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Introducing hesitant fuzzy equations and determining market equilibrium price*

by

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Abstract: A vast majority of research has been performed in the field of hesitant fuzzy sets (HFSs), involving the introduction of some properties, operations, relations and modifications of such sets or considering the application of HFSs in MCDM (multicriteria decision making). On the other hand, no research has been performed in the field of fully hesitant fuzzy equations. Therefore, in this paper, fully hesitant fuzzy equations and dual hesitant fuzzy equations are introduced. First, a method is proposed to solve one-element hesitant fuzzy equations. Then, the proposed method is extended to solve n-element hesitant fuzzy equations effectively. Moreover, to show the applicability of the proposed method, it is used to solve a real world problem. Thus, the proposed method is applied to determine market equilibrium price. Also, some other numerical examples are presented to better show the performance of the proposed method.

Keywords: hesitant fuzzy sets, hesitant fuzzy equations, dual hesitant fuzzy equations, membership degree

1. Introduction

Linear equations are commonly applied in various fields of science, such as engineering, physics, computer science, technology, business, and economics. Yet, in reality, many systems involve data that are not deterministic. Therefore, the parameters of such systems are often non-deterministic and the uncertainty must be considered in modeling these systems. One approach to uncertainty quantification is to consider a fuzzy set as an encoding of uncertainty (see, for instance, Khalili Goodarzi, Taghinezhad and Nasseri, 2014; Nasseri et al., 2014; Taghi-Nezhad, 2019; Taleshian, Fathali and Taghi-Nezhad, 2018; Babakordi, Allahviranloo and Adabitabarfirozja, 2016; or Allahviranloo and Babakordi,

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2017). Fuzzy set theory is considered in different areas, and many new results are being continuously obtained, such as those reported, for instance, in Viattchenin, Owsiński and Kacprzyk (2018), Begnini et al. (2018), Kalshetti and Dixit (2018), or Hesamian (2017).

The initial investigations of interest here focused on solving fuzzy linear equations (see, e.g., Buckley, 1991). Subsequent studies dealt with various techniques for solving fuzzy equations and systems of fuzzy equations. Thus, for instance, the Newton method and some extensions, such as the steepest descent method, and then evolutionary algorithms, neural nets and other iterative methods can be used for this purpose (see, in particular, Abbasbandy and Asady, 2004; Buckley, Feuring and Hayashi, 2002; Amirfakharian, 2012; Noor'ani et al., 2011; Farahani, Nehi and Paripour, 2016; or Babakordi and Firozja, 2020). The most recent research in this area concerns the ranking method for solving dual fuzzy polynomial equations. In the method mentioned, fuzzy sets require the specification of membership degree for each element in the reference set; whereas the hesitant fuzzy sets (HFS) permit the designer to include some hesitation on this value, see Torra (2010) and Torra and Narukawa (2009), who introduced the concept of HFS. The HFSs allow the membership degree to acquire some different possible crisp values between zero and one.

Recently, HFSs have gained the attention of researchers, prompting them to apply HFSs, in particular, to multi-criteria decision-making (MCDM) problems. For instance, a number of studies on the aggregation operators of HFSs and their extensions were conducted in Zhu, Xu and Xia (2011), Xia, Xu and Chen (2013), Zhang et al. (2014), Zhou (2014), Tang et al. (2018), as well as Farhadinia and Herrera-Viedma (2018). The correlation coefficient, distance, and correlation measurement of HFSs were developed in Yu, Wu and Zhou (2011) and in Farhadinia (2014a, b). Wang et al. (2014) proposed an outranking approach with the use of HFSs to solve the MCDM problems. Peng et al. (2017) introduced an MCDM approach with hesitant interval-valued intuitionistic fuzzy sets (HIVIFSs), which are an extension of the dual HFSs. Chen, Xu and Xia (2013) generalized the concept of HFS to interval-valued hesitant fuzzy set (IVHFS) and proposed some aggregation operators. At more or less the same time, Wei and Zhao (2013) defined the hesitant interval-valued fuzzy sets (HIVFSs) and developed Einstein operations on them. Further, Wei and Zhao (2013) defined also a series of hesitant interval-valued fuzzy aggregation operators for the MCDM problems, based on algebraic operations. However, regardless whether IVHFSs or HIVFSs are involved, the membership degrees of an element in a given set are represented by several possible interval values. IVHFSs and HIVFSs are both extensions of HFSs and IVFSs, and they are essentially of the same nature. Both concepts are generalized forms of HFSs and can be reduced to the latter when the upper and the lower limits of the possible interval values are the same. This, naturally, means that HFSs are a special case of IVHFSs or HIVFSs. Several related studies were also conducted based on IVHFSs or HIVFSs. Thus, for example, Zhu et al. (2014) developed some

Einstein aggregation operators with hesitant interval-valued fuzzy information and applied them to MCDM problems (compare Wei and Zhao, 2013).

In 2017, based on a two-stage optimization and multiplicative consistency, the priority vector and consistency of hesitant fuzzy linguistic (HFL) preference relation were discussed by Peng et. al. (2017).

