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Parameter estimation of MIMO two-dimensional ARMAX model based on IGLS method*

by

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Abstract: This paper presents an iterative method for the unbiased identification of linear Multiple-Input Multiple-Output (MIMO) discrete two-dimensional (2D) systems. The system discussed here has Auto-Regressive Moving-Average model with exogenous inputs (ARMAX model). The proposed algorithm functions on the basis of the traditional Iterative Generalized Least Squares (IGLS) method. In summary, this paper proposes a two-dimensional Multiple-Input Multiple-Output Iterative Generalized Least Squares (2DMIGLS) algorithm to estimate the unknown parameters of the ARMAX model. Finally, simulation results show the efficiency and accuracy of the presented algorithm in estimating the unknown parameters of the model in the presence of colored noise.

Keywords: ARMAX model, discrete 2D systems, iterative identification, MIMO systems, parameter estimation

1. Introduction

Natural phenomena and physical processes such as electromagnetic waves, sound diffusion, heat transfer, moisture spreading, power transmission lines, chemical processes, biological systems, image processing, wireless networks and so on mostly have mathematical models with functions of more than one independent variable, such as time and space, see, for instance, Marszalek (1984), Fornasini and Marchesini (1978), Bracewell (1995), Kaczorek (1985), Wellstead, Zarrop

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and Duncan (1999), Young, Garnier and Gilson (2008), Zeinali and Shafiee (2016), Ding, Du and Li (2015), Amiri Mehra et al. (2020), Sadeghi, Shafiee and Shafieirad (2013), Shafieirad and Shafiee (2008), Abbasi et al. (2020). In fact, the phenomena in question are both distributed in space and depend on time, which means that the input signals (main input and noise) and output signals are functions of time and space. Whenever the signals depend on two independent variables, such as space (in two dimensions) or space-time, the system is called a two-dimensional (2D) system. Actually, the majority of physical systems have more than one input and output, this situation being referred to through the notion of Multiple-Input Multiple-Output systems, or the frequently used abbreviation MIMO.

There are two basic procedures of modelling the natural phenomena and physical processes. The first one is analytical modeling or white-box modeling, within which the model is constructed by using physical laws. With this approach, the model becomes often too complicated, and therefore the respective system becomes too difficult to analyze and control. The second is experimental modeling or system identification, which includes the black-box and grey-box modeling. The experimental modeling is performed using the input-output sampled data. If the model structure is known a priori and the identification problem is limited to estimating the values of the unknown coefficients of the model, then the respective identification task is called parameter estimation.*

There are many developments, both scientific and pragmatic, in the field of parameter estimation of 2D systems. Thus, in particular, in Ding, Du and Li (2015) and Chen and Kao (1979), the parameters of 2D Auto-Regressive with Exogenous input (ARX) and Finite Impulse Response (FIR) models are estimated based on the Least Square (LS) method. In Ali, Chughtai and Werner (2010), Shafieirad, Shafiee and Abedi (2013), and Shafieirad, Shafiee and Abedi (2014a), the one-dimensional Instrumental Variable (IV) techniques are extended for identification of 2D systems. The parameter estimation of 2D systems with the state-space model is presented in Fraanje and Verhaegen (2005), Wang et al. (2017), and in Zhao et al. (2017). Also, the recursive identification of 2D continuous systems with Box-Jenkins (BJ) model is presented in Shafieirad, Shafiee and Abedi (2014b). However, to the best of our knowledge, parameter estimation of Multiple-Input Multiple-Output discrete 2D ARMAX (MIMO 2D ARMAX) model has not been fully investigated. Due to MIMO structure and existing two independent variables in MIMO 2D ARMAX, parameter estimation appears to be challenging and has motivated us to carry out the present study.

This paper is organized as follows: In Section 2, the MIMO 2D ARMAX model is described. Then, in Section 3, the one-dimensional iterative generalized

*Actually, these two basic approaches are not necessarily totally "disjoint", both because parts of the model can be constructed with the use of any of the two approaches, and because in the analytic approach it is almost always necessary to estimate some of the parameters, which remain "outside" of the theory used to construct the model (eds.).

least square method is extended for parameter estimation of MIMO 2D ARMAX model. Finally, the simulation results are presented to show the accuracy and efficiency of the proposed algorithm.

2. Model description

In this section, the MIMO 2D ARMAX model and the regression equation of the system that is shown in Fig. 1, are presented.

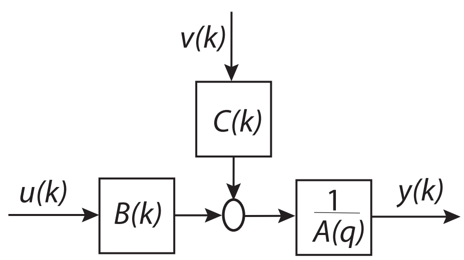

Figure 1. The ARMAX model block diagram

2.1. System model

A linear MIMO 2D discrete system with ARMAX model is expressed in the form of a partial difference equation (PDE) as follows:

$$\begin{aligned}&\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} A_{i,j}y\left(k-i,p-j\right)=\\&\sum_{i=0}^{m_1}\sum_{j=0}^{m_2} B_{i,j}u\left(k-i,p-j\right)+\sum_{i=0}^{q_1}\sum_{j=0}^{q_2} C_{i,j}v\left(k-i,p-j\right)\end{aligned}\qquad(1)$$

where $y=[y_1,y_2,...,y_n]^T\in R^{n\times 1}$, $u=[u_1,u_2,...,u_m]^T\in R^{m\times 1}$, and $v=[v_1,v_2,...,v_n]^T\in R^{n\times 1}$ are output, input and white noise vectors, respectively. The unknown coefficient matrices (i.e. ARMAX model parameters) in (1) are defined as follows:

$$A_{i,j}=[a_{i,j}^1,a_{i,j}^2,...,a_{i,j}^n]^T\in R^{n\times n};\ i=0,1,...,n_1,\ \ j=0,1,...,n_2$$

$$B_{i,j}=[b_{i,j}^1,b_{i,j}^2,...,b_{i,j}^n]^T\in R^{n\times m};\ i=0,1,...,m_1,\ \ j=0,1,...,m_2$$

