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Neumann boundary optimal control problems governed by parabolic variational equalities*

by

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Abstract: We consider a heat conduction problem S with mixed boundary conditions in an n-dimensional domain Ω with regular boundary and a family of problems S_α with also mixed boundary conditions in Ω, where $\alpha > 0$ is the heat transfer coefficient on the portion of the boundary Γ_1. In relation to these state systems, we formulate *Neumann boundary* optimal control problems on the heat flux q which is definite on the complementary portion Γ_2 of the boundary of Ω. We obtain existence and uniqueness of the optimal controls, the first order optimality conditions in terms of the adjoint state and the convergence of the optimal controls, the system state and the adjoint state when the heat transfer coefficient α goes to infinity. Furthermore, we formulate particular *boundary* optimal control problems on a real parameter λ, in relation to the parabolic problems S and S_α and to mixed elliptic problems P and P_α. We find an explicit form for the optimal controls, we prove monotony properties and we obtain convergence results when the parameter time goes to infinity.

Keywords: parabolic variational equalities, optimal control, mixed boundary conditions, optimality conditions, convergence

1. Introduction

Following Gariboldi and Tarzia (2008), Menaldi and Tarzia (2007), and Tarzia, Bollo and Gariboldi (2020), we will study some Neumann boundary parabolic and elliptic optimal control problems. We consider a bounded domain Ω in $\mathbb{R}^n$,

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whose regular boundary Γ consists of the union of the two disjoint portions Γ_1 and Γ_2 with $|\Gamma_1| > 0$ and $|\Gamma_2| > 0$. We denote with $|\Gamma_i| = meas(\Gamma_i)$ (for $i = 1, 2$), the $(n-1)$-dimensional Hausdorff measure of the portion Γ_i on Γ. Let $[0, T]$ be a time interval, for a $T > 0$. We present the following heat conduction problems S and S_α (for each parameter $\alpha > 0$) respectively, with mixed boundary conditions (we denote by $u(t)$ to the function $u(\cdot, t)$):

$$\frac{\partial u}{\partial t} - \Delta u = g \quad \text{in } \Omega \qquad u\big|_{\Gamma_1} = b \qquad -\frac{\partial u}{\partial n}\bigg|_{\Gamma_2} = q \qquad u(0) = v_b \tag{1}$$

$$\frac{\partial u}{\partial t} - \Delta u = g \quad \text{in } \Omega \qquad -\frac{\partial u}{\partial n}\bigg|_{\Gamma_1} = \alpha(u-b) \qquad -\frac{\partial u}{\partial n}\bigg|_{\Gamma_2} = q \qquad u(0) = v_b, \tag{2}$$

where u is the temperature in $\Omega \times (0, T)$, g is the internal energy in Ω, b is the temperature on Γ_1 for (1) and the temperature of the external neighborhood of Γ_1 for (2), $v_b = b$ on Γ_1, q is the heat flux on Γ_2, and $\alpha > 0$ is the heat transfer coefficient on Γ_1 through a Robin condition, which satisfy the hypothesis: $g \in \mathcal{H} = L^2(0, T; L^2(\Omega))$, $q \in \mathcal{Q} = L^2(0, T; L^2(\Gamma_2))$ and $b \in H^{\frac{1}{2}}(\Gamma_1)$. In addition, $v_b \in H^1(\Omega)$ is the initial temperature for (1) and (2), respectively.

Let u and u_α be the unique solutions to the parabolic problems (1) and (2), whose variational formulations are given by (Menaldi and Tarzia, 2007):

$$\left\{ \begin{array}{l} u - v_b \in L^2(0, T; V_0), \qquad u(0) = v_b \quad \text{and} \quad \dot{u} \in L^2(0, T; V_0') \\ \text{such that} \quad \langle \dot{u}(t), v \rangle + a(u(t), v) = L(t, v), \quad \forall v \in V_0, \end{array} \right. \tag{3}$$

$$\left\{ \begin{array}{l} u_\alpha \in L^2(0, T; V), \qquad u_\alpha(0) = v_b \quad \text{and} \quad \dot{u}_\alpha \in L^2(0, T; V') \\ \text{such that} \quad \langle \dot{u}_\alpha(t), v \rangle + a_\alpha(u_\alpha(t), v) = L_\alpha(t, v), \quad \forall v \in V, \end{array} \right. \tag{4}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between the functional space (V or V_0) and its dual space (V' or V_0') and

$$\begin{array}{l} V = H^1(\Omega)\,; \qquad V_0 = \{v \in V : \, v\big|_{\Gamma_1} = 0\}\,; \qquad Q = L^2(\Gamma_2); \quad H = L^2(\Omega) \\ (g, h)_H = \displaystyle\int_\Omega gh \, dx; \quad (q, \eta)_Q = \int_{\Gamma_2} q\eta \, d\gamma; \\ a(u, v) = \displaystyle\int_\Omega \nabla u \nabla v dx; \quad a_\alpha(u, v) = a(u, v) + \alpha \int_{\Gamma_1} uv d\gamma \\ L(t, v) = (g(t), v)_H - (q(t), v)_Q; \quad L_\alpha(t, v) = L(t, v) + \alpha \displaystyle\int_{\Gamma_1} bv d\gamma. \end{array}$$

All data, g, q, b, v_b and the domain Ω with the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ are assumed to be sufficiently smooth so that the problems (1) and (2) admit variational solutions in Sobolev spaces. The existence and uniqueness of the solutions to the variational equalities (3) and (4), are well known, see, for example Brézis (1972), Chrysafinos and Hou (2017), Duvant and Lions (1972), or Gunzburger and Hou (1992).

Let $\mathcal{H} = L^2(0,T;H)$ be with norm $||.||_{\mathcal{H}}$ and internal product $(g,h)_{\mathcal{H}} = \int_0^T (g(t),h(t))_H dt$, and the space $\mathcal{Q} = L^2(0,T;Q)$, with norm $||.||_{\mathcal{Q}}$ and internal product $(q,\eta)_{\mathcal{Q}} = \int_0^T (q(t),\eta(t))_Q dt$.

For the sake of simplicity, for a Banach space X and $1 \le p \le \infty$, we will often use $L^p(X)$ instead of $L^p(0,T;X)$.

If we denote by u_q and $u_{\alpha q}$ the unique solution to the problems (3) and (4), respectively, we formulate the following *boundary* optimal control problems for the heat flux q as control variable, see Gariboldi and Tarzia (2008), Lions (1968), Tröltzsch (2010):

$$\text{find} \quad \overline{q} \in \mathcal{Q} \quad \text{such that} \quad J(\overline{q}) = \min_{q\in\mathcal{Q}} J(q), \tag{5}$$

$$\text{find} \quad \overline{q}_\alpha \in \mathcal{Q} \quad \text{such that} \quad J_\alpha(\overline{q}_\alpha) = \min_{q\in\mathcal{Q}} J_\alpha(q), \tag{6}$$

where the cost functionals $J : \mathcal{Q} \to \mathbb{R}_0^+$ and $J_\alpha : \mathcal{Q} \to \mathbb{R}_0^+$ are given by:

$$\text{i) } J(q) = \frac{1}{2}\|u_q - z_d\|_{\mathcal{H}}^2 + \frac{M}{2}\|q\|_{\mathcal{Q}}^2, \text{ ii) } J_\alpha(q) = \frac{1}{2}\|u_{\alpha q} - z_d\|_{\mathcal{H}}^2 + \frac{M}{2}\|q\|_{\mathcal{Q}}^2 \tag{7}$$

with $z_d \in \mathcal{H}$ given and M being a positive constant.

In Gariboldi and Tarzia (2008), the authors studied boundary optimal control problems on the heat flux q in mixed elliptic problems and they proved existence, uniqueness and asymptotic behavior of the optimal solutions, when the heat transfer coefficient goes to infinity. Similar results were obtained in Gariboldi and Tarzia (2015) for simultaneous distributed-boundary optimal control problems on the internal energy g and the heat flux q in mixed elliptic problems. In Menaldi and Tarzia (2007), convergence results were proven for heat conduction problems in relation to distributed optimal control problems on the internal energy g as a control variable. Parabolic control problem with Robin boundary conditions were considered in Bergounioux and Tröltzsch (1999), Boukrouche and Tarzia (2013), Chrysafinos, Gunzburger and Hou (2006), Menaldi and Tarzia (2007), and Tarzia, Bollo and Gariboldi (2020). Other papers on the subject are Ben Belgacem, El Fekih and Raymond (2003), Sener and Subasi (2015), Sweilam and Abd-Elal (2003), Wang and Yan (2019).

In this paper, our main goal is to study the existence and uniqueness of solutions and the asymptotic behaviour of the optimal control problems (5) and (6), when $\alpha \to \infty$. Moreover, motivated by Gonzalez and Tarzia (1990) we try find explicit solutions for the optimal controls and a relationship between elliptic and parabolic boundary optimal control problems, when the time goes to infinity. In this way, we consider the following family of optimization problems on the heat flux dependent of a real parameter:

For fixed $q_0 \in \mathcal{Q}$, we define $\mathcal{Q}_0 = \{\lambda q_0 : \lambda \in \mathbb{R}\} \subset \mathcal{Q}$, and we formulate the following *real Neumann parabolic boundary* optimal control problems, for each

$T > 0$ and $\alpha > 0$:

$$\text{find} \quad \overline{\lambda}(T) \in \mathbb{R} \quad \text{such that} \quad H_T(\overline{\lambda}(T)) = \min_{\lambda \in \mathbb{R}} H_T(\lambda), \tag{8}$$

$$\text{find} \quad \overline{\lambda}_\alpha(T) \in \mathbb{R} \quad \text{such that} \quad H_{\alpha T}(\overline{\lambda}_\alpha(T)) = \min_{\lambda \in \mathbb{R}} H_{\alpha T}(\lambda), \tag{9}$$

where

$$H_T(\lambda) = J(\lambda q_0) \quad \text{and} \quad H_{\alpha T}(\lambda) = J_\alpha(\lambda q_0). \tag{10}$$

Moreover, we consider the elliptic mixed problems P and P_α, for each $\alpha > 0$, Gariboldi and Tarzia (2008, 2015):

$$-\Delta u = g \quad \text{in } \Omega \qquad u\big|_{\Gamma_1} = b \qquad -\frac{\partial u}{\partial n}\bigg|_{\Gamma_2} = q \tag{11}$$

$$-\Delta u = g \quad \text{in } \Omega \qquad -\frac{\partial u}{\partial n}\bigg|_{\Gamma_1} = \alpha(u - b) \qquad -\frac{\partial u}{\partial n}\bigg|_{\Gamma_2} = q, \tag{12}$$

whose variational equalities are given by

$$a(u, v) = L(v), \quad \forall v \in V_0, \ u \in K \tag{13}$$

$$a_\alpha(u_\alpha, v) = L_\alpha(v), \quad \forall v \in V, \ u_\alpha \in V \tag{14}$$

with $K = v_0 + V_0$ for a given $v_0 = b$ in Γ_1. For fixed $q_0^* \in Q$, we define $Q_0 = \{\lambda q_0^* : \lambda \in \mathbb{R}\} \subset Q$, and we formulate the following *real Neumann elliptic boundary* optimal control problems, for each $\alpha > 0$:

$$\text{find} \quad \overline{\lambda} \in \mathbb{R} \quad \text{such that} \quad H(\overline{\lambda}) = \min_{\lambda \in \mathbb{R}} H(\lambda), \tag{15}$$

$$\text{find} \quad \overline{\lambda}_\alpha \in \mathbb{R} \quad \text{such that} \quad H_\alpha(\overline{\lambda}_\alpha) = \min_{\lambda \in \mathbb{R}} H_\alpha(\lambda), \tag{16}$$

where

$$H(\lambda) = J^*(\lambda q_0^*) \quad \text{and} \quad H_\alpha(\lambda) = J_\alpha^*(\lambda q_0^*), \tag{17}$$

with $J^* : Q \to \mathbb{R}_0^+$ and $J_\alpha^* : Q \to \mathbb{R}_0^+$ given by Gariboldi and Tarzia (2008):

$$\text{i) } J^*(q) = \frac{1}{2}\left\| u_{\infty q} - z_d \right\|_H^2 + \frac{M}{2}\left\| q \right\|_Q^2, \text{ ii) } J_\alpha^*(q) = \frac{1}{2}\left\| u_{\infty\alpha q} - z_d \right\|_H^2 + \frac{M}{2}\left\| q \right\|_Q^2 \tag{18}$$

where $u_{\infty q}$ and $u_{\infty\alpha q}$ are the unique solutions to the variational equalities (13) and (14), respectively, $z_d \in H$ is given and M is a positive constant.