Then, a method was proposed by Liu and Zhang (2020) that converts the original decision matrix, expressed by the hesitant fuzzy linguistic term sets (HFLTSs) into the evidence matrix with HFLTSs. The same authors also developed a weight-determining model for MADM problems with HFL information (Liu and Zhang, 2020). A new group decision making (GDM) method with hesitant fuzzy linguistic preference relations (HFLPRs) was proposed by Zhang and Chen (2020). First, a consensus checking method was proposed to measure the consensus level of individual HFLPRs. Then, a definition of acceptable consensus was introduced. The generalized interval probability hesitant fuzzy linguistic IOWA weighted average (GVIOWAWA) operator was proposed by Xian and Guo (2020) to aggregate the uncertain linguistic information with incomplete reliability. The GVIOWAWA operator enables the decision makers to select the appropriate parameters according to their needs. Then, the interval probability hesitant fuzzy linguistic TOPSIS (IPHFL-TOPSIS) based on the interval probability hesitant fuzzy linguistic Euclidean distance was established by Xian and Guo (2020). The IPHFL-TOPSIS model is shown in Xian and Guo (2020) to effectively and objectively help businesses find the strategic cooperation supplier. The focus of Boyaci (2020) is on the selection of eco-friendly cities in Turkey, according to the criteria such as average PM_{10} measurement values at the air quality measurement stations, forest area per km^2, and percentage of population receiving waste services, using the hesitant fuzzy linguistic term set (HFLTS)-based additive ratio assessment (ARAS) method.

Although a vast majority of the investigations mentioned have been performed in the field of HFSs, these studies only introduce some properties, operations, relations and modifications of HFSs or consider their application uniquely in MCDM. In the literature, only Ranjbar and Effati (2019) extended significantly the domain of application of HFSs and used them in a linear programming problem. Since no significant research has been performed in the field of hesitant fuzzy equations, in this paper, hesitant fuzzy equations in the form of half hesitant fuzzy equation, partial fuzzy equation, fully hesitant fuzzy equation and hesitant dual fuzzy equations are investigated.

The present paper is structured as follows: in Section 2, the preliminaries are presented. In Section 3, hesitant fuzzy equations are introduced. In Section 4, a method is proposed to solve one-element hesitant fuzzy equations. The proposed method is extended to n-element hesitant fuzzy equations in Section 5. Then, in Section 6, an economic application of dual hesitant fuzzy equations is presented. Namely, the market equilibrium price that can be modeled using a hesitant fuzzy equation, is determined. Some numerical examples are presented in Section 7 that verify the effectiveness of the proposed method. Finally, the conclusions are presented in Section 8.

2. The preliminaries

In this section, the required notations, basic concepts and some necessary definitions are reviewed. The concept of the HFSs is used extensively throughout this paper, therefore; first, some basic definitions of HFSs are presented in the following.

Definition 1 (Torra, 2010) *Let X be a fixed set. An HFS on X is expressed in terms of a function that, when applied to X, returns a subset of [0,1], containing a finite number of elements. For better understanding, Xia, Xu and Chen (2013) expressed HFS in terms of the following symbolic notation:*

$$A = \{< x, h_A(x) > |x \in X\} \tag{1}$$

where $h_A(x)$ is some set of values from $[0\,,1]$, corresponding to the possible membership degrees of the element $x \in X$ regarding the set A. For convenience, we shall refer to $h_A(x)$ as to a hesitant fuzzy element (HFE). Some of the operations on the HFEs, which are defined in Torra (2010) and Xia, Xu and Chen (2013), are as follows:

$$\begin{aligned} &h^c = \cup_{\gamma\in h}\{1-\gamma\}, \\ &h^\lambda = \cup_{\gamma\in h}\{\gamma^\lambda\}, \\ &h^\lambda = \cup_{\gamma\in h} 1-(1-\gamma)\lambda \\ &h_1 \cup h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2} max\{\gamma_1,\gamma_2\} \\ &h_1 \cap h_2 = \cap_{\gamma_1\in h_1,\gamma_2\in h_2} min\{\gamma_1,\gamma_2\} \\ &h_1 + h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2}\{\gamma_1+\gamma_2-\gamma_1\gamma_2\}\, h_1 \times h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2}\{\gamma_1\gamma_2\}. \end{aligned} \tag{2}$$

3. Hesitant fuzzy equations

In this section, two different types of hesitant fuzzy equations are defined.

Definition 2 *The equation $AX = B$ is called fully hesitant fuzzy equation, when A and B are known hesitant fuzzy sets and X is an unknown hesitant fuzzy set. $AX + B = CX + D$ is called dual hesitant fuzzy equation, when A, B, C and D are known hesitant fuzzy sets and X is an unknown hesitant fuzzy set.*

In the following, fully hesitant fuzzy equations and dual hesitant fuzzy equations are solved for the cases, in which all the involved hesitant fuzzy sets have equal number of elements. First, the method is defined for one-element hesitant fuzzy sets, and then it is extended to n-element hesitant fuzzy sets.

4. The proposed method for solving one-element hesitant fuzzy equation

4.1. The definitions pertaining to the solutions

Let us start with the four already known solution types, proposed by the researchers active in interval analysis (Fiedler et al., 2006; Lodwick, 1990; Stolfi and de Figueriredo, 1997) for fully and dual hesitant fuzzy equation. Then, the method for solving each type of hesitant fuzzy equation is described.

Definition 3 *Consider hesitant fuzzy element h_x. It is said that $h_x \in [ab]$ when for each $h \in h_x$ there is $a \leq h \leq b$.*

Definition 4 (United solution set (USS)) *The united solution set of the fully hesitant fuzzy equation $\{< a, h_a >\}\{< x, h_x >\} = \{< b, h_b >\}$ is denoted by $X_{\exists\exists}$, and is defined as follows:*

$$X_{\exists\exists} = \left\{x' \in h_x \in [0,1] : (\exists h \in h_a)\left(\exists\, h' \in h_b\right) \; s.t. \; h\,x' = h'\right\}$$
$$= \{h_x \in [0,1] : (h_a \times\, h_x) \cap h_b\} \neq \emptyset\}.$$

Definition 5 (Tolerable solution set (TSS)) *A tolerable solution set of the fully hesitant fuzzy equation $\{< a, h_a >\}\{< x, h_x >\} = \{< b, h_b >\}$ is denoted by $X_{\forall\exists}$ and is defined as follows:*

$$X_{\forall\exists} = \left\{x' \in h_x \in [0,1] : (\forall h \in h_a)\left(\exists\, h' \in h_b\right) \; s.t. \; h\,x' = h'\right\}$$
$$= \{h_x \in [0,1] : h_a \times\, h_x \subseteq h_b\}.$$

