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Voting and MCDM: the pedagogy of the Saari triangle*

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Abstract: The essay has a twofold objective: primarily, to present an application of voting theory as a possible evaluation method, and concurrently, to offer a pedagogic framework, based on that very application. Evaluation and certain notions of preference and value have common semantic roots. By equating preference and choice, we end up amidst social choice (SC) theory and voting methods, also manageable as joint decisions in multiple-criteria decision making (MCDM). With the aid of the Saari triangle some essential differences of pairwise and positional voting rules for up to three alternatives can be depicted. A voting or decision rule does not necessarily follow the true preferences of the actors, but may mirror the problematics of the chosen rule. The Saari triangle makes it possible to visualize some paradoxical results in the exemplary evaluations of digital websites through an imaginary case description via voting and MCDM. As candidates and voters in SC are put to stand for alternatives and criteria in MCDM, the methodological and pedagogical goals of the study are achieved.

Keywords: social choice, voting, Saari triangle, MCDM, pedagogy

1. Introduction

1.1. General background

The concept of evaluation is connected with the notions of preference, value, choice, and decision-making. The two value concepts of "better" and "equal in value to", are strict preference and indifference – symbolized as $\succ$ or P, and $\sim$ or I, respectively. The preference relation A$\succ$B, as an expression of value, means that A is more valuable than B, while A$\sim$B means that A and B are equally valued in numeral (cardinal or ordinal) terms. The basic notion

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of preference refers to subjective comparative evaluation (Hansson and Grüne-Yanoff, 2017). Preferences can also be incomparable or incomplete (Pini et al., 2011; Rabinowicz, 2012).

Social choice (SC) theory includes several models, which aggregate individual inputs (e.g., preferences, votes) into collective outputs (e.g., preferences, collective decisions). Two French mathematicians, Borda and Condorcet, were the pioneers of SC in the 18$^{\text{th}}$ century, continued with important developments during the 19$^{\text{th}}$ century, while later on the scientific community has been challenged by the Arrovian tradition (see Arrow, 1951). In recent years, the emphasis of the relevant research has been on computational social choice, see List (2013) and Brandt et al. (2016). There is a wide range of methods and tools designed to be utilized in SC, further applied in multi-criteria decision-making (MCDM) (Brams and Fishburn, 2002; Nurmi and Meskanen, 2000; Schoop and Kilgour, 2017).

1.2. Methodological preliminaries and assumptions

Different voting methods presume different types of information from the voters as input. The inputs, given by the voters, are called ballots, e.g., in the form of a ranking of the set of candidates. Let $N = \{1, 2, \ldots, n\}$ be a finite set of voters (treated as criteria in MCDM), and X a finite set of m candidates (alternatives in MCDM), with $m \geq 2$ ($|X| = m = 3$ in this essay). A binary relation over the set X describes the relative merits of any two outcomes in X with respect to some criterion. Voters have rational preferences, i.e. each voter's preference relation $\succ$ is a strict order (complete, transitive, and strict). No voter is able to order alternatives in a cycle. A (preference) profile P is a listing of all voters and their preferences (Saari, 2021; Pacuit, 2019; Zahid, 2012).

A voting procedure is defined by two characteristics: 1) the type of a vote (a ballot), and 2) the aggregation rule, by which the votes are counted to find the winner or the resulting preference order. Upon denoting by $L(X)$ the set of all linear orderings on X, by $W(X)$ the set of all weak orderings on X (i.e. allowing for ties), and by $C(X)$ the set of all complete orderings on X (i.e. accounting for all the candidates), the inclusion relations $L(X) \subseteq W(X) \subseteq C(X)$ hold. A profile P is a function P: $N \to L(X)$. $L(X)^N$ denotes the set of all profiles. A social choice function (a voting rule), denoted *SCF*, is defined as a mapping from (admissible) ballot profiles to subsets of (feasible) candidates, formally *SCF*: $L(X)^N \to X$. In other words, *SCF* simply takes a profile of voter preferences as input and produces a winner as output (Brams and Fishburn, 2012; Saari, 2021; Zahid, 2012). To repeat, in this paper, the voting rules are assumed to take strict preferences as inputs, and return one candidate (alternative) − the winner. Multiple winners are theoretically possible, but not taken here into account.

