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Partial observability of finite dimensional linear systems*

by

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Abstract: In this work, we consider the partial observability problem for finite dimensional dynamical linear systems that are not necessarily observable. For that purpose we introduce the so called "observable subspaces" and "partial observability" to find a way to reconstruct the observable part of the system state. Some characterizations of "observable subspaces" have been provided. The reconstruction of the orthogonal projection of the state on the observable subspace is obtained. We give some examples to illustrate our theoretical approach.

Keywords: partial observation, linear system, observable subspace, system state reconstruction

1. Introduction

A dynamical system can be considered as composed of multiple objects interacting with each other. Mathematically, this interaction can be represented by a model, constituted by equations. The system is linked to its environment through input elements (physical elements acting on the system) and output elements (measurements or observations). The analysis of several concepts is necessary to better understand a given dynamical system and its functioning in order to optimize its use. Among the fundamental concepts, constituting the basis for the analysis of systems are those of controllability, observability, stability and stabilizability (Bellman and Kalaba, 1964; Aizerman and Gantmacher, 1970; Bongiorno, 1964; Bridgeland, 1964; Gilbert, 1963; Ho and Kalman, 1966; Lee and Markus, 1967; Bryson and Ho, 1969).

The observation problem consists in extracting the state of the system by using the output equation and the dynamics of the system. In the case of

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a non-observable system, we will never be able to extract the totality of the system state; that is why we have opted for the partial observability. This kind of observability consists in observing and extracting the reconstructible part of the system state from the output equation unless the system is not fully observable. In this paper, we focus just on the observability problem for finite dimensional linear systems.

The partial analysis of a dynamical system is necessary when the system is not observable or controllable. It is worth mentioning here the Kalman Decomposition (Kalman, 1962, 1963): Kalman decomposed the state space into a direct sum of four vector subspaces, based on the two notions, those of observability and controllability. He formulated the canonical form of the corresponding equations. The proof of the state space decomposition theorem was provided by Kalman (Kalman, 1963). In particular, when considering the properties of controllability and observability, several possibilities of this decomposition are indicated (see Kalman, 1963). Despite this, a complete and rigorous proof has still not been produced.

Partial observation will be very useful when we concentrate our attention only on very specific parameters or a combination of state parameters. In this case, the study can be concentrated on the desired parameters or a combination of parameters. We cite as an example the work done by Bichara, Cozic and Iggir (2012), in which the authors use an observer that relies only on the available measurable data, concerning the peripheral infected erythrocytes, $y_1 + y_2$, and provides estimates of the sequestered ones, $y_3+y_4+y_5$, that cannot be measured by clinical methods.

One can be interested in partial observation when dealing with a system, in which many parameters act, making it impossible to observe the system. As an example, it is impossible to do all medical tests and analyses in order to know the health state of the patient, that is why we limit our concentration and interest on the measurement of only some parameters, such as temperature and pressure of the patient, to know approximately his state of health. This fact confronts us with the observation of a system that is not necessarily totally observable, which pushes us to dealing with only a part of the health state of the patient.

The "partial" analysis is also necessary when the system under study does not fulfill the standard operating conditions, i.e., when the system contains an ambiguity that prevents the behavior of the system from being known, so that it can be used and controlled. We cite as examples, among many others: incomplete measurement systems, complex systems and large systems. In this context, several approaches have been adopted for the partial analysis of the system. We cite the work by Boukhobza et al. (2009), who proposed a method based on a graph theoretical approach. There is also a quantitative study of the two notions of observability and controllability, made by Kang and Xu (2009).

In their work, they used dynamic optimization and its calculation methods as a tool to quantitatively define and measure observability and controllability.

In the Kalman's works, Kalman (1962, 1963), it is shown, without a rigorous proof, however, that the space of system states can be decomposed into two subspaces. The first part is the observable part and the second part is the non–observable one. From the examples given, we believe that Kalman's results can be verified and proven. In order to see this, we study the partial observability problem from the algebraic point of view. In this context we have introduced a definition of a simple notion that gives the possibility of characterizing and extracting the totality of the observable parts of the system space states. We have introduced a new concept, which we call observable subspace.

By a generalization of the observation condition on Kalman's matrix, we have characterized all observable subspaces, which gives us the possibility of studying the observability of a given subspace without taking into consideration the rest of the system state space.

By using the characterizations and the introduced definition we have proven that the state space is a direct sum of two subspaces. The first one is *totally non observable* and the second one is *totally observable* and this is consistent with Kalman's result. The observable subspace constitutes the largest subspace, in which we can reconstruct the system states. Three steps have to be taken to partially reconstruct the state of the system. The first one is to calculate the orthogonal projection of the initial state on the largest observable subspace (this is what we have called the "visible part of the initial state"), the second one is the calculation of the system state generated by the visible part of the initial state and the third one is the calculation of the orthogonal projection of the system state on the largest observable subspace of the system (this is what we call the "visible part of the system state"). By this procedure, it has been shown that the orthogonal projection of the system state over an observable subspace remains in the observed subspace over time interval. This fact allows us to reconstruct the visible part of the state of the system just by knowing the visible part of the initial state.

This paper is organized as follows: in the first part, we give the problem statement. In the second part, we characterize our problem and give some preliminary results, which are used thereafter. In the third part, we give the definition of an observable subspace, as well as some characterizations and an illustrative example, and in the fourth part, we explain the entire procedure and the theoretical approach followed for the partial reconstruction of the system state.

2. The preliminaries and the problem statement

2.1. An example and problem outline

Consider the system given by the following equations:

$$\begin{cases} z_1'(t) = z_1(t) & , \quad z_1(0) = z_{01} \\ z_2'(t) = z_1(t) + z_2(t) & , \quad z_2(0) = z_{02} \\ z_3'(t) = z_3(t) & , \quad z_3(0) = z_{03} \end{cases} \tag{1}$$

with the output equation

$$y(t) = z_1(t) + z_2(t) + z_3(t). \tag{2}$$

The solution is given by

$$\left| \begin{array}{l} z_1(t) = z_{01}e^t \\ z_2(t) = z_{02}e^t + tz_{01}e^t \\ z_3(t) = z_{03}e^t \end{array} \right. ,$$

and the output function is then given by

$$y(t) = (t+1)\, z_{01}e^t + z_{02}e^t + z_{03}e^t. \tag{3}$$

Consider the initial states given by

$$z_0 = \begin{pmatrix} 1 \\ \beta + 1 \\ 1 - \beta \end{pmatrix}, \ \beta \in \mathbb{R}.$$

Then, the corresponding solution is given by

$$z(t) = \begin{pmatrix} e^t \\ e^t(\beta + 1) + te^t \\ -e^t(\beta - 1) \end{pmatrix}$$

and we have the same output $y(t) = e^t(t+3)$, $\forall t \geqslant 0$ and for any β in $\mathbb{R}$. This system is therefore not observable.

However, if we fix t_1 and t_2 so that $0 < t_1 < t_2 < T$ and for $t = t_1$ and $t = t_2$ in (3) and by taking the difference, we obtain

$$z_{01} = \frac{y^{\text{mes}}(t_1)\, e^{-t_1} - y^{\text{mes}}(t_2)\, e^{-t_2}}{t_1 - t_2},$$

which makes it possible to determine the expression for the first component of the state

$$z_1(t) = z_{01}e^t = \frac{y^{\text{mes}}(t_1)\, e^{t-t_1} - y^{\text{mes}}(t_2)\, e^{t-t_2}}{t_1 - t_2}$$

in a unique way, while the system is not observable. Let us denote by e_1 the first vector of the canonical basis, and since $\langle z(t), e_1\rangle = z_1(t)$, then we obtain the following vector

$$\langle z(t), e_1\rangle e_1,$$

which can be written down as follows

$$e_1 \langle e_1, z(t)\rangle = e_1 e_1^{\mathrm{T}} z(t) = \mathrm{P}_H z(t),$$

where

$$\mathrm{P}_H = e_1 e_1^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is the orthogonal projection on the vector subspace $H = \overline{\{e_1\}}$.

2.2. Problem statement

Consider the linear system, given by the following equation:

$$\begin{cases} \dot{z}(t) = Az(t), & t_0 < t < T \\ z(t_0) \in \mathbb{R}^n, \end{cases}, \quad A \in \mathcal{M}_n(\mathbb{R}), \tag{4}$$

and the output equation

$$y(t) = Cz(t),\ t \in [t_0, T],\ C \in \mathcal{M}_{q,n}(\mathbb{R}). \tag{5}$$

In terms of the initial state, we have

$$y(t) = Ce^{(t-t_0)A} z(t_0).$$

(P) : If the system is unobservable, can it be observed in a given subspace? if so, can we characterize it and reconstruct a part of the system's state?

To answer these questions, we introduce the concept of partial observability.

3. Partial observability

3.1. Some useful results

In this subsection we present some results that will be used throughout. We introduce the observability operator, given by

$$\mathcal{K} : z_0 \in \mathbb{R}^n \longrightarrow y(.) \in \mathrm{L}^2[t_0, T; \mathbb{R}^q],$$

whose adjoint operator

$$\mathcal{K}^* : \mathrm{L}^2\left[t_0, T; \mathbb{R}^q\right] \longrightarrow \mathbb{R}^n$$

can be written down as

$$\mathcal{K}^*\eta = \int_{t_0}^{T} e^{(t-t_0)A^{\mathrm{T}}} C^{\mathrm{T}} \eta(t)\,\mathrm{d}t,\ \ \eta \in \mathrm{L}^2\left[t_0, T; Y\right].$$

We denote by $\mathbf{M}$ the following matrix

$$\mathbf{M} \equiv \mathcal{K}^*\mathcal{K} = \int_{t_0}^{T} e^{(t-t_0)A^{\mathrm{T}}} C^{\mathrm{T}} C e^{(t-t_0)A} \mathrm{d}t \in \mathcal{M}_n(\mathbb{R}).$$

Remark 1 *1. The matrix $\mathbf{M}$ is symmetric and positive semi-definite.*

2. The system is observable if and only if the matrix $\mathbf{M}$ is positive definite.

3. We have

$$\mathrm{Im}\left(\mathcal{K}^*\right) = \mathrm{Im}\left(\mathbf{M}\right),\ \ Ker\left(\mathcal{K}\right) = Ker\left(\mathbf{M}\right)$$

and

$$\mathbb{R}^n = \mathrm{Im}\left(\mathbf{M}\right) \oplus Ker\left(\mathbf{M}\right). \tag{6}$$

Remark 2 *Any vector $x \in \mathbb{R}^n$ can be decomposed, in a unique fashion, in the following form*

$$x = x_1 + x_2 \tag{7}$$

with $x_1 \in \mathrm{Im}\left(\mathbf{M}\right)$ and $x_2 \in Ker\left(\mathbf{M}\right)$; x_1 being the orthogonal projection of x on $\mathrm{Im}\left(\mathbf{M}\right)$ and x_2 the orthogonal projection of x on $Ker\left(\mathbf{M}\right)$. These components verify

$$\langle x_1, x_2 \rangle = 0\ ,\ \ \|x_1\| \leqslant \|x\|\ \text{ and }\ \|x_2\| \leqslant \|x\|. \tag{8}$$

Lemma 1 *Let R be a $m \times n$ matrix, $T > t_0$ and $x \in \mathbb{R}^n$. Then, the following properties are equivalent:*

1. *$Re^{(t-t_0)A}x = 0,\ \forall t \in [t_0, T]$;*
2. *$RA^k x = 0,\ \forall k \in \mathbb{N}$;*
3. *$Re^{tA}x = 0,\ \forall t \in \mathbb{R}$;*
4. *$RA^{k-1}x = 0,\ 1 \leqslant k \leqslant n$.*

Proof The implications **3) $\Longrightarrow$ 1)** and **2) $\Longrightarrow$ 4)** are equivalent.