Definition 6 (Controllable solution set (CSS)) *A controllable solution set of the fully hesitant fuzzy equation $\{< a, h_a >\}\{< x, h_x >\} = \{< b, h_b >\}$ is denoted by $X_{\exists\forall}$ and is defined as follows:*

$$X_{\exists\forall} = \left\{x' \in h_x \in [0,1] : (\exists h \in h_a)\left(\forall\, h' \in h_b\right) \; s.t. \; h\,x' = h'\right\}$$
$$= \{h_x \in [0,1] : h_a \times\, h_x \supseteq h_b\}.$$

For dual hesitant fuzzy equation, we have the corresponding analogous definitions as follows:

Definition 7 (United solution set (USS)) *A united solution set of the dual hesitant fuzzy equation*

$$\{< a, h_a >\}\{< x, h_x >\} + \{< b, h_b >\} = \{< c, h_c >\}\{< x, h_x >\} + \{< d, h_d >\}$$

is denoted by $X_{\exists\exists\exists\exists}$ and is defined as follows:

$$X_{\exists\exists\exists\exists} = \left\{x' \in h_x \in [0,1] : (\exists h \in h_a)\left(\exists\, h' \in h_b\right)(\exists h'' \in h_c)\left(\exists h''' \in h_d\right) \right.$$
$$\left. s.t. \; h\,x' + h' = h''\,x' + h''' \right\}$$
$$= \{h_x \in [0,1] : (h_a \times h_x + h_b) \cap (h_c \times h_x + h_d) \neq \emptyset\}.$$

Definition 8 (Tolerable solution set (TSS)) *A tolerable solution set of the dual hesitant fuzzy equation*

$$\{< a, h_a >\}\{< x, h_x >\}+\{< b, h_b >\} = \{< c, h_c >\}\{< x, h_x >\}+\{< d, h_d >\},$$

is denoted by $X_{\forall\forall\exists\exists}$ *and is defined as follows:*

$$X_{\forall\forall\exists\exists} = \Big\{x^{'} \in h_x \in [0,1] : (\forall h \in h_a)\left(\forall h^{'} \in h_b\right)(\exists h'' \in h_c)\left(\exists h^{'''} \in h_d\right) \\ s.t.\ h\,x^{'} + h' = h''\,x^{'} + h''^{'}\Big\} = \{h_x \in [0,1] : (h_a \times h_x + h_b) \subseteq (h_c \times h_x + h_d)\}.$$

Definition 9 (Controllable solution set (CSS)) *A controllable solution set of the dual hesitant fuzzy equation*

$$\{< a, h_a >\}\{< x, h_x >\}+\{< b, h_b >\} = \{< c, h_c >\}\{< x, h_x >\}+\{< d, h_d >\},$$

is denoted by $X_{\exists\exists\forall\forall}$ *and is defined as follows:*

$$X_{\exists\exists\forall\forall} = \Big\{x^{'} \in h_x \in [0,1] : (\exists h \in h_a)\left(\exists h^{'} \in h_b\right)(\forall h'' \in h_c)\left(\forall h^{'''} \in h_d\right) \\ s.t.\ h\,x^{'} + h' = h''\,x^{'} + h''^{'}\Big\} = \{h_x \in [0,1] : (h_a \times h_x + h_b) \supseteq (h_c \times h_x + h_d)\}.$$

We shall now turn to the definitions, specifying the solutions we are looking for:

Definition 10 *A hesitant fuzzy set* $X = \{< x, h_x >\}$ *is called the solution of the fully hesitant fuzzy equation* $\{< a, h_a >\}\{< x, h_x >\} = \{< b, h_b >\}$*, when* h_x *fulfils the conditions of USS, CSS or TSS and* $ax = b$*.*

Definition 11 *A hesitant fuzzy element* $X = \{< x, h_x >\}$ *is called the solution of dual hesitant fuzzy equation* $\{< a, h_a >\}\{< x, h_x >\} + \{< b, h_b >\} = \{< c, h_c >\}\{< x, h_x >\} + \{< d, h_d >\}$*, when* h_x *fulfils the conditions of USS, CSS or TSS and* $ax + b = cx + d$*.*

4.2. Solving a fully hesitant fuzzy equation

In the fully hesitant fuzzy equation (3), assume that $A = \{< a, \{h_1, h_2, \ldots, h_n\} >\}$ and $B = \{< b, \{h'_1, h'_2, \ldots, h'_m\} >\}$. There is an unknown fully fuzzy hesitant vector h_x, which can be obtained using the following relation: $X = \left\{< \frac{b}{a}, h_x >\right\}$. For determining h_x, it is assumed that $h_t = \{h''_1, h''_2, \ldots, h''_{mn}\}$. Then, to obtain $h''_1, h''_2, \ldots, h''_{mn}$, the following equations are introduced:

$$h_i.h''_j = h'_k,\ \forall\ 1 \le i \le n,\ 1 \le j \le nm,\ 1 \le k \le m. \tag{3}$$

By solving the above equations and thereby calculating h_t, the value of h_x can be determined as follows:

$$h_x \subseteq \{h''_i \in h_t : h''_i \le 1, 1 \le i \le mn\} \tag{4}$$

and if maximum hesitancy is considered, there is:

$$h_x^M = \{h''_i \in h_t \,: h''_i \,\leq 1,\, 1 \leq i \leq mn\}\,.$$

To better understand the performance of the proposed method, one can see further on the Examples 7.1 and 7.2.