The simplest class of voting procedures is that of two-candidate elections. The majority rule always presupposes voting over pairs of alternatives: candidate A beats candidate B if and only if, in a pairwise comparison, a majority of voters prefers A to B. We speak of a Condorcet winner when we deal with a candidate, which, in a pairwise comparison, defeats every other candidate. A Condorcet loser is a candidate, which, in a pairwise comparison, is defeated by all other candidates. The absence of a Condorcet winner is called the Condorcet paradox. The majority rule may produce cycles, which contradicts the assumption of the transitivity of rational preferences (Brandt et al., 2016; Brams and Fishburn, 2002).

The three positional rules, which are applied in this work, are described as follows. According to the plurality (Pl) rule ("vote for one") each voter votes for a single candidate: the candidate with the most votes wins. The antiplurality (Apl) rule ("vote for two") states that each voter votes for two candidates, and the candidate having obtained the most votes is the winner. The third positional rule here utilized is the Borda Count (BCo). The Borda Count is also a scoring rule, as it calculates a score based on weights specifically determined by voters' rankings. In the case of n candidates, each voter ranks the candidates by giving n-1 points to the most preferred candidate, n-2 points to the candidate ranked second, ..., 1 point to the candidate ranked second to last, and 0 points to the bottom-ranked candidate. The candidate with the highest score wins, see Section 2 (Pacuit, 2019; Saari, 2008).

The societal outcomes (the aggregates) produced by the positional methods can change radically when candidates (alternatives) are dropped or added. With fixed voter profiles, by varying the choice of positional methods, multiple contradictory outcomes can be generated. Different methods, used by the same group of sincere (i.e. non-strategic) voters, can end up with different outcomes. As voting rules are prototypes for various aggregation rules, same kinds of inconsistencies may appear, e.g., in engineering decisions, statistics and MCDM (Nurmi and Meskanen, 2000; Saari, 2008).

2. The Saari triangle as a method

The methodological essence of the Saari triangle dates back to the Borda–Condorcet debate. Borda preferred a positional procedure, which gives a complete ranking from the best to the worst candidate. On the contrary, Condorcet suggested a pairwise majority ranking, where one candidate beats every other candidate, being the Condorcet winner. The pros and cons of these methods have brought an endless academic discussion (see, e.g., Brams and Fishburn, 2002; Saari, 1999).

According to Saari, social choice and voting theory are most complex in their nature. The standard way to graph the voting outcomes in the simplest case of three alternatives fails because the six-dimensional profile space does not admit a basic graphical expression. Profiles for six alternatives define a 6!=720 dimensional space. An election with ten candidates – possible in presidential primaries in the USA – encompasses the 10! = 3 628 800 dimensional space. With the aforementioned examples Saari raises awareness of the (computational) complexity of the voting contexts, the "curse of dimensionality" (Saari, 2008).

Counter-arguments, promoting the "blessing of dimensionality" have been presented, based on methodological advances. The degree of dimensionality has an impact on, for instance, the distribution of data points in the decision space and feasibility of algorithmic solutions (Gorban, Makarov and Tyukin, 2014). In theory and practice, input-output systems, with the processes of aggregation methods as mediators, have many inconsistencies and restrictions both on the input and the output side. Brams and Fishburn (2002) repeatedly use expressions "admissible ballots", "admissible strategies", and "feasible candidates". As mentioned in Section 1 of this paper, definitions of incomparability and incompleteness of preferences are essential in the whole input-output system (Pini et al., 2011). In linear programming, feasibility constraints restrict reasonable options (referred to as loss of options) (Brandt et al., 2016). Various near-synonymic concepts are a part of the (in)comparability-feasibility problem (Pini et. al., 2011; Endriss, 2018; Gerasimou, 2018). Rapid progress, made in the disciplines of computational social choice and artificial intelligence triggered off an ethical discussions on the proper ways of design and use of the respective technologies (Baum, 2020).

