sciendo
Control and Cybernetics
vol. 50 (2021) No. 2
pages: 315-333
DOI: 10.2478/candc-2021-0016

Model of a minimal risk portfolio under hybrid uncertainty*

by

Alexander V. Yazenin and Ilia S. Soldatenko

Tver State University, Zhelyabova 33, Tver, Russia

Abstract: The article is devoted to the development and study of a model of a minimal risk portfolio under conditions of hybrid uncertainty of possibilistic-probabilistic type. In this model, the interaction of fuzzy parameters is described by both the strongest and the weakest triangular norms. The formula for variance of a portfolio is given that allows for estimating its risk. Models of acceptable portfolios are based on the principle of expected possibility or on the basis of fulfilling the restriction on the possibility/necessity and probability of the level of portfolio return that is acceptable to an investor. Equivalent deterministic analogues of the models are constructed and their solution methods are developed. Theorems describing a set of investment opportunities are proven. The obtained results are demonstrated on a model example.

Keywords: minimum risk portfolio, hybrid uncertainty, possibility, necessity, strongest t-norm, weakest t-norm, fuzzy random variable

1. Introduction

The article presents the architecture of some models for optimization problems under conditions of hybrid uncertainty of possibilistic-probabilistic type and some indirect methods for their solving, complementing the results previously obtained by Yazenin (1991, 1997, 2007), Yazenin and Shefova (2010), Yazenin and Soldatenko (2018), as well as Egorova and Yazenin (2018).

In a number of relevant papers, devoted to the problem of portfolio selection, only the situation, when the interaction of fuzzy factors is described by the strongest triangular norm (t-norm) has been studied (see, for example, Xu and Zhou, 2011). In our work, attention is paid to the study of situations, when the interaction of fuzzy model factors is described by both the strongest and

*Submtted: September 2020; Accepted: February 2021. This paper is an amended version of the paper, presented at the BOS/SOR2020 conference in December 2020.

the weakest t-norms, which allows us to assess the range of risk changes and the behavior of a set of acceptable portfolios, that is, to manage more adequately the uncertainty when making investment decisions. In order to remove probabilistic uncertainty from the acceptable portfolio model, the principle based on the expected possibility is used. Uncertainty of possibilistic (fuzzy) type is removed by imposing requirements for the possibility/necessity of fulfilling restrictions on the acceptable level of expected profitability of the portfolio. The relationship between the models of acceptable portfolios of different architectures is established and investigated. Theoretical results and conclusions are confirmed by numerical calculations.

2. The necessary concepts and notations

In the context of works by Nahmias (1979), Yazenin (2016), Feng, Hu and Shu (2001), Dubois and Prade (1988), Nguyen and Walker (1997), Mesiar (1997) and Hong (2001), we introduce a number of definitions and concepts from the theory of possibilities. Let further $(\Gamma,\ \mathrm{P}(\Gamma),\ \tau)$ and $(\Omega,\ \mathrm{B},\ \mathbf{P})$ be the possibility and probability spaces, respectively, in which Ω is the space of elementary events $\omega \in \Omega$, Γ is the model space with elements $\gamma \in \Gamma$, B is the σ-algebra of events, $\mathrm{P}(\Gamma)$ is the set of all subsets of Γ, $\tau \in \{\pi,\ \nu\}$, π and ν are measures of possibility and necessity, respectively, and $\mathbf{P}$ is the probability measure; E^n is n-dimensional Euclidean space, $E^n_+ = \{x \in E^n : x \geq \emptyset\}$.

We define a fuzzy random variable and its distribution as follows (see Yazenin and Wagenknecht, 1996, Yazenin, 2016).

Definition 1 *Fuzzy random variable $Y(\omega,\gamma)$ is a real function $Y{:}\Omega\times\Gamma\to\mathrm{E}^1$ σ-measurable for each fixed γ, and*

$$\mu_Y(\omega,t)=\pi\{\gamma\in\Gamma{:}Y(\omega,\gamma)=t\},\ \forall t\in\mathrm{E}^1$$

is called its distribution function.

From Definition 1 it follows that the distribution function of a fuzzy random variable depends on a random parameter, i.e. it is a random function.

Definition 2 *Let $Y(\omega,\ \gamma)$ be a fuzzy random variable. Its expected value $\mathbf{E}[Y]$ is a fuzzy value that has a possibility distribution function*

$$\mu_{\mathbf{E}[Y]}(t)=\pi\{\gamma\in\Gamma{:}\ \mathbf{E}[Y(\omega,\gamma)]=t\},\ \forall t\in E^1$$

where $\mathbf{E}$ is the mathematical expectation operator

$$\mathbf{E}[Y(\omega,\gamma)]=\int_\Omega Y(\omega,\gamma)\,\mathbf{P}(d\omega).$$

The distribution function of the expected value of a fuzzy random variable is no longer dependent on the random parameter and therefore is fuzzy.

We define, following Feng, Hu and Shu (2001), the second-order moments. Let X and Y be fuzzy random variables.

Definition 3 *The covariance of fuzzy random variables X and Y is defined as follows:*

$$cov\left(X,Y\right)=\frac{1}{2}\int_0^1\left(cov\left(X_\omega^-\left(\alpha\right),Y_\omega^-\left(\alpha\right)\right)+cov\left(X_\omega^+\left(\alpha\right),Y_\omega^+\left(\alpha\right)\right)\right)d\alpha,$$

where $X_\omega^-\left(\alpha\right), Y_\omega^-\left(\alpha\right), X_\omega^+\left(\alpha\right), Y_\omega^+\left(\alpha\right)$ are the boundaries of α-level sets of fuzzy variables X_ω, Y_ω, respectively.

Definition 4 *The variance of a fuzzy random variable Y is*

$$\mathbf{D}\left[Y\right]=cov\left(Y,Y\right). \tag{1}$$

The mathematical expectation, variance, and covariance of fuzzy random variables, determined in accordance with the considered approach, inherit the main properties of similar characteristics of the real random variables.

An LR-type distribution is often used for modeling fuzzy variables (see, for example, Dubois and Prade, 1988). This distribution, for a fuzzy variable $Y\left(\gamma\right)$ is usually written as $\mu_Y\left(t\right)=\left[\underline{m},\overline{m},\underline{d},\overline{d}\right]_{LR}$. Further on we will simply write $Y\left(\gamma\right)=\left[\underline{m},\overline{m},\underline{d},\overline{d}\right]_{LR}$. Here $\underline{m},\overline{m}$ are left and right boundaries of the tolerance interval, $\underline{d},\overline{d}$ are the coefficients of fuzziness, while $\underline{m}\leq\overline{m}$ and $\underline{d}>0$, $\overline{d}>0$, $L(t)$ and $R\left(t\right)$ are left and right shape functions for the possibility distribution.