Theorem 1 *Equation (3) does not have a non-empty TSS.*

Proof. Assume

$$A = \{< a, \{h_1,\, h_2, \ldots, h_n\} >\} \text{ and } B = \{< b, \{h'_1,\, h'_2, \ldots, h'_m\} >\}\,,$$

after calculating the value of h_x from (6) and the value of $\{h_1,\, h_2, \ldots, h_n\} \times h_x$, by applying the multiplication definition, provided in (2), item 7, it can be seen that the following always holds:

$$\{h_1,\, h_2, \ldots, h_n\} \times h_x \supset \left\{h_1^{'},\, h_2^{'}, \ldots, h_m^{'}\right\}.$$ □

Theorem 2 *Assume that*

$$A = \{< a, \{h_1,\, h_2, \ldots, h_n\} >\}\,, \quad B = \{< b, \{h'_1,\, h'_2, \ldots, h'_m\} >\}\,.$$

If there exists only one $h' \in h_a$ *and one* $h'' \in h_b$ *such that* $\frac{h''}{h'} \leq 1$, *or if a one-element subset of* $\left\{\frac{h'_j}{h_i} \,:\, \frac{h'_j}{h_i} \leq 1,\, \forall 1 \leq i \leq n,\, 1 \leq j \leq m\right\}$ *is considered, then equation (3) has a fuzzy solution.*

Proof Because of these assumptions, h_x has only one member, therefore the proof is complete. □

4.3. Solving a dual hesitant fuzzy equation

In order to solve the fully dual hesitant fuzzy equation (4), assume that

$$A = \{< a, \{h_1,\, h_2, \ldots, h_n\} >\}\,,$$

$$B = \{< b, \{h'_1,\, h'_2, \ldots, h'_m\} >\}\,,$$

$$C = \{< c, \{h''_1,\, h''_2, \ldots, h''_k\} >\} \text{ and}$$

$$D = \{< d, \{h'''_1,\, h'''_2, \ldots, h'''_f\} >\}\,.$$

There is the variable $X = \{< x, h_x >\}$, which is calculated from the following relation: $X = \left\{< \frac{d-b}{a-c},\, h_x>\right\}$. In order to determine h_x, first h_t must be calculated from (7), then each subset of h_t that has values between 0 and 1 can be considered as representing the final value.

So, assume that $h_t = \{h''''_1,\ h''''_2, \dots, h''''_{nmkf}\}$, therefore, for determining $h''''_1,\ h''''_2, \dots, h''''_{nmkf}$ the following equation is introduced:

$$h_i.h''''_j + h'_l - h_i.h''''_j.h'_l = h''_{i'}.h''''_j + h'''_{j'} - h''_{i'}.h''''_j.h'''_{j'},$$
$$\forall 1 \le i \le n,\ 1 \le j \le nmkf,\ 1 \le l \le m,\ 1 \le i' \le k,\ 1 \le j' \le f. \quad (5)$$

By solving the above equations and calculating h_t, the value of h_t is defined as follows:

$$h_x \subseteq \{h''''_j \in h_t : h''''_j \le 1,\ 1 \le j \le nmkf\} \quad (6)$$

and the case of the maximum hesitancy is considered as follows:

$$h_x^M = \{h''''_j \in h_t : h''''_j \le 1,\ 1 \le j \le nmkf\}.$$

Theorem 3 *Equation (4) does not have non-empty TSS, CSS. It only has non-empty USS.*

Proof Assume that

$$A = \{< a, \{h_1,\ h_2, \dots, h_n\} >\},$$
$$B = \{< b, \{h'_1,\ h'_2, \dots, h'_m\} >\},$$
$$C = \{< c, \{h''_1,\ h''_2, \dots, h''_k\} >\}$$

and

$$D = \{< d, \{h'''_1,\ h'''_2, \dots, h'''_f\} >\},$$

there is

$$X = \left\{< \frac{d-b}{a-c},\ h_x>\right\}.$$

After calculating h_x from equation (6) and calculating $\{h_1,\ h_2, \dots, h_n\} \times h_x + \{h'_1,\ h'_2, \dots, h'_m\}$ and $\{h''_1,\ h''_2, \dots, h''_k\} \times h_x + \{h'''_1,\ h'''_2, \dots, h'''_f\}$, by applying the addition and multiplication definitions, presented in Definition 2.1, it can be seen that none of the following cases holds:

$$\{h_1,\ h_2, \dots, h_n\} \times h_x + \{h'_1,\ h'_2, \dots, h'_m\} \subseteq \{h''_1,\ h''_2, \dots, h''_k\} \times h_x + \{h'''_1,\ h'''_2, \dots, h'''_f\}$$

$$\{h_1,\ h_2, \dots, h_n\} \times h_x + \{h'_1,\ h'_2, \dots, h'_m\} \supseteq \{h''_1,\ h''_2, \dots, h''_k\}\, h_x + \{h'''_1,\ h'''_2, \dots, h'''_f\}$$

while the following always holds:

$$\left(\{h_1,\ h_2, \dots, h_n\} \times h_x + \left\{h_1^{'},\ h_2^{'}, \dots, h_m^{'}\right\}\right) \cap \left(\left\{h_1^{''},\ h_2^{''}, \dots, h_k^{''}\right\} \times h_{\ x} + \left\{h_1^{'''},\ h_2^{'''}, \dots, h_t^{'''}\right\}\right) \neq \emptyset.$$

□

5. The proposed method for solving the n-element hesitant fuzzy equations

In the previous section, a method was proposed for solving the hesitant fuzzy equations in which hesitant fuzzy sets have only one hesitant fuzzy element. The method and the solutions, presented there, can be extended as follows:

Assume that the goal is to solve the fully hesitant fuzzy equation $AX = B$, where

$$A = \{< a_1, h_{a_1} >, < a_2, h_{a_2} >, \ldots, < a_n, h_{a_n} >\} \text{ and}$$
$$B = \{< b_1, h_{b_1} >, < b_2, h_{b_2} >, \ldots, < b_n, h_{b_n} >\}$$

are known hesitant fuzzy sets and $X = \{< x_1, h_{x_1} >, < x_2, h_{x_2} >, \ldots, < x_n, h_{x_n} >\}$ is an unknown hesitant fuzzy set.