1) $\Longrightarrow$ 2) : If $Re^{(t-t_0)A}x = 0$, for all $t \in [t_0, T]$ then for all $t \in [t_0, T]$ we have

$$Rx + (t-t_0)\,RAx + \frac{(t-t_0)^2}{2} RA^2 x + \frac{(t-t_0)^3}{6} RA^3 x + \ldots + \frac{(t-t_0)^j}{j!} RA^j x + \ldots = 0.$$

The k^{th} derivative with respect to t gives

$$RA^k x + (t - t_0)\, RA^{k+1} x + \frac{(t-t_0)^2}{2} RA^{k+2} x$$
$$+\frac{(t-t_0)^3}{6} RA^{k+3} x + \ldots + \frac{(t-t_0)^j}{j!} RA^{k+j} x + \ldots = 0,$$

which, for $t = t_0$, becomes $RA^k x = 0$, for all $k \in \mathbb{N}$.

2) $\Longrightarrow$ 3) : If $RA^k x = 0$, for all $k \in \mathbb{N}$, then, for every $t \geqslant t_0$, we have

$$Re^{(t-t_0)A} x = Rx + (t-t_0)\, RAx + \frac{(t-t_0)^2}{2} RA^2 x$$
$$+\frac{(t-t_0)^3}{6} RA^3 x + \ldots + \frac{(t-t_0)^j}{j!} RA^j x + \ldots = 0.$$

4) $\Longrightarrow$ 2) : Let us assume that $RA^{k-1}x = 0$, $1 \leqslant k \leqslant n$; Cayley-Hamilton's theorem gives the decomposition

$$A^n = \sum_{j=0}^{n-1} \beta_{n,j} A^j, \ \beta_{n,j} \in \mathbb{R}.$$

We deduce (by recurrence) a similar decomposition of A^k

$$A^k = \sum_{j=0}^{n-1} \beta_{kj} A^j, \ \beta_{kj} \in \mathbb{R},$$

and then

$$RA^k x = \sum_{j=0}^{n-1} \beta_{kj} RA^j x = 0, \ \forall k \geqslant n. \qquad \square$$

Let us use the notation

$$\mathbf{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

We formulate the following proposition:

Proposition 1 *We have*

$$Ker(\mathbf{M}) = \bigcap_{k=1}^{n} Ker\left(CA^{k-1}\right) = Ker(\mathbf{O}), \tag{9}$$

and then $\mathrm{rg}(\mathbf{M}) = \mathrm{rg}(\mathbf{O})$.

$Ker(\mathbf{M})$ is stable by A^k with $k = 1, 2, 3, \ldots$ and then by $e^{(t-t_0)A}$, $\forall t \geqslant t_0$.

Proof Let $x \in \mathbb{R}^n$. We have $x \in \mathrm{Ker}\,(\mathbf{M}) = \mathrm{Ker}\,(\mathcal{K})$, equivalent to $Ce^{(t-t_0)A}x = 0$, for all $t \in [t_0, T]$, or (by taking $R = C$ in Lemma 1) $CA^{k-1}x = 0$, $1 \leqslant k \leqslant n$, which can be written as $x \in \bigcap_{k=1}^{n} \mathrm{Ker}\left(CA^{k-1}\right)$. This relation is equivalent to

$$\begin{bmatrix} Cx \\ CAx \\ \vdots \\ CA^{n-1}x \end{bmatrix} = 0,$$

or even $x \in \mathrm{Ker}\,(\mathbf{O})$ and the following properties result. By taking the orthogonal in the relation $\mathrm{Ker}\,(\mathbf{M}) = \mathrm{Ker}\,(\mathbf{O})$ we obtain $\mathrm{Im}\,(\mathbf{M}) = \mathrm{Im}\left(\mathbf{O}^{\mathrm{T}}\right)$ then $\mathrm{rg}\,(\mathbf{M}) = \mathrm{rg}\left(\mathbf{O}^{\mathrm{T}}\right) = \mathrm{rg}\,(\mathbf{O})$.

If $x \in \mathrm{Ker}\,(\mathbf{M}) = \mathrm{Ker}\,(\mathcal{K})$ then $Ce^{(t-t_0)A}x = 0$, for all $t \in [t_0, T]$, which, according to Lemma 1, is equivalent to $CA^jx = 0$, for all $j \in \mathbb{N}$,, then, for $k \in \mathbb{N}$ we have

$$CA^j\left(A^kx\right) = 0, \ \forall j \in \mathbb{N},$$

and then, for $j = 0$ we have $A^kx \in \mathrm{Ker}\,(\mathcal{K})$. From that, since

$$e^{(t-t_0)A}x = \sum_{k \geqslant 1} \frac{(t-t_0)^k}{k!} A^kx \in \mathrm{Ker}\,(\mathcal{K}),$$

the stability of $\mathrm{Ker}\,(\mathcal{K}) = \mathrm{Ker}\,(\mathbf{M})$ follows. □

For a subspace H of $\mathbb{R}^n$, let us introduce the square matrices, $\mathbf{G}_H$ of order n and $\mathbf{Q}_H$ of type $n^2 \times n$:

$$\mathbf{G}_H = \int_{t_0}^{T} e^{(t-t_0)A^{\mathrm{T}}} \left(\mathrm{P}_H\right)^{\mathrm{T}} \mathrm{P}_H e^{(t-t_0)A} \mathrm{d}t,$$

$$\mathbf{Q}_H = \begin{bmatrix} \mathrm{P}_H \\ \mathrm{P}_H A \\ \vdots \\ \mathrm{P}_H A^{n-1} \end{bmatrix}. \qquad . \tag{10}$$

Lemma 2 *We have, for every subspace H of $\mathbb{R}^n$,*

$$\begin{aligned} Ker\,(\mathbf{G}_H) &= \bigcap_{t_0 \leqslant t \leqslant T} Ker\left(\mathrm{P}_H e^{(t-t_0)A}\right), \\ &= \bigcap_{k=1}^{n} Ker\left(\mathrm{P}_H A^{k-1}\right), \\ &= Ker\,(\mathbf{Q}_H). \end{aligned} \tag{11}$$

Proof The proof is similar to that of Proposition 1, performed by taking P_H instead of C and by applying Lemma 1 with $R = \mathrm{P}_H$.

3.2. Partial observability

3.2.1. Definition

We consider the system,defined by (4) and (5). Let H be a subspace of $\mathbb{R}^n$, not reduced to $\{0\}$.

DEFINITION 1 *1. We say that the system (4) and (5) is observable on H during the time interval $[t_0, T]$ if two given states on $[t_0, T]$ that yield the same output have the same projection on H during $[t_0, T]$:*

$$(\ y(t) = \widetilde{y}(t)\,,\ \forall t \in [t_0, T]\ ) \Longrightarrow (\ \mathrm{P}_H z(t) = \mathrm{P}_H \widetilde{z}(t)\,,\ \forall t \in [t_0, T]\ )\,.$$

We will say, for simplicity, that H is observable on $[t_0, T]$.

2. We say that the system is partially observable during $[t_0, T]$ if it admits a subspace $H \neq \{0\}$ observable during $[t_0, T]$.

REMARK 3 *1. If the system is observable, then any vector subspace of H is observable.*

2. If the state $z(.)$ is located in an observable space: $z(t) \in H$, $\forall t \in [t_0, T]$, then it is the only single state evolving in H and it yields the same measure as given by $z(.)$:

$$\begin{aligned}&(y(t) = \widetilde{y}(t)\,,\ \forall t \in [t_0, T] \ \text{ and } z(t)\,, \widetilde{z}(t) \in H,\ \forall t \in [t_0, T])\\ &\Longrightarrow\ (z(t) = \widetilde{z}(t)\,,\ \forall t \in [t_0, T])\end{aligned}$$

This does not exclude the possibility that there exists a state $\widehat{z}(t)$ yielding the same measure as $z(t)$, and evolving outside of H (therefore distinct from $z(t)$).