$$C_{i,j}=[c_{i,j}^1,c_{i,j}^2,...,c_{i,j}^n]^T\in R^{n\times n};\ i=0,1,...,q_1,\ \ j=0,1,...,q_2$$

where $a_{i,j}^{lT}$, $b_{i,j}^{l^T}$, and $c_{i,j}^{l^T}$ are the l-th rows of matrices $A_{i,j}$, $B_{i,j}$, and $C_{i,j}$, respectively, in which $A_{00}=I$ and $C_{00}=I$. In here, n_1 and n_2 are the highest difference orders of output y in PDE (1) with respect to the variables k and p, respectively. Also m_1 and m_2 are the highest difference orders of input u in PDE (1) with respect to the variables k and p, respectively. The pair (n_1, n_2) is called the order of noise-free system and (q_1, q_2) is called the order of noise dynamics. Here, we suppose the model is proper, i.e. $n_1, n_2 \geq \max(m_1, m_2)$.

The dynamics of additive colored noise w is defined as follows:

$$\sum_{i=0}^{n} {}_1 \sum_{j=0}^{n_2} A_{i,j} w\left(k-i, p-j\right) = \sum_{i=0}^{q_1} \sum_{j=0}^{q_2} C_{i,j} v\left(k-i, p-j\right). \quad (2)$$

The output vector of MIMO 2D system can be written down in the following manner:

$$\begin{aligned} y\left(k,p\right) = -A\left(z_1^{-1}, z_2^{-1}\right) y\left(k,p\right) + \mathrm{B}\left(z_1^{-1}, z_2^{-1}\right) u\left(k,p\right) + \\ +C\left(z_1^{-1}, z_2^{-1}\right) v\left(k,p\right), \end{aligned} \quad (3)$$

where

$$y(k,p) = [y_1(k,p), y_2(k,p), ..., y_n(k,p)]^T \in R^{n\times 1}$$

and

$$u(k,p) = [u_1(k,p), u_2(k,p), ..., u_m(k,p)]^T \in R^{m\times 1}.$$

Further, the matrices A, B, and C constitute the polynomial matrices that form the main transfer matrix of the system, being equivalent to the transfer function in MIMO systems, and are defined as:

$$A\left(z_1^{-1}, z_2^{-1}\right) = \sum_{i=0}^{n_1} \sum_{j=0}^{n} {}_2 A_{i,j} z_1^{-i} z_2^{-j}; \quad i+j \neq 0$$

$$B\left(z_1^{-1}, z_2^{-1}\right) = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} B_{i,j} z_1^{-i} z_2^{-j}$$

$$C\left(z_1^{-1}, z_2^{-1}\right) = I + \sum_{i=0}^{q_1} \sum_{j=0}^{q_2} C_{i,j} z_1^{-i} z_2^{-j}; \quad i+j \neq 0,$$

where I is the unit matrix. Note that (3) is written in terms of combination of time-space (k,p) domain and complex frequency $\left(z_1^{-1}, z_2^{-1}\right)$ domain. Such notation is very common in the literature and is only used to show the output relationship with polynomials of system transfer matrix (Young, Garnier and Gilson, 2008; and Shafieirad, Shafiee and Abedi, 2014b).

2.2. Regression equation

The observation equation or the regression equation of MIMO 2D system can be written down as follows:

$$y\left(k,p\right)^T = \phi\left(k,p\right)^T \Theta + e\left(k,p\right)^T, \quad (4)$$

where the matrices ϕ and Θ are the variable and parameter matrices of the noise-free system, respectively, and $e(k,p)$ is the observation error term. The matrices in (4) are defined as follows:

$$\begin{aligned}\phi(k,p)^T = [&-y(k-1,p)^T, -y(k,p-1)^T, -y(k-1,p-1)^T, \ldots,\\ &-y(k-n_1,p-n_2)^T, u(k,p)^T, u(k-1,p)^T,\\ &u(k,p-1)^T, \ldots, u(k-m_1,p-m_2)^T]\end{aligned} \tag{5}$$

and

$$\Theta = \begin{bmatrix} a^1_{1,0} & a^2_{1,0} & \cdots & a^n_{1,0} \\ a^1_{0,1} & a^2_{0,1} & \cdots & a^n_{0,1} \\ a^1_{1,1} & a^2_{1,1} & \cdots & a^n_{1,1} \\ \vdots & \vdots & \ddots & \vdots \\ a^1_{n_1,n_2} & a^2_{n_1,n_2} & \cdots & a^n_{n_1,n_2} \\ b^1_{0,0} & b^2_{0,0} & \cdots & b^n_{0,0} \\ b^1_{1,0} & b^2_{1,0} & \cdots & b^n_{1,0} \\ \vdots & \vdots & \ddots & \vdots \\ b^1_{m_1,m_2} & b^2_{m_1,m_2} & \cdots & b^n_{m_1,m_2} \end{bmatrix}. \tag{6}$$

In the next section, the parameter estimation algorithm is presented.

3. 2D MIMO Iterative Generalized Least Square Algorithm (2DMIGLS)

In (4), if $e(k,p)$ is white noise, i.e. $C\left(z_1^{-1}, z_2^{-1}\right) = I$, the system model is converted to the ARX model and the unknown coefficient matrices can be estimated using Ordinary Least Square method (OLS) without any estimation bias. However, if $e(k,p)$ is colored noise, i.e. $C\left(z_1^{-1}, z_2^{-1}\right) \neq I$, this colored noise should be whitened before using the OLS method. In order to whiten the colored noise, matrix C must be known. If C is known, the colored noise $e(k,p)$ turns into white noise by left multiplying Eq. (3) with filter $C\left(z_1^{-1}, z_2^{-1}\right)^{-1}$, and then the OLS method will be applicable. In order to find C^{-1}, an estimation for $e(k,p)$ as $\hat{e}(k,p)$ must be generated first. This estimation can be obtained from $\hat{e} = y - \phi\hat{\Theta}$ by using the estimated matrix $\hat{\Theta}$, which is determined using the estimations $\hat{A}$ and $\hat{B}$.