Therefore, the following equation set is constructed:

$$\begin{array}{c} \{< a_1, h_{a_1} >\}\{< x_1, h_{x_1} >\} = \{< b_1, h_{b_1} >\} \\ \{< a_2, h_{a_2} >\}\{< x_2, h_{x_2} >\} = \{< b_2, h_{b_2} >\} \\ \cdot \\ \cdot \\ \cdot \\ \{< a_n, h_{a_n} >\}\{< x_n, h_{x_n} >\} = \{< b_n, h_{b_n} >\}. \end{array} \tag{7}$$

Then, each one of these equations is solved using the method proposed in the previous section. It is said that the fully hesitant equation $AX = B$ has hesitant fuzzy solution when each of the above equations has solution.

Assume now that the goal is to solve fully the dual hesitant fuzzy equation

$$AX + B = CX + D$$

where:

$$A = \{< a_1, h_{a_1} >, < a_2, h_{a_2} >, \ldots, < a_n, h_{a_n} >\},$$

$$B = \{< b_1, h_{b_1} >, < b_2, h_{b_2} >, \ldots, < b_n, h_{b_n} >\},$$

$$C = \{< c_1, h_{c_1} >, < c_2, h_{c_2} >, \ldots, < c_n, h_{c_n} >\}, \text{ and}$$

$$D = \{< d_1, h_{d_1} >, < d_2, h_{d_2} >, \ldots, < d_n, h_{d_n} >\}$$

are known hesitant fuzzy sets and

$$X = \{< x_1, h_{x_1} >, < x_2, h_{x_2} >, \ldots, < x_n, h_{x_n} >\}$$

is an unknown hesitant fuzzy set. Therefore, the following equation set is constructed:

$$\begin{aligned}
&\{< a_1, h_{a_1} >\}\{< x_1, h_{x_1} >\} + \{< b_1, h_{b_1} >\} = \\
&\qquad \{< c_1, h_{c_1} >\}\{< x_1, h_{x_1} >\} + \{< d_1, h_{d_1} >\} \\
&\{< a_2, h_{a_2} >\}\{< x_2, h_{x_2} >\} + \{< b_2, h_{b_2} >\} = \\
&\qquad \{< c_2, h_{c_2} >\}\{< x_2, h_{x_2} >\} + \{< d_2, h_{d_2} >\} \\
&. \\
&. \\
&. \\
&\{< a_n, h_{a_n} >\}\{< x_n, h_{x_n} >\} + \{< b_n, h_{b_n} >\} = \\
&\qquad \{< c_n, h_{c_n} >\}\{< x_n, h_{x_n} >\} + \{< d_n, h_{d_n} >\}. \qquad (8)
\end{aligned}$$

Then, each one of these equations is solved using the method proposed in the previous section. It is said that the fully dual hesitant equation $AX + B = CX + D$ has hesitant fuzzy solution when each of the above equations has a solution.

6. Application of hesitant dual fuzzy equation to determining equilibrium market price

Microeconomics is a science that discusses the economic behavior and performance of a unit of consumption (household), or of a unit of production (firm), or of a group of consumers and producers. For example: what does a firm produce? how does it produce its products? what price does it ask? or: what goods does a household buy? how much of them does it buy? etc.

It is very difficult and sometimes even impossible to answer these questions in a definitive manner, and this is especially true in the current situation, when the corona virus exerts a very disturbing influence on the world economy, and the decisions of the individual and group subjects are marked by a deep ambiguity and skepticism. Definitely, the resulting kinds of uncertainties ought to be accounted for in respective mathematical modelling. In the present section, we discuss this kind of issues.

6.1. Demand

Individual demand is the amount of goods that the buyer is willing and able to buy due to its price and the stability of other factors in a given period of time, see Baumol (1972).

Definition 12 (Baumol (1972)) *Demand is the maximum amount of goods that a person buys according to its price.*

Of course, it should be noted that there is, likewise, an analogous demand for services, such as passenger transport services.

Definition 13 *Baumol (1972) Need and demand are different from each other. We may need a lot of goods and services but we may not be able to turn this need into demand. For example, a person may need a plane, but does not dispose of monetary resources, necessary to buy one. Some of our needs become demand due to price, income, etc.*

The amount of demand for goods x is affected by the following factors:

$$Q_x^d = F(P_x, I, P_y, T, A_x, E_d, \ldots),$$

where:

P_x is the price of goods x.

P_y is the price of other goods.

I is consumer income or budget.

T is the consumer taste that can be derived from his needs, the source of which can be due to social customs and habits or, above all, the product of his values and beliefs.

A_x is advertising for goods x.

E_d is the factor of demand price expectations, so that the consumer demand is influenced by his expectations of the availability or non-availability of goods in the future and also his forecast of future price trends of this product.

If we keep factors affecting demand fixed in the above equation, except for the price of goods x, we can write:

$$Q_x^d = f(P_x).$$

The demand function can be, and actually is, expressed in different forms, one of them being, for instance, the form $Q_x^d = a + b\,P_x$ in which a and b are real numbers.

6.2. Supply

Individual supply is the amount of goods that the seller is willing and able to offer in the market in a given period of time due to its price and the stability of other factors, see Baumol (1972).

Definition 14 (Baumol (1972)) *The supply of a commodity is the maximum amount of that commodity that the seller offers according to the price of that commodity in the market.**

*Note that this definition assumes, according to the classical approach, perfect competition, as it does not account for the influence of the seller's offer on the market price (eds.).