We will use triangular norms (t-norms) to aggregate fuzzy information. These norms generalize the "min" operation, inherent in operations on fuzzy sets and fuzzy variables (see Nguyen and Walker, 1997). The following t-norms are of particular interest:

$$T_M\left(x,y\right)=\min\left\{x,y\right\} \textit{ and } T_W\left(x,y\right)=\begin{cases}\min\left\{x,y\right\}, \; if \max\left\{x,y\right\}=1,\\ 0, \; otherwise,\end{cases}$$

T_M is called the strongest, and T_W is called the weakest t-norm, since for any arbitrary t-norm T and $\forall x,y\in[0,1]$, the following inequality holds (see, for example, Nguyen and Walker, 1997):

$$T_W\left(x,y\right)\leq T\left(x,y\right)\leq T_M\left(x,y\right).$$

3. Mathematical models of a minimal risk portfolio under hybrid uncertainty

3.1. Portfolio return under hybrid uncertainty of possibilistic -probabilistic type

Under conditions of hybrid uncertainty of possibilistic-probabilistic type, the return on an investment portfolio can be represented by a fuzzy random function

$$R_p\left(w,\omega,\gamma\right)=\sum_{i=1}^{n} R_i\left(\omega,\gamma\right)w_i, \tag{2}$$

which is a linear function of equity shares $w = (w_1, \ldots, w_n)$ in the portfolio. Here $R_i\left(\omega,\gamma\right)$ are fuzzy random variables that model the returns of individual financial assets with the help of shift-scale representation (see Yazenin, 2016):

$$R_i\left(\omega,\gamma\right)=a_i\left(\omega\right)+\sigma_i\left(\omega\right)Z_i\left(\gamma\right). \tag{3}$$

Further, we assume that fuzzy variables $Z_i(\gamma) = \left[\underline{m}_i, \overline{m}_i, \underline{d}_i, \overline{d}_i\right]_{LR}$ in representation (3) are mutually T-related, where $T \in \{T_M, T_W\}$, and $a_i\left(\omega\right)$, $\sigma_i(\omega)$ are shift and scale coefficients – random variables defined on a probability space (Ω, B, $\mathbf{P}$), with $\underline{E}[\sigma_i\left(\omega\right)] \geq 0$. Then, the possibility distribution of the portfolio return (2) takes on the following form

$$R_p^T(w,\omega,\gamma)=\left[\underline{m}_{R_p}\left(w,\omega\right),\overline{m}_{R_p}\left(w,\omega\right),\underline{d}_{R_p^T}\left(w,\omega\right),\overline{d}_{R_p^T}\left(w,\omega\right)\right]_{LR}, \tag{4}$$

where

$$\underline{m}_{R_p}\left(w,\omega\right)=\sum\nolimits_{i=1}^{\mathrm{n}}\left(a_i\left(\omega\right)+\sigma_i\left(\omega\right)\underline{m}_i\right)w_i,$$
$$\overline{m}_{R_p}\left(w,\omega\right)=\sum\nolimits_{\mathrm{i}=1}^{\mathrm{n}}\left(a_i\left(\omega\right)+\sigma_i\left(\omega\right)\overline{m}_i\right)w_i$$

and the coefficients of fuzziness take on the form depending on the type of T:

$$\underline{d}_{R_p^M}\left(w,\omega\right)=\sum\nolimits_{i=1}^{n}\sigma_i\left(\omega\right)\underline{d}_i w_i,\;\; \overline{d}_{R_p^M}\left(w,\omega\right)=\sum\nolimits_{i=1}^{n}\sigma_i\left(\omega\right)\overline{d}_i w_i$$

when $T = T_M$, and

$$\underline{d}_{R_p^W}\left(w,\omega\right)=\max_{i=1,\ldots,n}\left\{\sigma_i\left(\omega\right)\underline{d}_i w_i\right\},\;\; \overline{d}_{R_p^W}\left(w,\omega\right)=\max_{i=1,\ldots,n}\left\{\sigma_i\left(\omega\right)\overline{d}_i w_i\right\}$$

in case of $T = T_W$. Further, we will denote $R_p^T(w,\omega,\gamma)$ as $R_p^M(w,\omega,\gamma)$ when $T = T_M$ and $R_p^W(w,\omega,\gamma)$ when $T = T_W$.

To remove the uncertainty of probabilistic type in accordance with the approach of Yazenin (2007) it is necessary to identify the possibility distribution of the mathematical expectation of the function $R_p^T(w,\omega,\gamma)$, that is, to calculate its parameters. The expected return in portfolio models is a fuzzy value for a fixed w. This follows from the results of the theorems, provided below (see, for example, Yazenin, 2016; Yazenin and Soldatenko, 2018).

Theorem 1 *Let $T{=}T_M$. Then, the expected portfolio return $\hat{R}_p^M(w,\gamma)$ is characterized by the possibility distribution function*

$$\hat{R}_p^M(w,\gamma) = \mathbf{E}\left[R_p^M(w,\omega,\gamma)\right] = \left[\underline{m}_{\hat{R}_p}(w), \overline{m}_{\hat{R}_p}(w), \underline{d}_{\hat{R}_p^M}(w), \overline{d}_{\hat{R}_p^M}(w)\right]_{LR}$$

where

$$\underline{m}_{\hat{R}_p}(w) = \sum\nolimits_{i=1}^{n} (\hat{a}_i + \hat{\sigma}_i \underline{m}_i) w_i, \quad \overline{m}_{\hat{R}_p}(w) = \sum\nolimits_{i=1}^{n} (\hat{a}_i + \hat{\sigma}_i \overline{m}_i) w_i$$

$$\underline{d}_{\hat{R}_p^M}(w) = \sum\nolimits_{i=1}^{n} \hat{\sigma}_i \underline{d}_i w_i,\ \overline{d}_{\hat{R}_p^M}(w) = \sum\nolimits_{i=1}^{n} \hat{\sigma}_i \overline{d}_i w_i,\ \hat{a}_i = \mathbf{E}\left[a_i(\omega)\right],\ \hat{\sigma}_i = \mathbf{E}\left[\sigma_i(\omega).\right]$$

Theorem 2 *Let $T{=}T_W$. Then, the expected portfolio return $\hat{R}_p^W(w,\gamma)$ is characterized by the possibility distribution function*

$$\hat{R}_p^W(w,\gamma) = \mathbf{E}\left[R_p^W(w,\omega,\gamma)\right] = \left[\underline{m}_{\hat{R}_p}(w), \overline{m}_{\hat{R}_p}(w),\ \underline{d}_{\hat{R}_p^W}(w),\ \overline{d}_{\hat{R}_p^W}(w)\right]_{LR}$$

where

$$\underline{d}_{\hat{R}_p^W}(w) = \mathbf{E}\left[\max_{\{i=1,\ldots,n\}} \{\sigma_i(\omega)\underline{d}_i w_i\}\right],\ \ \overline{d}_{\hat{R}_p^W}(w) = \mathbf{E}\left[\max_{\{i=1,\ldots,n\}} \{\sigma_i(\omega)\overline{d}_i w_i\}\right].$$

Remark 1 *The functions $\underline{d}_{\hat{R}_p^W}(w)$ and $\overline{d}_{\hat{R}_p^W}(w)$ can be calculated explicitly only for simple probabilities distributions of random components $a_i(\omega)$ and $\sigma_i(\omega)$. This is due to a large amount of calculations. In order to reduce the amount of calculations, we can use stochastic optimization methods, in particular the stochastic quasi-gradient method (see, for example, Ermolyev, 1976, and Egorova and Yazenin, 2017).*

3.2. Models of acceptable portfolios under hybrid uncertainty of possibilistic-probabilistic type

In accordance with the classical Markowitz (1952) approach, we need to construct a portfolio risk function in the minimal risk portfolio model. The expected return or portfolio return can be entered into a system of restrictions. Since the expected return on a portfolio in the case of fuzzy random data is a fuzzy value, then, in order to remove the uncertainty of possibilistic type in a system of restrictions that defines the set of acceptable portfolios, one can introduce a restriction on possibility/necessity of the level of expected return acceptable to an investor. Then, the generalized Markowitz model of acceptable portfolios can be represented as

$$F_p^{\tau\mathbf{E}}(\mathrm{w}) = \left\{ \begin{array}{c} \tau\left\{\hat{R}_p^T(\mathrm{w},\gamma)\ \mathrm{R}\ m_d\right\} \geq \alpha, \\ \sum_{i=1}^{n} \mathrm{w}_i = 1, \\ \mathrm{w} \in \mathrm{E}_+^n, \end{array} \right.$$

where $\hat{R}_p^T(w,\gamma)$ is the expected return, R – crisp relation $\{\geq, =\}$; $\alpha \in (0, 1]$, m_d – level of profitability, acceptable to an investor, $T \in \{T_M, T_W\}$.