The paper is structured as follows. In Section 2, we consider Neumann boundary optimal control problems on the heat flux q for heat conduction problems (1), (5) and (7i) and Neumann parabolic boundary optimal control problems on the heat flux q for (2), (6) and (7ii), for each $\alpha > 0$. We prove existence and uniqueness of the optimal controls and we give the first order optimality conditions. In Section 3, for fixed q, we prove asymptotic estimates and convergence results for the system states, the adjoint states and the optimal controls, when the heat transfer coefficient goes to infinity. In Section 4, we prove estimates between the optimal controls of the problems (5) and (6) and the second component of the simultaneous optimal controls of the problems studied in Tarzia, Bollo and Gariboldi (2020). In Section 5, for the real Neumann parabolic boundary optimal control problems (8), (9), (15) and (16) we prove the existence and uniqueness and we find explicit solutions for the optimal control $\overline{\lambda}(t)$, $\overline{\lambda_\alpha}(t)$, $\overline{\lambda}$ and $\overline{\lambda_\alpha}$, respectively. Moreover, monotonicity properties with respect to the data are also studied. Finally, in Section 6, convergence results of the solutions to the problems (3) to the solution to the problem (13) are obtained, when the parameter time $t \to \infty$.

2. Boundary optimal control problems for systems S and $\mathbf{S}_\alpha$

Here, we prove that the functionals J and J_α are strictly convex and Gâteaux differentiable in $\mathcal{Q}$. Moreover, we obtain the existence and uniqueness of the boundary optimal controls $\overline{q}$ and $\overline{q}_\alpha$ and we give the optimality conditions in terms of the adjoint states, for the optimal control problems (5) and (6), respectively.

Following Lions (1968), Menaldi and Tarzia (2007) and Tröltzsch (2010), we define the application $C : \mathcal{Q} \to L^2(V_0)$ such that $C(q) = u_q - u_0$, where u_0 is the solution of problem (3) for $q = 0$.

We consider $\Pi : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ and $\mathcal{L} : \mathcal{Q} \to \mathbb{R}$, defined by the expressions

$$\Pi(q, \eta) = (C(q), C(\eta))_{\mathcal{H}} + M(q, \eta)_{\mathcal{Q}} \quad \forall q, \eta \in \mathcal{Q}$$

$$\mathcal{L}(q) = (C(q), z_d - u_0)_{\mathcal{H}} \quad \forall q \in \mathcal{Q}$$

and we prove the following result

Lemma 1 *i) C is a linear and continuous functional.*

ii) Π is a bilinear, symmetric, continuous form and coercive in $\mathcal{Q}$.

iii) $\mathcal{L}$ is linear and continuous functional in $\mathcal{Q}$.

iv) J can be written down as:

$$J(q) = \frac{1}{2}\Pi(q, q) - \mathcal{L}(q) + \frac{1}{2}||u_0 - z_d||^2_{\mathcal{H}} \quad \forall q \in \mathcal{Q}.$$

v) J is a strictly convex functional on $\mathcal{Q}$, that is, $\forall q_1, q_2 \in \mathcal{Q}$, $\forall t \in [0, 1]$

$$(1 - t)J(q_2) + tJ(q_1) - J((1 - t)q_2 + tq_1) \geq \frac{Mt(1 - t)}{2}||q_2 - q_1||^2_{\mathcal{Q}}.$$

vi) There exists a unique optimal control $\overline{q} \in \mathcal{Q}$ *such that*

$$J(\overline{q}) = \min_{q\in\mathcal{Q}} J(q).$$

Proof It follows from Lions (1968) and Menaldi and Tarzia (2007), and

$$(1-t)J(q_2) + tJ(q_1) - J((1-t)q_2 + tq_1) = \\ \frac{t(1-t)}{2}\left[||u_{q_2} - u_{q_1}||^2_{\mathcal{H}} + M||q_2 - q_1||^2_{\mathcal{Q}}\right], \quad \forall q_1, q_2 \in \mathcal{Q}, \quad \forall\, t \in [0,1] \quad \square$$

Now, we define the adjoint state p_q, corresponding to the system (1) for each $q \in \mathcal{Q}$, as the unique solution of the following mixed parabolic problem:

$$-\frac{\partial p_q}{\partial t} - \Delta p_q = u_q - z_d \text{ in } \Omega, \qquad p_q\big|_{\Gamma_1} = 0, \qquad \frac{\partial p_q}{\partial n}\bigg|_{\Gamma_2} = 0, \qquad p_q(T) = 0,$$

whose variational formulation is given by

$$\begin{cases} p_q \in L^2(V_0),\ p_q(T) = 0 \quad \text{and } \dot{p}_q \in L^2(V_0') \text{ such that} \\ -\langle \dot{p}_q(t), v\rangle + a(p_q(t), v) = (u_q(t) - z_d(t), v)_H, \quad \forall v \in V_0, \end{cases} \tag{19}$$

and we consider the following properties of the functional J, following Lions (1968), Menaldi and Tarzia (2007), and Tarzia, Bollo and Gariboldi (2020).

Lemma 2 *i) The adjoint state* p_q *satisfies:*

$$(C(\eta), u_q - z_d)_{\mathcal{H}} = -(\eta, p_q)_{\mathcal{Q}}, \quad \forall q, \eta \in \mathcal{Q}.$$

ii) The functional J *is Gâteaux differentiable and* J' *is given by:*

$$\begin{aligned}\langle J'(q), \eta - q\rangle &= (u_\eta - u_q, u_q - z_d)_{\mathcal{H}} + M(q, \eta - q)_{\mathcal{Q}} \\ &= \Pi(q, \eta - q) - \mathcal{L}(\eta - q), \quad \forall q, \eta \in \mathcal{Q}.\end{aligned}$$

iii) The Gâteaux derivative of J *can be written as:*

$$J'(q) = Mq - p_q \quad \forall q \in \mathcal{Q}.$$

iv) The optimality condition for the optimal control problem (5) is given by

$$M\overline{q} - p_{\overline{q}} = 0 \quad \text{in } \mathcal{Q}.$$

Next, we define the application $C_\alpha : \mathcal{Q} \to L^2(V)$ such that $C_\alpha(q) = u_{\alpha q} - u_{\alpha 0}$, where $u_{\alpha 0}$ is the solution of the variational problem (4) for $q = 0$.

If we consider $\Pi_\alpha : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ and $\mathcal{L}_\alpha : \mathcal{Q} \to \mathbb{R}$, defined by

$$\Pi_\alpha(q, \eta) = (C_\alpha(q), C_\alpha(\eta))_{\mathcal{H}} + M(q, \eta)_{\mathcal{Q}} \quad \forall q, \eta \in \mathcal{Q}$$

$$\mathcal{L}_\alpha(q) = (C_\alpha(q), z_d - u_{\alpha 0})_{\mathcal{H}} \quad \forall q \in \mathcal{Q}$$

in a similar way to Lemmae 1 and 2, we have the following result:

LEMMA 3 *(i) There exists a unique optimal control $\overline{q}_\alpha \in \mathcal{Q}$ such that*

$$J_\alpha(\overline{q}_\alpha) = \min_{q \in \mathcal{Q}} J_\alpha(q).$$

(ii) The Gateaux derivative of J_α can be written as:

$$J'_\alpha(q) = Mq - p_{\alpha q} \quad \forall q \in \mathcal{Q}, \tag{20}$$

and the optimality condition for the optimal control problem (6) is given by:

$$M\overline{q}_\alpha - p_{\alpha \overline{q}_\alpha} = 0 \quad in\ \mathcal{Q} \tag{21}$$

where the adjoint state $p_{\alpha q}$ corresponds to (2) for each $q \in \mathcal{Q}$, as the unique solution of

$$-\frac{\partial p_{\alpha q}}{\partial t} - \Delta p_{\alpha q} = u_{\alpha q} - z_d \ \ in\ \Omega, \ -\frac{\partial p_{\alpha q}}{\partial n}\bigg|_{\Gamma_1} = \alpha p_{\alpha q}, \ \frac{\partial p_{\alpha q}}{\partial n}\bigg|_{\Gamma_2} = 0,\ p_{\alpha q}(T) = 0,$$

whose variational formulation is given by

$$\begin{cases} p_{\alpha q} \in L^2(V),\ p_{\alpha q}(T) = 0 \quad and\ \dot{p}_{\alpha q} \in L^2(V')\ such\ that \\ -\langle \dot{p}_{\alpha q}(t), v\rangle + a_\alpha(p_{\alpha q}(t), v) = (u_{\alpha q}(t) - z_d(t), v)_H,\ \forall v \in V \end{cases} \tag{22}$$

for each $\alpha > 0$.

3. Convergence of Neumann boundary optimal control problems when $\alpha \to \infty$

Now, for fixed $q \in Q$, we obtain estimates on $u_{\alpha q}$ and $p_{\alpha q}$ uniformly when $\alpha > 1$. Next, we prove strong convergence for q_α, $u_{\alpha q}$ and $p_{\alpha q}$, when α goes to infinity.

PROPOSITION 1 *(i) If u_q and $u_{\alpha q}$ are the unique solutions to the variational equalities (3) and (4), respectively, we have the estimation*

$$||\dot{u}_{\alpha q}||_{L^2(V_0')} + ||u_{\alpha q}||_{L^\infty(H)} + ||u_{\alpha q}||_{L^2(V)} + \sqrt{(\alpha-1)}||u_{\alpha q} - b||_{L^2(L^2(\Gamma_1))} \leq C \tag{23}$$

for all $\alpha > 1$, where the constant C depends only on the norms $||\dot{u}_q||_{L^2(V_0')}$, $||\dot{u}_q||_{L^2(V')}$, $||\nabla u_q||_{\mathcal{H}}$, $||u_q||_{L^2(V)}$, $||u_q||_{L^\infty(H)}$, $||g||_{\mathcal{H}}$, $||q||_{\mathcal{Q}}$ and the coerciveness constant λ_1 of the bilinear form a_1.