Apart from the price factor P_x, which is the most important variable affecting supply, other factors, such as production cost (TC, which includes the price of production institutions, etc.), P_y , the price of related goods, the level of technology or technical knowledge (T), E_s the factor of supply price expectations, etc., intervene in determining the supply of a product. In general, supply is related to profit. If the profit increases, the supply also increases, and vice versa; the profit, naturally, also depends on the above factors. Therefore, the general form of the individual supply function can be assumed to be as follows:

$$Q_x^s = F(P_x, T, C, P_y, T, E_s, \ldots).$$

If we assume other factors to be fixed in the above equation, except for the price of goods, we can write:

$$Q_x^s = f(P_x).$$

The above equation is called the supply function. Therefore, the supply function is a function that shows the relationship between the price of a commodity and the supply of the same commodity, assuming that other factors are fixed.

This function should specify, in particular, the following important characteristic quantities:

1. The minimum price, at which the supplier is willing to offer the goods.
2. The maximum amount offered for each price.

The supply function can be expressed in different forms without losing the essential features of the whole subject; in particular, it can appear in the form $Q_x^s = c + d\,P_x$, in which c and d are real numbers. If it is specified in this form, the characteristics, mentioned above, must be additionally also specified.

6.3. Equilibrium

Equilibrium is a state, in which there is no motivation, stimulus or force to change it. If we are not at equilibrium point, we tend to change the situation, see, e.g., Baumol (1972). The market equilibrium is attained at a point, at which the quantity of goods demanded is equal to the quantity of goods supplied, i.e., $Q_x^D = Q_x^S$. But, in practice, determining the values of the respective parameters, corresponding to this point, is heavily burdened by ambiguity and imprecision, which should be considered in mathematical modeling. The use of hesitant fuzzy numbers is very effective in such issues because it shows and covers doubts about the amount of demand due to price instability and economic conditions.

Definition 15 *We show the fuzzy price of hesitant commodity x with $p_x = \{< p,\ h_p >\}$ in which p is a real and negative number and h_p is a set of values belonging to* $[0, 1]$.

Example 1 *When we say that the fuzzy price of a product x is $p_x = \{< 3000\,, \{0.1, 0.4\} >\}$ tomans, it means that the price of the product is about 3000* *tomans, where it is not possible to determine the exact membership amount, and there are various possible amounts for some reasons, owing to which the correctness of each one is in doubt, and so the expert has considered different degrees for the price of 3000 tomans.*

DEFINITION 16 *If the price of commodity* x *is equal to* $p_x = \{< p,\ h_p >\}$, *then the hesitant fuzzy demand is as follows:*

$$Q_x^d = Ap_x + B \tag{9}$$

where $A = \{< a,\ h_a >\}$ *and* $B = \{< b,\ h_b >\}$ *and* a *and* b *are positive real numbers that must be defined in such a way that* $Q = ap + b$ *is a descending real function and demand function is shown with the maximum hesitancy through a* Q_x^{dM}.

DEFINITION 17 *If the price of commodity* x *is equal to* $p_x = \{< p,\ h_p >\}$, *then the hesitant fuzzy supply is as follows:*

$$Q_x^s = Cp_x + D \tag{10}$$

where $C = \{< c,\ h_c >\}$ *and* $D = \{< d,\ h_d >\}$ *and* c *and* d *are positive real numbers that must be defined so that* $Q = cp + d$ *is an ascending real function and demand function is shown with the maximum hesitancy through a* Q_x^{dM}.

Determining the equilibrium market price. To determine the equilibrium market price, we must solve the equation $Q_x^d = Q_x^s$, and hence we have:

$$Ap_x + B = Cp_x + D.$$

The above equation is a hesitant dual fuzzy equation, from which we obtain p_x using the method presented in Section 4.2. After calculating p_x^M (the equilibrium price of the market with the maximum hesitancy), the equilibrium point of the market can be calculated with the maximum hesitancy using formulas (9) and (10).

7. Numerical examples

In this section, the effectiveness and applicability of the proposed method is illustrated by solving some numerical examples.

EXAMPLE 2 *Consider the fully hesitant fuzzy equation* $AX = B$ *where* $A = \{< 3, \{0.3, 0.8\} >\}$ *and* $B = \{< 9, \{0.1, 0.5, 0.9\} >\}$. *There is* $X = \{< 3,\ h_x >\}$. *Also, consider* $h_t = \{h_1,\ h_2, \ldots, h_6\}$. *To determine* $h_1,\ h_2, \ldots, h_6$, *equations (5) are introduced as follows:*

$0.3h_1 = 0.1 \quad 0.3h_2 = 0.5 \quad 0.3h_3 = 0.9 \quad 0.8h_4 = 0.1 \quad 0.8h_5 = 0.5 \quad 0.8h_6 = 0.9$

By solving the above equations, it can be concluded that $h_t = \left\{\frac{1}{3}, \frac{5}{3}, 3,\ \frac{1}{8},\ \frac{5}{8},\ \frac{9}{8}\right\}$. *Therefore:*

$$h_x \subseteq \left\{\frac{1}{3}, \frac{1}{8}, \frac{5}{8}\right\}$$

and

$$h_x^M = \left\{\frac{1}{3}, \frac{1}{8}, \frac{5}{8}\right\}.$$

As a result, the solution of the equation is approximately 3. However, determining the exact value of the membership degree is not possible and involves hesitancy. Therefore, the solution of the equation, upon considering the maximum hesitancy is the following:

$$X^M = \left\{< 3,\ \left\{\frac{1}{3}, \frac{1}{8}, \frac{5}{8}\right\} >\right\}$$

and the general form of the solution is as follows:

$$X = \left\{\{< 3,\ h_x >\} : h_x \subseteq \left\{\frac{1}{3}, \frac{1}{8}, \frac{5}{8}\right\}\right\}.$$

Now, a decision maker can choose the final solution according to the conditioning and perception of the real world problem.