By comparing (3) with (4), we obtain:

$$e(k,p) = C\left(z_1^{-1}, z_2^{-1}\right) v(k,p). \tag{7}$$

According to (7), one can get:

$$C\left(z_1^{-1}, z_2^{-1}\right)^{-1} e\left(k, p\right) = v\left(k, p\right). \tag{8}$$

The filter $C\left(z_1^{-1}, z_2^{-1}\right)^{-1}$ is an Infinite Impulse Response (IIR) filter. Therefore, in order to estimate this filter, a FIR approximation can be formulated as follows:

$$C\left(z_1^{-1}, z_2^{-1}\right)^{-1} \triangleq D\left(z_1^{-1}, z_2^{-1}\right) \simeq I + \sum\nolimits_{i=0}^{r_1} \sum\nolimits_{j=0}^{r_2} D_{i,j} z_1^{-i} z_2^{-j}; \quad i, j \neq 0, \tag{9}$$

where $D_{i,j} = [d_{i,j}^1, d_{i,j}^2, ..., d_{i,j}^n]^T \in R^{n\times n}; \quad i = 0, 1, ..., r_1; \qquad j = 0, 1, ..., r_2.$

Since the filter $D\left(z_1^{-1}, z_2^{-1}\right)$ is an IIR filter, by inverting $D\left(z_1^{-1}, z_2^{-1}\right)$, the $C\left(z_1^{-1}, z_2^{-1}\right)$ coefficients are obtained as shown further on in this paper, in the simulation section.

By the substitution of $e\left(k, p\right)$ by $\hat{e}\left(k, p\right)$ in (8) and using (9) we obtain:

$$D\left(z_1^{-1}, z_2^{-1}\right) \hat{e}\left(k, p\right) = v\left(k, p\right). \tag{10}$$

The regression equation for $\hat{e}\left(k, p\right)$ can be obtained from (10) as follows:

$$\begin{aligned} \hat{e}\left(k, p\right) = -D_{10}\left(z_1^{-1}, z_2^{-1}\right) \hat{e}\left(k, p\right) - D_{01}\left(z_1^{-1}, z_2^{-1}\right) \hat{e}\left(k, p\right) - \\ -D_{r_1 r_2}\left(z_1^{-1}, z_2^{-1}\right) \hat{e}\left(k, p\right) + v\left(k, p\right). \end{aligned} \tag{11}$$

Equation (11) can be written down in the following manner:

$$\hat{e}\left(k, p\right)^T = \eta\left(k, p\right)^T \Upsilon + v\left(k, p\right)^T \tag{12}$$

in which

$$\eta(k, p)^T = [-\hat{e}\left(k-1, p\right)^T - \hat{e}\left(k, p-1\right)^T - \hat{e}\left(k-r_1, p-r_2\right)^T] \tag{13}$$

and

$$\Upsilon = \begin{bmatrix} \mathbf{d}_{1,0}^1 & \mathbf{d}_{1,0}^2 & \cdots & \mathbf{d}_{1,0}^n \\ \mathbf{d}_{0,1}^1 & \mathbf{d}_{0,1}^2 & \cdots & \mathbf{d}_{0,1}^n \\ \mathbf{d}_{1,1}^1 & \mathbf{d}_{1,1}^2 & \cdots & \mathbf{d}_{1,1}^n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{d}_{r_1,r_2}^1 & \mathbf{d}_{r_1,r_2}^2 & \cdots & \mathbf{d}_{r_1,r_2}^n \end{bmatrix}. \tag{14}$$

It is obvious that the observation error in (12) is white noise and the unknown coefficient matrices D_{ij} can be estimated using OLS.

Now, by considering (3) and assuming that the estimation of matrix D ($\hat{D}$=$\hat{C}^{-1}$) is known, we have the following regression equation:

$$\begin{aligned}&\hat{D}\left(z_1^{-1},z_2^{-1}\right)y\left(k,p\right)=\\&\quad-\hat{D}\left(z_1^{-1},z_2^{-1}\right)A\left(z_1^{-1},z_2^{-1}\right)y\left(k,p\right)+\hat{D}\left(z_1^{-1},z_2^{-1}\right)B\left(z_1^{-1},z_2^{-1}\right)u\left(k,p\right)\\&\quad+v\left(k,p\right).\end{aligned}\tag{15}$$

Equation (15) can be written down in a new form as:

$$y_f\left(k,p\right)=-A\left(z_1^{-1},z_2^{-1}\right)y_f\left(k,p\right)+\mathrm{B}\left(z_1^{-1},z_2^{-1}\right)u_f\left(k,p\right)+v\left(k,p\right),\tag{16}$$

where the terms y_f and u_f are the prefiltered output and input vectors, respectively. In fact, the output and input of the system are filtered by $\hat{D}\left(z_1^{-1},z_2^{-1}\right)$.

Now, the observation error in (16) is whitened by prefiltering (3) using the filter $\hat{D}$=$\hat{C}^{-1}$ and, as previously mentioned, the OLS method can be applied to estimate the unknown parameter matrices **A** and **B**.

The same computations can be repeated again using the new estimations $\hat{A}$ and $\hat{B}$ to obtain the new estimation of $\hat{D}$ and then using $\hat{D}$, the new estimations of $\hat{A}$ and $\hat{B}$ can be produced. This procedure is repeated iteratively until the convergence criterion is satisfied.

The parameter estimation process is summarized as 2DMIGLS algorithm, shown below:

2DMIGLS ALGORITHM

Start (set $i=0$)

1. **Sampling** : Generate $(N_1+1)\times(N_2+1)$ sampled input-output data $(0\le k\le N_1, 0\le p\le N_2\ )$
2. **Output and variable matrices**: Form the output matrix Y and the variable matrix Φ using output and input data for all values of k,p, as follows:

$$Y=\begin{bmatrix}y\left(0,0\right)^T\\y\left(1,0\right)^T\\\vdots\\y\left(N_1,N_2\right)^T\end{bmatrix},\quad\Phi=\begin{bmatrix}\phi\left(0,0\right)^T\\\phi\left(1,0\right)^T\\\vdots\\\phi\left(N_1,N_2\right)^T\end{bmatrix}.$$

3. **Initial estimation**: Compute the initial estimation of $\hat{\Theta}_0$ using the OLS method as follows:

$$\hat{\Theta}_0=\left(\Phi^T\Phi\right)^{-1}\Phi^TY$$

and set the initial value of $\hat{D}_0$ equal to I.
4. **Error estimation**: Compute $\hat{e}=y-\phi\hat{\Theta}_i$.