The following theorems allow us to construct the equivalent deterministic analogues of acceptable portfolio models (see, for example, Yazenin, 2016; Yazenin and Soldatenko, 2018).

Theorem 3 *Let in the model constraint $F_p^{\tau\mathbf{E}}$ $\tau='\pi'$, R = '$\geq$'. Then, the equivalent deterministic model of acceptable portfolios has the form:*

$$F_p^{\pi\mathbf{E}}(w) = \begin{cases} \sum_{i=1}^n (\hat{a}_i+\hat{\sigma}_i\overline{m}_i)\, w_i+\overline{d}_{\hat{R}_p^T}(w) * R^{-1}(\alpha) \geq m_d, \\ \sum_{i=1}^n w_i = 1, \\ w \in E_+^n. \end{cases}$$

Theorem 4 *Let in the model of acceptable portfolios $F_p^{\tau\mathbf{E}}\tau$ = 'ν', R = '$\geq$'. Then, the equivalent deterministic model of acceptable portfolios takes the form:*

$$F_p^{\nu\mathbf{E}}(w) = \begin{cases} \sum_{i=1}^n (\hat{a}_i+\hat{\sigma}_i\underline{m}_i)\, w_i-\underline{d}_{\hat{R}_p^T}(w) * L^{-1}(1-\alpha) \geq m_d, \\ \sum_{i=1}^n w_i = 1, \\ w \in E_+^n. \end{cases}$$

From Theorems 3 and 4 we get

Corollary 1 $F_p^{\nu\mathbf{E}}(w) \subseteq F_p^{\pi\mathbf{E}}(w)$.

Proof Since $\overline{m}_i \geq \underline{m}_i$ and $w \in \mathrm{E}_+^n$, then

$$\sum_{i=1}^n (\hat{a}_i+\hat{\sigma}_i\overline{m}_i)\, w_i \geq \sum_{i=1}^n (\hat{a}_i+\hat{\sigma}_i\underline{m}_i)\, w_i.$$

Based on the assumptions made earlier, $[\sigma_i(\omega)] \geq 0$, $i=1,\ldots,n$, therefore $\overline{d}_{\hat{R}_p^T}(w)$, $\underline{d}_{\hat{R}_p^T}(w) \geq 0$. Considering that $R^{-1}(\alpha)$, $L^{-1}(1-\alpha) > 0$, we obtain

$$\sum_{i=1}^n (\hat{a}_i+\hat{\sigma}_i\overline{m}_i)\, w_i+\overline{d}_{\hat{R}_p^T}(w) * R^{-1}(\alpha) \geq$$
$$\sum_{i=1}^n (\hat{a}_i+\hat{\sigma}_i\underline{m}_i)\, w_i-\underline{d}_{\hat{R}_p^T}(w) * L^{-1}(1-\alpha)$$

for both the strongest and the weakest t-norms. Therefore, we get the statement of the corollary. □

In the case when the portfolio return (2) is included in the system of restrictions, the hybrid uncertainty can be removed by imposing a limit on the possibility/necessity and probability of an acceptable level of return. Formally, the mathematical model of such a constraint can be written as:

$$F_p^{\tau\mathbf{P}}(w) = \begin{cases} \tau\{\mathbf{P}\{R_p(w,\omega,\gamma)\ R\, m_d\} \geq p_0\} \geq \alpha_0, \\ \sum_{i=1}^n w_i = 1, \\ w \in E_+^n, \end{cases}$$

where $\mathbf{P}$ is the probability measure, and $p_0 \in (0, 1]$ is the probability level.

For further analysis we will need the following notations and concepts. We denote by $t = (t_1, \ldots, t_n)$ – a vector whose components are possible values of fuzzy variables $Z_1(\gamma), \ldots, Z_n(\gamma)$, respectively. For the strongest t-norm with the possibility of $\mu_{Z_i}(t_i)$, the return of the i-th financial asset is a random variable

$$Z_i^{t_i}(w) = a_i(\omega) + \sigma_i(\omega)\, t_i,$$

and

$$R_p^t(w,\omega) = \sum_{i=1}^{n} (a_i(\omega) + \sigma_i(\omega)\, t_i)\, w_i$$

is the return on the portfolio with the possibility of $\mu_p(t) = \min_{1 \le i \le n} \{\mu_{Z_i}(t_i)\}$.

Then, following Yazenin (2016), with the possibility of $\mu_p(t)$, the expected return and risk of the portfolio are determined by the formulas

$$m_{R_p}(w,t) = \mathbf{E}\left[R_p^t(w,\omega)\right] = \sum\nolimits_{i=1}^{n} (\hat{a}_i + \hat{\sigma}_i t_i)\, w_i$$

and

$$d_{R_p}(w,\, t) = \mathbf{E}\left[\left(R_p^t(w,\omega) - m_{R_p}(w,\, t)\right)^2\right]$$

respectively.

Using standard transformations we obtain the following formula for the variance with the possibility of $\mu_p(t)$ (see, for example, Yazenin, 2016):

$$d_{R_p}(w,t) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}(t_i,t_j) w_i w_j$$

in which

$$c_{ij}(t_i,t_j) = C_{a_i a_j} + C_{a_j \sigma_i} t_j + C_{\sigma_j a_i} t_i + C_{\sigma_i \sigma_j} t_i t_j,$$

$$C_{a_i a_j} = cov(a_i, a_j),\, C_{a_j\sigma_i} = cov(a_j, \sigma_i),\, C_{\sigma_j a_i} = cov(\sigma_j, a_i),\, C_{\sigma_i\sigma_j} = cov(\sigma_i, \sigma_j).$$

Function $d_{R_p}(w,\, t)$ has the properties that are due to the properties of the covariance matrix $\mathbf{C}$ with elements $c_{ij}(t_i, t_j)$:

- $d_{R_p}(w,\, t)$ is a convex function on w for a fixed t;
- for any vectors w and t the function $d_{R_p}(w,\, t)$ is nonnegative;
- $d_{R_p}(w,\, t)$ is a convex function on t for a fixed w.

The function $d_{R_p}(w,\, t)$ can be written with the help of matrix $\mathbf{C}$ as

$$d_{R_p}(w,\, t) = (\mathbf{C}\ w,\, w).$$

We proceed to the construction of an equivalent deterministic analogue of the model $F_p^{\tau\mathbf{P}}(w)$. We prove the following Lemma, based on the technique developed by Yazenin and Shefova (2010) and Yazenin and Soldatenko (2019).