3. According to the definition above, H is observable and this is equivalent to the fact that

$$(y(t) = (\mathcal{K}z_0)(t) = 0, \forall t \in [t_0, T]) \Longrightarrow (z(t) = 0, \forall t \in [t_0, T])\,. \quad (12)$$

3.2.2. Example

Let us use the example from the preliminaries, described by the following equations:

$$\left\{\begin{array}{lcl} z_1'(t) = z_1(t) & , & z_1(0) = z_{01} \\ z_2'(t) = z_1(t) + z_2(t) & , & z_2(0) = z_{02} \\ z_3'(t) = z_3(t) & , & z_3(0) = z_{03} \end{array}\right. \quad (13)$$

with the output equation

$$y(t) = z_1(t) + z_2(t) + z_3(t)\,, \quad (14)$$

for which

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}.$$

In this case, we have

$$\mathrm{Ker}(\mathcal{K}) = \mathrm{vect}\{e_2 - e_3\}$$

and

$$\mathrm{Im}(\mathcal{K}^*) = \mathrm{vect}\{e_1, e_2 + e_3\}.$$

Let $z_0 \in \mathrm{Ker}(\mathcal{K})$, so that there is an $\alpha \in \mathbb{R}$ such that

$$z_0 = \alpha(e_2 - e_3) = \begin{pmatrix} 0 \\ \alpha \\ -\alpha \end{pmatrix}.$$

Let

$$H = \mathrm{vect}\,\{e_1\}\,.$$

Since

$$e^{At} = \begin{bmatrix} e^t & 0 & 0 \\ te^t & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix},$$

we have

$$\begin{aligned} & P_H e^{At} z_0 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ te^t & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix} \begin{pmatrix} 0 \\ \alpha \\ -\alpha \end{pmatrix} \\ &= \begin{bmatrix} e^t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ \alpha \\ -\alpha \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \end{aligned}$$

for all $t \in [0, T]$, which implies that H is observable.

The subspace H is generated by the vector e_1, so the definition has a meaning, because we have already extracted the projection of the state on H.

4. The characterization and the properties of partial observability

In this section, we present some properties and characterization results for partial observability.

Proposition 2 *H is observable on $[t_0, T]$ if, and only if,*

$$Ker(\mathbf{M}) \subseteq \bigcap_{t_0 \leqslant t \leqslant T} Ker\left(\mathrm{P}_H e^{(t-t_0)A}\right). \tag{15}$$

Proof Let $z_0 \in \mathbb{R}^n$ be an initial state producing the state z. On the one hand ($y(t) = 0,\ \forall t \in [t_0, T]$) is equivalent to $\mathbf{M} z_0 = 0$; on the other hand ($\mathrm{P}_H z(t) = 0,\ \forall t \in [t_0, T]$) is equivalent to

$$\left(\ \mathrm{P}_H e^{(t-t_0)A} z_0 = 0,\ \forall t \in [t_0, T]\ \right),$$

or to

$$z_0 \in \mathrm{Ker}\left(\ \mathrm{P}_H e^{(t-t_0)A}\right),\ \forall t \in [t_0, T],$$

which can be reduced to $z_0 \in \bigcap_{t_0 \leqslant t \leqslant T} \mathrm{Ker}\left(\mathrm{P}_H e^{(t-t_0)A}\right)$. H is therefore observable if and only if, for all $z_0 \in Z$,

$$\mathcal{K} z_0 = 0 \quad \Longrightarrow \quad z_0 \in \bigcap_{t_0 \leqslant t \leqslant T} \mathrm{Ker}\left(\mathrm{P}_H e^{(t-t_0)A}\right),$$

or $\mathrm{Ker}(\mathbf{M}) \subseteq \bigcap_{t_0 \leqslant t \leqslant T} \mathrm{Ker}\left(\mathrm{P}_H e^{(t-t_0)A}\right)$. □

Remark 4 *The system is observable if and only if every subspace of $\mathbb{R}^n$ is observable.*

Proposition 3 *The following propositions are equivalent:*

1. *H is observable ;*
2. $\bigcap_{k=1}^{n} Ker\left(CA^{k-1}\right) \subseteq \bigcap_{k=1}^{n} Ker\left(\mathrm{P}_H A^{k-1}\right)$ *;*
3. $Ker(\mathbf{M}) \subseteq Ker(\mathbf{G}_H)$ *;*
4. $Ker(\mathbf{O}) \subseteq Ker(\mathbf{Q}_H)$.

Proof With the Proposition 2, H is observable if and only if,

$$\mathrm{Ker}(\mathbf{M}) \subseteq \bigcap_{t_0 \leqslant t \leqslant T} \mathrm{Ker}\left(\mathrm{P}_H e^{(t-t_0)A}\right),$$

this statement being equivalent, according to Lemma 2, to

$\bigcap_{k=1}^{n} \mathrm{Ker}\left(CA^{k-1}\right) \subseteq \bigcap_{k=1}^{n} \mathrm{Ker}\left(\mathrm{P}_H A^{k-1}\right)$ or to $\mathrm{Ker}(\mathbf{O}) \subseteq \mathrm{Ker}(\mathbf{Q}_H)$,
which is equivalent also to $\mathrm{Ker}(\mathbf{M}) \subseteq \mathrm{Ker}(\mathbf{G}_H)$ □

Remark 5 *We can deduce from this proposition that a subspace H is not observable if*

$$\mathrm{rg}\left(\mathbf{Q}_H\right) > \mathrm{rg}\left(\mathbf{O}\right).$$

Indeed, inclusion $Ker\left(\mathbf{O}\right) \subseteq Ker\left(\mathbf{Q}_H\right)$ yields, by taking the orthogonal, $\mathrm{Im}\left(\left[\mathbf{Q}_H\right]^{\mathrm{T}}\right) \subseteq \mathrm{Im}\left(\mathbf{O}^{\mathrm{T}}\right)$.

Then, $\mathrm{rg}\left(\left[\mathbf{Q}_H\right]^{\mathrm{T}}\right) \leqslant \mathrm{rg}\left(\mathbf{O}^{\mathrm{T}}\right)$*, which gives* $\mathrm{rg}\left(\mathbf{Q}_H\right) \leqslant \mathrm{rg}\left(\mathbf{O}\right)$.

We deduce the following properties:

Proposition 4 *Let H_1 and H_2 be two vector subspaces of $\mathbb{R}^n$.*

1. *If $H_1 \subseteq H_2$ and H_2 is observable, then H_1 is observable.*
2. *If H_1 or H_2 is observable, then $H_1 \cap H_2$ is observable.*
3. *If H_1 and H_2 are observable, then their sum $H_1 + H_2$ is also observable.*

Proof 1) H_2 being observable, we have

$$\mathrm{Ker}\left(\mathbf{M}\right) \subseteq \bigcap_{t_0 \leqslant t \leqslant T} \mathrm{Ker}\left(\mathrm{P}_{H_2} e^{(t-t_0)A}\right),$$

and with $H_1 \subseteq H_2$ we have $\mathrm{Ker}\left(\mathrm{P}_{H_2}\right) = \left(H_2\right)^{\perp} \subseteq \left(H_1\right)^{\perp} = \mathrm{Ker}\left(\mathrm{P}_{H_1}\right)$.

Then, for $\xi \in \mathrm{Ker}\left(\mathrm{P}_{H_2} e^{(t-t_0)A}\right)$, we have $e^{(t-t_0)A}\xi \in \mathrm{Ker}\left(\mathrm{P}_{H_2}\right) \subseteq \mathrm{Ker}\left(\mathrm{P}_{H_1}\right)$, and then $\xi \in \mathrm{Ker}\left(\mathrm{P}_{H_1} e^{(t-t_0)A}\right)$. This proves the inclusion

$$\mathrm{Ker}\left(\mathrm{P}_{H_2} e^{(t-t_0)A}\right) \subseteq \mathrm{Ker}\left(\mathrm{P}_{H_1} e^{(t-t_0)A}\right).$$

Hence,

$$\begin{aligned} \mathrm{Ker}\left(\mathbf{M}\right) &\subseteq \bigcap_{t_0 \leqslant t \leqslant T} \mathrm{Ker}\left(\mathrm{P}_{H_1} e^{(t-t_0)A}\right) \\ &\subseteq \bigcap_{t_0 \leqslant t \leqslant T} \mathrm{Ker}\left(\mathrm{P}_{H_2} e^{(t-t_0)A}\right), \end{aligned}$$

and then H_1 is observable.

2) If H_1 is observable, then $H_1 \cap H_2 \subseteq H_1$ is also observable from item 1 above.

3) H_1 and H_2 being observable, we have

$$\mathrm{Ker}\left(\mathbf{M}\right) \subseteq \bigcap_{k=1}^{n} \mathrm{Ker}\left(\mathrm{P}_{H_1} A^{k-1}\right)$$

and

$$\mathrm{Ker}\,(\mathbf{M}) \subseteq \bigcap_{k=1}^{n} \mathrm{Ker}\left(\mathrm{P}_{H_2} A^{k-1}\right).$$

If $x \in \mathrm{Ker}\,(\mathbf{M})$, then for $k \in \{1, \ldots, n\}$ we have

$$\mathrm{P}_{H_1} A^{k-1} x = \mathrm{P}_{H_2} A^{k-1} x = 0.$$

Then

$$A^{k-1} x \in \mathrm{Ker}\,(\mathrm{P}_{H_1}) \cap \mathrm{Ker}\,(\mathrm{P}_{H_2}) = \mathrm{Ker}\,(\mathrm{P}_{H_1+H_2})$$

and, subsequently, $\mathrm{P}_{H_1+H_2} A^{k-1} x = 0$. This gives

$$\mathrm{Ker}\,(\mathbf{M}) \subseteq \bigcap_{k=1}^{n} \mathrm{Ker}\left(\mathrm{P}_{H_1+H_2} A^{k-1}\right),$$

i.e., $H_1 + H_2$ is observable. □

Corollary 1 *Every observable subspace is contained in* $\mathrm{Im}\,(\mathbf{M})$.

Proof Let H be a subspace of $\mathbb{R}^n$. For all $t \in [t_0, T]$ we have

$$\bigcap_{t_0 \leqslant s \leqslant T} \mathrm{Ker}\left(\mathrm{P}_H e^{(s-t_0)A}\right) \subseteq \mathrm{Ker}\left(\mathrm{P}_H e^{(t-t_0)A}\right).$$

By taking $t = t_0$ we get $\bigcap_{t_0 \leqslant t \leqslant T} \mathrm{Ker}\left(\mathrm{P}_H e^{(t-t_0)A}\right) \subseteq \mathrm{Ker}\,(\mathrm{P}_H)$. If H is observable, then $\mathrm{Ker}\,(\mathbf{M}) \subseteq \mathrm{Ker}\,(\mathrm{P}_H) = H^{\perp}$ or $H \subseteq [\mathrm{Ker}\,(\mathbf{M})]^{\perp} = \mathrm{Im}\,(\mathbf{M})$ □.

Proposition 5 *The subspace*

$$\mathrm{Im}\,(\mathbf{M})$$

is observable.