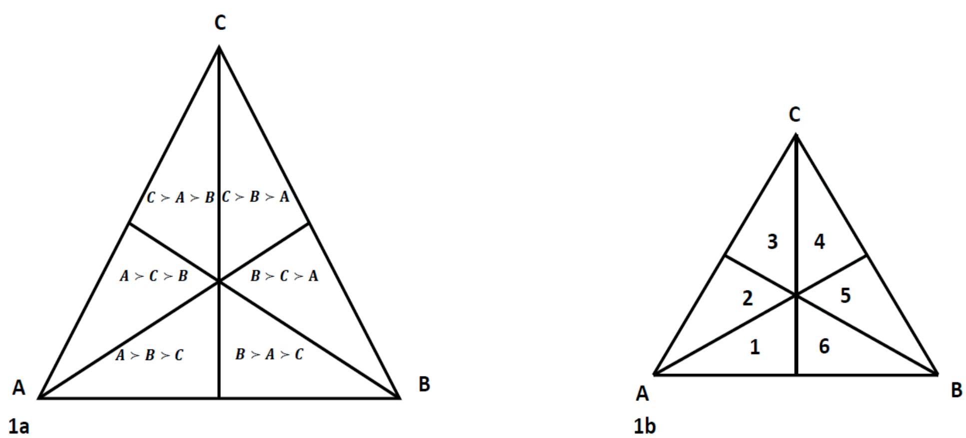

Figure 1. a. The Saari triangle (restructured from notations and figures in Saari, 1999. Figure 1.b. The six (3!=6) voter types (Saari 1992; Saari 1994)

The Saari triangle is a geometric profile representation (i.e. an equilateral triangle simplex). In Fig. 1a, each candidate, A, B and C, is placed at the vertex of the triangle. Each point in the triangle defines (corresponds to) a ranking based on its distance to each vertex, according to the rule "the closer the better". Thus, the triangle is divided into thirteen ranking regions, six of which are the strict ones, represented by the small triangles, and the remaining seven ones are parts of lines, with at least one tie (meaning: a tie ≡ indifference ≡∼). Accordingly, the six possible, strict preference rankings are:

1. A ≻B ≻ C; 2. A ≻ C ≻ B; 3. B ≻ A ≻ C ; 4. B ≻ C ≻ A; 5. C ≻ A ≻ B; 6. C ≻ B ≻ A

(see Hansson and Grüne-Yanoff, 2017). The Saari triangle actually constitutes two interpretative applications of the plurality ternary diagrams: 1) the point-share triangle and 2) the profile triangle (Eggers, 2020).

Thus, Fig. 2, as an instantiation of the Saari triangle, presents the essential properties of the point-share and profile triangles in the same context. Further, Fig. 2 depicts the way the number of voters with a specific ranking is related to the associated ranking regions, corresponding to the example profile from Table 1. For instance, the number 4 in the upper small triangle of the example case is in the region nearest to the C vertex, next nearest to B and farthest from A, i.e. the ranking is C≻B≻A.

The midpoint of the Saari triangle represents a complete tie between the candidates (alternatives) with equal shares of votes for each. The median line, originating from any of the vertices and dividing the opposite side of the triangle into the parts of equal length, represents a tie (e.g., the median from A to BC represents a tie B∼C).

Table 1. An example of a preference profile (Saari, 2008)

An example of a preference profile			
Number of votes	Ranking	Number of votes	Ranking
5	A≻B≻C	4	C≻B≻A
2	A≻C≻B	2	C≻A≻B
4	B≻C≻A	0	B≻A≻C

A voter's type is defined by the ranking of the candidates, which, in the case of three candidates, results in six (3!=6) voter types, as shown in Fig. 1b. Each voter is assumed to have a strict linear ordering of the candidates. A profile determines the number of voters of each type. If $p(j)$ denotes the fraction of all voters that are of the j^{th} type, j = 1,..., 6, then a (normalized) profile is the vector $p = [p(1), p(2), p(3), p(4), p(5), p(6)]$, i.e. a rational point in the simplex $Si(3!{=}6)$. According to Saari (1994), most points in $Si(6)$ have irrational values, which can be identified using the limits of integer points or from the profiles of weighted voting systems. By defining an election as a mapping from all points in $Si(6)$, much wider applications are available. The election outcomes of the unanimity profiles are the vertices of the representation triangle, and the space of the normalized election outcomes is the full simplex $Si(3)$ (Saari, 1994; 1999).