5. **Observation error regression**: Compute the variable matrix of error:

$$\hat{E} = \begin{bmatrix} \hat{e}\,(0,0)^T \\ \hat{e}\,(1,0)^T \\ \vdots \\ \hat{e}\,(N_1, N_2)^T \end{bmatrix}, \quad \Psi = \begin{bmatrix} \eta\,(0,0)^T \\ \eta\,(1,0)^T \\ \vdots \\ \eta\,(N_1, N_2)^T \end{bmatrix}.$$

6. **Noise dynamics estimation**: Compute the estimation $\hat{\Upsilon}_{i+1}$ using the OLS method as follows:

$$\hat{\Upsilon}_{i+1} = \left(\Psi^T \Psi\right)^{-1} \Psi^T \hat{E}.$$

7. **Prefiltering**: Prefilter sampled input and output data using filter $\hat{D}_{i+1}$.
8. **Parameter estimation**: Estimate the unknown parameters of system using prefiltered data and the OLS method as follows:

$$\hat{\Theta}_{i+1} = \left(\Phi_f^T \Phi_f\right)^{-1} \Phi_f^T Y_f.$$

9. **Convergence criteria**: If the following convergence condition is satisfied, the algorithm ends. Otherwise, set $i = i + 1$ and return to Step 4:

$$\left\| \begin{bmatrix} \hat{\Theta} \\ \hat{\Upsilon} \end{bmatrix}_{i+1} - \begin{bmatrix} \hat{\Theta} \\ \hat{\Upsilon} \end{bmatrix}_i \right\| < \varepsilon,$$

in which ε is a predefined threshold.

End

Remark 1 Note that in Step 3, the initial value of $\hat{D}_0$ is set to $\mathbf{I}$ in order to compute $\hat{\Upsilon}_0$. In Step 6, according to (14), $\hat{\Upsilon}_i$ contains the coefficients of $\hat{D}_i$. Also, in Step 8, Y_f and Φ_f have the same forms as Y and Φ, with the difference that u and y are replaced by u_f and y_f, respectively.

Remark 2 The convergence criterion in the IGLS method can be considered as constituted by the error between the real and the estimated value of parameters (estimation error). Thus, if the estimation error is less than a certain upper bound, called the convergence threshold, and does not exceed this threshold in the next few iterations, the convergence condition will be assumed to be met. However, the final accuracy of the estimation will be independent of the choice of the order of approximated filter $D(z^{-1})$, and the process will continue until the convergence criterion will be satisfied. Moreover, the only effect of the order of approximated filter $D(z^{-1})$ concerns the convergence rate. It means that the higher-order approximation (and as a result, the more accurate filter) has higher convergence rate. Also, approximation of $D(z_1^{-1}, z_2^{-1})$ with the lower order (r_1, r_2) is not necessarily a reason for bias, because the convergence condition must be satisfied anyway, see Norton (2009). It has been extended to the 2D and MIM IO model and the criteria of convergence and error have not been changed. The above discussion can be explained in the form of following basic example. Assume $C\left(z^{-1}\right) = 1 + c_1 z^{-1}$. Then, using the Maclaurin Series we have:

$$D\left(z^{-1}\right) = \frac{1}{C\left(z^{-1}\right)} = \frac{1}{1 + c_1 z^{-1}} =$$
$$\sum_{n=0}^{\infty} (-1)^n \left(c_1 z^{-1}\right)^n = 1 - c_1 z^{-1} + \left(c_1 z^{-1}\right)^2 - \left(c_1 z^{-1}\right)^3 + \dots$$

For the stability of filter $D\left(z^{-1}\right)$ the roots of $C\left(z^{-1}\right)$ must be inside the unit circle. So, in the above example we should have $|c_1| < 1$. Therefore, it can be concluded that the terms with higher powers have small coefficients and can be eliminated.

Remark 3 The coefficient vector of $C\left(z^{-1}\right)$ represents the state of the identification error at the moment k . If the zeros of this polynomial lie outside of the unit circle, in the time domain, the error tends to an infinite value at $t \to \infty$, which indicates that the system is unstable and the estimation method is incorrect, because the purpose of estimation is to reduce the error, specifically, as concerns the LS method for finding $e_k = C\left(z^{-1}\right) v_k$. Therefore, we conclude that the polynomial zeros of the estimation error must always lie inside the unit circle, so that the poles of $D\left(z^{-1}\right)$ also lie inside the unit circle, and the filter is stable.

4. Simulation results

In this section, a numerical example is considered, serving to evaluate the efficiency of 2DMIGLS Algorithm. Consider the following MIMO 2D ARMAX model with two inputs and two outputs:

$$\begin{bmatrix} y_1(k,p) \\ y_2(k,p) \end{bmatrix} =$$

$$\begin{bmatrix} 0.6z_1^{-1} + 0.2z_2^{-2} + 0.1z_1^{-1}z_2^{-2} & 0.5z_1^{-1} + 0.1z_2^{-2} + 0.05z_1^{-1}z_2^{-2} \\ -0.8z_1^{-1} - 0.25z_2^{-2} - 0.2z_1^{-1}z_2^{-2} & 0.01z_1^{-1} + 0.33z_2^{-2} + 0.15z_1^{-1}z_2^{-2} \end{bmatrix} \times$$