Proof Let $z_0 \in \mathrm{Ker}\,(\mathbf{M})$. By using Proposition 1 we have $CA^{k-1} z_0 = 0$, $1 \leqslant k \leqslant n$,, which gives, with Lemma 1, $CA^k z_0 = 0$, $\forall k \in \mathbb{N}$, and $Ce^{sA} z_0 = 0$, $\forall s \in \mathbb{R}$.

Let us take $t \in [t_0, T]$. For $s = (\tau - t_0) + (t - t_0)$ we get $Ce^{(\tau-t_0)A}\left[e^{(t-t_0)A} z_0\right] = 0$, $\forall \tau \in \mathbb{R}$. In particular, for $\tau \in [t_0, T]$, we get

$$Ce^{(\tau-t_0)A}\left[e^{(t-t_0)A} z_0\right] = 0, \quad \forall \tau \in [t_0, T].$$

Then, $e^{(t-t_0)A} z_0 \in \mathrm{Ker}\,(\mathcal{K}) = \mathrm{Ker}\,(\mathbf{M})$, and subsequently

$$\mathrm{P}_{\mathrm{Im}(\mathbf{M})}\left[e^{(t-t_0)A} z_0\right] = 0$$

and so for all $t \in [t_0, T]$. This shows that $\mathrm{Im}\,(\mathbf{M})$ is observable. □

Theorem 1 $\mathrm{Im}(\mathbf{M})$ *is the largest observable subspace (i.e. it contains all the observable subspaces).*

Proof This is the direct result of Corollary 1 and the preceding proposition.

Remark 6 *1. Every subspace H of $\mathbb{R}^n$ is an orthogonal direct sum $H = H_0 \oplus H_1$, with H_0 observable and H_1 unobservable, and containing no observable subspace. This decomposition is unique.*

2. Subspaces H_0 and H_1 are given by

$$H_0 = H \cap \mathrm{Im}(\mathbf{M}), \quad H_1 = H \cap Ker(\mathbf{M}).$$

Corollary 2 *The subspace $H = \mathrm{Im}\left(C^{\mathrm{T}}\right)$ is observable.*

Proof We have $H^{\perp} = \left[\mathrm{Im}\left(C^{\mathrm{T}}\right)\right]^{\perp} = \mathrm{Ker}(C)$. Let $z_0 \in \mathrm{Ker}(\mathbf{M})$, then for all $t \in [t_0, T]$, $Ce^{(t-t_0)A} z_0 = 0$. Then, $e^{(t-t_0)A} z_0 \in \mathrm{Ker}(C) = H^{\perp}$, which shows that its projection on H is null, i.e., $\mathrm{P}_H e^{(t-t_0)A} z_0 = 0$. H is then observable. □

Remark 7 *We deduce from these two corollaries that $Ker(\mathbf{M}) \subseteq Ker(C)$.*

5. Partial reconstruction of the state

We will present in this section of the paper the proposed method for extracting the reconstructible part of the system state from the observable subspace.

Let us then consider the equations (4) and (5), with the initial state $z(t_0) \in \mathbb{R}^n$ "unknown", and let $y^{\mathrm{mes}}(.) \in \mathrm{L}^2[t_0, T; \mathbb{R}^q]$ be a measurement obtained on $[t_0, T]$ by a state $z(t)$, generated by the initial state $z(t_0) \in \mathbb{R}^n$ unknown.

5.1. Affine space of initial states

In this subsection, we consider the space of initial states, in which we can find all the initial states that give a given measurement. Let us note that the projection of the state of the system on an observable subspace H is unique and its determination can only be made through the following equation

$$\mathcal{K} z(t_0) = y^{\mathrm{mes}}. \tag{16}$$

Definition 2 *We call the initial states, according to the measurement y^{mes}, the set of all initial states that give this measure*

$$X_{\mathrm{mes}} = \{z_0 \in \mathbb{R}^n \;/\; \mathcal{K} z_0 = y^{\mathrm{mes}}\}. \tag{17}$$

This set is always not empty, since $z(t_0) \in X_{\mathrm{mes}}$. We will then assume that

$$X_{\mathrm{mes}} \neq \emptyset. \tag{18}$$

Proposition 6 *The vectors of* X_{mes} *are characterized by the equation*

$$\mathbf{M}z_0 = \mathcal{K}^* y^{\mathrm{mes}}.$$

Proof If $z_0 \in X_{\mathrm{mes}}$ then $\mathcal{K}z_0 = y^{\mathrm{mes}}$ and then $\mathbf{M}z_0 = \mathcal{K}^*\mathcal{K}z_0 = \mathcal{K}^* y^{\mathrm{mes}}$.

Conversely, let $z_0 \in \mathbb{R}^n$ such that $\mathbf{M}z_0 = \mathcal{K}^* y^{\mathrm{mes}}$. $\mathcal{K}$ being bounded, the function

$$\phi(\xi) = \frac{1}{2}\left\|\mathcal{K}\xi - y^{\mathrm{mes}}\right\|^2 \quad , \quad \xi \in \mathbb{R}^n,$$

is twice differentiable and

$$\begin{aligned} \phi'(\xi).h &= \left(\mathcal{K}\xi - y^{\mathrm{mes}}, \mathcal{K}h\right) \\ &= \left\langle \left(\mathcal{K}^*\mathcal{K}\right)\xi - \mathcal{K}^* y^{\mathrm{mes}}, h\right\rangle \\ &= \left\langle \mathbf{M}\xi - \mathcal{K}^* y^{\mathrm{mes}}, h\right\rangle. \end{aligned}$$

Now, z_0 cancels the differential of ϕ, since

$$\phi'(z_0).h = \left\langle \mathbf{M}\xi - \mathcal{K}^* y^{\mathrm{mes}}, h\right\rangle = 0, \forall h \in \mathbb{R}^n.$$

The second derivative of ϕ is

$$\phi''(\xi).(h,k) = \langle \mathbf{M}k, h\rangle, \; h,k \in \mathbb{R}^n.$$

Then, ϕ is convex, since

$$\phi''(\xi).(h,h) = 2\langle \mathbf{M}h, h\rangle \geqslant 0, \; \forall h \in \mathbb{R}^n.$$

Therefore, z_0 is a minimum of ϕ on $\mathbb{R}^n : \phi(z_0) \leqslant \phi(\xi), \forall \xi \in \mathbb{R}^n$, which gives

$$\left\|\mathcal{K}z_0 - y^{\mathrm{mes}}\right\|^2 \leqslant \left\|\mathcal{K}\xi - y^{\mathrm{mes}}\right\|^2 \quad , \quad \forall \xi \in \mathbb{R}^n.$$

By taking $\xi \in X_{\mathrm{mes}}$ (which is not empty) we get $\left\|\mathcal{K}z_0 - y^{\mathrm{mes}}\right\|^2 \leqslant 0$, and subsequently $\mathcal{K}z_0 = y^{\mathrm{mes}}$. □

Proposition 7 *1. For any vector* $z_0 \in X_{\mathrm{mes}}$*, we have*

$$X_{\mathrm{mes}} = z_0 + Ker(\mathbf{M}). \tag{19}$$

2. X_{mes} *is a closed convex set of* $\mathbb{R}^n$*.*
3. *The initial states of* X_{mes} *generate the system states, whose projections coincide on any observable subspace* H*:*

$$\forall z_0, \widetilde{z}_0 \in X_{\mathrm{mes}} \; : \; \mathrm{P}_H e^{(t-t_0)A} z_0 = \mathrm{P}_H e^{(t-t_0)A} \widetilde{z}_0 \; \textit{on} \; [t_0, T]. \tag{20}$$

Proof 1) Let $z_0 \in X_{\mathrm{mes}}$, then $\xi \in X_{\mathrm{mes}}$ if and only if $\mathcal{K}\xi = y^{\mathrm{mes}}$ or $\mathcal{K}\xi = \mathcal{K}z_0$. Hence, $\mathcal{K}(\xi - z_0) = 0$ or $\xi - z_0 \in \mathrm{Ker}(\mathcal{K}) = \mathrm{Ker}(\mathbf{M})$. Then we cane write $\xi \in z_0 + \mathrm{Ker}(\mathbf{M})$.

2) $\mathrm{Ker}(\mathbf{M})$ is a vector subspace of $\mathbb{R}^n$. So, it is a closed convex of $\mathbb{R}^n$ and it is then the same for $z_0 + \mathrm{Ker}(\mathbf{M}) = X_{\mathrm{mes}}$.

3) Let $z_0, \widetilde{z}_0 \in X_{\mathrm{mes}}$. The states they generate $z(t) = e^{(t-t_0)A} z_0$ and $\widetilde{z}(t) = e^{(t-t_0)A}\widetilde{z}_0$ give the same output $y(.) = \widetilde{y}(.)$, since H is observable

$$\mathrm{P}_H e^{(t-t_0)A} z_0 = \mathrm{P}_H e^{(t-t_0)A} \widetilde{z}_0, \; \forall t \in [t_0, T]. \qquad \square$$

Remark 8 *We deduce from this proposition and Proposition 1 that* X_{mes} *is independent of the initial time* t_0*, of the final instant* T *and also of the duration of the measurement* $(T - t_0)$.

6. The visible part of the initial state and its reconstruction

6.1. The preliminaries

In this section, we will reconstruct the so-called visible part of the initial state from the output equation. We know that X_{mes} is a closed convex set, which implies that X admits a single element with minimal norm.