Lemma 1 *Let in the constraint model $F_p^{\tau\mathbf{P}}(w)$ random parameters be normally distributed: $a_i(\omega)\in N_p\left(\hat{a}_i,\hat{d}_{a_i}\right),\sigma_i(\omega)\in N_p\left(\hat{\sigma}_i,\hat{d}_{\sigma_i}\right)$, $i=1,\dots,n$; fuzzy parameters $Z_1(\gamma),\dots,Z_N(\gamma)$ be T_M-related, $\mu_p(t)\geq\alpha_0$. Then, with the possibility of $\mu_p(t)$, the system of restrictions $F_p^{\tau\mathbf{P}}(w)$ is equivalent to the system*

$$F_p^{\mu\mathbf{P}}(w)=\left\{\begin{array}{c} m_{R_p}(w,t)+\beta_0\sqrt{d_{R_p}(w,t)}\geq m_d,\\ \sum_{i=1}^n w_i=1,\\ w\in E_+^n,\end{array}\right.$$

where β_0 – is a solution to the equation $\mathrm{F}_0^1(x)=1-p_0$, and $\mathrm{F}_0^1(x)$ – is a function of the standard normal probability distribution.

Proof It is clear that with the possibility of $\mu_p(t)$

$$R_p(w,\omega,t)\sim N_p(m_{R_p}(w,t),\, d_{R_p}(w,t)),$$

N_p is a class of normal probabilities distributions. Then, with the possibility of $\mu_p(t)$ we have:

$$\mathbf{P}\left\{R_p^t(w,\omega)\geq m_d\right\}=\mathbf{P}\left\{\frac{R_p^t(w,\omega)-m_{R_p}(w,t)}{\sqrt{d_{R_p}(w,t)}}\geq\frac{m_d-m_{R_p}(w,t)}{\sqrt{d_{R_p}(w,t)}}\right\}=$$

$$=1-F_0^1\left(\frac{m_d-m_{R_p}(w,t)}{\sqrt{d_{R_p}(w,t)}}\right)\geq p_0\rightarrow\frac{m_d-m_{R_p}(w,t)}{\sqrt{d_{R_p}(w,t)}}\leq\beta_0.$$

Hence, as a result, we get the inequality indicated in the statement of the theorem:

$$m_{R_p}(\mathrm{w},t)+\beta_0\sqrt{d_{R_p}(\mathrm{w},t)}\geq m_d. \qquad \square$$

Remark 2 *With $p_0>0.5$, the solution of equation $F_0^1(x)=1-p_0$ is a negative number $\beta_0<0$. Therefore, in this case, the set of acceptable portfolios, defined by the system $F_p^{\mu\mathbf{P}}(w)$, will be convex, since the function $m_{R_p}(w,t)+\beta_0\sqrt{d_{R_p}(w,t)}$ is concave.*

Remark 3 *With $p_0=0.5$, the solution of equation is $\beta_0=0$. Hence, the system $F_p^{\mu\mathbf{P}}(w)$ takes on the form*

$$F_p^{\mu\mathbf{P}}(w)=\left\{\begin{array}{c} \sum_{i=1}^n(\hat{a}_i+\hat{\sigma}_i t_i)\,w_i\geq m_d,\\ \sum_{i=1}^n w_i=1,\\ w\in E_+^n.\end{array}\right.$$

The Lemma proven above allows us to prove a theorem with the help of which an equivalent deterministic analogue of the system $F_p^{\tau\mathbf{P}}(w)$ can be constructed.

Theorem 5 *Let in the constraint model $F_p^{\tau\mathbf{P}}(w)$ random parameters be normally distributed: $a_i(\omega)\in N_p\left(\hat{a}_i,\hat{d}_{a_i}\right)$, $\sigma_i(\omega)\in N_p\left(\hat{\sigma}_i,\hat{d}_{\sigma_i}\right)$, $i=1,\ldots,n$; fuzzy parameters $Z_1(\gamma),\ldots,Z_N(\gamma)$ be T_M-related, $\tau='\pi'$. Then, the system of restrictions $F_p^{\tau\mathbf{P}}(w)$ is equivalent to the system*

$$F_p^{\pi\mathbf{P}}(w)=\begin{cases} m_{R_p}(w,t^+)+\beta_0\sqrt{d_{R_p}(w,t^+)}\geq m_d,\\ \sum_{i=1}^n w_i=1,\\ w\in E_+^n,\end{cases}$$

where

$$m_{R_p}\left(w,t^+\right)=\sum_{i=1}^n\left(\hat{a}_i+\hat{\sigma}_i t_i^+\right)w_i,$$

$$d_{R_p}\left(w,t^+\right)=\sum_{i=1}^n\sum_{j=1}^n c_{ij}(t_i^+,t_j^+)w_iw_j,$$

and t_i^+,t_j^+ are right borders of α_0-level sets of fuzzy variables $Z_i(\gamma),Z_j(\gamma)$, respectively.

Proof We introduce the following notation in order to simplify the formulas

$$G(w,t_1,\ldots,t_n)=m_{R_p}(w,t)+\beta_0\sqrt{d_{R_p}(w,t)}.$$

Then the system of restrictions $F_p^{\mu\mathbf{P}}(w)$ can be rewritten in the form

$$F_p^{\mu\mathbf{P}}(w)=\begin{cases} G(w,t_1,\ldots,t_n)\geq m_d,\\ \sum_{i=1}^n w_i=1,\\ w\in E_+^n,\end{cases}$$

and the system of restrictions $F_p^{\pi\mathbf{P}}(w)$, based on Lemma 1, in the form

$$F_p^{\pi\mathbf{P}}(w)=\begin{cases} \pi\{\gamma\in\Gamma: G(w,Z_1(\gamma),\ldots,Z_n(\gamma))\geq m_d\}\geq\alpha_0,\\ \sum_{i=1}^n w_i=1,\\ w\in E_+^n.\end{cases}$$

Let us reduce the restriction on possibility from the system $F_p^{\pi\mathbf{P}}(w)$ to the equivalent deterministic one. We have (based on the properties of the measure of possibility and the definition of the distribution function, see Yazenin, 2016):

$$\pi\{\gamma\in\Gamma: G(w,Z_1(\gamma),\ldots,Z_n(\gamma))\geq m_d\}=$$

$$=\pi\left\{\bigcup_{y\geq m_d}(\gamma\in\Gamma: G(w,Z_1(\gamma),\ldots,Z_n(\gamma))=y)\right\}=$$

$$=\sup_{y\geq m_d}\pi\{\gamma\in\Gamma: G(w,Z_1(\gamma),\ldots,Z_n(\gamma))=y\}=$$

$$= \sup_{y \geq m_d} \pi \left\{ \bigcup_{\substack{(u_1, \ldots, u_n): \\ G(w, u_1, \ldots, u_n) = y}} (\gamma \in \Gamma : Z_1(\gamma) = u_1, \ldots, Z_n(\gamma) = u_n) \right\} =$$

$$= \sup_{y \geq m_d} \sup_{\substack{(u_1, \ldots, u_n): \\ G(w, u_1, \ldots, u_n) = y}} = y\pi \{\gamma \in \Gamma : Z_1(\gamma) = u_1, \ldots, Z_n(\gamma) = u_n\}$$

$$= \sup_{y \geq m_d} \sup_{\substack{(u_1, \ldots, u_n): \\ G(w, u_1, \ldots, u_n) = y}} = y\mu_{Z_1}, \ldots, Z_n(u_1, \ldots, u_n) =$$

$$= \sup_{y \geq m_d} \sup_{\substack{(u_1, \ldots, u_n): \\ G(w, u_1, \ldots, u_n) = y}} \min_{1 \leq i \leq n} = y\{\mu_{Z_i}(u_i)\} \geq \alpha_0.$$