The concept of a procedure line is central to understanding the theoretical and practical aspects of the Saari triangle, especially the formation of positional voting paradoxes based on a fixed preference profile. Saari (1994) pinpoints that the reason for paradoxes lies in the differences between the Pl and Apl methods. The former ignores a voter's second ranked candidate, while the latter treats the same candidate as a top-ranked one. The geometrical explanation for the differences is that the line segment of (normalized) voting vectors $w(s)$, s belonging to [0,$1/2$] (in three dimensions), has the normalized voting vectors $w(0)$ = (1, 0, 0) and $w(1/2)$= ($1/2$, $1/2$, 0) as their endpoints. Of these, $w(0)$ represents the Pl vote and $w(1/2)$ the Apl vote. The normalized form for BCo is $w(1/3)$= (2/3, 1/3, 0) (Saari, 1994; 1992). Further, any point in the convex set (hull) $CH[w(s)]$ is an election outcome for some profile, and the weights for the convex representation of this point define the associated profile. The convex hulls relate to each other as follows: $CH[w(1/2)] \subseteq CH[w(s)] \subseteq CH[w(0)] = Si(3)$, i.e. the representation triangle. The two limiting convex hulls are $CH[w(0)]$ and $CH[w(1/2\ )]$. The Pl outcome is the function $f[p,\ w(0)]$, the possible values of which cover every point in the representation triangle [$=Si(3)$]. The Apl outcome is the function $f[p,w(1/2)]$: its possible values are situated in the open, small triangle connecting the middle points of the edges of the representation triangle (Saari, 1994).

Saari (1994) defines $Sup(p)$ ={all election rankings of 3 candidates that can arise from profile p with changes in the positional voting method}. With certain assumptions [$w(s1) \neq w(s2)$], using the barycentric point I of the full simplex $Si(3)$ and the ball $B(I,r)$, with radius r>0, it can be proven that by selecting any two election points $q(j)$, $j = 1, 2$, in the ball B, there is a one-dimensional line of profiles $L[w(s1),\ w(s2)]$ ($q1$, $q2$) $\in Si(6)$. If p belongs to $L[w(s1),\ w(s2)]$ ($q1$, $q2$), then $f[p, w(s1)] = q(1)$ and $f[p, w(s2)] = q(2)$, and further, for a profile p, belonging to $Si(6)$ and $w(s)$, the election vector $f[p,w(s)]$, belonging to $Si(3)$, is on the line segment, connecting $f[(p,w(0)]$ and $f[p, w(1/2)]$. More precisely,

$f[p, w(s)] = (1 - s)f[p, w(0)] + 2sf[p, w(1/2)]$; where s belongs to $[0, 1/2]$.

The line segment, defined above, is called the procedure line $PL(p)$ for the profile p (Saari, 1994; 1992). By varying $w(s)$ and holding a profile p fixed, the function $f(p$, -) becomes a linear mapping of the voting vectors. The set of positional voting vectors is the convex hull, defined by the two vertex voting vectors, $w(0)$, $w(1/2)$. The procedure line of the election outcomes with vertices $f[p,w(0)]$ and $f[p,w(1/2\ )]$ is a (linear) transformation from the line of voting vectors with vertices $w(0)$, $w(1/2)$, steered by the linear mapping. The transformation emphasizes the way, in which the different $w(s)$ election outcomes along the procedure line reflect the weight each $w(s)$ puts on the voter's second-ranked candidate. The properties of the procedure line completely determine the properties of $Sup(p)$ (Saari, 1994).