Definition 3 1. *We call the visible part of the initial state of the system corresponding to the measurement* y^{mes} *on* $[t_0, T]$*. Denote by* $z^\star(t_0)$ *the element of* X_{mes} *which minimizes the norm:*

$$z^\star(t_0) \in X_{\mathrm{mes}} \text{ and } \|z^\star(t_0)\| \leqslant \|z_0\|, \ \forall z_0 \in X_{\mathrm{mes}}. \tag{21}$$

2. *Consider* $z^\star(t)$*, the system state generated by the visible part of the initial state:*

$$z^\star(t) = e^{(t-t_0)A} z^\star(t_0), t \geqslant t_0.$$

Theorem 2 1. *We have*

$$\mathrm{Im}(\mathbf{M}) \cap X_{\mathrm{mes}} = \{ \ z^\star(t_0) \ \}. \tag{22}$$

2. $z^\star(t_0)$ *coincides with the orthogonal projection of* $z(t_0)$ *on* $\mathrm{Im}(\mathbf{M})$.
3. *All vectors of* X_{mes} *have* $z^\star(t_0)$ *as orthogonal projection on* $\mathrm{Im}(\mathbf{M})$*, i.e.,*

$$\forall z_0 \in X_{\mathrm{mes}} \ : \ \mathrm{P}_{\mathrm{Im}(\mathbf{M})}(z_0) = z^\star(t_0).$$

Proof 1(a) We have $z^\star(t_0) \in X_{\mathrm{mes}}$. It is sufficient to prove that $z^\star(t_0) \in \mathrm{Im}(\mathbf{M})$. On the one hand, the projection of $z^\star(t_0)$ on $\mathrm{Im}(\mathbf{M})$ is characterized by $\mathrm{P}_{\mathrm{Im}(\mathbf{M})}[z^\star(t_0)] \in \mathrm{Im}(\mathbf{M})$ and

$$z^\star(t_0) - \mathrm{P}_{\mathrm{Im}(\mathbf{M})}[z^\star(t_0)] \in [\mathrm{Im}(\mathbf{M})]^\perp = \mathrm{Ker}(\mathbf{M}).$$

Then, $\mathrm{P}_{\mathrm{Im}(\mathbf{M})}[z^\star(t_0)] \in z^\star(t_0) + \mathrm{Ker}(\mathbf{M}) = X_{\mathrm{mes}}$, which shows that this projection is in X_{mes}. On the other hand this projection, according to (8), implies $\left\|\mathrm{P}_{\mathrm{Im}(\mathbf{M})}[z^\star(t_0)]\right\| \leqslant \|z^\star(t_0)\|$. The uniqueness of the minimal norm element on X_{mes} gives $\mathrm{P}_{\mathrm{Im}(\mathbf{M})}[z^\star(t_0)] = z^\star(t_0)$, which implies $z^\star(t_0) \in \mathrm{Im}(\mathbf{M})$, i.e., $z^\star(t_0) \in X_{\mathrm{mes}} \cap \mathrm{Im}(\mathbf{M})$.

1(b) The intersection $X_{\mathrm{mes}} \cap \mathrm{Im}(\mathbf{M})$ cannot have more than one element: if $z_0, \widetilde{z}_0 \in X_{\mathrm{mes}} \cap \mathrm{Im}(\mathbf{M})$, then $\mathcal{K} z_0 = \mathcal{K} \widetilde{z}_0 = y^{\mathrm{mes}}$, which implies $z_0 - \widetilde{z}_0 \in$

$\mathrm{Ker}\,(\mathbf{M})$. But $z_0 - \widetilde{z}_0 \in \mathrm{Im}\,(\mathbf{M})$, thus $z_0 - \widetilde{z}_0 \in \mathrm{Ker}\,(\mathbf{M}) \cap \mathrm{Im}\,(\mathbf{M}) = (0)$ consequently $z_0 = \widetilde{z}_0$. All of the above give $X_{\mathrm{mes}} \cap \mathrm{Im}\,(\mathbf{M}) = \{z^\star(t_0)\}$.

2 Let $z_0 \in X_{\mathrm{mes}} = z^\star(t_0) + \mathrm{Ker}\,(\mathbf{M})$. Then, $\xi \equiv z_0 - z^\star(t_0) \in \mathrm{Ker}\,(\mathbf{M})$, which gives $z_0 = z^\star(t_0) + \xi$ with $\xi \in \mathrm{Ker}\,(\mathbf{M})$ and $z^\star(t_0) \in \mathrm{Im}\,(\mathbf{M})$, i.e., it is the decomposition of z_0 on $\mathrm{Ker}\,(\mathbf{M})$ and $\mathrm{Im}\,(\mathbf{M})$.

3 The proof results directly from 1. and 2. □

We deduce from this theorem that $z^\star(t_0)$ is the unique vector of $\mathrm{Im}\,(\mathbf{M})$ which, as an initial state for the system (4) and (5), gives the measure y^{mes}. We also deduce that, since any vector $z_0 \in \mathbb{R}^n$ is decomposed as

$$z_0 = \mathrm{P}_{\mathrm{Im}(\mathbf{M})} z_0 + \mathrm{P}_{\mathrm{Ker}(\mathbf{M})} z_0,$$

any vector $z_0 \in X_{\mathrm{mes}}$ is decomposed as

$$z_0 = z^\star(t_0) + \xi_0 \quad \text{with } \xi_0 \in \mathrm{Ker}\,(\mathbf{M}),$$

with ξ_0 varying with z_0.

Reconstruction of the visible part of the initial state:

When the system is observable, the equation

$$\mathbf{M} z(t_0) = \mathcal{K}^* y^{\mathrm{mes}} \tag{23}$$

is used to get the system initial state

$$z(t_0) = \mathbf{M}^{-1} \mathcal{K}^* y^{\mathrm{mes}}.$$

In the general case, $\mathbf{M}$ is not invertible. We denote by $(\mathbf{M})^+$ the Moore-Penrose pseudo-inverse of $\mathbf{M}$ (see Ben-Israel and Greville, 2003). Then

$$z^\star(t_0) = (\mathbf{M})^+ \mathcal{K}^* y^{\mathrm{mes}}.$$

Indeed, if we put $\widehat{z_0} = (\mathbf{M})^+ \mathcal{K}^* y^{\mathrm{mes}}$, then $\widehat{z_0}$ is a solution of $\mathbf{M} z_0 = \mathcal{K}^* y^{\mathrm{mes}}$ and it is of minimal norm among all the solutions of (23). We have then $\widehat{z_0} \in X_{\mathrm{mes}}$ and $\widehat{z_0}$ has a minimal norm in X_{mes}. The uniqueness of the minimal norm element gives $\widehat{z_0} = z^\star(t_0)$. By writing explicitly the expressions of the matrices we obtain the immediate following proposition.

PROPOSITION 8 *The visible part of the initial state has the form*

$$z^\star(t_0) = \left[\int_{t_0}^{T} e^{(\tau - t_0)A^{\mathrm{T}}} C^{\mathrm{T}} C e^{(\tau - t_0)A} \mathrm{d}\tau\right]^+ \int_{t_0}^{T} e^{(s - t_0)A^{\mathrm{T}}} C^{\mathrm{T}} y^{\mathrm{mes}}(s)\,\mathrm{d}s. \tag{24}$$

REMARK 9 *According to Ben-Israel and Greville (2003), we have*

$$\mathbf{M}^+ = \lim_{\delta \to 0^+} \left[\mathbf{M}^2 + \delta I_n\right]^{-1} \mathbf{M}$$

and

$$\mathbf{M}^{+} = \lim_{\delta \to 0^{+}} \mathbf{M} \left[\mathbf{M}^{2} + \delta I_{n}\right]^{-1},$$

which can be used to approximate $z^{\star}(t_0)$ by fixing $\delta > 0$ close enough to 0:

$$z^{\star}(t_0) \approx \left[\mathbf{M}^{2} + \delta I_{n}\right]^{-1} \mathbf{M}\mathcal{K}^{*} y^{\mathrm{mes}}$$

or

$$z^{\star}(t_0) \approx \mathbf{M} \left[\mathbf{M}^{2} + \delta I_{n}\right]^{-1} \mathcal{K}^{*} y^{\mathrm{mes}}.$$

6.2. Reconstruction of the state on an observable subspace

6.2.1. The preliminary results

In this part, we are interested in reconstructing the projection of the system's state on an observable subspace using the visible part of the initial state.

Proposition 9 *For any observable H vector space, we have:*

- *$H \cap X_{\mathrm{mes}} = \emptyset$;*
- *or $H \cap X_{\mathrm{mes}} = \{z^{\star}(t_0)\}$.*

Proof If H is observable, then it is contained in $\mathrm{Im}(\mathbf{M})$, and then

$$H \cap X_{\mathrm{mes}} \subseteq \mathrm{Im}(\mathbf{M}.) \cap X_{\mathrm{mes}} = \{z^{\star}(t_0)\}.$$

Consequently, we have $H \cap X_{\mathrm{mes}} = \emptyset$ or $H \cap X_{\mathrm{mes}} = \{z^{\star}(t_0)\}$. □

The unknown initial state therefore decomposes in a unique way

$$z(t_0) = z^{\star}(t_0) + z^{\perp}(t_0)$$

with

$$z^{\star}(t_0) = \mathrm{P}_{\mathrm{Im}(\mathbf{M})}(z(t_0)) \text{ and } z^{\perp}(t_0) = \mathrm{P}_{\mathrm{Ker}(\mathbf{M})}(z(t_0)).$$

We denote by $z^{\star}(t)$ the system status generated by the visible part of the initial state :

$$z^{\star}(t) = e^{(t-t_0)A} z^{\star}(t_0), \quad t \geqslant t_0.$$

Since the initial states $z(t_0)$ and $z^{\star}(t_0)$ are both in X_{mes}, then the states they generate share the same measure. Then, these states have the same orthogonal projection on H:

$$\mathrm{P}_{H} e^{(t-t_0)A} z(t_0) = \mathrm{P}_{H} e^{(t-t_0)A} z^{\star}(t_0), \quad t \geqslant t_0.$$

We have the following proposition:

PROPOSITION 10 *The projection of the state on an observable vector subspace H is given by the orthogonal projection of the state generated by the visible part of the initial state on H:*

$$\mathrm{P}_H z(t) = \mathrm{P}_H z^\star(t)\,, \quad \forall t \in [t_0, T]\,.$$

Let H be a given observable subspace.