(ii) For fixed $q \in \mathcal{Q}$ we have $u_{\alpha q} \to u_q$ strongly in $L^2(V) \cap L^\infty(H)$ and $\dot{u}_{\alpha q} \to \dot{u}_q$ strongly in $L^2(V_0')$, when $\alpha \to \infty$.

Proof (i) Taking $v = u_{\alpha q}(t) - u_q(t) \in V$ in the variational equation (4), and taking into account that $u_q(t)\big|_{\Gamma_1} = b$, by using Young's inequality and integrating between $[0,T]$, we obtain

$$\frac{1}{2}||u_{\alpha q}(T) - u_q(T)||_H^2 + \frac{\lambda_1}{2}||u_{\alpha q} - u_q||_{L^2(V)}^2 + (\alpha - 1)||u_{\alpha q} - b||_{L^2(L^2(\Gamma_1))}^2$$
$$\leq \frac{2}{\lambda_1}[||g||_{\mathcal{H}}^2 + \|\gamma_0\|^2||q||_{\mathcal{Q}}^2 + ||\nabla u_q||_{\mathcal{H}}^2 + ||\dot{u}_q||_{L^2(V')}^2],$$

where γ_0 is the trace operator on Γ.

Here, we prove that there exists a positive constant K, independent of α, which depends on

$$K = K(\lambda_1, ||u_q||_{L^\infty(H)}, ||u_q||_{L^2(V)}, ||g||_{\mathcal{H}}, ||q||_{\mathcal{Q}}, ||\nabla u_q||_{\mathcal{H}}, ||\dot{u}_q||_{L^2(V')})$$

such that for all $\alpha > 1$, we have:

$$||u_{\alpha q}||_{L^\infty(H)} + ||u_{\alpha q}||_{L^2(V)} + \sqrt{(\alpha - 1)}||u_{\alpha q} - b||_{L^2(L^2(\Gamma_1))} \leq K. \tag{24}$$

Next, by taking $v \in V_0$ in the variational equality (4) and subtracting the variational equality (3), we have

$$(\dot{u}_{\alpha q}(t) - \dot{u}_q(t), v)_H \leq ||u_q(t) - u_{\alpha q}(t)||_V ||v||_{V_0} \quad \forall v \in V_0,$$

and integrating in $[0,T]$, we obtain

$$||\dot{u}_{\alpha q} - \dot{u}_q||_{L^2(V_0')} \leq ||u_q - u_{\alpha q}||_{L^2(V)}.$$

Next, by using (24), we conclude that there exists a positive constant $C = C(K, ||\dot{u}_q||_{L^2(V_0')})$ such that (23) holds.

(ii) Let there be a fixed $q \in \mathcal{Q}$, we consider a sequence $\{u_{\alpha_n q}\}$ in $L^2(V) \cap L^\infty(H)$ and by estimation (23), we have that

$$||u_{\alpha_n q}||_{L^2(V)} \leq C \text{ and } ||\dot{u}_{\alpha_n q}||_{L^2(V_0')} \leq C,$$

therefore, there exists a subsequence $\{u_{\alpha_n q}\}$ which is weakly convergent to $w_q \in L^2(V)$ and weakly* in $L^\infty(H)$ and there exists a subsequence $\{\dot{u}_{\alpha_n q}\}$ which is weakly convergent to $\dot{w}_q \in L^2(V_0')$. Now, from the third term of left hand side of (24) and the weak lower semicontinuity of the norm in $L^2(L^2(\Gamma_1))$, we have that $w_q = b$ on Γ_1 and therefore $w_q - v_b \in L^2(V_0)$. Next, we prove that w_q satisfies

$$\langle \dot{w}_q(t), v\rangle + a(w_q(t), v) = L(t, v), \quad \forall v \in V_0$$

and $w_q(0) = v_b$ with $\dot{w}_q \in L^2(V_0')$.

Therefore, by the uniqueness of the solution of the variational problem (3), we obtain $w_q = u_q$. That is, when $\alpha \to \infty$ we have

$$u_{\alpha q} \rightharpoonup u_q \text{ in } L^2(V), \quad u_{\alpha q} \overset{*}{\rightharpoonup} u_q \text{ in } L^\infty(H) \quad \text{and} \quad \dot{u}_{\alpha q} \rightharpoonup \dot{u}_q \text{ in } L^2(V_0').$$

Now, we have

$$\frac{1}{2}||u_{\alpha q}(T)-u_q(T)||_H^2+\lambda_1||u_{\alpha q}-u_q||_{L^2(V)}^2+(\alpha-1)||u_{\alpha q}-u_q||_{L^2(L^2(\Gamma_1))}^2$$
$$\leq \int_0^T \{L(t,u_{\alpha q}(t)-u_q(t))-a(u_q(t),u_{\alpha q}(t)-u_q(t))-\langle \dot{u}_q(t),u_{\alpha q}(t)-u_q(t)\rangle\}dt$$

and by using the weak convergence of $u_{\alpha q}$ to u_q, we prove the strong convergence in $L^2(V)$. Next, taking into account that

$$||u_{\alpha q}-u_q||_{L^2(L^2(\Gamma_1))}^2 \leq \frac{1}{\alpha-1}\int_0^T \{L(t,u_{\alpha q}(t)-u_q(t))-a(u_q(t),u_{\alpha q}(t)-u_q(t))$$
$$-\langle \dot{u}_q(t),u_{\alpha q}(t)-u_q(t)\rangle\}dt$$

and the weak convergence of $u_{\alpha q}$ to u_q, we prove the strong convergence in $L^2(L^2(\Gamma_1))$. Now, from the variational equalities (3) and (4), we have

$$||\dot{u}_{\alpha q}-\dot{u}_q||_{L^2(V_0')}^2 \leq ||u_q-u_{\alpha q}||_{L^2(V)}^2 \to 0, \quad \text{when} \quad \alpha\to\infty.$$

We deduce that $\dot{u}_{\alpha q}$ is strongly convergent to $\dot{u}_q$ in $L^2(V_0')$. Finally, we have

$$||u_{\alpha q}-u_q||_{L^\infty(H)}^2 \leq 2(||g||_{\mathcal{H}}+\|\gamma_0\|||q||_{\mathcal{Q}}+||u_q||_{L^2(V_0)}+||\dot{u}_q||_{\mathcal{H}})||u_{\alpha q}-u_q||_{L^2(V)}$$

and from the strong convergence of $u_{\alpha q}$ to u_q in $L^2(V)$, we prove that $u_{\alpha q}$ is strongly convergent to u_q in $L^\infty(H)$, when $\alpha\to\infty$. □

PROPOSITION 2 *(i) If p_q and $p_{\alpha q}$ are the unique solutions to the variational equalities (19) and (22), respectively, we have the estimation*

$$||\dot{p}_{\alpha q}||_{L^2(V_0')}+||p_{\alpha q}||_{L^\infty(H)}+||p_{\alpha q}||_{L^2(V)}+\sqrt{(\alpha-1)}||p_{\alpha q}||_{L^2(L^2(\Gamma_1))}\leq C \quad (25)$$

for all $\alpha>1$, where the constant C depends on the norms $||\dot{p}_q||_{L^2(V_0')}$, $||\dot{p}_q||_{L^2(V')}$, $||\nabla p_q||_{\mathcal{H}}$, $||p_q||_{L^2(V)}$, $||p_q||_{L^\infty(H)}$, $||g||_{\mathcal{H}}$, $||q||_{\mathcal{Q}}$, $||z_d||_{\mathcal{H}}$, $||\dot{u}_q||_{L^2(V')}$, $||\nabla u_q||_{\mathcal{H}}$, $||u_q||_{L^2(V)}$, $||u_q||_{L^\infty(H)}$ and of the coerciveness constant λ_1.

(ii) For fixed $q\in\mathcal{Q}$, we have that $p_{\alpha q}\to p_q$ strongly in $L^2(V)\cap L^\infty(H)$ and $\dot{p}_{\alpha q}\to\dot{p}_q$ strongly in $L^2(V_0')$, when $\alpha\to\infty$.

PROOF Let there be fixed $q\in\mathcal{Q}$, the estimation (25) follows from analogous reasoning to that of Proposition 1. We have that there exists a subsequence $\{p_{\alpha_n q}\}$, which is weakly convergent to $\eta_q\in L^2(V)$ and weakly* in $L^\infty(H)$. From the weak semicontinuity of the norm, we have that $\eta_q=0$ on Γ_1 and therefore $\eta_q\in L^2(V_0)$. Moreover, η_q satisfies

$$-\langle\dot{\eta}_q(t),v\rangle+a(\eta_q(t),v)=(u_q(t)-z_d(t),v)_H \quad \forall v\in V_0$$

and $\eta_q(T)=0$ with $\dot{\eta}_q \in L^2(V_0')$. Therefore, by the uniqueness of the solution of the variational problem (19), we obtain $\eta_q = p_q$ and when $\alpha \to \infty$, we have

$$p_{\alpha q} \rightharpoonup p_q \text{ in } L^2(V), \quad p_{\alpha q} \overset{*}{\rightharpoonup} p_q \text{ in } L^\infty(H) \quad \text{and} \quad \dot{p}_{\alpha q} \rightharpoonup \dot{p}_q \text{ in } L^2(V_0').$$

Finally, the strong convergence of $p_{\alpha q}$ to p_q in $L^2(V) \cap L^\infty(H)$ and of $\dot{p}_{\alpha q}$ to $\dot{p}_q$ in norm $L^2(V_0')$ is obtained in a similar way to that in Proposition 1. □

Now, we consider the boundary optimal control problems (5) and (6) and our goal is to prove the following theorem:

Theorem 1 *Let $\overline{q}$ and $\overline{q}_\alpha$ be the unique solutions of the optimal control problems (5) and (6), respectively. Then, we have that $\overline{q}_\alpha \to \overline{q}$ strongly in $\mathcal{Q}$, when the parameter $\alpha \to \infty$. Moreover, the system state and the adjoint state satisfy $(u_{\alpha\overline{q}_\alpha}, \dot{u}_{\alpha\overline{q}_\alpha}) \to (u_{\overline{q}}, \dot{u}_{\overline{q}})$ and $(p_{\alpha\overline{q}_\alpha}, \dot{p}_{\alpha\overline{q}_\alpha}) \to (p_{\overline{q}}, \dot{p}_{\overline{q}})$ strongly in $L^2(V) \times L^2(V_0')$.*

Proof We will establish the proof in three steps.