Later on, Saari has restructured the condition of the parameter s to range from the interval $0 \leq s \leq 1/2$ to the interval $0 \leq s \leq 1$. As Saari (2008) puts it: "A positional election assigns points to candidates based on how each voter positions them on the ballot: it is defined by the voting vector [$w(1)$,$w(2)$,$w(3)$], with $w(1) \geq w(2) \geq w(3)$ and $w(1) \geq w(3)$, where $w(j)$ points are assigned to the candidate a voter positions at the j^{th} place. To normalize these rules, let $w(3) = 0$ and divide the weights by $w(1)$ to obtain $w(s) = [w(1)/w(1), w(2)/w(1), 0] = (1,\ s,\ 0)$ for $0 \leq s \leq 1$." And further on: "A candidate's election tally is her plurality tally plus s times the number of voters who have her second ranked." To sum up, procedure lines are straight lines, which have the Pl and Apl outcomes as their endpoints, and all other (infinitely many) positional outcomes are between the two, with the BCo outcome situated two thirds from the Pl endpoint towards the Apl endpoint (i.e. on the two-dimensional plane) (Saari and Barney, 2003).

The procedure line is a tool to analyze the positional voting paradoxes (with three candidates or alternatives). It is important to notice that near I, belonging to $Si(3)$, all thirteen ranking regions are very near to each other. For example, the complete tie can be broken in twelve different ways. Two election outcomes (tallies) can vary very minutely between each other, and still it is possible to place the procedure line in positions where $f[p,w(s1)]$ and $f[p,w(s2)]$ have specific rankings of their own. Geometry clarifies the discrepancy, where close election outcomes lead to different election rankings. A theorem, constructed by Saari (1994, Theorem 2.4.3), says that the j^{th} election ranking is in $Sup(p)$ if and only if (the procedure line) $PL(p) \cap R(j)$ is not an empty set, where $R(j)$ is the reversed ranking. Further, on the condition that the parameter k is an integer, $1 \leq k \leq 7$, there exists a profile p such that $Sup(p)$ has precisely k rankings; and vice versa, if $Sup(p)$ has k rankings, then $1 \leq k \leq 7$. When k is an integer, $1 \leq k \leq 4$, there exists a profile p so that $Sup(p)$ has exactly k strict rankings; and conversely, if $Sup(p)$ has k strict rankings, then $0 \leq k \leq 4$. If the election ranking of the plurality and antiplurality vote are the reversals of each other, then $Sup(p)$ has either three or seven entries. In the case of three entries, the entries of $Sup(p)$ are the rankings of the plurality vote, the antiplurality vote, and I. The election ranking for p is the same for all choices of $w(s)$ if and only if the plurality and antiplurality rankings are the same (SIC! primarily tie rankings are omitted in this summary) (Saari, 1994).

The larger is the value of the parameter k, the more conflicting it is to make valid conclusions about the results. Interestingly, when a profile p is fixed, the values of probability of the voting outcomes depend on whether the value of k is odd or even. When k is even, the probability is zero (k = 2, 4 or 6). The probability to get seven different positional outcomes is 0.06. The probability to get only one positional outcome ($k = 1$) is 0.31, the probability to get three different positional outcomes is 0.44, and that of getting five different positional outcomes is 0.19 (i.e. 0.06 + 0.31 + 0.44 + 0.19 = 1) (Saari and Tataru, 1999).

In the example case, which is depicted in Fig. 2, the procedure line determines the profile to produce seven different election rankings (the parameter $k = 7$), four of which are strict ones and three are ties. This is the maximum variation when positional rules are applied in the context of three candidates (alternatives). The minimum number is one ($k = 1$), meaning that the procedure line is situated only in one strict ranking region. In the case of three ranking outcomes ($k = 3$), the procedure line crosses over at least one median line, and the ranking outcomes are two strict ones and a tie between two candidates (alternatives). In the three-valued case there is one exception: when the line passes through the midpoint of the triangle, all median lines are traversed, producing a complete tie, and the two strict rankings reverse each other. Five different outcomes ($k = 5$) produce three strict rankings and two ties. With the even ranking outcomes from two to six ($k = 2, 4, 6$) either of the two endpoints of the procedure line is a tie (or indifference) (Saari, 1999; Saari and Tataru, 1999; Saari and Barney, 2003).