From $\min_{1 \geq i \geq n} \{\mu_{Z_i}(u_i)\} \geq \alpha_0$ it follows that $\mu_{Z_i}(u_i) \geq \alpha_0, i = 1, \ldots, n$. Since the function $G(w, t_1, \ldots, t_n)$ is continuous on the parameters $u_1, \ldots, u_n$, and α_0-levels of fuzzy variables $Z_1(\gamma), \ldots, Z_n(\gamma)$ are bounded segments of the number line (i.e., compact), then the corresponding supremums are reached. Therefore, the equivalent deterministic constraint takes the form:

$$G\left(w, t_1^+, \ldots, t_n^+\right) \geq m_d$$

or, by virtue of the accepted notations:

$$m_{R_p}\left(w, t^+\right) + \beta_0 \sqrt{d_{R_p}(w, t^+)} \geq m_d. \qquad \square$$

Remark 4 *With $p_0 = 0.5$ and $\tau =' \pi'$ the system of restrictions $F_p^{\pi \mathbf{P}}(w)$ takes on the form:*

$$F_p^{\pi \mathbf{P}}(w) = \begin{cases} \sum_{i=1}^n (\hat{a}_i + \hat{\sigma}_i t_i) w_i \geq m_d, \\ \sum_{i=1}^n w_i = 1, \\ w \in E_+^n. \end{cases}$$

The following theorem establishes a connection between the constraint models $F_p^{\tau \mathbf{E}}(w)$ and $F_p^{\tau \mathbf{P}}(w)$.

Theorem 6 *Let in the constraint models $F_p^{\tau \mathbf{E}}(w)$ and $F_p^{\tau \mathbf{P}}(w)$ random parameters be normally distributed: $a_i(\omega) \in \mathrm{N}_p\left(\hat{a}_i, \hat{d}_{a_i}\right)$, $\sigma_i(\omega) \in \mathrm{N}_p\left(\hat{\sigma}_i, \hat{d}_{\sigma_i}\right)$, $i = 1, \ldots, n$, fuzzy parameters $Z_1(\gamma), \ldots, Z_N(\gamma)$ be T_M-related, $\tau =' \pi'$. Then, if in the model $F_p^{\tau \mathbf{P}}(w)$ the probability level $p_0 = 0.5$, then $F_p^{\pi \mathbf{E}}(w) = F_p^{\pi \mathbf{P}}(w)$.*

Proof Given that $t_i^+ = \overline{m}_i + \overline{d}_i R^{-1}(\alpha)$, we get the following:

$$\sum\nolimits_{i=1}^{n} \left(\hat{a}_i + \hat{\sigma}_i t_i^+\right) w_i = \sum\nolimits_{i=1}^{n} \left(\hat{a}_i + \hat{\sigma}_i (\overline{m}_i + \overline{d}_i R^{-1}(\alpha))\right) w_i =$$
$$= \sum\nolimits_{i=1}^{n} \left(\hat{a}_i + \hat{\sigma}_i \overline{m}_i\right) w_i + \sum\nolimits_{i=1}^{n} \hat{\sigma}_i \overline{d}_i R^{-1}(\alpha) w_i =$$
$$= \sum\nolimits_{i=1}^{n} \left(\hat{a}_i + \hat{\sigma}_i \overline{m}_i\right) w_i + \overline{d}_{\hat{R}_p^T}(w) R^{-1}(\alpha).$$

That is what we needed to prove. □

Remark 5 *Theorem 6 determines the conditions, under which the second-order moments in the $F_p^{\pi \mathbf{P}}(w)$ model do not affect the formation of the set of acceptable portfolios, which leads to the coincidence of the sets of acceptable portfolios, defined by the $F_p^{\pi \mathbf{E}}(w)$ and $F_p^{\pi \mathbf{P}}(w)$ models.*

3.3. Assessment of portfolio risk with hybrid uncertainty

In accordance with the indicated approach to determining the second-order moments, we can determine the variance of the portfolio to assess its risk. We need to obtain the formulas (variances) for the case of both the strongest and the weakest t-norms.

With $T = T_M$ formula (1) takes the form

$$D_p^M(w) = \frac{1}{2} \int_0^1 \left(\mathbf{D}\left[R_p^{M-}(w,\omega,\alpha)\right] + \mathbf{D}\left[R_p^{M+}(w,\omega,\alpha)\right]\right) d\alpha,$$

where $R_p^{M-}(w,\omega,\alpha)$ and $R_p^{M+}(w,\omega,\alpha)$ are the left and right boundaries of α-level set of fuzzy random variable $R_p^M(w,\omega,\alpha)$:

$$R_p^{M-}(w,\omega,\alpha) = \sum\nolimits_{i=1}^{n} \left(a_i(\omega) + \sigma_i(\omega) \underline{m}_i\right) w_i - \sum\nolimits_{i=1}^{n} \sigma_i(\omega) \underline{d}_i w_i * L^{-1}(\alpha).$$

$$R_p^{M+}(w,\omega,\alpha) = \sum\nolimits_{i=1}^{n} \left(a_i(\omega) + \sigma_i(\omega) \overline{m}_i\right) w_i + \sum\nolimits_{i=1}^{n} \sigma_i(\omega) \overline{d}_i w_i * R^{-1}(\alpha).$$

If all random parameters of distributions are independent, then after standard transformations we get the formula for the variance:

$$D_p^M(w) = \sum_{i=1}^{n} \Bigg(\mathbf{D}\left[a_i(\omega)\right] + \frac{1}{2}\mathbf{D}\left[\sigma_i(\omega)\right] \Bigg(\underline{m}_i^2 + \overline{m}_i^2 + 2\overline{m}_i \overline{d}_i \int_0^1 R^{-1}(\alpha)\, d\alpha - 2\underline{m}_i \underline{d}_i$$
$$\int_0^1 L^{-1}(\alpha)\, d\alpha + \overline{d}_i^2 \int_0^1 \left(R^{-1}(\alpha)\right)^2 d\alpha + \underline{d}_i^2 \int_0^1 \left(L^{-1}(\alpha)\right)^2 d\alpha\Bigg)\Bigg) w_i^2.$$

Note that if in fuzzy random variables the fuzzy components are given by LR-type fuzzy numbers with the same left and right shapes and coefficients of fuzziness, i.e. $L(\alpha) = R(\alpha) = S(\alpha), \forall\alpha$ and $\overline{m}_i = \underline{m}_i = m_i$, $\overline{d}_i = \underline{d}_i = d_i$, $i = 1, \ldots, n$, then the variance formula can be simplified:

$$D_p^M(w) = \sum_{i=1}^{n} \left(\mathbf{D}[a_i(\omega)] + \mathbf{D}[\sigma_i(\omega)] \left(m_i^2 + d_i^2 \int_0^1 \left(S^{-1}(\alpha)\right)^2 d\alpha \right) \right) w_i^2.$$

Example 1 In case when shift and scale coefficients $a_i(\omega)$, $\sigma_i(\omega)$ are uniformly distributed over the segment [0,1] and independent, and the shape function $S(t) = \max\{0, 1-t\}$, $t \geq 0$, we obtain the following formula for the variance:

$$D_p^M(w) = \frac{1}{12} \sum_{i=1}^{n} \left(m_i^2 + \frac{1}{3} d_i^2 + 1 \right) w_i^2.$$