$$\times \begin{bmatrix} y_1(k,p) \\ y_2(k,p) \end{bmatrix} +$$

$$+ \begin{bmatrix} 0.5 + 0.1z_1^{-1} + 0.8z_1^{-3}z_2^{-2} & -0.3 - 0.8z_1^{-1} - 0.7z_1^{-3}z_2^{-2} \\ 0.4 + 1.1z_1^{-1} + 0.1z_1^{-3}z_2^{-2} & 0.23 - 1.5z_1^{-1} - 0.1z_1^{-3}z_2^{-2} \end{bmatrix} \begin{bmatrix} u_1(k,p) \\ u_2(k,p) \end{bmatrix} +$$

$$\begin{bmatrix} 1 + 0.2z_1^{-1}z_2^{-1} & -0.1z_1^{-1}z_2^{-1} \\ -0.1z_1^{-1}z_2^{-1} & 1 + 0.6z_1^{-1}z_2^{-1} \end{bmatrix} \begin{bmatrix} v_1(k,p) \\ v_2(k,p) \end{bmatrix}$$

where $u_1(k,p)$ and $u_2(k,p)$ are system inputs and $y_1(k,p)$ and $y_2(k,p)$ are system outputs. Further, $v_1(k,p)$ and $v_2(k,p)$ are white Gaussian noise with zero mean and variance σ_v^2. In order to simulate the above system, the input signals are assumed to be white Gaussian noise with zero mean and variance 1.

To evaluate the efficiency and accuracy of the 2DMIGLS Algorithm, two different values for noise variance are chosen. Also, we consider the effect of the sampled data number on estimation accuracy. Finally, to evaluate the stability and robustness of 2DMIGLS Algorithm, the Monte-Carlo analysis will be carried out.

4.1. Sampled data number and noise variance effects

For purposes of studying the effect of the sampled data number on the values of estimated parameters, the input-output signals are sampled with two different sampling rates, namely $N_1 = N_2 = 150$ and $N_1 = N_2 = 250$. Also, in order to check the effect of noise variance value on parameter estimation, two cases are considered. For the first, σ_v^2 is assumed to be equal 1, while for the second one the assumed value is 16. For both cases the convergence threshold ε is assumed to be 0.001.

Table 1 shows the estimated parameters using the proposed algorithm. The $a_{i,j}^{(l)(r)}$ shows the entry of l^{th} row and r^{th} column in $\mathrm{A}_{i,j}$. As it is explained in Section 2, $\mathrm{A}_{i,j}$ indicates the corresponding coefficients of system output vectors with i shifts in dimension 1 and j shifts in dimension 2.

In the last row of Table 1, in order to show the high accuracy of estimations, the values of the relative error $\delta = \frac{\|\hat{\theta}-\theta\|}{\|\theta\|}$ are presented, where θ is the real parameter vector and $\hat{\theta}$ corresponds to the estimated parameter vector. It is clear that the relative error is about 1.5%, which is indeed very small.

According to Table 1, one can see that as the variance of noise increases, the estimation accuracy decreases. On the other hand, as the number of sampled data increases, the estimation accuracy becomes, of course, higher.

The magnitudes of the error, understood as the difference between the system outputs and the model outputs for the step input with amplitude 1 and 2 for the first and second inputs are shown in Figs. 2 and 3, respectively.

4.2. Output error

In this section, two sinusoidal inputs are applied and the error between the real and the estimated outputs is analyzed. The number of sampled data is $N_1 = N_2 = 250$. The variance of noise is assumed to be equal to 1.

Table 1. 2DMIGLS results for various sampled data number and different noise variance values

Parameter	**Variance=1**		**Variance=16**		**Real value**
	N=150	***N=250***	***N=150***	***N=250***	
$a_{1,0}^{(1)(1)}$	0.5961	0.5985	0.5902	0.6019	0.6
$a_{1,0}^{(1)(2)}$	0.5000	0.4963	0.4971	0.5010	0.5
$a_{0,2}^{(1)(1)}$	0.1994	0.2033	0.2050	0.1997	0.2
$a_{0,2}^{(1)(2)}$	0.1049	0.1009	0.0864	0.0943	0.1
$a_{1,2}^{(1)(1)}$	0.0864	0.1011	0.1134	0.1021	0.1
$a_{1,2}^{(1)(2)}$	0.0499	0.0486	0.0464	0.0485	0.05
$b_{0,0}^{(1)(1)}$	0.4887	0.4957	0.5085	0.5017	0.5
$b_{0,0}^{(1)(2)}$	-0.2820	-0.2970	-0.3005	-0.3070	-0.3
$b_{0,0}^{(1)(0)}$	0.0842	0.1021	0.0972	0.0854	0.1
$b_{1,0}^{(1)(2)}$	-0.7990	-0.8059	-0.7874	-0.8052	-0.8
$b_{3,2}^{(1)(1)}$	0.8048	0.7976	0.7984	0.8003	0.8
$b_{3,2}^{(1)(2)}$	-0.7019	-0.7025	-0.7077	-0.6917	-0.7
$a_{1,0}^{(2)(1)}$	-0.8017	-0.8015	-0.8005	-0.7979	-0.8
$a_{1,0}^{(2)(2)}$	0.0093	0.0094	0.0143	0.0099	0.01
$a_{0,2}^{(2)(1)}$	-0.2456	-0.2514	-0.2445	-0.2474	-0.25
$a_{0,2}^{(2)(2)}$	0.3264	0.3304	0.3398	0.3304	0.33
$a_{1,2}^{(2)(1)}$	-0.1949	-0.2013	-0.2136	-0.2055	-0.2
$a_{1,2}^{(2)(2)}$	0.1520	0.1472	0.1508	0.1498	-0.15
$b_{0,0}^{(2)(1)}$	0.4103	0.3982	0.3944	0.3808	0.4
$b_{0,0}^{(2)(2)}$	0.2274	0.2331	0.2344	0.2192	0.23
$b_{1,0}^{(2)(1)}$	1.1066	1.1042	1.1107	1.1077	1.1
$b_{1,0}^{(2)(2)}$	-1.5019	-1.5038	-1.4992	-1.5048	-1.5
$b_{3,2}^{(2)(1)}$	0.0995	0.0996	0.1159	0.0957	0.1
$b_{3,2}^{(2)(2)}$	-0.1071	-0.0975	-0.0998	-0.1226	-0.1
$c_{1,1}^{(1)(1)}$	0.1971	0.1987	0.1902	0.2017	0.2
$c_{1,1}^{(1)(2)}$	-0.1062	-0.0996	-0.0976	-0.1021	-0.1
$c_{1,1}^{(2)(1)}$	-0.1171	-0.0991	-0.0872	-0.1121	-0.1
$c_{1,1}^{(2)(2)}$	0.5871	0.5891	0.5864	0.6114	0.6
(%) δ	1.2935	0.4876	1.4557	1.4500	