For a given voting procedure, each choice option receives a number of points reflecting its ranking. (As a reminder, the equation by Saari is repeated here: $w(s)$ $[w(1)/w(1), w(2)/w(1), 0] = (1, s, 0)$ for $0 \leq s \leq 1$.) The positional rule of "vote for one" is the plurality rule (Pl), represented by the vector (1, 0, 0), the parameter $s = 0$; the positional rule of "vote for two" is the antiplurality rule (Apl), represented by the vector (1, 1, 0), with $s = 1$. The Borda Count (BCo) can be expressed as the vector (2, 1, 0), which becomes (1, $^1/_2$, 0) after normalization, with $s = {}^1/_2$. The numbers of votes from Table 1 are visualized accordingly in Fig. 2. For example, to tally the positional-method ballots, B is top-ranked in the two regions nearest to B as the vertex, make the addition 0+4=4. Next, B is the second ranked in the two adjacent regions, containing 5 and 4. B's final tally is counted $(0+4)+s(5+4) = 4+9s$, as shown in Fig. 2 (Saari, 1999, 2008; Saari and Barney, 2003).

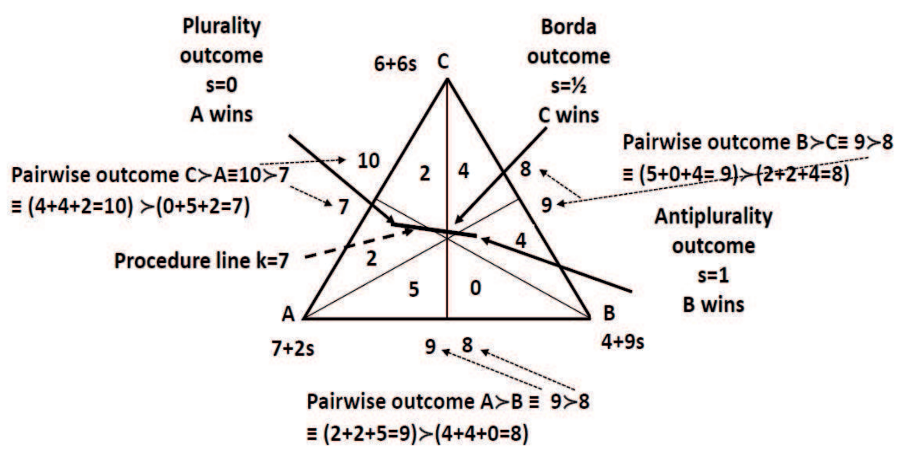

Figure 2. An instantiation of the Saari triangle (Saari, 2008)

The numbers outside the equilateral triangle in Fig. 2 are the results of pairwise comparisons, which are summed up from the given votes in the three ranking regions on the same side of the median line bordering the two candidates (alternatives) in comparison. For example, B gets 5+0+4 = 9 votes compared to C's 2+2+4 = 8 votes, with the conclusion B≻C. In the example case, pairwise outcomes end up in a cycle. No rational choice can be made (Saari, 2008).

Two kinds of symmetries in the representation triangle are responsible for all the differences involved in the pairwise and positional ranking outcomes (this observation extending to any number of candidates!). The Condorcet triplet has no effect on positional rankings, but can produce pairwise cycles due to the deficient recognition of transitive preferences. The reversal symmetry explains all the paradoxes in positional settings, except for the Borda Count. Saari lists several interrelated conclusions, based on the aforementioned methodological symmetries and paradoxes of voting outcomes. A plethora of voting methods do not identify all relevant data, with lost information as a consequence (Saari, 1999, 2008; Saari and Barney, 2003).

3. The example case applied in the MCDM context

Fig. 1a,b and Fig. 2, along with Table 1 in the previous section present the basic theoretical construction of the Saari triangle, with an example case, established by a fixed preference profile. The example case shows that when voting according to the Pl rule, A has the winning score, whereas the Apl rule sets B as the winner, and C wins in the case of the BCo rule.