Example 3 *The goal is to solve the fully hesitant fuzzy equation* $AX = B$ *where*
$A = \{< 2, \{0.5, 0.7, 0.9\} >, < 6, \{0.8, 0.6\} >, < 9, \{0.3, 0.6, 0.4\} >,$
$< 5, \{0.8, 0.7, 0.5 >, < 2, \{0.8, 0.9\} >\}$
and
$B = \{< 3, \{0.1, 0.4\} >, < 18, \{0.3, 0.5, 0.2\} >, < 36, \{0.1, 0.3\} >,$
$< 25, \{0.6, 0.2 >, < 12, \{0.3, 0.4, 0.5\} >\}.$

To find the solution of
$AX = B$ $(X = \{< x_1, h_{x_1} >, < x_2, h_{x_2} >, < x_3, h_{x_3} >, < x_4, h_{x_4} >\})$,

the following equations are constructed:

a) $\{< 2, \{0.5, 0.7, 0.9\} >\}\, \{< x_1, h_{x_1} >\} = \{< 3, \{0.1, 0.4\} >\}$

b) $\{< 6, \{0.8, 0.6\}\} > \{< x_2, h_{x_2} >\}$
$= \{< 18, \{0.3, 0.5, 0.2\} >\}\, \{< 6, \{0.8, 0.6\}\} > \{< x_2, h_{x_2} >\}$
$= \{< 18, \{0.3, 0.5, 0.2\} >\}$

c) $\{< 9, \{0.3, 0.6, 0.4\} >\}\, \{< x_3, h_{x_3} >\} = \{< 36, \{0.1, 0.3\} >\}$

d) $\{< 5, \{0.8, 0.7, 1 >\}\, \{< x_4, h_{x_4} >\} = \{< 25, \{0.6, 0.2 >\}$

e) $< 2, \{0.8, 0.9\} > \{< x_5, h_{x_5} >\} = \{< 12, \{0.3, 0.4, 0.5\} >\}$

To solve equation (a), there is:

$$2x_1 = 3 \Longrightarrow \; x_1 = \frac{3}{2}$$

and to find $h_{x_1} = \{h_1, h_2, \ldots, h_6\}$*, the following equations are solved:*

$$0.5\,h_1 = 0.1\,, \qquad 0.7\,h_3 = 0.1, \;\; 0.9\,h_5 = 0.1$$

$$0.5\,h_2 = 0.4, \qquad 0.7\,h_4 = 0.4, \qquad 0.9\,h_6 = 0.4.$$

Therefore, $h_x = \left\{\frac{1}{5}\,, \frac{4}{5}\,, \frac{1}{7}, \frac{4}{7}, \frac{1}{9}, \frac{4}{9}\right\}$*. Hence, the solution of equation (a) is*

$$< \frac{3}{2} \left\{ \frac{1}{5}\,, \frac{4}{5}\,, \frac{1}{7}, \frac{4}{7}, \frac{1}{9}, \frac{4}{9} \right\} > \quad .$$

In a similar way, the solutions of equations (b) to (e) are obtained, respectively, as:

$$(b)\; < 3, \left\{\tfrac{3}{8}\,, \tfrac{3}{6}\,, \tfrac{5}{8}, \tfrac{5}{6}, \tfrac{2}{8}, \tfrac{2}{6}\right\} >,$$

$$(c) < 4, \left\{\tfrac{1}{3}\,, \tfrac{1}{6}\,, \tfrac{1}{4}, 1, \tfrac{3}{6}, \tfrac{3}{4}\right\} >,$$

$$(d) < 5, \left\{\tfrac{6}{8}\,, \tfrac{2}{8}\,, \tfrac{6}{7}, \tfrac{2}{7}, \tfrac{6}{10}, \tfrac{2}{10}\right\} >$$

and

$$(e) < 6, \left\{\tfrac{3}{8}\,, \tfrac{4}{8}\,, \tfrac{5}{8}, \tfrac{3}{9}, \tfrac{4}{9}, \tfrac{5}{9}\right\} >.$$

Finally, upon rounding to two decimal places, the solution of equation $AX = B$ *can be presented as follows:*

$$\begin{aligned} X = \{ &< 1.5, \{0.11, 0.14, 0.2, 0.44, 0.57, 0.8\} >, \\ &< 3, \{0.25, 0.33, 0.38, 0.5, 0.62, 0.83\} >, \\ &< 4, \{0.17, 0.25, 0.33, 0.5, 0.75, 1\} >, \\ &< 5, 0.2, 0.25, 0.29, 0.6, 0.75, 0.86\} >, \\ &< 6, 0.33, 0.38, 0.44, 0.5, 0.56, 0.62 >\} \end{aligned}$$

The set X *is depicted in Fig. 1.*

Example 4 *Assume*

$$Q_x^s = \{< -20, \{0.1, 0.4\} >\} + \{< 4, \{0.3, 0.8\} >\}\, p_x$$

and the hesitant fuzzy demand function

$$Q_x^d = \{< 100, \{0.5, 0.7\} >\} + \{< -2, \{0.1, 0.5\} >\}\, p_x.$$

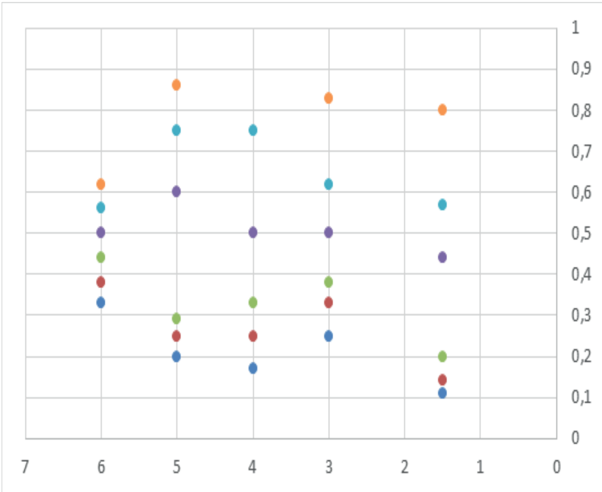

Figure 1. Representation of X from Example 3

The goal is to calculate the hesitant fuzzy market equilibrium price. The equilibrium condition is $Q_x^s = Q_x^d$. Therefore, the following equation must be solved:

$$\{< -20, \{0.1, 0.4\} >\} + \{< 4, \{0.3, 0.8\} >\}\, p_x \\ = \{< 100, \{0.5, 0.7\} >\} + \{< -2, \{0.1, 0.5\} >\}\, p_x.$$