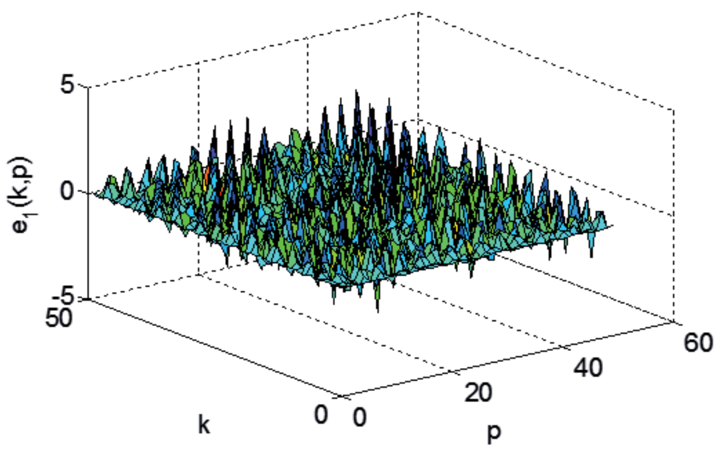

Figure 2. Error between the first system output and the model for the step input

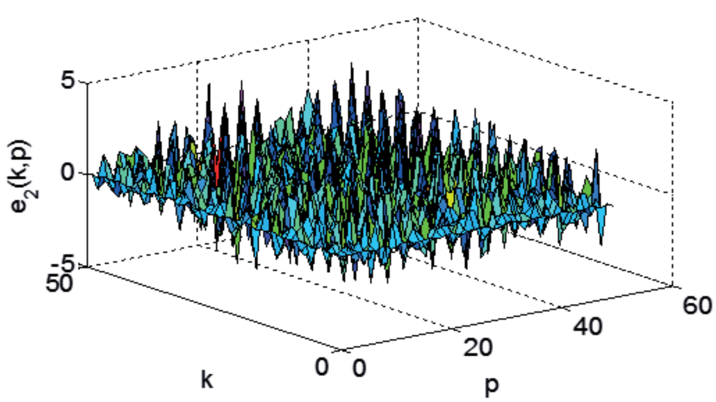

Figure 3. Error between the second system output and the model for the step input

The inputs are as follows:

$$u_1(k,p) = \sin(2k)\cos(0.5p)$$
$$u_2(k,p) = -\sin(3k)\cos(0.2p).$$

The output estimation error for y_1 and y_2 is shown in Figs. 4 and 5, respectively. It is clear that the error bound is very small (less than 5), and that the estimated system has been able to regenerate the output signal.

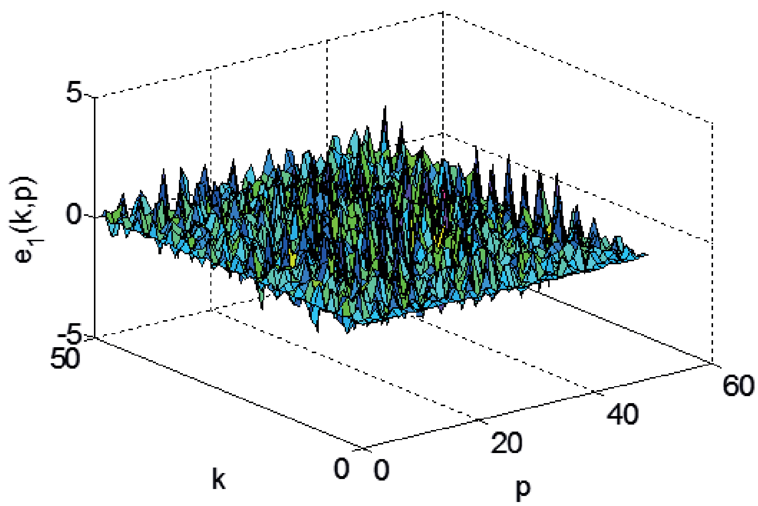

Figure 4. Output error for y_1

Since the decoupling condition in MIMO systems is in general not satisfied, therefore the system cannot be identified as two separate SISO systems. Therefore, in order to show the effect of interaction between input-output loops in the real and identified MIMO system, the unit step input is applied to the system without delay for the first input and the delayed unit step input for the second input, as this is illustrated in Figs. 6 and 7, respectively. Also, the effect of interaction on the behavior of the system and the identified model was studied for the case with the number of 150 data samples in each dimension and the standard deviation of the noise equal to 4, the respective results being presented in Figs. 8 and 9.

4.3. Monte-Carlo simulation

In the framework of the Monte-Carlo analysis, the sampling and parameter estimation process has been performed for 50 times. Table 2 shows the mean and standard deviation values of estimations in 50 simulations. Monte-Carlo simulation has been repeated two times, with two different numbers of data samples, i.e. $N_1 = N_2 = 150$ and $N_1 = N_2 = 250$.