Table 2. Weighted criteria of the websites A, B, and C

Weighted criteria of the websites A, B and C			
Weights	Criteria	Weights	Criteria
5	Usability (US)	4	Interoperability (I)
2	Flexibility (F)	2	Effectiveness (E)
4	Reliability (R)	0	Unspecified (UN)

The choice of the rule is crucial for reaching "the right decisions" (Saari, 1999, 2008). These results can be applied in the context of MCDM as candidates stand for alternatives and voters for criteria (Nurmi and Meskanen, 2000). Referring to the example case of Fig. 2, the alternatives A, B and C could represent, say, three (imaginary) digital websites of service providers in healthcare (Vehko, Ruotsalainen and Hyppönen, 2019), as represented for this new interpretation in Table 2 and Fig. 3. The criteria, with the weight of importance could be, e.g., as this is shown in Table 2: 1) overall usability of digital services (US), with the weight 5 (= 5 votes); 2) flexibility between digital and face-to-face services (F), with the weight 2 (= 2 votes); 3) reliability of information (R), with the weight 4 (= 4 votes); 4) interoperability of the information systems (I), with the weight 4 (= 4 votes); 5) effectiveness (of the services) (E), with the weight 2 (= 2 votes), and (potentially many) other unspecified criteria (UN), with a zero weight (= 0 votes).

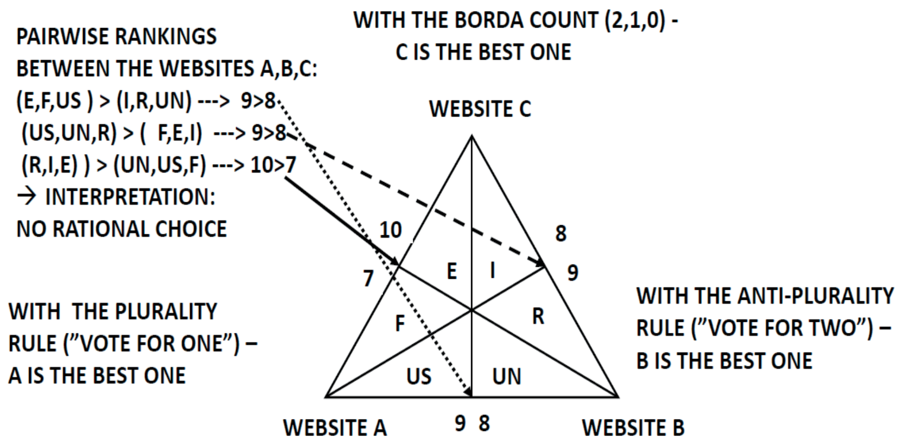

Figure 3. The websites of A, B, and C with the six criteria in an MCDM context

The conclusive decision (or the best website among those evaluated) depends on the voting method chosen – and evidently, on the process of naming the justified criteria to be assessed and analyzed. The irrational, pairwise cycles reside also in the MCDM context (Nurmi and Meskanen, 2000). The preferential order in a pairwise manner between two triplets of criteria – e.g. (E,F,US) >(I,R,UN) – cannot be reasonably interpreted because of the irrational cyclic result produced by the initial voting procedure. But one question remains! Which one is the best digital website? Saari himself prefers the Borda Count to the plurality and antiplurality rules: he would choose the website C (Saari, 1999, 2008).

4. Conclusion

The mathematical constitution of the Saari triangle is most challenging, and requires a thorough knowledge of the area to be fully understood. The geometrical analysis of the formation of inconsistencies in both voting and MCDM benefits the whole community of researchers and appliers. The visualizations of the Saari triangle are appropriate especially for non-mathematicians. Everyone interested in testing the properties of the Saari triangle as a digital user interface can make acquaintance with it via a web-based application (Romney, Tan and Tang, 2016). The leap from voting to MCDM may contain debatable interpretations to be further discussed.

The pedagogical relevance of the Saari triangle could be empirically tested. In the same spirit of pedagogy, Saari (2019) has most interestingly analyzed his relation to the Arrovian tradition. Solutions to trans- and interdisciplinary problems presented by Saari (2019) can be summarized in the known adage: "The whole can differ from the sum of its parts."

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