The hesitant fuzzy price is $p_x = \{< 20, h_{p_x} >\}$ and for determining h_{p_x}, it is assumed that $h_t = \{h_1, h_2, \ldots, h_{16}\}$. Also, equations in (7) are constructed as the following ones:

$$0.3h_1 + 0.1 - 0.03h_1 = 0.1h_1 + 0.5 - 0.05h_1\,,$$

$$0.3h_2 + 0.1 - 0.03h_2 = 0.1h_2 + 0.7 - 0.07h_2\,,$$

$$0.3h_3 + 0.1 - 0.03h_3 = 0.5h_3 + 0.5 - 0.25h_3\,,$$

$$0.3h_4 + 0.1 - 0.03h_4 = 0.5h_4 + 0.7 - 0.35h_4\,,$$

$$0.3h_2 + 0.1 - 0.03h_2 = 0.1h_2 + 0.7 - 0.07h_2\,,$$

$$0.3h_5 + 0.4 - 0.12h_5 = 0.1h_5 + 0.5 - 0.05h_5\,,$$

$$0.3h_6 + 0.4 - 0.12h_6 = 0.1h_6 + 0.7 - 0.07h_6,$$

$$0.3h_7 + 0.4 - 0.12h_7 = 0.5h_7 + 0.5 - 0.25h_7,$$

$$0.3h_8 + 0.4 - 0.12h_8 = 0.3h_8 + 0.7 - 0.35h_8,$$

$$0.8h_9 + 0.1 - 0.08h_9 = 0.1h_9 + 0.5 - 0.05h_9,$$

$$0.8h_{10} + 0.1 - 0.08h_{10} = 0.1h_{10} + 0.7 - 0.07h_{10},$$

$$0.8h_{11} + 0.1 - 0.08h_{11} = 0.5h_{11} + 0.5 - 0.25h_{11},$$
$$0.8h_{12} + 0.1 - 0.08h_{12} = 0.5h_{12} + 0.7 - 0.35h_{12},$$
$$0.8h_{13} + 0.4 - 0.32h_{13} = 0.1h_{13} + 0.5 - 0.05h_{13},$$
$$0.8h_{14} + 0.4 - 0.32h_{14} = 0.1h_{14} + 0.7 - 0.07h_{14},$$
$$0.8h_{15} + 0.4 - 0.32h_{15} = 0.5h_{15} + 0.5 - 0.25h_{15},$$
$$0.8h_{16} + 0.4 - 0.32h_{16} = 0.5h_{16} + 0.7 - 0.35h_{16}.$$

By solving the above equations, the following result is obtained:

$$h_t = \left\{ \frac{2}{11}, \frac{5}{2}, 20, 5, \frac{10}{13}, 2, -\frac{10}{7}, 10, \frac{40}{67}, \frac{60}{69}, \frac{40}{47}, \frac{60}{57}, \frac{10}{43}, \frac{2}{3}, \frac{10}{23}, \frac{10}{11} \right\}.$$

Therefore, from (6) h_{p_x} is obtained as:

$$h_{p_x} \subseteq \left\{ \frac{2}{11}, \frac{10}{13}, \frac{40}{67}, \frac{60}{69}, \frac{40}{47}, \frac{10}{43}, \frac{2}{3}, \frac{10}{23}, \frac{10}{11} \right\}.$$

As a result, the market equilibrium price is as follows:

$$p_x = \left\{ \{< 20, h_{p_x} >\} : h_{p_x} \subseteq \left\{ \frac{2}{11}, \frac{10}{13}, \frac{40}{67}, \frac{60}{69}, \frac{40}{47}, \frac{10}{43}, \frac{2}{3}, \frac{10}{23}, \frac{10}{11} \right\}, [??] \right\}$$

Therefore, the market equilibrium price is approximately 20 tomans. However, determining the exact value of the membership degree is not possible. There are various possible values such that their accuracy is not certain. Hence, different degrees are considered. Finally, a decision maker can choose the final solution according to the perception of the real world problem.

8. Conclusion

Models of many economic problems, such as determining market equilibrium price, can be formulated in terms of linear equations. In many of these problems the parameters are ambiguous and uncertain, and this fact must be taken into consideration when modeling the respective equations. Whenever the parameters are hesitant fuzzy sets, one has to deal with hesitant fuzzy equations. As the world economy has faced recently a severe crisis, involving abruptly increasing degrees of uncertainty, hesitancy must be considered when, for instance, defining the market price in mathematical representation. Therefore, in this paper the tolerable solution set and the controllable solution set are defined in order to introduce and solve effectively the fully hesitant fuzzy equation $AX{=}B$ and the dual hesitant fuzzy equation $AX + B = CX + D$. Also, an application of hesitant fuzzy equation in determining the market equilibrium price is presented as a practical example. It is obvious that there can be more general studies on other types of problems that lead to one type of linear equation systems. At the end, various numerical examples are solved using the proposed method to show the simplicity and the effectiveness of this method.

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