We now define the variance for the t-norm T_W. To do this, we will again use formula (1) to find the covariance of two fuzzy random variables. For the weakest t-norm, the formula for finding the variance takes the form:

$$D_p^W(w) = \frac{1}{2} \int_0^1 \left(\mathbf{D}\left[R_p^{W-}(w,\omega,\alpha)\right] + \mathbf{D}\left[R_p^{W+}(w,\omega,\alpha)\right] \right) d\alpha,$$

where $R_p^{W-}(w,\omega,\alpha)$ and $R_p^{W+}(w,\omega,\alpha)$ are respectively left and right boundaries of α-level set of portfolio return – fuzzy random variable $R_p^W(w,\omega,\gamma)$:

$$R_p^{W-}(w,\omega,\alpha) = \sum\nolimits_{i=1}^{n} \left(a_i(\omega) + \sigma_i(\omega)\underline{m}_i\right) w_i - \max_{i=1,\ldots,n} \left\{\sigma_i(\omega)\,\underline{d}_i w_i\right\} * L^{-1}(\alpha)$$

$$R_p^{W+}(w,\omega,\alpha) = \sum\nolimits_{i=1}^{n} \left(a_i(\omega) + \sigma_i(\omega)\overline{m}_i\right) w_i + \max_{i=1,\ldots,n} \left\{\sigma_i(\omega)\,\bar{d}_i w_i\right\} * R^{-1}(\alpha).$$

After appropriate transformations, the final formula for variance, which allows us to determine the risk of the portfolio, has the form:

$$\begin{aligned}
&D_p^W(w) = \frac{1}{2} \sum_{i=1}^{n} w_i^2 \left(\mathbf{D}\left[a_i(\omega) + \sigma_i(\omega)\,\underline{m}_i\right] + \mathbf{D}\left[a_i(\omega) + \sigma_i(\omega)\,\overline{m}_i\right] \right) + \\
&+ \frac{1}{2}\mathbf{D}\left[\max_{i=1,\ldots,n} \left\{\sigma_j(\omega)\,\overline{d}_j w_j\right\}\right] \int_0^1 \left(R^{-1}(\alpha)\right)^2 d\alpha + \\
&+ \frac{1}{2}\mathbf{D}\left[\max_{i=1,\ldots,n} \left\{\sigma_j(\omega)\,\underline{d}_j w_j\right\}\right] \int_0^1 \left(L^{-1}(\alpha)\right)^2 d\alpha \\
&+ \sum_{1 \leq i < j \leq n} w_i w_j \left(cov\left(\left(a_i(\omega) + \sigma_i(\omega)\,\underline{m}_i\right), \left(a_j(\omega) + \sigma_j(\omega)\,\underline{m}_j\right) \right) + \right. \\
&\left. cov\left(\left(a_i(\omega) + \sigma_i(\omega)\,\overline{m}_i\right), \left(a_j(\omega) + \sigma_j(\omega)\,\overline{m}_j\right) \right) \right) + \\
&\sum\nolimits_{i=1}^{n} w_i \left(\int_0^1 R^{-1}(\alpha)\, d\alpha\, cov\left(\left(a_i(\omega) + \sigma_i(\omega)\overline{m}_i\right), \max_{i=1,\ldots,n} \left\{\sigma_j(\omega)\,\overline{d}_j w_j\right\} \right) - \right. \\
&\left. \int_0^1 L^{-1}(\alpha)\, d\alpha\, cov\left(\left(a_i(\omega) + \sigma_i(\omega)\,\underline{m}_i\right), \max_{i=1,\ldots,n} \left\{\sigma_j(\omega)\,\underline{d}_j w_j\right\} \right) \right).
\end{aligned}$$

If all random parameters of distributions are independent, then:

$$D_p^W(w) = \frac{1}{2}\sum_{i=1}^{n} w_i^2 \left(2\mathbf{D}\left[a_i(\omega)\right] + \mathbf{D}\left[\sigma_i(\omega)\right]\left(\underline{m}_i^2 + \overline{m}_i^2\right)\right) +$$
$$+\frac{1}{2}\mathbf{D}\left[\max_{i=1,\ldots,n}\left\{\sigma_j(\omega)\overline{d}_j w_j\right\}\right]\int_0^1 \left(R^{-1}(\alpha)\right)^2 d\alpha +$$
$$+\frac{1}{2}\mathbf{D}\left[\max_{i=1,\ldots,n}\left\{\sigma_j(\omega)\underline{d}_j w_j\right\}\right]\int_0^1 \left(L^{-1}(\alpha)\right)^2 d\alpha +$$
$$+\sum\nolimits_{i=1}^{n} w_i \left(\int_0^1 R^{-1}(\alpha)\, d\alpha cov\left(\left(a_i(\omega) + \sigma_i(\omega)\overline{m}_i\right), \max_{i=1,\ldots,n}\left\{\sigma_j(\omega)\overline{d}_j w_j\right\}\right)\right.$$
$$\left. - \int_0^1 L^{-1}(\alpha)\, d\alpha cov\left(\left(a_i(\omega) + \sigma_i(\omega)\underline{m}_i\right), \max_{i=1,\ldots,n}\left\{\sigma_j(\omega)\underline{d}_j w_j\right\}\right)\right)$$

We note that if in all distributions the fuzzy component is given by LR-type fuzzy numbers with the same left and right shape functions and coefficients of fuzziness, then the terms with covariance are mutually eliminated and the variance formula can be simplified:

$$D_p^W(w) = \sum_{i=1}^{n} w_i^2 \left(\mathbf{D}\left[a_i(\omega)\right] + +\mathbf{D}\left[\sigma_i(\omega)\right] m_i^2\right) +$$
$$+\mathbf{D}\left[\max_{i=1,\ldots,n}\left\{\sigma_j(\omega) d_j w_j\right\}\right]\int_0^1 \left(S^{-1}(\alpha)\right)^2 d\alpha. \tag{5}$$

Example 2 Let the shift and scale coefficients $a_i(\omega), \sigma_i(\omega)$ be uniformly distributed over the segment [0,1] and independent. Then we get the following formula for the variance:

$$D_p^W(w) = \frac{1}{2}\sum_{i=1}^{n} w_i^2 \left(\frac{1}{12}\left(\underline{m}_i^2 + \overline{m}_i^2\right) + \frac{1}{6}\right)$$
$$+\frac{1}{2}\left(EMax2\left(\overline{d}w\right) - \left(EMax\left(\overline{d}w\right)\right)^2\right)\int_0^1 \left(R^{-1}(\alpha)\right)^2 d\alpha +$$
$$\frac{1}{2}\left(EMax2\left(\underline{d}w\right) - \left(EMax\left(\underline{d}w\right)\right)^2\right)\int_0^1 \left(L^{-1}(\alpha)\right)^2 d\alpha +$$
$$+\sum\nolimits_{i=1}^{n} w_i \left(\int_0^1 R^{-1}(\alpha)\, d\alpha \overline{m}_i \left(\mathbf{E}\left[\sigma_i(\omega)\max_{j=1,\ldots,n}\left\{\sigma_j(\omega)\overline{d}_j w_j\right\}\right] - \frac{1}{2}EMax\left(\overline{d}w\right)\right) -\right.$$
$$\left. - \int_0^1 L^{-1}(\alpha)\, d\alpha \underline{m}_i \left(\mathbf{E}\left[\sigma_i(\omega)\max_{j=1,\ldots,n}\left\{\sigma_j(\omega)\underline{d}_j w_j\right\}\right] - \frac{1}{2}EMax\left(\underline{d}w\right)\right)\right)$$