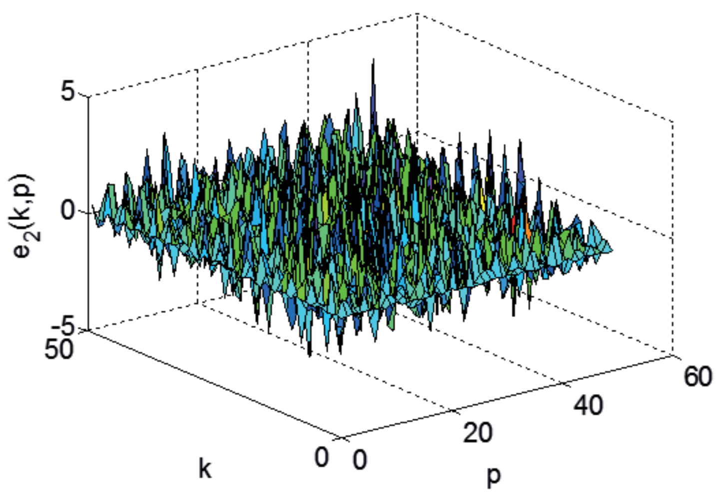

Figure 5. Output error for y_2

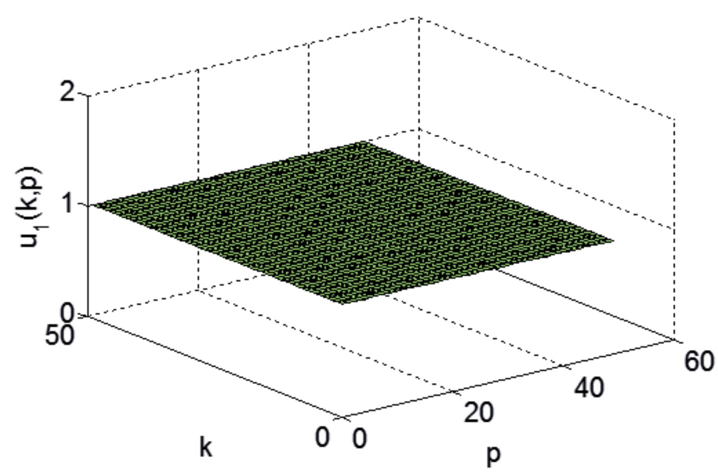

Figure 6. Unit step input without delay for the first input

Table 2. Monte Carlo results for 2DMIGLS algorithm

Parameter	**Variance=1**		**Variance=16**		**Real value**
	SD	***Mean***	***SD******	***Mean***	
$a_{1,0}^{(1)(1)}$	0.0053	0.5992	0.0024	0.5990	0.6
$a_{1,0}^{(1)(2)}$	0.0043	0.4988	0.0027	0.4995	0.5
$a_{0,2}^{(1)(1)}$	0.0070	0.1994	0.0040	0.1995	0.2
$a_{0,2}^{(1)(2)}$	0.0067	0.0998	0.0029	0.0995	0.1
$a_{1,2}^{(1)(1)}$	0.0115	0.1005	0.0066	0.1003	0.1
$a_{1,2}^{(1)(2)}$	0.0086	0.0504	0.0050	0.0499	0.05
$b_{0,0}^{(1)(1)}$	0.0303	0.4976	0.0123	0.4990	0.5
$b_{0,0}^{(1)(2)}$	0.0231	-0.2969	0.0182	-0.3009	-0.3
$b_{0,0}^{(1)(0)}$	0.0262	0.0996	0.0128	0.0975	0.1
$b_{1,0}^{(1)(2)}$	0.0273	-0.8019	0.0141	-0.8059	-0.8
$b_{3,2}^{(1)(1)}$	0.0249	0.7945	0.0150	0.7993	0.8
$b_{3,2}^{(1)(2)}$	0.0315	-0.7005	0.0168	-0.7015	-0.7
$a_{1,0}^{(2)(1)}$	0.0038	-0.7991	0.0027	-0.8001	-0.8
$a_{1,0}^{(2)(2)}$	0.0037	0.0109	0.0025	0.0096	0.01
$a_{0,2}^{(2)(1)}$	0.0054	-0.2509	0.0036	-0.2499	-0.25
$a_{0,2}^{(2)(2)}$	0.0058	0.3307	0.0037	0.3306	0.33
$a_{1,2}^{(2)(1)}$	0.0102	-0.2035	0.0065	-0.2021	-0.2
$a_{1,2}^{(2)(2)}$	0.0054	0.1496	0.0042	0.1491	-0.15
$b_{0,0}^{(2)(1)}$	0.0260	0.4027	0.0135	0.3994	0.4
$b_{0,0}^{(2)(2)}$	0.0200	0.2266	0.0121	0.2321	0.23
$b_{1,0}^{(2)(1)}$	0.0226	1.0990	0.0154	1.1001	1.1
$b_{1,0}^{(2)(2)}$	0.0188	-1.5031	0.0149	-1.5026	-1.5
$b_{3,2}^{(2)(1)}$	0.0214	0.1077	0.0162	0.1012	0.1
$b_{3,2}^{(2)(2)}$	0.0187	-0.1032	0.0148	-0.0991	-0.1
$c_{1,1}^{(1)(1)}$	0.0077	0.1993	0.0045	0.1996	0.2
$c_{1,1}^{(1)(2)}$	0.0189	-0.1012	0.0152	-0.0992	-0.1
$c_{1,1}^{(2)(1)}$	0.0176	-0.1041	0.0141	-0.0983	-0.1
$c_{1,1}^{(2)(2)}$	0.0055	0.5994	0.0023	0.5992	0.6
(%) δ	0.6475	3.1984	0.3706	1.9290	

*SD: standard deviation

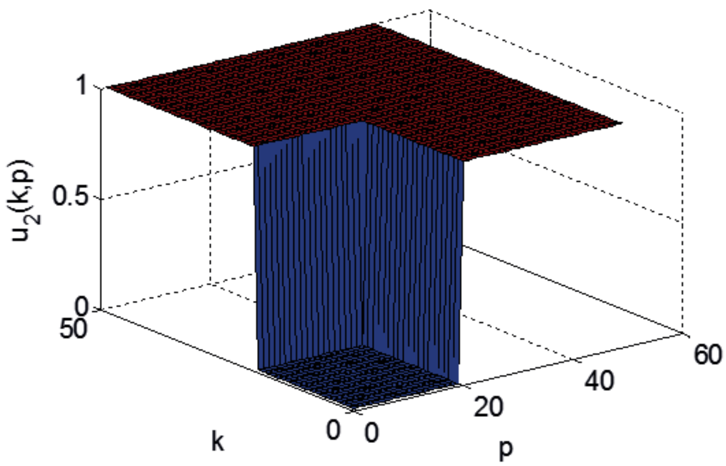

Figure 7. Delayed unit step input for the second input

According to Table 2, we can conclude that as the number of data samples is increased, the mean of the estimated parameters converges to real values. Also, increase in the number of data samples leads to lower standard deviation.