Step 1. From the estimation (23) for $q=0$, there exists a constant $C_1 > 0$ such that

$$||u_{\alpha 0}||_{\mathcal{H}} \leq ||u_{\alpha 0}||_{L^2(V)} \leq C_1, \ \forall \alpha > 1,$$

and from $J_\alpha(\overline{q}_\alpha) \leq J_\alpha(0)$, we have

$$\frac{1}{2}||u_{\alpha\overline{q}_\alpha} - z_d||^2_{\mathcal{H}} + \frac{M}{2}||\overline{q}_\alpha||^2_{\mathcal{Q}} \leq \frac{1}{2}||u_{\alpha 0} - z_d||^2_{\mathcal{H}}.$$

Therefore, there exist positive constants C_2 and C_3 such that

$$||u_{\alpha\overline{q}_\alpha}||_{\mathcal{H}} \leq C_2 \quad \text{and} \quad ||\overline{q}_\alpha||_{\mathcal{Q}} \leq C_3, \quad \forall \alpha > 1.$$

Now, by an analogous reasoning to that for the estimates (23) and (25), there exist positive constants C_4 and C_5 such that, for all $\alpha > 1$, we obtain

$$||u_{\alpha\overline{q}_\alpha}||_{L^2(V)} + ||\dot{u}_{\alpha\overline{q}_\alpha}||_{L^2(V_0')} + \sqrt{(\alpha-1)}||u_{\alpha\overline{q}_\alpha} - b||_{L^2(L^2(\Gamma_1))} \leq C_4$$

$$||p_{\alpha\overline{q}_\alpha}||_{L^2(V)} + ||\dot{p}_{\alpha\overline{q}_\alpha}||_{L^2(V_0')} + \sqrt{(\alpha-1)}||p_{\alpha\overline{q}_\alpha}||_{L^2(L^2(\Gamma_1))} \leq C_5.$$

From the previous estimations, we have that there exist $f \in \mathcal{Q}$, $\mu \in L^2(V)$, $\dot{\mu} \in L^2(V_0')$, $\rho \in L^2(V)$ and $\dot{\rho} \in L^2(V_0')$ such that

$$\overline{q}_\alpha \rightharpoonup f \in \mathcal{Q}, \quad u_{\alpha\overline{q}_\alpha} \rightharpoonup \mu \in L^2(V), \quad \dot{u}_{\alpha\overline{q}_\alpha} \rightharpoonup \dot{\mu} \in L^2(V_0'),$$

$$p_{\alpha\overline{q}_\alpha} \rightharpoonup \rho \in L^2(V), \quad \dot{p}_{\alpha\overline{q}_\alpha} \rightharpoonup \dot{\rho} \in L^2(V_0').$$

Step 2. Taking into account the weak convergence of $u_{\alpha\overline{q}_\alpha}$ to μ in $L^2(V)$ and the estimation $\sqrt{(\alpha-1)}||u_{\alpha\overline{q}_\alpha} - b||_{L^2(L^2(\Gamma_1))} \leq C_4$, in a similar way to that of Proposition 1, we obtain that $\mu = u_f$. Moreover, for the adjoint state, we have that $p_{\alpha\overline{q}_\alpha}$ is weakly convergent to ρ in $L^2(V)$ and from the estimation

$\sqrt{(\alpha-1)}||p_{\alpha\overline{q}_\alpha}||_{L^2(L^2(\Gamma_1))} \leq C_5$, in a similar way to that of Proposition 2, we obtain that $\rho = p_f$. Therefore, we have $u_{\alpha\overline{q}_\alpha} \rightharpoonup u_f$ in $L^2(V)$ and $p_{\alpha\overline{q}_\alpha} \rightharpoonup p_f$ in $L^2(V)$. Now, the optimality condition for the optimal control problem (6) is given by $(M\overline{q}_\alpha - p_{\alpha\overline{q}_\alpha}, \eta)_{\mathcal{Q}} = 0$, $\forall \eta \in \mathcal{Q}$, and taking into account that

$$p_{\alpha\overline{q}_\alpha} \rightharpoonup p_f \quad \text{in} \quad L^2(V) \quad \text{and} \quad \overline{q}_\alpha \rightharpoonup f \quad \text{in} \quad \mathcal{Q},$$

we obtain $-p_f + Mf = 0$, and by the uniqueness of the optimal control we deduce that $f = \overline{q}$. Therefore, $u_f = u_{\overline{q}}$, $p_f = p_{\overline{q}}$, $\dot{u}_f = \dot{u}_{\overline{q}}$ and $\dot{p}_f = \dot{u}_{\overline{q}}$.

Step 3. We have, for all $q \in \mathcal{Q}$

$$J(\overline{q}) = \frac{1}{2}||u_{\overline{q}} - z_d||^2_{\mathcal{H}} + \frac{M}{2}||\overline{q}||^2_{\mathcal{Q}} \leq \liminf_{\alpha\to\infty}\left[\frac{1}{2}||u_{\alpha\overline{q}_\alpha} - z_d||^2_{\mathcal{H}} + \frac{M}{2}||\overline{q}_\alpha||^2_{\mathcal{Q}}\right] \leq$$

$$\limsup_{\alpha\to\infty}\left[\frac{1}{2}||u_{\alpha\overline{q}_\alpha} - z_d||^2_{\mathcal{H}} + \frac{M}{2}||\overline{q}_\alpha||^2_{\mathcal{Q}}\right] \leq \limsup_{\alpha\to\infty} J_\alpha(q) =$$

$$\lim_{\alpha\to\infty}\left[\frac{1}{2}||u_{\alpha q} - z_d||^2_{\mathcal{H}} + \frac{M}{2}||q||^2_{\mathcal{Q}}\right] = \frac{1}{2}||u_q - z_d||^2_{\mathcal{H}} + \frac{M}{2}||q||^2_{\mathcal{Q}} = J(q).$$

By taking infimum on q, all the above inequalities become equalities and therefore

$$\lim_{\alpha\to\infty}\left[||u_{\alpha\overline{q}_\alpha} - z_d||^2_{\mathcal{H}} + M||\overline{q}_\alpha||^2_{\mathcal{Q}}\right] = ||u_{\overline{q}} - z_d||^2_{\mathcal{H}} + M||\overline{q}||^2_{\mathcal{Q}}$$

that is

$$\lim_{\alpha\to\infty}||(\sqrt{M}\overline{q}_\alpha, u_{\alpha\overline{q}_\alpha} - z_d)||^2_{\mathcal{Q}\times\mathcal{H}} = ||(\sqrt{M}\overline{q}, u_{\overline{q}} - z_d)||^2_{\mathcal{Q}\times\mathcal{H}}.$$

The previous equality, the convergence $\overline{q}_\alpha \rightharpoonup \overline{q}$ in $\mathcal{Q}$ and $u_{\alpha\overline{q}_\alpha} \rightharpoonup u_{\overline{q}}$ in $L^2(V)$ imply that $(\overline{q}_\alpha, u_{\alpha\overline{q}_\alpha}) \to (\overline{q}, u_{\overline{q}})$ strongly in $\mathcal{Q}\times\mathcal{H}$, when $\alpha\to\infty$.
Finally, if we take $v = u_{\alpha\overline{q}_\alpha}(t) - u_{\overline{q}}(t) \in V$ in (4) for $u = u_{\alpha\overline{q}_\alpha}$ and if we call $z_\alpha = u_{\alpha\overline{q}_\alpha} - u_{\overline{q}}$, we have

$$\lambda_1||z_\alpha(t)||^2_V \leq (g(t) - \dot{u}_{\overline{q}}(t), z_\alpha(t))_H - (\overline{q}_\alpha(t), z_\alpha(t))_Q - a(u_{\overline{q}}(t), z_\alpha(t)).$$

Integrating between 0 and T and taking into account that $z_\alpha \rightharpoonup 0$ weakly in $L^2(V)$, z_α is bounded independently of α, and $\overline{q}_\alpha \to \overline{q}$ strongly in $\mathcal{Q}$ when $\alpha\to\infty$, following Boukrouche and Tarzia (2013), we obtain

$$\int_0^T [(g(t) - \dot{u}_{\overline{q}}(t), z_\alpha(t))_H - (\overline{q}_\alpha(t), z_\alpha(t))_Q - a(u_{\overline{q}}(t), z_\alpha(t))]\, dt \to 0.$$

Next, we have $\lim\limits_{\alpha\to\infty}||z_\alpha||_{L^2(V)} = 0$. From the variational equalities (3) and (4), we have

$$(\dot{z}_\alpha(t), v)_H + a(z_\alpha(t), v) = (\overline{q}(t) - \overline{q}_\alpha(t), v)_Q, \quad \forall v \in V_0.$$

Therefore, $||\dot{z}_\alpha(t)||^2_{V_0'} \le 2||z_\alpha(t)||^2_V + 2||\gamma_0||^2||\overline{q}(t) - \overline{q}_\alpha(t)||^2_Q$ and by integrating on $[0,T]$, we obtain

$$||\dot{z}_\alpha||^2_{L^2(V_0')} \le 2||z_\alpha||^2_{L^2(V)} + 2||\gamma_0||^2||\overline{q} - \overline{q}_\alpha||^2_{\mathcal{Q}}.$$

Since $\overline{q}_\alpha \to \overline{q}$ strongly in $\mathcal{Q}$ and $u_{\alpha\overline{q}_\alpha} \to u_{\overline{q}}$ strongly in $L^2(V)$ when $\alpha \to \infty$, we can say that $\dot{z}_\alpha \to 0$ strongly in $L^2(V_0')$, that is, $\dot{u}_{\alpha\overline{q}_\alpha} \to \dot{u}_{\overline{q}}$ strongly in $L^2(V_0')$. In a similar way, we prove that $(p_{\alpha\overline{q}_\alpha}, \dot{p}_{\alpha\overline{q}_\alpha}) \to (p_{\overline{q}}, \dot{p}_{\overline{q}})$ strongly in $L^2(V) \times L^2(V_0')$, when $\alpha \to \infty$. □

4. Estimation relations between the optimal controls

In this section, we obtain the estimation relations between the solutions of some Neumann boundary optimal control problems and the solutions of the simultaneous distributed-boundary optimal control problems studied in Tarzia, Bollo and Gariboldi (2020).

4.1. Estimations with respect to the problem S

We consider the Neumann boundary optimal control problem

$$\text{find} \quad \overline{q} \in \mathcal{Q} \quad \text{such that} \quad J_1(\overline{q}) = \min_{q\in\mathcal{Q}} J_1(q) \qquad \text{for fixed} \quad g \in \mathcal{H}, \tag{26}$$

where J_1 is the cost functional, given in (5), plus the constant $\frac{M_1}{2}||g||^2_{\mathcal{H}}$, that is, $J_1 : \mathcal{Q} \to \mathbb{R}^+_0$ is given by

$$J_1(q) = \frac{1}{2}||u_q - z_d||^2_{\mathcal{H}} + \frac{M_1}{2}||g||^2_{\mathcal{H}} + \frac{M}{2}||q||^2_{\mathcal{Q}} \qquad (\text{fixed} \quad g \in \mathcal{H}),$$

where u_q is the unique solution of the problem (3) for fixed g.

Remark 1 *The functional $J^+(g,q)$, defined in Tarzia, Bollo and Gariboldi (2020), see (7), and the functional J_1 previously defined, satisfy the following elemental estimation*

$$J^+(\overline{\overline{g}}, \overline{\overline{q}}) \le J_1(\overline{q}), \quad \forall g \in \mathcal{H}.$$

In the following theorem we obtain estimations between the solution of the boundary optimal control problem (26) and the second component of the solution to simultaneous distributed-boundary optimal control problem studied in Tarzia, Bollo and Gariboldi (2020).

Theorem 2 *If $(\overline{\overline{g}}, \overline{\overline{q}}) \in \mathcal{H} \times \mathcal{Q}$ is the unique solution to the distributed-boundary optimal control problem in Tarzia, Bollo and Gariboldi (2020), see (7), and $\overline{q}$ is the unique solution to the optimal control problem (26), then*

$$||\overline{q} - \overline{\overline{q}}||_{\mathcal{Q}} \le \frac{||\gamma_0||}{\lambda_0 M}||u_{\overline{\overline{g}}\,\overline{\overline{q}}} - u_{g\,\overline{q}}||_{\mathcal{H}} \quad \forall g \in \mathcal{H} \tag{27}$$

where γ_0 is the trace operator with $||\gamma_0|| = \sup\limits_{v\in V-\{0\}} \frac{||\gamma_0(v)||_{\mathcal{Q}}}{||v||_V}$ and λ_0 the coerciveness constant of the bilinear form a.