Since the initial states $z(t_0)$ and $z^\star(t_0)$ are both in X_{mes}, then the states they generate share the same result measure, and these states therefore have the same orthogonal projection on H:

$$\mathrm{P}_H e^{(t-t_0)A} z(t_0) = \mathrm{P}_H e^{(t-t_0)A} z^\star(t_0)\,, \quad t \geqslant t_0.$$

By using the expression of the visible part of the initial state, we can reconstruct the projection of the initial state on an observable subspace H using the following algorithm:

ALGORITHM 1 *The orthogonal projection of the state on an observable subspace H is obtained as follows:*

1. *Reconstruction of the visible part of the initial system state*

$$z^\star(t_0) = \left[\int_{t_0}^{T} e^{(\tau-t_0)A^{\mathrm{T}}} C^{\mathrm{T}} C e^{(\tau-t_0)A} \mathrm{d}\tau\right]^{+} \int_{t_0}^{T} e^{(s-t_0)A^{\mathrm{T}}} C^{\mathrm{T}} y^{\mathrm{mes}}(s)\,\mathrm{d}s;$$

2. *Reconstruction of the state generated by the visible part of the initial state*

$$z^\star(t) = e^{(t-t_0)A} z^\star(t_0)\,, \quad t \geqslant t_0;$$

3. *Projection of the visible part of the system state onto H*

$$\mathrm{P}_H z(t) = \mathrm{P}_H z^\star(t)\,, \quad t \geqslant t_0.$$

REMARK 10 *Im(M) constitutes the largest observable subspace, so reconstructing the initial state on Im(M) is equivalent to reconstructing the totality of the visible part of the system's state.*

6.2.2. Reconstruction of the state on Im(M)

We call the visible part of the state, denoted $z^{\mathbf{v}}(t)$, the projection of the state $z(t)$ on $\mathrm{Im}(\mathbf{M})$:

$$z^{\mathbf{v}}(t) = \mathrm{P}_{\mathrm{Im}(\mathbf{M})} z(t)$$

THEOREM 3 *The visible part of the state is determined by*

$$z^{\mathbf{v}}(t) = \mathrm{P}_{\mathrm{Im}(\mathbf{M})} z^\star(t)\,,\ t \geqslant t_0.$$

The orthogonal projection of the state on an observable subspace H is determined by

$$\mathrm{P}_H z(t) = \mathrm{P}_H z^{\mathbf{v}}(t) = \mathrm{P}_H z^{\star}(t), \ t \geqslant t_0.$$

Proof $\mathrm{Im}(\mathbf{M})$ being observable and $z(t_0), z^{\star}(t_0) \in X_{\mathrm{mes}}$, we have, for all $t \geqslant t_0$

$$\mathrm{P}_{\mathrm{Im}(\mathbf{M})} z(t) = \mathrm{P}_{\mathrm{Im}(\mathbf{M})} z^{\star}(t)$$

and equality follows.

By applying on $z^{\mathbf{v}}(t)$ the orthogonal projection matrix on H, we get:

$$\mathrm{P}_H z^{\mathbf{v}}(t) = \mathrm{P}_H \mathrm{P}_{\mathrm{Im}(\mathbf{M})} z^{\star}(t) = \mathrm{P}_H \mathrm{P}_{\mathrm{Im}(\mathbf{M})} z(t),$$

for all $t \geqslant t_0$. But H is a subspace of $\mathrm{Im}(\mathbf{M})$, then $\mathrm{P}_H \mathrm{P}_{\mathrm{Im}(\mathbf{M})} = \mathrm{P}_{\mathrm{Im}(\mathbf{M})} \mathrm{P}_H = \mathrm{P}_H$, which gives:

$$\mathrm{P}_H z^{\mathbf{v}}(t) = \mathrm{P}_H z^{\star}(t) = \mathrm{P}_H z(t),$$

for all $t \geqslant t_0$. And the equality follows. □

6.2.3. Reconstruction of the orthogonal projection on a vector

Let us suppose that we want to determine a quantity

$$q(t) = \alpha_1 z_1(t) + \ldots + \alpha_n z_n(t)$$

with the α_j being all not null. This quantity is written down as $q(t) = v^{\mathrm{T}} z(t)$, with $v^T = \begin{pmatrix} \alpha_1, & \ldots, & \alpha_n \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^n$. We take for H the subspace, generated by the vector v. So, the quantity $q(t)$ is unique if and only if H is observable or $v \in \mathrm{Im}(\mathbf{M})$. The orthogonal projection matrix associated to this selection of H is

$$\mathrm{P}_H = \frac{1}{(v^{\mathrm{T}} v)} v v^{\mathrm{T}}.$$

By noting that $v^{\mathrm{T}} \mathrm{P}_H = v^{\mathrm{T}}$, we have

$$q(t) = v^{\mathrm{T}} z(t) = v^{\mathrm{T}} \mathrm{P}_H z(t) = v^{\mathrm{T}} \mathrm{P}_H z^{\star}(t) = v^{\mathrm{T}} z^{\star}(t).$$

We have the following proposition:

Proposition 11 *The quantity $q(t) = v^{\mathrm{T}} z(t)$ is unique if and only if $v \in \mathrm{Im}(\mathbf{M})$. In this case it can be estimated by*

$$q(t) = v^{\mathrm{T}} z^{\star}(t), \ t \geqslant t_0.$$

Next we discuss an application, illustrated by the following example:

EXAMPLE 1 *Let us consider a 3-dimensional system, governed by the state equation $z' = Az$ and the output equation $y = Cz$, with*

$$A = \begin{bmatrix} 9 & 5 & 1 \\ -10 & -3 & 1 \\ 4 & -2 & -3 \end{bmatrix},\ C = \begin{pmatrix} 1 & \frac{3}{2} & 1 \end{pmatrix},\ T = 1.$$

Then

$$e^{tA} = \begin{bmatrix} e^t\left(9t^2+8t+1\right) & te^t\left(9t+5\right) & \frac{1}{2}te^t\left(9t+2\right) \\ -2te^t\left(9t+5\right) & e^t - 18t^2e^t - 4te^t & -te^t\left(9t-1\right) \\ 2te^t\left(9t+2\right) & 2te^t\left(9t-1\right) & e^t\left(9t^2-4t+1\right) \end{bmatrix},$$

$$\mathbf{M} = \frac{1}{16}\begin{bmatrix} 20e^2-68 & 18e^2-78 & 8e^2-44 \\ 18e^2-78 & 18e^2-90 & 9e^2-51 \\ 8e^2-44 & 9e^2-51 & 5e^2-29 \end{bmatrix}.$$

Hence, we have

$$\operatorname{Im}(\mathbf{M}) = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} \right\} \quad \textit{and} \quad Ker(\mathbf{M}) = \left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \right\}.$$

For $z(0) = \begin{pmatrix} 5 & 0 & 2 \end{pmatrix}^{\mathrm{T}}$*; we obtain*

$$y(t) = (7-18t)\,e^t,\ 0 \leqslant t \leqslant 1.$$

Then

$$\begin{aligned} \mathcal{K}^* y &= \int_0^1 \begin{pmatrix} -e^t(3t-1) - \frac{3}{2}e^t(2t-1) - \frac{1}{2}e^t(3t-2) \end{pmatrix}^T (7-18t)\,e^t \mathrm{d}t \\ &= \frac{1}{8}\begin{pmatrix} 58e^2-214 \\ 54e^2-246 \\ 25e^2-139 \end{pmatrix}. \end{aligned}$$

1. *Reconstruction of the visible part of the initial state:*

$$\begin{aligned} X_{\mathrm{mes}} &= \{z_0 \ / \ \mathbf{M}z_0 = K^* y\} \\ &= \left\{ \begin{pmatrix} \frac{1}{2}\beta+4 \\ 2-\beta \\ \beta \end{pmatrix} \ / \ \beta \in \mathbb{R} \right\}, \end{aligned}$$

$$M^+ = \frac{1}{729\left(e^4-6e^2+1\right)} \begin{bmatrix} 64\left(41e^2-221\right) & 64\left(82-19e^2\right) & 32\left(385-79e^2\right) \\ 64\left(82-19e^2\right) & 16\left(53e^2-125\right) & 16\left(91e^2-289\right) \\ 32\left(385-79e^2\right) & 16\left(91e^2-289\right) & 32\left(85e^2-337\right) \end{bmatrix}$$

$$z^\star(0) = M^+ K^* y = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}.$$

2. *Reconstruction of the state, generated by the visible part of the initial state:*

$$\begin{aligned}
z^{\star}(t) &= exp\left(t\begin{bmatrix} 9 & 5 & 1 \\ -10 & -3 & 1 \\ 4 & -2 & -3 \end{bmatrix}\right)\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 2e^{t}\left(27t^{2}+21t+2\right) \\ -2e^{t}\left(54t^{2}+24t-1\right) \\ 12te^{t}\left(9t+1\right) \end{pmatrix}.
\end{aligned}$$

$z^{\star}(t) \notin \mathrm{Im}(\mathbf{M})$ *since*

$$\begin{aligned}
z^{\star}(t)^{T}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} &= \begin{pmatrix} 2e^{t}\left(27t^{2}+21t+2\right) \\ -2e^{t}\left(54t^{2}+24t-1\right) \\ 12te^{t}\left(9t+1\right) \end{pmatrix}^{T}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \\
&= 162te^{t}\left(3t+1\right) \neq 0,\ \forall t > 0.
\end{aligned}$$

3. *The projection of the system state on Im(M) (visible part of the system state):*
The orthogonal projection matrix on $\mathrm{Im}(\mathbf{M})$ *is given by*

$$\begin{aligned}
\mathrm{P}_{\mathrm{Im}(\mathbf{M})} &= \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}^{T} + \frac{1}{18}\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}^{T} \\
&= \frac{1}{9}\begin{bmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{bmatrix}
\end{aligned}$$

and the visible part of the state is given by

$$\begin{aligned}
z^{\mathbf{v}}(t) &= \mathrm{P}_{\mathrm{Im}(\mathbf{M})} z^{\star}(t) \\
&= \frac{1}{9}\begin{bmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{bmatrix}\begin{pmatrix} 2e^{t}\left(27t^{2}+21t+2\right) \\ -2e^{t}\left(54t^{2}+24t-1\right) \\ 12te^{t}\left(9t+1\right) \end{pmatrix} \\
&= \begin{pmatrix} 4e^{t}\left(1+6t\right) \\ 2e^{t}\left(1-6t\right) \\ -24te^{t} \end{pmatrix}.
\end{aligned}$$

Remark 11 *Note that for* $t = t_0$

$$z^{\mathbf{v}}(t_0) = z^{\star}(t_0),$$

but for $t > t_0$, *we have*

$$z^{\mathbf{v}}(t) \neq z^{\star}(t),$$

which is well verified in the previous example.