and the formula (5) under the corresponding assumptions and for $S(t) = \max\{0, 1-t\}$, $t \geq 0$ takes on the form:

$$D_p^W(w) = \frac{1}{12}\sum_{i=1}^{n} w_i^2 \left(m_i^2 + 1\right) + \frac{1}{3}\left(EMax2\,(dw) - \left(EMax\,(dw)\right)^2\right)$$

where

$$EMax\,(dw) := \mathbf{E}\left[\max_{j=1,\ldots,n}\{\sigma_i\,(\omega)\,d_iw_i\}\right] =$$
$$\sum_{i=1}^{n}\frac{(dw)_{(i)}^{n-i+1}}{(n-i+1)\,(n-i+2)\,(dw)_{(i+1)}\cdots(dw)_{(n)}},$$

$$EMax2\,(dw) := \mathbf{E}\left[\max_{j=1,\ldots,n}\left(\{\sigma_i\,(\omega)\,d_iw_i\}\right)^2\right] =$$
$$\sum_{i=1}^{n}\frac{2\,(dw)_{(i)}^{n-i+2}}{(n-i+2)\,(n-i+3)\,(dw)_{(i+1)}\cdots(dw)_{(n)}}$$

and $(dw)_{(1)}, \ldots, (dw)_{(n)}$ is an ascending permutation of elements $\{d_1w_1, \ldots, d_nw_n\}$.

3.4. Minimum risk portfolio models

Based on the results presented in Sections 3.1, 3.2, and 3.3, the minimum risk portfolio models can be written down as:

$$D_p^T\,(w) \to min, \tag{6}$$

$$w \in F_p\,(w) \tag{7}$$

where $F_p\,(w) \in \left\{F_p^{\pi\mathbf{E}}, F_p^{\nu\mathbf{E}}, F_p^{\pi\mathbf{P}}, F_p^{\nu\mathbf{P}}\right\}$. Further, we assume that the minimum risk portfolio models use the same t-norm in the criteria and constraints. Let us now move on to their illustrative investigations.

4. Minimal risk portfolio under hybrid uncertainty and numerical calculations

We consider an example of two-dimensional portfolio ($n = 2$).

Let $Z_1 = [2.2,\,2.2,\,0.3,\,0.3]_{LR}$, $Z_2 = [1.2,\,1.2,\,0.4,\,0.4]_{LR}$, $L\,(t) = R\,(t) = \max\{0,\,1-t\}, t \geq 0$, $\alpha = 0.75$.

Recall that all $a_i\,(\omega)\,\sigma_i\,(\omega)$ are independent random variables with a uniform distribution on the segment [0,1]. We first specify the minimum risk portfolio models for the weakest t-norm. Under the assumptions made, the equivalent deterministic analogue of the minimum risk portfolio (6)-(7) in the context of the possibility measure takes the form:

$$\frac{73}{150}w_1^2 + \frac{61}{300}w_2^2 + \frac{1}{3}\left(EMax2\,(dw) - (EMax\,(dw))^2\right) \to min$$

$$F_p^{\pi\mathbf{E}}(w) = \left\{\begin{array}{c} 1.6w_1 + 1.1w_2 + 0.25 * EMax\,(dw) \geq m_d, \\ w_1 + w_2 = 1, \\ w_1, w_2 \geq 0, \end{array}\right.$$

and in the context of the necessity measure:

$$\frac{73}{150}w_1^2+\frac{61}{300}w_2^2+\frac{1}{3}\left(EMax2\,(dw)-(EMax\,(dw))^2\right)\rightarrow min,$$

$$F_p^{\nu\mathbf{E}}(w)=\begin{cases}1.6w_1+1.1w_2-0.75*EMax\,(dw)\geq m_d,\\ w_1+w_2=1,\\ w_1,w_2\geq 0.\end{cases}$$

It is not possible to write out the formulas for $EMax2\,(dw)$ and $EMax\,(dw)$ numerically, since they depend on the specific values of w_1 and w_2.

Let us now consider the same problem for the strongest t-norm. Under the assumptions made, the equivalent deterministic analogue of the minimum risk portfolio (6)-(7) in the context of the possibility measure takes the form:

$$\frac{587}{1200}w_1^2+\frac{187}{900}w_2^2\rightarrow min\,,$$

$$F_p^{\pi\mathbf{E}}(w)=\begin{cases}1.6375w_1+1.15w_2\geq m_d,\\ w_1+w_2=1,\\ w_1,w_2\geq 0,\end{cases}$$

and in the context of the necessity measure:

$$\frac{587}{1200}w_1^2+\frac{187}{900}w_2^2\rightarrow min,$$

$$F_p^{\nu\mathbf{E}}(w)=\begin{cases}1.4875w_1+0.95w_2\geq m_d,\\ w_1+w_2=1,\\ w_1,w_2\geq 0.\end{cases}$$

Figure 1 shows a set of quasi-efficient (i.e., efficient with a given possibility) portfolio estimates in accordance with respective models and initial data presented above.

The first thing to note in Fig. 1 is the behavior of quasi-efficient portfolio estimates in different contexts. In the context of possibility we have an optimistic decision making model, while in the context of necessity, we have a pessimistic one, which, for a given level of expected return, gives a significantly higher risk.

Secondly, as one can see in the context of possibility measure, the weakest t-norm, which has the property of reducing the uncertainty (see, for example Yazenin, 2016), narrows the scope of feasible solutions and makes the model more "strict" or "cautious", i.e. the risk at a fixed rate of return is increased slightly. In the context of necessity, the model behaves in the "opposite" way. Thus, we can say that the weakest t-norm reduces the "level of optimism" in the optimistic model and reduces the level of "pessimism" in the pessimistic model.

Let us now consider the effect of the random component in the above models on the portfolio risk in the case of the strongest t-norm in the context of

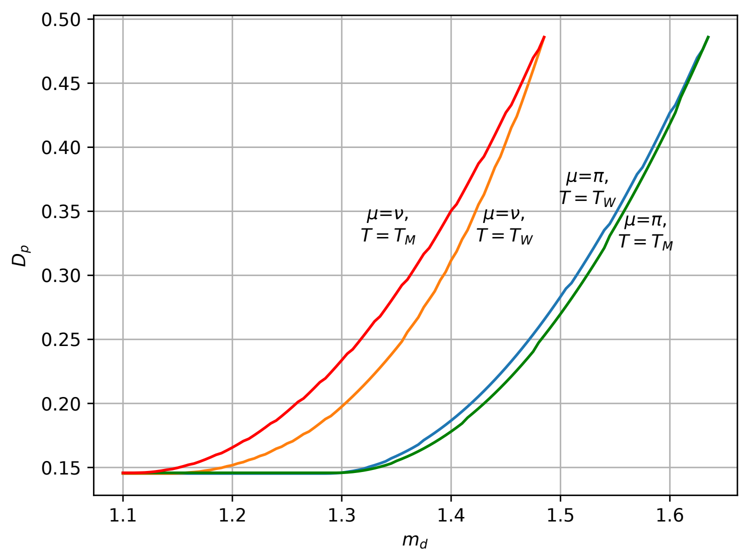

Figure 1. Sets of quasi-efficient portfolios depending on the measure of possibility/necessity and the t-norm

the possibility measure. We analyze how the solution of the problem behaves for different values of the parameters of the scaling coefficients σ_i of uniform distribution. Let all of them have a mathematical expectation equal to 0.5, and the lengths of the segments be assumed equal to 1, 0.75, 0.5, 0.25, and 0.01, respectively, that is, we will gradually reduce the spread (variance) of a random variable from a certain maximum level to an "almost non-random" value, with all potential values being concentrated in a very small neighborhood of 0.5. Figure 2 shows the results of numerical experiments.