Proof By the optimality condition for $\overline{q}$, for fixed $g \in \mathcal{H}$, we have

$$(M\overline{q} - p_{g\,\overline{q}}, \eta)_{\mathcal{Q}} = 0, \quad \forall \eta \in \mathcal{Q},$$

and then by taking $\eta = \overline{\overline{q}} - \overline{q}$ we obtain

$$(M\overline{q} - p_{g\,\overline{q}}, \overline{\overline{q}} - \overline{q})_{\mathcal{Q}} = 0. \tag{28}$$

On the other hand, if we take $h = 0 \in \mathcal{H}$ in the optimality condition for $(\overline{\overline{g}}, \overline{\overline{q}})$, given in Tarzia, Bollo and Gariboldi (2020), we have $(M\overline{\overline{q}} - p_{\overline{\overline{g}}\,\overline{\overline{q}}}, \eta)_{\mathcal{Q}} = 0$, $\forall \eta \in \mathcal{Q}$; next, by taking $\eta = \overline{q} - \overline{\overline{q}}$, we obtain

$$(-M\overline{\overline{q}} + p_{\overline{\overline{g}}\,\overline{\overline{q}}}, \overline{\overline{q}} - \overline{q})_{\mathcal{Q}} = 0. \tag{29}$$

By adding the expressions (28) and (29), we have

$$\Big(M(\overline{q} - \overline{\overline{q}}) + (p_{\overline{\overline{g}}\,\overline{\overline{q}}} - p_{g\,\overline{q}}), \overline{\overline{q}} - \overline{q}\Big)_{\mathcal{Q}} = 0.$$

So, by the Cauchy-Schwarz inequality and the trace theorem, we have

$$||\overline{\overline{q}} - \overline{q}||_{\mathcal{Q}} \le \frac{||\gamma_0||}{M} ||p_{\overline{\overline{g}}\,\overline{\overline{q}}} - p_{g\,\overline{q}}||_{L^2(V)}.$$

Now, if we prove that

$$||p_{\overline{\overline{g}}\,\overline{\overline{q}}} - p_{g\,\overline{q}}||_{L^2(V)} \le \frac{1}{\lambda_0} ||u_{\overline{\overline{g}}\,\overline{\overline{q}}} - u_{g\,\overline{q}}||_{\mathcal{H}}$$

the estimation (27) holds. In fact, by the variational equality for the adjoint state given in Tarzia, Bollo and Gariboldi (2020), see (5), for $g = \overline{\overline{g}}$ and $q = \overline{\overline{q}}$, we have

$$-\langle \dot{p}_{\overline{\overline{g}}\,\overline{\overline{q}}}(t), v\rangle + a(p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t), v) = (u_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - z_d(t), v)_H, \quad \forall v \in V_0,$$

and for fixed $g \in \mathcal{H}$ and $q = \overline{q}$

$$-\langle \dot{p}_{g\,\overline{q}}(t), v\rangle + a(p_{g\,\overline{q}}(t), v) = (u_{g\,\overline{q}}(t) - z_d(t), v)_H, \quad \forall v \in V_0.$$

Subtracting these equations, we obtain

$$-\langle \dot{p}_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - \dot{p}_{g\,\overline{q}}(t), v\rangle + a(p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - p_{g\,\overline{q}}(t), v) = (u_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - u_{g\,\overline{q}}(t), v)_H, \quad \forall v \in V_0.$$

By replacing $v = p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - p_{g\,\overline{q}}(t) \in V_0$ and using the fact that

$$2\langle \dot{p}_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - \dot{p}_{g\,\overline{q}}(t), p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - p_{g\,\overline{q}}(t)\rangle = \frac{d}{dt} ||p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - p_{g\,\overline{q}}(t)||_H^2,$$

we obtain

$$
\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}||p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t)-p_{g\,\overline{q}}(t)||_H^2+\lambda_0||\nabla\left(p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t)-p_{g\,\overline{q}}(t)\right)||_H^2\\
&\leq ||u_{\overline{\overline{g}}\,\overline{\overline{q}}}(t)-u_{g\,\overline{q}}(t)||_H||p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t)-p_{g\,\overline{q}}(t)||_H.
\end{aligned}
$$

and by using Young inequality for $\epsilon=\lambda_0$, we have

$$
\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}||p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t)-p_{g\,\overline{q}}(t)||_H^2+\lambda_0||p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t)-p_{g\,\overline{q}}(t)||_V^2\\
&\leq \frac{1}{2\lambda_0}||u_{\overline{\overline{g}}\,\overline{\overline{q}}}(t)-u_{g\,\overline{q}}(t)||_H^2+\frac{\lambda_0}{2}||p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t)-p_{g\,\overline{q}}(t)||_V^2.
\end{aligned}
$$

Then

$$
-\frac{d}{dt}||p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t)-p_{g\,\overline{q}}(t)||_H^2+\lambda_0||p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t)-p_{g\,\overline{q}}(t)||_V^2\leq\frac{1}{\lambda_0}||u_{\overline{\overline{g}}\,\overline{\overline{q}}}(t)-u_{g\,\overline{q}}(t)||_H^2.
$$

By integrating between 0 and T, and using that $p_{\overline{\overline{g}}\,\overline{\overline{q}}}(T)=p_{g\,\overline{q}}(T)=0$, we deduce that

$$
||p_{\overline{\overline{g}}\,\overline{\overline{q}}}(0)-p_{g\,\overline{q}}(0)||_H^2+\lambda_0||p_{\overline{\overline{g}}\,\overline{\overline{q}}}-p_{g\,\overline{q}}||_{L^2(V)}^2\leq\frac{1}{\lambda_0}||u_{\overline{\overline{g}}\,\overline{\overline{q}}}-u_{g\,\overline{q}}||_{\mathcal{H}}^2
$$

i.e.

$$
||p_{\overline{\overline{g}}\,\overline{\overline{q}}}-p_{g\,\overline{q}}||_{L^2(V)}\leq\frac{1}{\lambda_0}||u_{\overline{\overline{g}}\,\overline{\overline{q}}}-u_{g\,\overline{q}}||_{\mathcal{H}},
$$

and then (27) holds. □

Now, if we consider the distributed optimal control problem

$$
\text{find}\quad \overline{g}\in\mathcal{H}\quad\text{such that}\quad J_2(\overline{g})=\min_{g\in\mathcal{H}}J_2(g)\qquad\text{for fixed}\quad q\in\mathcal{Q},\tag{30}
$$

where J_2 is the cost functional given in Menaldi and Tarzia (2007) plus the constant $\frac{M}{2}||q||_{\mathcal{Q}}^2$, that is, $J_2:\mathcal{H}\to\mathbb{R}_0^+$ is given by

$$
J_2(g)=\frac{1}{2}||u_g-z_d||_{\mathcal{H}}^2+\frac{M_1}{2}||g||_{\mathcal{H}}^2+\frac{M}{2}||q||_{\mathcal{Q}}^2\qquad(\text{fixed}\quad q\in\mathcal{Q}),
$$

where u_g is the unique solution of the problem (3) for fixed q, we can prove the following corollary.

Corollary 1 *If $(\overline{g},\overline{q})\in\mathcal{H}\times\mathcal{Q}$ is the unique solution to the simultaneous optimal control problem, studied in Tarzia, Bollo and Gariboldi (2020), see (5), $\overline{g}$ is the unique solution to the distributed optimal control problem (30) for fixed q $(q=\overline{\overline{q}})$, and $\overline{q}$ is the unique solution to the problem (26) for fixed g $(g=\overline{\overline{g}})$, then $\overline{g}=\overline{\overline{g}}$ and $\overline{q}=\overline{\overline{q}}$.*

PROOF If we take $h = \overline{\overline{g}} - \overline{g}$ in the optimality condition for the problem, given in Menaldi and Tarzia (2007), we have

$$\left(M_1\overline{g} + p_{\overline{g}\,\overline{q}}, \overline{\overline{g}} - \overline{g}\right)_{\mathcal{H}} = 0. \tag{31}$$

On the other hand, if we consider $h = \overline{\overline{g}} - \overline{g}$ and $\eta = 0$ in the optimality condition for the simultaneous optimal control problem studied in Tarzia, Bollo and Gariboldi (2020), we obtain

$$\left(M_1\overline{\overline{g}} + p_{\overline{\overline{g}},\overline{\overline{q}}}, \overline{\overline{g}} - \overline{g}\right)_{\mathcal{H}}. \tag{32}$$

Upon subtracting (31) and (32), we deduce that

$$\left(M_1\overline{g} + p_{\overline{g}\,\overline{q}} - M_1\overline{\overline{g}} - p_{\overline{\overline{g}},\overline{\overline{q}}}, \overline{\overline{g}} - \overline{g}\right)_{\mathcal{H}} = 0,$$

and therefore

$$\left(p_{\overline{g}\,\overline{q}} - p_{\overline{\overline{g}},\overline{\overline{q}}}, \overline{\overline{g}} - \overline{g}\right)_{\mathcal{H}} - M_1\left(\overline{\overline{g}} - \overline{g}, \overline{\overline{g}} - \overline{g}\right)_{\mathcal{H}} = 0.$$

Next, by using the fact that

$$\left(p_{\overline{g},\overline{q}} - p_{\overline{\overline{g}},\overline{\overline{q}}}, \overline{\overline{g}} - \overline{g}\right)_{\mathcal{H}} = -||u_{\overline{g}\,\overline{q}} - u_{\overline{\overline{g}}\,\overline{\overline{q}}}||^2_{\mathcal{H}}$$

we have

$$-||u_{\overline{g}\,\overline{q}} - u_{\overline{\overline{g}}\,\overline{\overline{q}}}||^2_{\mathcal{H}} = M_1||\overline{\overline{g}} - \overline{g}||^2_{\mathcal{H}}.$$

Hence, we deduce that $||\overline{\overline{g}} - \overline{g}||^2_{\mathcal{H}} = 0$, and therefore $\overline{g} = \overline{\overline{g}}$.

In a similar way we prove that $\overline{q} = \overline{\overline{q}}$. □

4.2. Estimations with respect to the problem $\mathbf{S}_\alpha$

For each $\alpha > 0$, we consider the following optimal control problem

$$\text{find} \quad \overline{q}_\alpha \in \mathcal{Q} \quad \text{such that} \quad J_{1\alpha}(\overline{q}_\alpha) = \min_{q \in \mathcal{Q}} J_{1\alpha}(q), \tag{33}$$

where $J_{1\alpha} : \mathcal{Q} \to \mathbb{R}_0^+$ is given by

$$J_{1\alpha}(q) = \frac{1}{2}||u_{\alpha q} - z_d||^2_{\mathcal{H}} + \frac{M_1}{2}||g||^2_{\mathcal{H}} + \frac{M}{2}||q||^2_{\mathcal{Q}} \qquad (\text{fixed} \quad g \in \mathcal{H}),$$

that is, $J_{1\alpha}$ is the functional (6) plus the constant $\frac{M_1}{2}||g||^2_{\mathcal{H}}$, and $u_{\alpha q}$ is the unique solution of the problem (4) for fixed g.