5. Conclusion

In this paper, a method was considered for parameter estimation of linear MIMO 2D discrete system with ARMAX model. The presented method focused on generating an unbiased estimation by colored noise without focusing on noise dynamics estimation directly. However, the inverse of noise dynamics was estimated for filtering and whitening, and therefore the noise dynamics was obtained. Simulation results showed that the proposed algorithm estimates the model parameters accurately. It was also shown that although the increase in amplitude and variance of noise leads to lowered accuracy, with a larger number of data samples, the estimation accuracy can be improved. Having real data, model order determination is the key first step for modeling any dynamic systems, particularly the two-dimensional processes that can be considered for future works.

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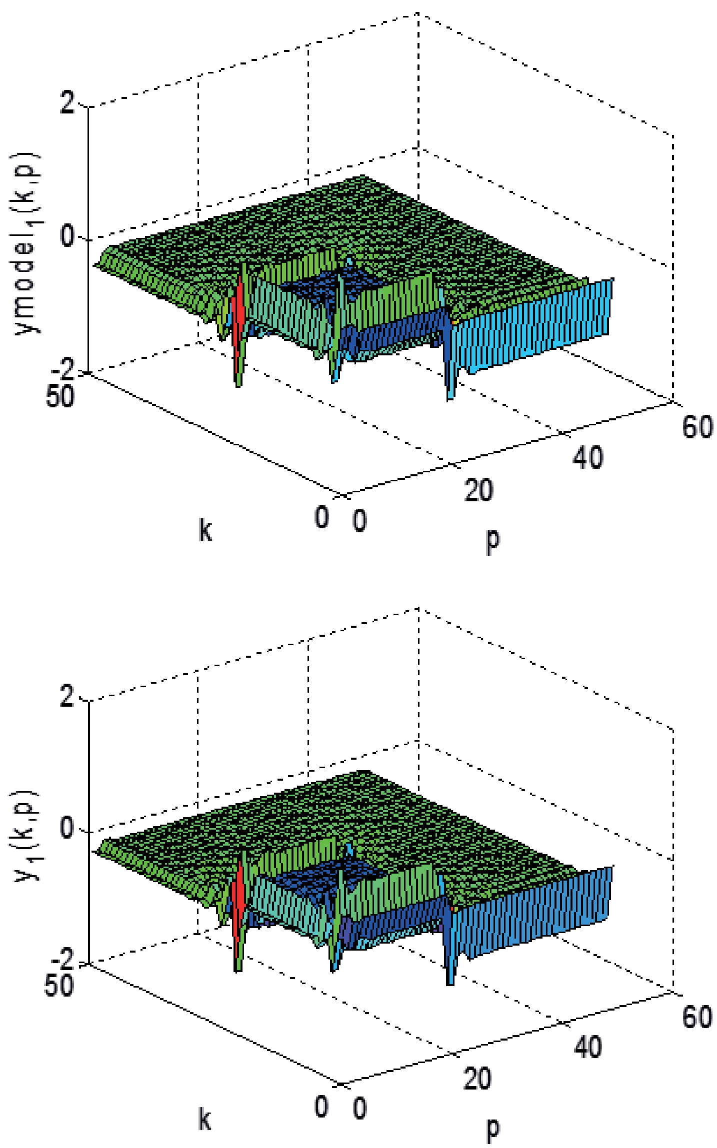

Figure 8. The first output of the system and the 2DMIGLS model

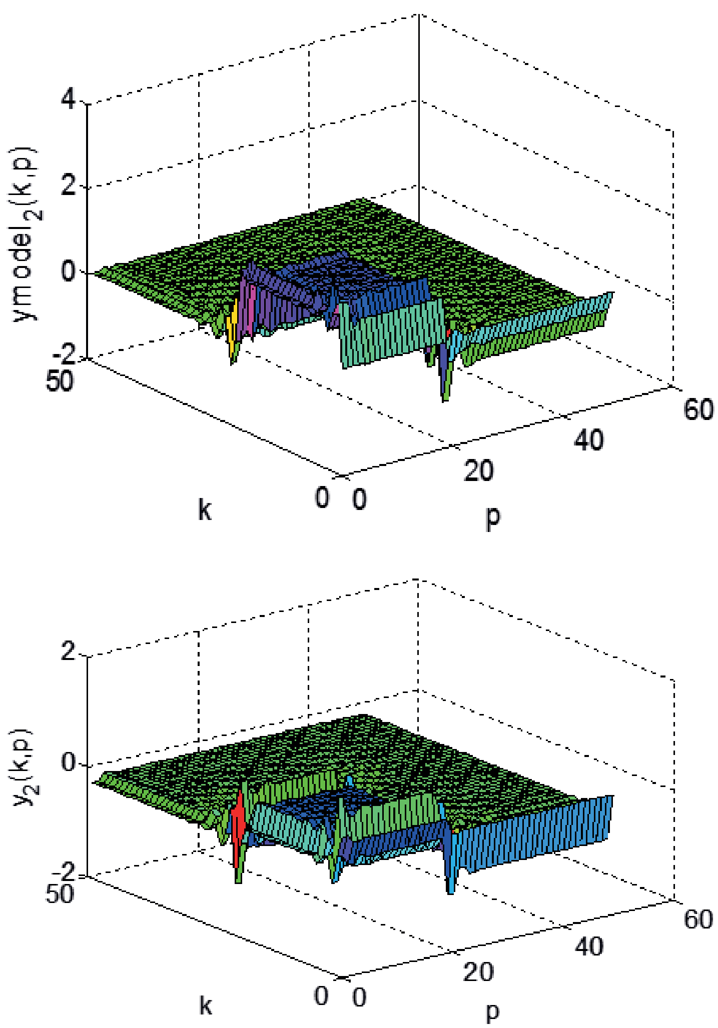

Figure 9. The second output of the system and the 2DMIGLS model

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