The results of the experiment, shown in Fig. 2, are consistent with our expectations: lower uncertainty of the probabilistic type in the problem conditioning leads to more guaranteed result and, accordingly, to lower risk levels at the same level of return.

5. Summary

In this paper, a comprehensive study of the architecture of mathematical models of the minimal risk portfolio has been carried out. For extremal t-norms (the weakest and the strongest) in the context of possibility/necessity, the properties of models of acceptable portfolios are studied depending on the decision-making

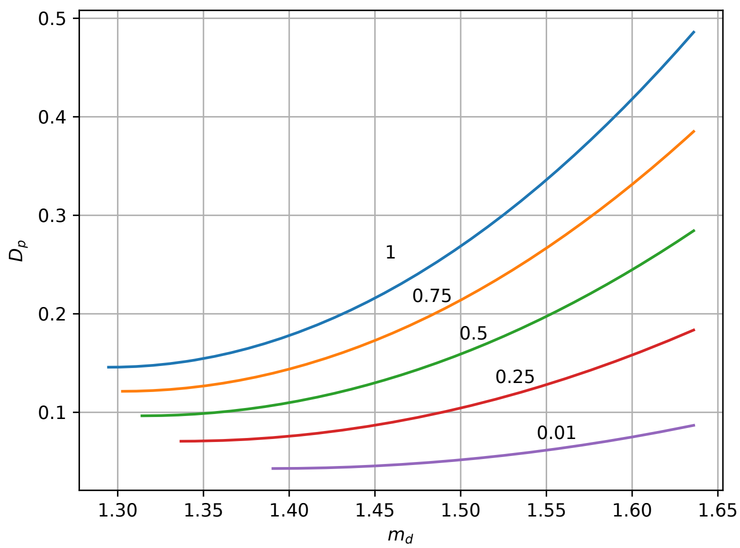

Figure 2. Influence of the random parameter σ_{i} (scaling) on the set of quasi-efficient portfolios in the case of the strongest t-norm in the context of the measure of possibility

principles used in conditions of hybrid uncertainty of possibilistic-probabilistic type.

Based on the approach of Feng, Hu and Shu (2001), formulas for assessing portfolio risk are specified in the contexts of the strongest and weakest t-norms. The obtained theoretical results and conclusions are consistent with illustrative numerical calculations.

In terms of further research, we intend to generalize the results of the article to the case when the acceptable level of portfolio return for an investor is a fuzzy value associated with the portfolio return by a fuzzy relation (see Gordeev and Yazenin, 2006). This will allow for a more "soft" and adequate modeling of preferences of an investor.

5.1. Acknowledgment

The study here reported was funded by RFBR, project number 20-01-00669-a.

References

Egorova, Y. E. and Yazenin, A. V. (2017) The problem of possibilistic-probabilistic optimization. *Journal of Computer and Systems Sciences International*, **56**, 4, 652-667. https://doi.org/10.1134/S1064230717040086

Egorova, Yu. E. and Yazenin, A. V. (2018) A method for minimum risk portfolio optimization under hybrid uncertainty. *Journal of Physics: Conference Series*, **973**, 012033. http://doi.org/10.1088/1742-6596/973/1/012033

Ermolyev, Yu. M. (1976) *Stochastic programming methods*. Fizmatlit Nauka Publ., Moscow.

Dubois, D. and Prade, H. (1988) *Théorie des possibilités: application à la représentation des connaissances en informatique*. Masson, Paris.

Feng, Y., Hu, L. and Shu, H. (2001) The variance and covariance of fuzzy random variables and their applications. *Fuzzy Sets and Systems*, **120**, 3, 487–497. https://doi.org/10.1016/S0165-0114(99)00060-3

Gordeev, R.N. and Yazenin, A.V. (2006) A method for solving a problem of possibilistic programming. *Journal of Computer and Systems Sciences International*, **45**, 3, 442–449. http://doi.org/10.1134/S1064230706030105

Hong, D.H. (2001) Parameter estimations of mutually T-related fuzzy variables. *Fuzzy Sets and Systems*, **123**, 1, 63–71. http://doi.org/10.1016/S0165-0114(00)00113-5

Markowitz, H.M. (1952) Portfolio selection. *The Journal of Finance*, **7**, 1, 77–91. http://doi.org/10.2307/2975974

Mesiar, R. (1997) Triangular-norm-based addition of fuzzy intervals. *Fuzzy Sets and Systems*, **9**1, 2, 231–237. http://doi.org/10.1016/S0165-0114(97)00143-7

Nahmias, S. (1979) Fuzzy variables in a random environment. In: M.M. Gupta, R.K. Ragade, R.R. Yager, eds., *Advances in Fuzzy Sets Theory and Applications*, NHCP, Amsterdam, 165–180.

Nguyen, H.T. and Walker, E.A. (1997) *A First Course in Fuzzy Logic*. CRC Press, Boca Raton.

Xu, J. and Zhou, X. (2011) Fuzzy-like multiple objective decision making. In: *Studies in Fuzziness and Soft Computing*, **263**. Springer, Berlin. http://doi.org/10.1007/978-3-642-16895-6

Yazenin, A.V. (1991) Linear programming with random fuzzy data. *Soviet Journal of Computer and Systems Sciences*, 3, (in Russian), 52-58.

Yazenin, A. and Wagenknecht, M. (1996) *Possibilistic Optimization*. Brandenburgische Technische Universität, Cottbus.

Yazenin, A.V. (1997) On a method of solving a problem of linear programming with random fuzzy data. *Journal of Computer and Systems Sciences International*, **36**, 5, 91-95.

Yazenin, A.V. (2007) Possibilistic-probabilistic models and methods of portfolio optimization. In: I. Batyrshin, J. Kacprzyk, eds., *Studies in Computational Intelligence*, **36**, Springer, Heidelberg, 241–259. http://doi.org/10.1007/978-3-540-36247-0_9

Yazenin, A.V. and Shefova, N.A. (2010) On a possibilistic-probabilistic model of a minimal risk portfolio. *Vestnik TvGU. Seriya: Prikladnaya Matematika* [*Herald of Tver State University. Series: Applied Mathematics*], **2** (17), (in Russian), 85-95.

Yazenin, A.V. (2016) *The basic concepts of the theory of possibility.* Fizmatlit Publ., Moscow (in Russian).

Yazenin, A. and Soldatenko, I. (2018) A portfolio of minimum risk in a hybrid uncertainty of a possibilistic-probabilistic type: comparative study. In: J. Kacprzyk, E. Szmidt, S. Zadrożny, K. Atanassov, M. Krawczak, eds., *Advances in Fuzzy Logic and Technology 2017.* EUSFLAT 2017, IWIFSGN 2017. *Advances in Intelligent Systems and Computing*, **643**, Springer, Cham, 551-563. https://doi.org/10.1007/978-3-319-66827-7_51

Yazenin, A. and Soldatenko, I. (2019) On the problem of possibilistic-probabilistic optimization with constraints on possibility/probability. In: S. Giove, F. Masulli, R. Fuller, eds., *Lecture Notes in Computer Science*, WILF 2018, *Advances in Intelligent Systems and Computing*, **11291**, Springer, Cham, 43-54.