A_n^* \\ s_1 B_{11} & s_2 B_{12} & \dots & s_n B_{1n} \\ \dots & \dots & \dots & \dots \\ s_1^{n-2} B_{11} & s_2^{n-2} B_{12} & \dots & s_n^{n-2} B_{1n} \end{bmatrix} .$$

Consider the product of the matrices

$$VB^T = \begin{bmatrix} 1 & 1 & . & 1 \\ s_1 & s_2 & . & s_n \\ \vdots & \vdots & & \vdots \\ s_1^{n-1} & s_2^{n-1} & . & s_n^{n-1} \end{bmatrix} \begin{bmatrix} A_1^* & s_1 A_1^* & s_1 B_{11} & . & s_1^{n-2} B_{11} \\ A_2^* & s_2 A_2^* & s_2 B_{12} & . & s_2^{n-2} B_{12} \\ \vdots & \vdots & \vdots & & \vdots\ \vdots \\ A_n^* & s_n A_n^* & s_n B_{1n} & . & s_n^{n-2} B_{1n} \end{bmatrix} .$$

Taking into account the fact that from the relations (2) and (4) for $t = 0$ we have $x^{(i-1)}(0) = c_i$, $i = 1, 2, \dots, n$ we obtain that

$$\left. \begin{aligned} A_1^* + A_2^* + \dots + A_n^* = {} & c_1^* \\ s_1 A_1^* + s_2 A_2^* + \dots + s_n A_n^* = {} & c_2 \\ \dots\dots\dots\dots\dots\dots\dots = {} & \dots \\ s_1^{n-1} A_n^* + s_2^{n-1} A_2^* + \dots + s_n^{(n-1)} A_n^* = {} & c_n \end{aligned} \right\} .$$

We can now express the product $VB^T$ in the following form

$$VB^T = \begin{bmatrix} c_1^* & c_2 & \beta_{13} & \beta_{14} & \dots & \beta_{1n} \\ c_2 & c_3 & \beta_{23} & \beta_{24} & \dots & \beta_{2n} \\ c_3 & c_4 & \beta_{33} & \beta_{34} & \dots & \beta_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{70}$$

where $c_{n+1} = x^n(0) = s_1^n A_1^* + s_2^n A_2^* + \ldots + s_n^n A_n^*$.

The elements $\beta_{ij}$ in the matrix $VB^T$ (70) can be written using (69) in the following form for $i = 1, 2, \ldots, n$ and $j = 3, 4, \ldots, n$

$$\beta_{ij} = \begin{bmatrix} s_1^{i-1} & s_2^{i-1} & \ldots & s_n^{i-1} \end{bmatrix} \begin{bmatrix} s_1^{j-2} B_{11} \\ s_2^{j-2} B_{12} \\ \vdots \\ s_n^{j-2} B_{1n} \end{bmatrix} =$$
$$= \begin{bmatrix} s_1^{i+j-3} & s_2^{i+j-3} & \ldots & s_n^{i+j-3} \\ s_1 & s_2 & \ldots & s_n \\ s_1^2 & s_2^2 & \ldots & s_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ s_1^{n-1} & s_2^{n-1} & \ldots & s_n^{n-1} \end{bmatrix}. \tag{71}$$

From the relations (71) it is evident that

$$\beta_{ij} = 0 \qquad \text{for} \quad i + j - 3 \le n - 1 \tag{72}$$

because in the matrix (71) some of the rows will repeat and the determinant of (71) will be equal to zero. The substitution of (72) into the matrix (70) yields the following determinant

$$|VB^T| = \begin{vmatrix} c_1^* & c_2 & 0 & 0 & \ldots & 0 \\ c_2 & c_3 & 0 & 0 & \ldots & 0 \\ c_3 & c_4 & \beta_{33} & \beta_{34} & \ldots & \beta_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ c_n & c_{n+1} & \beta_{n3} & \beta_{n4} & \ldots & \beta_{nn} \end{vmatrix} =$$

$$= c_1^* \begin{vmatrix} c_3 & 0 & 0 & \ldots & 0 \\ c_4 & \beta_{33} & \beta_{34} & \ldots & \beta_{3n} \\ c_5 & \beta_{43} & \beta_{44} & \ldots & \beta_{4n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{n+1} & \beta_{n3} & \beta_{nn} & \ldots & \beta_{nn} \end{vmatrix} - c_2 \begin{vmatrix} c_2 & 0 & 0 & \ldots & 0 \\ c_3 & \beta_{33} & \beta_{34} & \ldots & \beta_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_n & \beta_{n3} & \beta_{n4} & \ldots & \beta_{nn} \end{vmatrix} =$$

$$= c_1^* c_3 \begin{vmatrix} \beta_{33} & \beta_{34} & \ldots & \beta_{3n} \\ \beta_{43} & \beta_{44} & \ldots & \beta_{4n} \\ \ldots & \ldots & \ldots & \ldots \\ \beta_{n3} & \beta_{n4} & \ldots & \beta_{nn} \end{vmatrix} - c_2^2 \begin{vmatrix} \beta_{33} & \beta_{34} & \ldots & \beta_{3n} \\ \beta_{43} & \beta_{44} & \ldots & \beta_{4n} \\ \ldots & \ldots & \ldots & \ldots \\ \beta_{n3} & \beta_{n4} & \ldots & \beta_{nn} \end{vmatrix} =$$

$$(c_1^* c_3 - c_2^2) \begin{vmatrix} \beta_{33} & \beta_{34} & \ldots & \beta_{3n} \\ \beta_{43} & \beta_{44} & \ldots & \beta_{4n} \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}.$$

From the relations (72) we have

$$\left.\begin{array}{lll} \beta_{3,j}=0 & \text{for} & j=1,2,\ldots,n-1 \\ \beta_{4,j}=0 & \text{for} & j=1,2,\ldots,n-2 \\ \beta_{5,j}=0 & \text{for} & j=1,2,\ldots,n-3 \\ \ldots\ldots & \ldots & \ldots \\ \beta_{n-1,j}=0 & \text{for} & j=3 \end{array}\right\} \tag{73}$$

and from (71) we obtain

$$\beta_{3n}=\beta_{4(n-1)}=\beta_{5(n-2)}=\ldots=\beta_{n3}=(-1)^{n-1}s_1s_2\ldots s_nV=\beta \ . \tag{74}$$

Taking into account (73) and (74) we can write

$$\begin{aligned} \left|VB^T\right| = & \ (-1)^{n-1}\left(c_1^*c_3-c_2^2\right)\begin{vmatrix} 0 & 0 & \ldots & 0 & \beta \\ 0 & 0 & \ldots & \beta & \beta_{4n} \\ 0 & 0 & \ldots & \beta_{5n-1} & \beta_{5n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \beta & \beta_{n-4} & \beta_{n-5} & \ldots & \beta_{nn} \end{vmatrix} \\ = & \ (-1)^{n-2}\left(c_1^*c_3-c_2^2\right)(s_1s_2\ldots s_n)^{n-2}\ V^{n-2} \ . \end{aligned} \tag{75}$$

From (75) we have that the determinant

$$|B|=\left|B^T\right|=(-1)^{n-2}\left(c_1^*c_3-c_2^2\right)(s_1s_2\ldots s_n)^{n-2}\ V^{n-3} \ . \tag{76}$$

Finally, using (68) and (76) we obtain the main result

$$\Delta=\frac{(c_1-c_1^*)^{n-2}}{V}\,(-1)^{n-2}\left(\prod_{j=1}^{n}s_j\right)^{n-2}\left(c_1^*c_3-c_2^2\right)$$

and the relation (47) is proved.

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