Remark 2 *The functional J_α^+ defined in Tarzia, Bollo and Gariboldi (2020), see (8), and the functional $J_{1\alpha}$, previously defined, satisfy the following estimate*

$$J_\alpha^+(\overline{\overline{g}}_\alpha, \overline{\overline{q}}_\alpha) \leq J_{1\alpha}(\overline{q}_\alpha), \quad \forall g \in \mathcal{H}.$$

Estimation relation between the solution of the Neumann boundary optimal control problem (33) and the second component of the solution to the simultaneous optimal control problem, studied in Tarzia, Bollo and Gariboldi (2020), is given in the following theorem, whose proof is omitted.

Theorem 3 *If $(\overline{\overline{g}}_\alpha, \overline{\overline{q}}_\alpha) \in \mathcal{H} \times \mathcal{Q}$ is the unique solution of the simultaneous optimal control problem from Tarzia, Bollo and Gariboldi (2020), see (6), and $\overline{q}_\alpha$ is the unique solutions to the optimal control problem (33), then we have:*

$$||\overline{q}_\alpha - \overline{\overline{q}}_\alpha||_{\mathcal{Q}} \leq \frac{||\gamma_0||}{\lambda_\alpha M} ||u_{\alpha \overline{\overline{g}}_\alpha \overline{\overline{q}}_\alpha} - u_{\alpha\, g\, \overline{q}_\alpha}||_{\mathcal{H}} \quad \forall g \in \mathcal{H}$$

with $\lambda_\alpha = \lambda_1 \min\{1, \alpha\}$ and λ_1 the coerciveness constant of the bilinear form a_1.

If we consider the following distributed optimal control problem, for each $\alpha > 0$:

$$\text{find} \quad \overline{g}_\alpha \in \mathcal{H} \quad \text{such that} \quad J_{2\alpha}(\overline{g}_\alpha) = \min_{g \in \mathcal{H}} J_{2\alpha}(g), \tag{34}$$

where $J_{2\alpha} : \mathcal{H} \to \mathbb{R}_0^+$ is given by

$$J_{2\alpha}(g) = \frac{1}{2}||u_{\alpha g} - z_d||_{\mathcal{H}}^2 + \frac{M_1}{2}||g||_{\mathcal{H}}^2 + \frac{M}{2}||q||_{\mathcal{Q}}^2 \qquad (\text{fixed} \quad q \in \mathcal{Q}),$$

that is, $J_{2\alpha}$ is the functional, studied in Menaldi and Tarzia (2007) plus the constant $\frac{M}{2}||q||_{\mathcal{Q}}^2$, and $u_{\alpha g}$ is the unique solution of the problem (4) for fixed q, we give the following corollary, whose proof is omitted.

Corollary 2 *If $(\overline{\overline{g}}_\alpha, \overline{\overline{q}}_\alpha) \in \mathcal{H} \times \mathcal{Q}$ is the unique solution of the simultaneous optimal control problem studied in Tarzia, Bollo and Gariboldi (2020), see (6), $\overline{g}_\alpha$ is the unique solution of the problem (34) for fixed q ($q = \overline{\overline{q}}_\alpha$), and $\overline{q}_\alpha$ is the unique solution of the problem (33) for fixed g ($g = \overline{\overline{g}}_\alpha$), then $\overline{g}_\alpha = \overline{\overline{g}}_\alpha$ and $\overline{q}_\alpha = \overline{\overline{q}}_\alpha$.*

5. Real Neumann boundary optimal control problems

In this section, we consider the non-stationary real-boundary optimal control problems (8) and (9) and the stationary real-boundary optimal control problems (15) and (16). We prove existence and uniqueness of the solutions to these optimal control problems and monotonicity results are also obtained.

5.1. Real Neumann boundary optimal control problem in relation to the parabolic system S

If we consider the real-boundary optimal control problem (8) and we denote by u_{bqg} the unique solution of the variational equality (3) for the data b, q and g, and we take $q = \lambda q_0$ for fixed $q_0 \in \mathcal{Q}$ ($q_0 \neq 0$) and $\lambda \in \mathbb{R}$, we can prove that

$$u_{bqg}(t) = u_{b\lambda g}(t) = u_b(t) + u_q(t) + u_g(t), \quad \forall x \in \Omega,$$

where u_b is the unique solution to the parabolic variational equality

$$\begin{cases} u - v_b \in L^2(V_0), \qquad u(0) = v_b \quad \text{and} \quad \dot{u} \in L^2(V_0') \\ \text{such that} \quad \langle \dot{u}(t), v\rangle + a(u(t), v) = 0, \quad \forall v \in V_0, \end{cases} \tag{35}$$

u_q is the unique solution to the parabolic variational equality

$$\begin{cases} u \in L^2(V_0), \qquad u(0) = 0 \quad \text{and} \quad \dot{u} \in L^2(V_0') \\ \text{such that} \quad \langle \dot{u}(t), v\rangle + a(u(t), v) = -\lambda (q_0(t), v)_Q, \quad \forall v \in V_0, \end{cases} \tag{36}$$

and u_g is the unique solution to the parabolic variational equality

$$\begin{cases} u \in L^2(V_0), \qquad u(0) = 0 \quad \text{and} \quad \dot{u} \in L^2(V_0') \\ \text{such that} \quad \langle \dot{u}(t), v\rangle + a(u(t), v) = (g(t), v)_H, \quad \forall v \in V_0. \end{cases} \tag{37}$$

We note that, by the linearity, we can prove that $u_q(t) = \lambda u_{q_0}(t)$, where u_{q_0} is the unique solution of (3) for $q = q_0$ and $b = g = 0$.

Next, for each $T > 0$, the functional $H_T(\lambda)$ can be written down as

$$H_T(\lambda) = \frac{1}{2}\int_0^T\int_\Omega (u_b(t) + \lambda u_{q_0}(t) + u_g(t) - z_d(t))^2 dxdt + \frac{M\lambda^2}{2}\int_0^T\int_{\Gamma_2} q_0^2(t) d\gamma dt$$

therefore, $H_T(\lambda) = \lambda^2 A(T) + \lambda B(T) + C(T)$, where

$$A(T) = \frac{M}{2}\int_0^T\int_{\Gamma_2} q_0^2(t) d\gamma dt + \frac{1}{2}\int_0^T\int_\Omega u_{q_0}^2(t) dxdt$$

$$B(T) = \int_0^T\int_\Omega u_{q_0}(t)(u_b(t) + u_g(t) - z_d(t)) dxdt$$

$$C(T) = \frac{1}{2}\int_0^T\int_\Omega (u_b(t) + u_g(t) - z_d(t))^2 dxdt.$$

Here, taking into account that

$$
\begin{aligned}
&4A(T)C(T)\\
&=\left(M\int_0^T\int_{\Gamma_2} q_0^2(t)d\gamma dt+\int_0^T\int_\Omega u_{q_0}^2(t)dxdt\right)\left(\int_0^T\int_\Omega (u_b(t)+u_g(t)-z_d(t))^2dxdt\right)\\
&>\left(\int_0^T\int_\Omega u_{q_0}^2(t)dxdt\right)\left(\int_0^T\int_\Omega (u_b(t)+u_g(t)-z_d(t))^2dxdt\right)\\
&=||u_{q_0}||^2_{\mathcal{H}}||u_b+u_g-z_d||^2_{\mathcal{H}}\geq (u_{q_0},u_b+u_g-z_d)^2_{\mathcal{H}}\\
&=\left(\int_0^T\int_\Omega u_{q_0}(t)(u_b(t)+u_g(t)-z_d(t))dxdt\right)^2=(B(T))^2
\end{aligned}
$$

we deduce that $(B(T))^2-4A(T)C(T)<0$, and since $A(T)>0$, because $q_0\neq 0$, then there exists a unique $\overline{\lambda}(T)\in\mathbb{R}$ for each $T>0$, such that it satisfies the problem (8), whose solution is given by the following expression:

$$
\overline{\lambda}(T)=-\frac{B(T)}{2A(T)}=-\frac{\int_0^T\int_\Omega u_{q_0}(t)(u_b(t)+u_g(t)-z_d(t))dxdt}{M\int_0^T\int_{\Gamma_2} q_0^2(t)d\gamma dt+\int_0^T\int_\Omega u_{q_0}^2(t)dxdt}. \tag{38}
$$

Therefore, we have proven the following property.

Theorem 4 *For each $T>0$, there exists a unique solution $\overline{\lambda}(T)\in\mathbb{R}$ to the optimization problem (8).*

Now, we will prove some monotonicity properties.

Proposition 3 *Let $q_1=\lambda_1 q_0$ and $q_2=\lambda_2 q_0$ ($q_0>0$), with $\lambda_2\leq\lambda_1$ and $g_1\leq g_2$, then $u_{b\lambda_1 g_1}\leq u_{b\lambda_2 g_2}$ in $\Omega\times[0,T]$.*

Proof We define $w=u_{b\lambda_1 g_1}-u_{b\lambda_2 g_2}$ and we take $v=-w^+(t)\in V$ for $u_{b\lambda_1 g_1}$ in (3) and $v=w^+(t)\in V$ for $u_{b\lambda_2 g_2}$ in (3) (for the regularity of w^+ see Kinderlehrer and Stampacchia, 2000). Upon adding the variational equalities, we obtain

$$
\begin{aligned}
&\langle \dot{u}_{b\lambda_2 g_2}(t)-\dot{u}_{b\lambda_1 g_1}(t),w^+(t)\rangle+a(u_{b\lambda_2 g_2}(t)-u_{b\lambda_1 g_1}(t),w^+(t))\\
&\qquad=\int_\Omega (g_2(t)-g_1(t))w^+(t)dx+(\lambda_1-\lambda_2)\int_{\Gamma_2} q_0(t)w^+(t)d\gamma.
\end{aligned}
$$

Next,

$$
\begin{aligned}
-\langle \dot{w^+}(t),w^+(t)\rangle-a(w^+(t),w^+(t))&=\int_\Omega (g_2(t)-g_1(t))w^+(t)dx\\
&\quad+(\lambda_1-\lambda_2)\int_{\Gamma_2} q_0(t)w^+(t)d\gamma.
\end{aligned}
$$

Now, using the fact that

$$\langle \dot{w^+}(t), w^+(t)\rangle = \frac{1}{2}\frac{d}{dt}||w^+(t)||^2_H$$

and integrating between 0 and T, we prove that

$$\frac{1}{2}\left(||w^+(T)||^2_H - ||w^+(0)||^2_H\right) + \int\limits_0^T ||w^+(t)||^2_{V_0} dt \leq \int\limits_0^T \int\limits_\Omega (g_1(t) - g_2(t))w^+(t)dxdt$$
$$+ (\lambda_2 - \lambda_1) \int\limits_0^T \int\limits_{\Gamma_2} q_0(t)w^+(t)d\gamma dt.$$