6.3. Partial reconstruction where A is symmetric with single spectrum

6.3.1. Introductory results

We assume that A is symmetric with single spectrum. This case is quite interesting, since we can find an orthonormal basis $\{w_1, \ldots, w_n\}$ of $\mathbb{R}^n$ formed of eigenvectors of A :

$$Aw_i = \lambda_i w_i,\ \langle w_i, w_j \rangle = \delta_{ij},\ \lambda_i \neq \lambda_j \quad \text{if } i \neq j. \tag{25}$$

These vectors satisfy

$$\begin{aligned} & e^{(t-t_0)A} w_j = e^{(t-t_0)\lambda_j} w_j, \\ & (\mathcal{K} w_j)(t) = C e^{(t-t_0)A} w_j = e^{(t-t_0)\lambda_j} C w_j. \end{aligned} \tag{26}$$

$$\begin{cases} \langle \mathbf{M} w_j, w_k \rangle & = \dfrac{e^{(T-t_0)(\lambda_j+\lambda_k)} - 1}{\lambda_j + \lambda_k} \langle C w_j, C w_k \rangle \\ \langle \mathbf{M} w_j, w_j \rangle & = \dfrac{e^{2(T-t_0)\lambda_j} - 1}{2\lambda_j} \left\| C w_j \right\|^2. \end{cases} \tag{27}$$

Let us denote by m the number (which can be null) of these vectors that verify $Cw_i \neq 0$. Without loss of generality, it can be assumed that

$$Cw_i \neq 0 \quad \text{for all } i \leqslant m \quad \text{and} \quad Cw_i = 0 \quad \text{for all } i > m.$$

Proposition 12 *Let r be the rank of $\mathbf{M}$. We have*

- $\{w_1, \ldots, w_r\}$ *is the basis of* $\operatorname{Im}(\mathbf{M})$ *;*
- $\{w_{r+1}, \ldots, w_n\}$ *is the basis of* $Ker(\mathbf{M})$.

Proof For $i > m$ we have $Cw_i = 0$ and (26) gives $\mathcal{K} w_i = 0$ or $w_i \in \operatorname{Ker}(\mathbf{M})$. This proves that $\{w_{m+1}, \ldots, w_n\} \subseteq \operatorname{Ker}(\mathbf{M})$. Conversely, let us take $u \in \operatorname{Ker}(\mathbf{M}) = \operatorname{Ker}(\mathcal{K})$, which can be decomposed as:

$$u = \sum_{j=1}^{n} u_j w_j.$$

We have $\mathcal{K} u = \sum\limits_{j=1}^{n} u_j \mathcal{K} w_j = \sum\limits_{j=1}^{m} u_j \mathcal{K} w_j = 0$, then

$$\sum_{j=1}^{m} u_j e^{(t-t_0)\lambda_j} C w_j = 0, \quad t \in [t_0, T],$$

since λ_j are pairwise distinct, which gives, for all $j \leqslant m$, $u_j C w_j = 0$, and then $u_j = 0$, $j = 1, \ldots, m$. This gives $u = \sum_{j=m+1}^{n} u_j w_j$, which is in the space $\{w_{m+1}, \ldots, w_n\}$ generated by w_j, $m+1 \leqslant j \leqslant n$. Finally

$$\operatorname{Ker}(\mathbf{M}) = \{w_{m+1}, \ldots, w_n\}.$$

In particular, $\{w_{m+1}, \ldots, w_n\}$ is a basis of $\operatorname{Ker}(\mathbf{M})$ and then $\dim(\operatorname{Ker}(\mathbf{M})) = n - m$; yet $\dim(\operatorname{Ker}(\mathbf{M})) = n - r$, and then $m = r$. On the other hand

$$\begin{aligned} \operatorname{Im}(\mathbf{M}) &= [\operatorname{Ker}(\mathbf{M})]^{\perp} \\ &= [\operatorname{Span}\{w_{r+1}, \ldots, w_n\}]^{\perp} \\ &= \operatorname{Span}\{w_1, \ldots, w_r\}, \end{aligned}$$

which implies that $\{w_1, \ldots, w_r\}$ is a basis of $\operatorname{Im}(\mathbf{M})$. □

The following characterization follows:

PROPOSITION 13 *A subspace H is observable if and only if either $r = n$ or $r < n$ and*

$$\forall h \in H \ : \ \langle h, w_j \rangle = 0 \quad j = r+1, \ldots, n.$$

6.3.2. Partial reconstruction of the visible part of the state

Let

$$z(t_0) = \sum_{j=1}^{n} x_j w_j.$$

Then

$$z(t) = e^{(t-t_0)A} z(t_0) = \sum_{j=1}^{n} e^{(t-t_0)\lambda_j} x_j w_j,$$

and the vectors of X_{mes} are of the form

$$z_0 = \sum_{j=1}^{r} x_j w_j + \sum_{j=r+1}^{n} \eta_j w_j, \quad \eta_j \in \mathbb{R}, \quad j = r+1, \ldots, n.$$

Since the orthogonal projection matrix on $\operatorname{Im}(\mathbf{M})$ is

$$\mathrm{P}_{\operatorname{Im}(\mathbf{M})} = \sum_{j=1}^{r} w_j w_j^{\mathrm{T}},$$

the minimal initial state is given by

$$z^{\star}(t_0) = \mathrm{P}_{\operatorname{Im}(\mathbf{M})} z(t_0) = \sum_{j=1}^{n} x_j \mathrm{P}_{\operatorname{Im}(\mathbf{M})} w_j = \sum_{j=1}^{r} x_j w_j.$$

It satisfies $\mathbf{M}z^{\star}(t_0) = \mathcal{K}^{*}y$, and

$$\sum_{j=1}^{r} x_j \mathbf{M} w_j = \int_{t_0}^{T} e^{(s-t_0)A^{\mathrm{T}}} C^{\mathrm{T}} y(s)\,\mathrm{d}s.$$

The scalar product by w_k, $1 \leqslant k \leqslant r$, gives

$$\begin{aligned}\sum_{j=1}^{r} x_j \langle \mathbf{M} w_j, w_k \rangle &= \left\langle \int_{t_0}^{T} e^{(s-t_0)A^{\mathrm{T}}} C^{\mathrm{T}} y(s)\,\mathrm{d}s, w_k \right\rangle \\ &= \int_{t_0}^{T} \left\langle y(s), Ce^{(s-t_0)A} w_k \right\rangle \mathrm{d}s \\ &= \int_{t_0}^{T} e^{(s-t_0)\lambda_k} \langle y(s), Cw_k \rangle\,\mathrm{d}s,\end{aligned}$$

which leads to

$$\sum_{j=1}^{r} \frac{e^{(T-t_0)(\lambda_j+\lambda_k)} - 1}{\lambda_j + \lambda_k} \langle Cw_j, Cw_k \rangle x_j$$

$$= \int_{t_0}^{T} e^{(s-t_0)\lambda_k} \langle y(s), Cw_k \rangle\,\mathrm{d}s, \quad 1 \leqslant k \leqslant r.$$

The x_j, $1 \leqslant j \leqslant r$, are therefore the solutions in $\mathbb{R}^r$ of the linear system

$$\sum_{j=1}^{r} D_{kj} x_j = d_k\ , \quad 1 \leqslant k \leqslant r,$$

i.e., $Dx = d$ with

$$D_{jk} = D_{kj} = \frac{e^{(T-t_0)(\lambda_j+\lambda_k)} - 1}{\lambda_j + \lambda_k} \langle Cw_j, Cw_k \rangle\,, \tag{28}$$

$$d_k = \int_{t_0}^{T} e^{(s-t_0)\lambda_k} \langle y(s), Cw_k \rangle\,\mathrm{d}s,\ 1 \leqslant j,k \leqslant r. \tag{29}$$

The square matrix $D \in \mathcal{M}_r(\mathbb{R})$ is invertible. Indeed, for $\eta \in \mathbb{R}^r$, we have

$$\langle D\eta, \eta \rangle = \sum_{j,k=1}^{r} D_{jk}\eta_j\eta_k,$$

$$= \sum_{j,k=1}^{r} \eta_j\eta_k \int_{t_0}^{T} e^{(s-t_0)\left(\lambda_j+\lambda_k\right)}\mathrm{d}s \left\langle Cw_j, Cw_k \right\rangle,$$

$$= \int_{t_0}^{T} \left\langle \sum_{j=1}^{r} \eta_j e^{(s-t_0)\lambda_j} Cw_j, \sum_{k=1}^{r} \eta_k e^{(s-t_0)\lambda_k} Cw_k \right\rangle \mathrm{d}s,$$

$$= \int_{t_0}^{T} \left\| \sum_{j=1}^{r} \eta_j e^{(s-t_0)\lambda_j} Cw_j \right\|^2 \mathrm{d}s \geqslant 0,$$

and if $\langle D\eta, \eta \rangle = 0$, we obtain

$$\sum_{j=1}^{r} \eta_j e^{(s-t_0)\lambda_j} Cw_j = 0,\ \forall s \in [t_0, T].$$

The elements of the sequence of functions $s \in [t_0, T] \to e^{(s-t_0)\lambda_j}$ are linearly independent in $\mathrm{L}^2(t_0, T)$, so $\eta_j Cw_j = 0$ for $j = 1, \ldots, r$, since $Cw_j \neq 0$, $\forall j \leqslant r$, then $\eta_j = 0$, $j = 1, \ldots, r$. D is therefore positive definite and, consequently, invertible.

Proposition 14 *The visible part of the initial state is given by*

$$z^\star(t_0) = \sum_{j=1}^{r} x_j w_j,$$

where $x = (x_1, \ldots, x_r)^T$ is the unique solution in $\mathbb{R}^r$ of the linear system

$$Dx = d$$

where the matrix $D = (D_{jk})_{1 \leqslant j,k \leqslant r}$, the coefficients D_{jk} are given by (28), the vector $d = (d_j)_{1 \leqslant j \leqslant r}$, and the coefficients d_j are given by (29).