Since $w^+(0) = (u_{b\lambda_1 g_1} - u_{b\lambda_2 g_2})^+(0) = \max\{0, (u_{b\lambda_1 g_1} - u_{b\lambda_2 g_2})(0)\} = 0$,

$$\frac{1}{2}||w^+(T)||^2_H + \int\limits_0^T ||w^+(t)||^2_{V_0} dt \leq \int\limits_0^T \int\limits_\Omega (g_1(t) - g_2(t))w^+(t)dxdt$$
$$+ (\lambda_2 - \lambda_1) \int\limits_0^T \int\limits_{\Gamma_2} q_0(t)w^+(t)d\gamma dt \leq 0,$$

by using the facts that $g_1 \leq g_2$, $\lambda_2 \leq \lambda_1$ and $q_0 > 0$, here $||w^+||^2_{L^2(V_0)} = 0$, then $w^+ \equiv 0$ in $\Omega \times [0, T]$ and therefore $w \leq 0$ in $\Omega \times [0, T]$, that is, the thesis holds. □

Corollary 3 *If $q_1 = \lambda_1 q_0$ and $q_2 = \lambda_2 q_0$ ($q_0 > 0$), with $\lambda_2 \leq \lambda_1$, then $u_{b\lambda_1 g} \leq u_{b\lambda_2 g}$ in $\Omega \times [0, T]$.*

Proof This results from taking $g = g_1 = g_2$ in the proof of Proposition 3. □

Remark 3 *The previous monotonicity properties are still true if we consider $\lambda_1 \leq \lambda_2$ with $q_0 < 0$.*

5.2. Real Neumann boundary optimal control problems in relation to the parabolic $\mathbf{S}_\alpha$

If we consider the real Neumann boundary optimal control problem (9) and for each $\alpha > 0$ we denote by $u_{\alpha bqg}$ the unique solution of the variational equality (4) for data b, q and g, and we take $q = \lambda q_0$ for fixed $q_0 \in Q$ ($q_0 \neq 0$) and $\lambda \in \mathbb{R}$, we can see that

$$u_{\alpha bqg}(t) = u_{\alpha b\lambda g}(t) = u_{\alpha b}(t) + u_{\alpha q}(t) + u_{\alpha g}(t), \quad \forall x \in \Omega,$$

where $u_{\alpha b}, u_{\alpha q}, u_{\alpha g}$ are the unique solutions of (4) for $q = g = 0$, $b = g = 0$ and $b = q = 0$, respectively.

We note that, by the linearity, we can see that $u_{\alpha q}(t) = \lambda u_{\alpha q_0}(t)$, where $u_{\alpha q_0}$ is the unique solution of (4) for $q = q_0$ and $b = g = 0$. Next, for each $T > 0$, the functional $H_{\alpha T}(\lambda)$ can be written down as

$$H_{\alpha T}(\lambda) = \frac{1}{2}\int_0^T \int_\Omega (u_{\alpha b}(t) + \lambda u_{\alpha q_0}(t) + u_{\alpha g}(t) - z_d(t))^2 dxdt$$
$$+ \frac{M\lambda^2}{2}\int_0^T \int_{\Gamma_2} q_0^2(t) d\gamma dt = \lambda^2 A_\alpha(T) + \lambda B_\alpha(T) + C_\alpha(T),$$

where

$$A_\alpha(T) = \frac{M}{2}\int_0^T \int_{\Gamma_2} q_0^2(t) d\gamma dt + \frac{1}{2}\int_0^T \int_\Omega u_{\alpha q_0}^2(t) dxdt$$

$$B_\alpha(T) = \int_0^T \int_\Omega u_{\alpha q_0}(t)(u_{\alpha b}(t) + u_{\alpha g}(t) - z_d(t)) dxdt$$

$$C_\alpha(T) = \frac{1}{2}\int_0^T \int_\Omega (u_{\alpha b}(t) + u_{\alpha g}(t) - z_d(t))^2 dxdt.$$

Now, taking into account the fact that $A_\alpha(T) > 0$, in a similar way as in the previous subsection, we can prove that $(B_\alpha(T))^2 - 4A_\alpha(T)C_\alpha(T) < 0$ and therefore there exists a unique solution $\overline{\lambda}_\alpha(T) \in \mathbb{R}$, for each $\alpha > 0$ and for each $T > 0$, for the optimal control problem (9), which is given by the following expression:

$$\overline{\lambda}_\alpha(T) = -\frac{\int_0^T \int_\Omega u_{\alpha q_0}(t)(u_{\alpha b}(t) + u_{\alpha g}(t) - z_d(t)) dxdt}{M \int_0^T \int_{\Gamma_2} q_0^2(t) d\gamma dt + \int_0^T \int_\Omega u_{\alpha q_0}^2(t) dxdt}. \tag{39}$$

Therefore, we have proven the following property.

Theorem 5 *For each $\alpha > 0$ and $T > 0$, there exists a unique solution $\overline{\lambda}_\alpha(T) \in \mathbb{R}$ to the optimization problem (9).*

Now, in a similar way to that from the previous subsection, we can prove the following monotonicity properties, whose proof is omitted.

Proposition 4 *For each $\alpha > 0$, if $q_1 = \lambda_1 q_0$ and $q_2 = \lambda_2 q_0$ ($q_0 > 0$), with $\lambda_2 \le \lambda_1$, $g_1 \le g_2$, $b_1 \le b_2$ on Γ_1 and initial conditions $v_{b_1} \le v_{b_2}$, then $u_{\alpha b_1 \lambda_1 g_1} \le u_{\alpha b_2 \lambda_2 g_2}$ in $\Omega \times [0, T]$.*

5.3. Real Neumann boundary optimal control problem in relation to the elliptic system P

Here, we consider the stationary real Neumann boundary optimal control problem (15). If we denote by $u_{\infty bqg}$ the unique solution of the variational equality (13) for data b, q and g. and if we consider $q = \lambda q_0^*$ for fixed $q_0^* \in Q$ $(q_0^* \neq 0)$ and $\lambda \in \mathbb{R}$, we can see that

$$u_{\infty bqg} = u_{\infty b\lambda g} = u_{\infty b} + u_{\infty q} + u_{\infty g}, \quad \forall x \in \Omega,$$

where $u_{\infty b}, u_{\infty q}, u_{\infty g}$ are the unique solutions of the variational equality (13) for $q = g = 0$, $b = g = 0$ and $b = q = 0$, respectively.

Now, we take into account that $u_{\infty q} = \lambda u_{\infty q_0^*}$, where $u_{\infty q_0^*}$ is the solution of (13) for $q = q_0^*$ and $b = g = 0$, and so the functional $H(\lambda)$ can be written down as

$$H(\lambda) = \frac{1}{2}\int_\Omega (u_{\infty b} + \lambda u_{\infty q_0^*} + u_{\infty g} - z_d)^2 dx + \frac{M\lambda^2}{2}\int_{\Gamma_2} (q_0^*)^2 d\gamma.$$

Therefore, $H(\lambda) = \lambda^2 A + \lambda B + C$, where

$$A = \frac{M}{2}\int_{\Gamma_2} (q_0^*)^2 d\gamma + \frac{1}{2}\int_\Omega u^2_{\infty q_0^*} dx, \quad B = \int_\Omega u_{\infty q_0^*}(u_{\infty b} + u_{\infty g} - z_d) dx$$

$$C = \frac{1}{2}\int_\Omega (u_{\infty b} + u_{\infty g} - z_d)^2 dx.$$

Here, since $B^2 - 4AC < 0$, there exists a unique $\overline{\lambda}_\infty \in \mathbb{R}$ such that it solves the problem (15), that is

$$\overline{\lambda}_\infty = -\frac{B}{2A} = -\frac{\int_\Omega u_{\infty q_0^*}(u_{\infty b} + u_{\infty g} - z_d) dx}{M\int_{\Gamma_2} (q_0^*)^2 d\gamma + \int_\Omega u^2_{\infty q_0^*} dx} \tag{40}$$

Therefore, we have proven the following theorem.

THEOREM 6 *There exists a unique solution $\overline{\lambda}_\infty \in \mathbb{R}$ to the optimization problem (15).*

Now, we will give some monotonicity property, whose proof is omitted.

PROPOSITION 5 *Let $q_1 = \lambda_1 q_0^*$ and $q_2 = \lambda_2 q_0^*$ $(q_0^* > 0)$, with $\lambda_2 \leq \lambda_1$ and $g_1 \leq g_2$, then $u_{\infty b\lambda_1 g_1} \leq u_{\infty b\lambda_2 g_2}$.*

5.4. Real Neumann boundary optimal control problems in relation to the elliptic system $\mathbf{P}_\alpha$

We consider the stationary real Neumann boundary optimal control problem (16) and we denote by $u_{\infty\alpha bqg}$ the unique solution to the variational equality (14) for the data b, q and g. If we consider $q = \lambda q_0^*$ for fixed $q_0^* \in Q$ ($q_0^* \neq 0$) and $\lambda \in \mathbb{R}$, we can see that

$$u_{\infty\alpha bqg} = u_{\infty\alpha b\lambda g} = u_{\infty\alpha b} + u_{\infty\alpha q} + u_{\infty\alpha g}, \quad \forall x \in \Omega,$$

where $u_{\infty\alpha b}, u_{\infty\alpha q}, u_{\infty\alpha g}$ are the unique solutions of the variational equality (14) for $q = g = 0$, $b = g = 0$ and $b = q = 0$, respectively.

Now, taking into account the fact that $u_{\infty\alpha q} = \lambda u_{\infty\alpha q_0^*}$, where $u_{\infty\alpha q_0^*}$ is the solution of (14) for $q = q_0^*$ and $b = g = 0$, the functional $H_\alpha(\lambda)$ can be written down as

$$H_\alpha(\lambda) = \frac{1}{2}\int_\Omega (u_{\infty\alpha b}+\lambda u_{\infty\alpha q_0^*}+u_{\infty\alpha g}-z_d)^2 dx + \frac{M\lambda^2}{2}\int_{\Gamma_2}(q_0^*)^2 d\gamma = \lambda^2 A_\alpha + \lambda B_\alpha + C_\alpha,$$

where

$$A_\alpha = \frac{M}{2}\int_{\Gamma_2}(q_0^*)^2 d\gamma + \frac{1}{2}\int_\Omega u^2_{\infty\alpha q_0^*} dx, \quad B_\alpha = \int_\Omega u_{\infty\alpha q_0^*}(u_{\infty\alpha b} + u_{\infty\alpha g} - z_d) dx$$

$$C_\alpha = \frac{1}{2}\int_\Omega (u_{\infty\alpha b} + u_{\infty\alpha g} - z_d)^2 dx.$$

Here, since $B_\alpha^2 - 4A_\alpha C_\alpha < 0$, there exists a unique $\overline{\lambda}_\alpha \in \mathbb{R}$, which satisfies the optimization problem (16), that is

$$\overline{\lambda}_\alpha = -\frac{B_\alpha}{2A_\alpha} = -\frac{\int_\Omega u_{\infty\alpha q_0^*}(u_{\infty\alpha b} + u_{\infty\alpha g} - z_d)dx}{M\int_{\Gamma_2}(q_0^*)^2 d\gamma + \int_\Omega u^2_{\infty\alpha q_0^*} dx}. \tag{41}$$

Therefore, we have proven the following theorem.

Theorem 7 *For each $\alpha > 0$, there exists a unique solution $\overline{\lambda}_\alpha \in \mathbb{R}$ to the optimization problem (16).*

Now, we will give some monotonicity property, whose proof is omitted.

Proposition 6 *For each $\alpha > 0$, if $q_1 = \lambda_1 q_0^*$ and $q_2 = \lambda_2 q_0^*$ ($q_0^* > 0$), with $\lambda_2 \leq \lambda_1$, $g_1 \leq g_2$, $b_1 \leq b_2$ on Γ_1, then $u_{\alpha b_1\lambda_1 g_1} \leq u_{\alpha b_2\lambda_2 g_2}$ in Ω.*

6. Asymptotic behaviour of solutions when $\mathbf{t} \to +\infty$

In this section, we study the convergence of the solutions to the problem (3) for fixed data $b \in H^{\frac{1}{2}}(\Gamma_1)$, $q \in \mathcal{Q}$ and $g \in \mathcal{H}$, to the solution to the problem (13) for the same $b \in H^{\frac{1}{2}}(\Gamma_1)$ and fixed $q \in Q$ and $g \in H$, when $t \to +\infty$. Here, for the sake of simplicity, we denote by u_∞ the unique solution to the variational equality (13) for data $q_\infty \in Q$ and $g_\infty \in H$.

If we define

$$F_1(t) = e^{\lambda_0 t}||g(t) - g_\infty||_H^2, \qquad F_2(t) = e^{\lambda_0 t}||\gamma_0||^2||q(t) - q_\infty||_Q^2$$

with $g \in \mathcal{H}$, $q \in \mathcal{Q}$, λ_0 the coerciveness constant of the bilinear form a and γ_0 the trace operator, we can prove the following theorem.

Theorem 8 *If $b \in H^{\frac{1}{2}}(\Gamma_1)$, $q \in \mathcal{Q}$, $g \in \mathcal{H}$, $F_1 \in L^1(0,\infty)$ and $F_2 \in L^1(0,\infty)$, then*

$$||u_{bqg}(t) - u_\infty||_H^2 \leq ||u_{bqg}(0) - u_\infty||_H^2 e^{-\lambda_0 t} + \frac{2e^{-\lambda_0 t}}{\lambda_0}\left(||F_1||_{L^1(0,\infty)} + ||F_2||_{L^1(0,\infty)}\right)$$

and therefore

$$\lim_{t\to+\infty} u_{bqg}(t) = u_\infty \quad \textit{in } H \textit{ strong (exponentially).}$$

Proof If we consider $w(t) = u_{bqg}(t) - u_\infty$, we have that $w(t) \in V_0$, $w(0) = v_b - u_\infty$ and $\dot{w} = \dot{u}_{bqg}$. Therefore, by taking $v = w(t)$ in the variational equalities (3) and (13), respectively, and subtracting them, we obtain

$$\langle \dot{w}(t), w(t)\rangle + a(w(t), w(t)) = \langle L(t) - L_\infty, w(t)\rangle,$$

that is

$$\frac{1}{2}\frac{d}{dt}||w(t)||_H^2 + \lambda_0||w(t)||_V^2 \leq (g(t) - g_\infty, w(t))_H - (q(t) - q_\infty, w(t))_Q.$$

Next,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||w(t)||_H^2 + \lambda_0||w(t)||_V^2 &\leq ||g(t) - g_\infty||_H||w(t)||_V + ||q(t) - q_\infty||_Q||w(t)||_Q \\ &\leq \left(||g(t) - g_\infty||_H + ||\gamma_0||||q(t) - q_\infty||_Q\right)||w(t)||_V \\ &\leq \frac{\lambda_0}{2}||w(t)||_V^2 \\ &\quad + \frac{1}{2\lambda_0}\left(||g(t) - g_\infty||_H + ||\gamma_0||||q(t) - q_\infty||_Q\right)^2,\end{aligned}$$

where γ_0 is the trace operator. Then,

$$\frac{1}{2}\frac{d}{dt}||w(t)||_H^2 + \frac{\lambda_0}{2}||w(t)||_H^2 \leq \frac{1}{\lambda_0}\left(||g(t) - g_\infty||_H^2 + ||\gamma_0||^2||q(t) - q_\infty||_Q^2\right).$$

Next, if we put $F(t) = ||g(t) - g_\infty||_H^2 + ||\gamma_0||^2 ||q(t) - q_\infty||_Q^2$, we have

$$\frac{d}{dt}\left(||w(t)||_H^2\right) e^{\lambda_0 t} + \lambda_0 ||w(t)||_H^2 e^{\lambda_0 t} \leq \frac{2}{\lambda_0} F(t) e^{\lambda_0 t}$$

or, equivalently,

$$\frac{d}{dt}\left(||w(t)||_H^2 e^{\lambda_0 t}\right) \leq \frac{2}{\lambda_0} F(t) e^{\lambda_0 t}.$$

Now, by integrating between 0 and t, we obtain

$$||w(t)||_H^2 e^{\lambda_0 t} - ||w(0)||_H^2 \leq \frac{2}{\lambda_0} \int_0^t F(\tau) e^{\lambda_0 \tau} d\tau,$$

therefore

$$||w(t)||_H^2 \leq ||w(0)||_H^2 e^{-\lambda_0 t} + \frac{2e^{-\lambda_0 t}}{\lambda_0} \int_0^t F(\tau) e^{\lambda_0 \tau} d\tau$$

and the thesis holds. □

Corollary 4 *If $b \in H^{\frac{1}{2}}(\Gamma_1)$, $F_1 \in L^1(0,\infty)$ and $F_2 \in L^1(0,\infty)$, with $g(t) = g_\infty \in H$ and $q(t) = q_\infty \in Q$, then we have*

$$||u_{bqg}(t) - u_\infty||_H \leq ||u_{bqg}(0) - u_\infty||_H e^{-\frac{\lambda_0}{2} t}. \tag{42}$$

Proof This results directly from Theorem 8. □

Remark 4 *We note that if we consider the hypothesis: there exist $m \in (0, \lambda_0)$ and $C_1 = const. > 0$ such that*

$$\lim_{t\to+\infty} \frac{||F_1||_{L^1(0,t)} + ||F_2||_{L^1(0,t)}}{e^{mt}} \leq C_1$$

then we can prove the asymptotic behaviour obtained in Theorem 8.

Corollary 5 *a) If $b \in H^{\frac{1}{2}}(\Gamma_1)$, with $q(t) = q_\infty \in Q$ and there exist $m \in (0, \lambda_0)$ and $C_2 = const. > 0$ such that $\lim_{t\to+\infty} \frac{||F_1||_{L^1(0,t)}}{e^{mt}} \leq C_2$, then*

$$\lim_{t\to+\infty} ||u_{bqg}(t) - u_\infty||_H = 0.$$

b) If $b \in H^{\frac{1}{2}}(\Gamma_1)$, with $g(t) = g_\infty \in H$ and there exist $m \in (0, \lambda_0)$ and $C_3 = const. > 0$ such that $\lim_{t\to+\infty} \frac{||F_2||_{L^1(0,t)}}{e^{mt}} \leq C_3$, then

$$\lim_{t\to+\infty} ||u_{bqg}(t) - u_\infty||_Q = 0.$$

Remark 5 *An open problem is to study whether (38) is convergent to (40) and (39) is convergent to (41) (for each* $\alpha > 0$*), when* $T \to +\infty$*. We hope that these convergences do not happen. This conjecture is based on the fact that a function* $g(t)$ $(\forall t > 0)$ *can be strongly convergent in* H *to a function* g_∞*, but* g *is not necessarily strongly convergent to the same function* g_∞ *in* $\mathcal{H}$*, which is shown in the following counterexample.*

Example 1 *Assume* $\Omega = (0,1)$ *and* $g(t) = g_\infty + e^{-t}$*, then we have that*

$$\int_0^1 (g(t) - g_\infty)^2 dx = \int_0^1 (e^{-t})^2 dx = e^{-2t} \to 0, \quad \text{if } t \to +\infty,$$

and

$$\int_0^t \int_0^1 (g(\tau) - g_\infty)^2 dx d\tau = \int_0^t (e^{-\tau})^2 d\tau = \frac{1}{2}(1 - e^{-2t}) \to \frac{1}{2}, \quad \text{if } t \to +\infty.$$

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References

Ben Belgacem, F., El Fekih, H. and Raymond, J. P. (2003) A penalized Robin approach for solving a parabolic equation with nonsmooth Dirichlet boundary conditions. *Asymptotic Anal.*, 34, 121–136.

Bergounioux, M. and Tröltzsch, F. (1999) Optimal control of semilinear parabolic equations with state-constraints of Bottleneck type. *ESAIM: Control, Optim. Calc. Var.* 4, 595–608.

Boukrouche, M. and Tarzia, D. A. (2013) Convergence of optimal control problems governed by second kind parabolic variational inequalities. *J. Control Theory Appl.* 11, 422–427.

Brézis, H. (1972) Problèmes unilatéraux. *Journal de Mathématiques Pures et Appliquées*, **51**(1), 1–162.

Chrysafinos, K., Gunzburger, M. D. and Hou L. S. (2006) Semidiscrete approximations of optimal Robin boundary control problems constrained by semilinear parabolic PDE. *J. Math. Anal. Appl.* 323, 891–912.

Chrysafinos, K. and Hou, L. S. (2017) Analysis and approximations of the evolutionary Stokes equations with inhomogeneous boundary and divergence data using a parabolic saddle point formulation. *ESAIM: Mathematical Modelling and Numerical Analysis* 51, 1501–1526.

Duvaut, G. and Lions, J. L. (1972) *Les inéquations en mécanique et en physique*. Paris: Dunod.

Gariboldi, C. M. and Tarzia, D. A. (2008) Convergence of boundary optimal control problems with restrictions in mixed elliptic Stefan-like problems. *Adv. in Diff. Eq. and Control Processes*, **1**(2), 113-132.

Gariboldi, C. M. and Tarzia, D. A. (2015) Existence, uniqueness and convergence of simultaneous distributed-boundary optimal control problems. *Control and Cybernetics*, **44**, 5–17.

Gonzalez, R. L. V. and Tarzia, D. A. (1990) Optimization of heat flux in domains with temperature constraints. *Journal of Optimization Theory and Applications*, **65**(2), 245–256.

Gunzburger, M. D. and Hou, S. L. (1992) Treating inhomogeneous essential boundary conditions in finite element methods and the calculation of boundary stresses. *SIAM J. Numer. Anal.* **29**(2), 390–424.

Kinderleher, D. and Stampacchia, G. (2000) *An Introduction to Variational Inequalities and Their Applications.* SIAM, Philadelphia.

Lions, J.L. (1968) *Contrôle optimal de systèmes gouvernés par des équations aux drives partielles.* Dunod, Paris.

Menaldi, J. and Tarzia, D. A. (2007) A distributed parabolic control with mixed boundary conditions. *Asymptotic Analysis* 52, 227–241.

Sener, S. S. and Subasi, M. (2015) On a Neumann boundary control in a parabolic system. *Boundary Value Problems*, 2015:166, 1–12.

Sweilam, N. H. and Abd-Elal, L. F. (2003) A computational approach for optimal control systems goberned by parabolic variational inequalities. *Journal of Computational Mathematics* **21**:6, 815–824.

Tarzia, D. A., Bollo, C. M. and Gariboldi, C. M. (2020) Convergence of simultaneous distributed-boundary parabolic optimal control problems. *Evolution Equations and Control Theory.* **9**(4), 1187–1201.

Tröltzsch, F. (2010) *Optimal Control of Partial Differential Equations. Theory, Methods and Applications.* American Math. Soc., Providence.

Wang, L. and Yan, Q. (2019) Optimal control problem for exact synchronization of parabolic system. *Mathematical Control and Related Fields* **9**(3), 411–424.