Once the linear system $Dx = d$ is solved and components $x_1, \ldots, x_r$ of $z^\star(t_0)$ *are* determined, we get

$$\begin{aligned} z^\star(t) &= e^{(t-t_0)A} z^\star(t_0) \\ &= \textstyle\sum_{j=1}^{r} x_j e^{(t-t_0)A} w_j \\ &= \textstyle\sum_{j=1}^{r} e^{(t-t_0)\lambda_j} x_j w_j. \end{aligned}$$

Let us note that $z^\star(t) \in \mathrm{Im}(\mathbf{M})$, hence $z^\star$ coincides with the visible part of the state

$$z^{\mathbf{v}}(t) = z^\star(t), \ \forall t \geqslant t_0.$$

On the other hand, we have

$$\|z(t)\|^2 = \sum_{j=1}^{n} e^{2\lambda_j t} |x_j|^2,$$

and

$$\|z^{\mathbf{v}}(t)\|^2 = \|z^\star(t)\|^2 = \sum_{j=1}^{r} e^{2\lambda_j t} |x_j|^2,$$

which implies that $\|z^{\mathbf{v}}(t)\|^2 \leqslant \|z(t)\|^2$, $\forall t \geqslant t_0$. The state, generated by the minimal initial state has a minimal norm among all states giving the same measure as $z(t)$, and it is in $\mathrm{Im}(\mathbf{M})$. We deduce from this the following proposition:

Theorem 4 *We assume that A is symmetric with single spectrum.*

1. *When the solution $x_1, \ldots, x_r$ of $Dx = d$, is determined, the visible part of the state is given by*

$$z^{\mathbf{v}}(t) = z^\star(t) = \sum_{j=1}^{r} e^{(t-t_0)\lambda_j} x_j w_j, \ \forall t \geqslant t_0.$$

2. *At each moment the visible part of the state has the minimal norm among all the states giving the same measurement:*

$$\|z^{\mathbf{v}}(t)\| \leqslant \left\| e^{(t-t_0)A} z_0 \right\|, \ \forall z_0 \in X_{\text{mes}}.$$

On the other hand, for any observable subspace H, we have

$$\mathrm{P}_H z(t) = \mathrm{P}_H z^\star(t_0) = \mathrm{P}_H z^{\mathbf{v}}(t_0) = \sum_{j=1}^{r} x_j \mathrm{P}_H w_j.$$

Knowledge of the components x_j, $1 \leqslant j \leqslant r$, of the minimal initial state can therefore be used to determine the orthogonal projection of the state on any observable subspace and the orthogonal projection on $\mathrm{Im}(\mathbf{M})$ gives us the visible part of the state.

Example 2 *Let us take*

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \ C = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}.$$

The system generated by the matrices A and C is unobservable, because

$$\mathcal{O}=\begin{bmatrix}1 & -1 & 3\\ -1 & 3 & -5\\ 1 & -1 & 3\end{bmatrix} \text{ and } \operatorname{rg}(\mathcal{O})=2.$$

A is symmetric with a single spectrum

$$w_1=\frac{\sqrt{3}}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}\leftrightarrow\lambda_1=1,$$

$$w_2=\frac{\sqrt{6}}{6}\begin{pmatrix}1\\-2\\1\end{pmatrix}\leftrightarrow\lambda_2=-2,$$

$$w_3=\frac{\sqrt{2}}{2}\begin{pmatrix}-1\\0\\1\end{pmatrix}\leftrightarrow\lambda_3=2.$$

We have $Cw_1\neq 0$, $Cw_2\neq 0$, $Cw_3=0$, then $\operatorname{Im}(M)=\{w_1,w_2\}$ and $Ker(M)=\{w_3\}$. The corresponding semi-group of A can be written down as

$$e^{tA}=\frac{1}{6}\begin{bmatrix}2e^{t}+e^{-2t}+3e^{2t} & 2e^{t}-2e^{-2t} & 2e^{t}+e^{-2t}-3e^{2t}\\ 2e^{t}-2e^{-2t} & 2e^{t}+4e^{-2t} & 2e^{t}-2e^{-2t}\\ 2e^{t}+e^{-2t}-3e^{2t} & 2e^{t}-2e^{-2t} & 2e^{t}+e^{-2t}+3e^{2t}\end{bmatrix}$$

and then

$$M=\frac{1}{18}\begin{bmatrix}\begin{pmatrix}e^{2T}-8e^{-T}\\-2e^{-4T}+9\end{pmatrix} & \begin{pmatrix}4e^{-T}+e^{2T}\\+4e^{-4T}-9\end{pmatrix} & \begin{pmatrix}e^{2T}-8e^{-T}\\-2e^{-4T}+9\end{pmatrix}\\ \begin{pmatrix}4e^{-T}+e^{2T}\\+4e^{-4T}-9\end{pmatrix} & \begin{pmatrix}16e^{-T}+e^{2T}\\-8e^{-4T}-9\end{pmatrix} & \begin{pmatrix}e^{2T}+4e^{-4T}\\+\frac{4}{e^{T}}-9\end{pmatrix}\\ \begin{pmatrix}e^{2T}-8e^{-T}\\-2e^{-4T}+9\end{pmatrix} & \begin{pmatrix}4e^{-T}+e^{2T}\\+4e^{-4T}-9\end{pmatrix} & \begin{pmatrix}e^{2T}-8e^{-T}\\-2e^{-4T}+9\end{pmatrix}\end{bmatrix}$$

and

$$\begin{aligned}\operatorname{Im}(M) &= \left\{\begin{pmatrix}x\\y\\x\end{pmatrix}\ \backslash\ x,y\in\mathbb{R}\right\},\\ &= Span\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\}\end{aligned}$$

and, further

$$
\begin{aligned}
Ker(M) &= \left\{ \begin{pmatrix} x \\ 0 \\ -x \end{pmatrix} \backslash x \in \mathbb{R} \right\}, \\
&= Span \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.
\end{aligned}
$$

Let us take the measure $y(t) = e^t + 4e^{-2t}$, which is the output related to the initial state

$$
z(0) = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}.
$$

Indeed,

$$
\begin{aligned}
& y(t) = Ce^{tA} z(0) = \\
& \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \left[\frac{1}{6} \begin{matrix} 2e^t + e^{-2t} + 3e^{2t} & 2e^t - 2e^{-2t} & 2e^t + e^{-2t} - 3e^{2t} \\ 2e^t - 2e^{-2t} & 2e^t + 4e^{-2t} & 2e^t - 2e^{-2t} \\ 2e^t + e^{-2t} - 3e^{2t} & 2e^t - 2e^{-2t} & 2e^t + e^{-2t} + 3e^{2t} \end{matrix} \right] \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \\
& = e^t + 4e^{-2t}.
\end{aligned}
$$

We have

$$
\begin{aligned}
z(0) &= \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \\
&= \sqrt{3} w_1 + \sqrt{6} w_2 + \sqrt{2} w_3 \\
&= \sqrt{3} \left(\tfrac{\sqrt{3}}{3} \right) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \sqrt{6} \left(\tfrac{\sqrt{6}}{6} \right) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \left(\sqrt{2} \right) \tfrac{\sqrt{2}}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},
\end{aligned}
$$

and then the orthogonal projection of $z(0)$ on $Im(M)$ is given by

$$
P_{Im(M)^*} z(0) = \sqrt{3} \left(\frac{\sqrt{3}}{3} \right) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \sqrt{6} \left(\frac{\sqrt{6}}{6} \right) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.
$$

We have

$$\begin{aligned}
D_{11} &= \int_0^T e^{t(\lambda_1+\lambda_1)}\mathrm{d}t \left\|Cw_1\right\|^2 = \frac{1}{6}\left(e^{2T}-1\right), \\
D_{22} &= \int_0^T e^{t(\lambda_2+\lambda_2)}\mathrm{d}t \left\|Cw_2\right\|^2 = \frac{2}{3}\left(1-e^{-4T}\right), \\
D_{12} &= D_{21} = \int_0^T e^{t(\lambda_1+\lambda_2)}\mathrm{d}t \left\langle Cw_1, Cw_2\right\rangle = \frac{2\sqrt{3}\sqrt{6}}{9}\left(1-e^{-T}\right),
\end{aligned}$$

$$d_1 = \int_0^T e^{\lambda_1 s}\left\langle y\left(s\right), Cw_1\right\rangle \mathrm{d}s = \frac{1}{3}\sqrt{3}\left(\frac{1}{2}e^{2T} - 4e^{-T} + \frac{7}{2}\right)$$

and

$$d_2 = \int_0^T e^{\lambda_2 s}\left\langle y\left(s\right), Cw_2\right\rangle \mathrm{d}s = \frac{2}{3}\sqrt{6}\left(2-e^{-4T}-e^{-T}\right),$$

which gives us the following system

$$\begin{aligned}
&\tfrac{1}{6}\left(e^{2T}-1\right)x_1 + \tfrac{\sqrt{8}}{3}\left(1-e^{-T}\right)x_2 = \tfrac{1}{\sqrt{3}}\left(\tfrac{1}{2}e^{2T}-4e^{-T}+\tfrac{7}{2}\right), \\
&\tfrac{\sqrt{8}}{3}\left(1-e^{-T}\right)x_1 + \tfrac{2}{3}\left(1-e^{-4T}\right)x_2 = \sqrt{\tfrac{8}{3}}\left(2-e^{-4T}-e^{-T}\right).
\end{aligned}$$

The only solution of this system is

$$(x_1, x_2) = \left(\sqrt{3}, \sqrt{6}\right),$$

then

$$\begin{aligned}
z^{\star}(0) &= x_1 w_1 + x_2 w_2 \\
&= \left(\sqrt{3}\right)\tfrac{\sqrt{3}}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \left(\sqrt{6}\right)\tfrac{\sqrt{6}}{6}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}.
\end{aligned}$$

We remark that $z^{\star}$ coincides with the orthogonal projection of $z(0)$ on Im(M).

According to item 1 of Theorem 4, the visible part of the state can be written *down as*

$$
\begin{aligned}
z^{\mathbf{v}}(t) &= z^{\star}(t) \\
&= e^{\lambda_1 t} x_1 w_1 + e^{\lambda_2 t} x_2 w_2 \\
&= e^{t}\left(\sqrt{3}\right)\frac{\sqrt{3}}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-2t}\left(\sqrt{6}\right)\frac{\sqrt{6}}{6}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} e^{t}+e^{-2t} \\ e^{t}-2e^{-2t} \\ e^{t}+e^{-2t} \end{pmatrix}.
\end{aligned}
$$

7. Conclusion and prospects

In this paper, we have considered the partial observability problem for finite dimensional linear systems. In this context, we have developed the notion of an observable subspace. We give some characterizations for the observable subspaces. The problem of the reconstruction of the so called visible part of the state was also discussed.

An important feature of partial observability concept is the possibility of effectively using a dynamical system, even if the system is not fully observable. We can also reconstruct the most important parameters of the system without worrying about the other parameters. These ideas may allow us to carry out the same study for the case of infinite dimensional systems, for the case of distributed parameter systems, or even for the case of semi-linear and non-linear systems.

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