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Introducing and solving the hesitant fuzzy system $A\,X = B$ *

by

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Abstract: In this paper the solution for hesitant fuzzy system as $A\,X = B$ is introduced where A is an $n \times n$ known hesitant fuzzy matrix, B is an $n \times 1$ known hesitant fuzzy vector and X is an $n \times 1$ unknown hesitant fuzzy vector.

First, L_∞-norm and L_1-norm of a hesitant fuzzy vector are introduced. Then, the concepts of hesitant fuzzy zero, 'almost equal' and 'less than' and 'equal' are defined for two hesitant fuzzy numbers. Finally, using a minimization problem; the hesitant fuzzy system is solved. At the end, some numerical examples are presented to show the effectiveness of the proposed method.

Keywords: L_∞-norm, L_1-norm, hesitant fuzzy number, hesitant fuzzy system, minimization problem

1. Introduction

Since human knowledge is limited and relative, the presence of ambiguity and uncertainty in the real world problems is inevitable. Scientists use different tools, such as fuzzy sets, fuzzy sets of type 2, intuitionistic fuzzy sets etc. to define such uncertainties, which are of non-statistical character. The use of these tools enables them to model ambiguity and uncertainty of the real-world problems by the membership values or/and the non-membership values. Since in some problems defining the membership value may yet involve hesitancy, in such class of problems, hesitant fuzzy sets (HFS) are dealt with. As a result, introducing hesitancy in mathematical models or equations opens both a new

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field and constitutes a challenge for researchers (see Buckley, 1991, and then Nasseri et al., 2014; Taghi-Nezhad, 2019; Taleshian, Fathali and Taghi-Nezhad, 2018; Babakordi, Allahviranloo and Adabitabarfirozja, 2016; Allahviranloo and Babakordi, 2017; Xu, 2015; Babakordi, 2020a).

In what follows we present a brief overview of the progress in the relevant research, directed towards the issues of importance for this paper.

The presence of hesitancy in almost every issue of real life makes it difficult to take decisions concerning various choices. Due to ambiguity and uncertainty of perception and information, a complex situation arises when an expert wants to decide on a non-trivial issue. With this in mind, Torra and Narukawa (2009) and then Torra (2010) introduced hesitant fuzzy sets and some relevant basic operations. Also, the relationship of these new concepts was studied with respect to intuitionistic fuzzy set and fuzzy multisets.

Some hesitant fuzzy aggregation operators, together with their application in decision making, were proposed by Xia and Xu (2011).

Thereafter, distance and similarity scales, along with a series of score function were introduced for hesitant fuzzy sets in Xu and Xia (2011) and in Farhadnia (2014).

In 2014, hesitant fuzzy soft sets were defined and their applications in multicriteria decision making were considered, and, further, fuzzy soft sets were combined with hesitant fuzzy sets and hesitant fuzzy soft sets. Further, some operations on hesitant fuzzy soft sets, based on Archimedean t-norm and Archimedean t-conorm were defined in Wang, Li and Chen (2014).

Various properties of hesitant fuzzy sets were proposed inVanita and Venkatesan (2016). Following this, the priority degrees for hesitant fuzzy sets were defined and their application in multiple attribute decision making was investigated by Lan et al. (2017). In this context, Xiao, Cai and Wang (2017) proposed a variety of multicriteria decision-making (MCDM) methods using hesitant fuzzy linguistic terms sets for renewable energy project evaluation.

Alcantud and Torra (2018) explained that each typical hesitant fuzzy set (HFS) can be established using a well-structured family of fuzzy sets. Also, they presented a uniformly typical HFS, while defining properties of HFS and families of cuts for HFSs. Finally, the first decomposition theorem for HFSs, together with two extension principles to extend crisp maps to maps between HFSs were proposed by them.

Thereafter, in the method proposed by Zhu and Xu (2018), it was assumed that the membership of x in a set A is discussed by two DMs. What if one assigns 0.5 and 0.6 while the other assigns 0.6 and 0.7? Which of these numbers should be picked? If one assumes that the decision makers (DMs) are inhomogeneous, then $\{0.5, 0.6, 0.7\}$ would be preferred. However, if they are assumed to be homogeneous, this representation of preferences loses the preference value of 0.6, assigned by one of the DMs. Moreover, the preference of 0.5 and 0.7 may not be identified by a different DMs, who could have a different importance in decision making. Therefore, the HFSs were extended to probability-hesitant fuzzy sets (P-HFSs). The probability-hesitant fuzzy preference relations (P-HFPRs) were developed, based on P-HFSs to provide a method for DMs to reach consensus on their preferences. And for the aforementioned example the probability-hesitant fuzzy element can have the form: $h_p = \{0.5\,(0.25)\,,\, 0.6\,(0.5)\,,\, 0.7(0.25)\}$, with probabilities following the values proposed, see the explanation below, this being a better representation of all the preferences without losing the problem's data.

(One issue that has a great impact on the understanding of decision makers' (DMs) behavior when making decisions is the interpretation of probability, and it can also include the consideration of the probabilistic preference of DMs in problem modeling. As can be seen from the example, 0.5 and 0.7 were considered by one DM and 0.6 by two DMs, and h_p shows this well.)

Furthermore, the hesitant fuzzy information was investigated for information fusion in decision making problems in Rodriguez, Xu and Martinez (2018).

In 2019, multiple criteria decision-making problem was considered, based on probabilistic interval-valued hesitant fuzzy sets in Sindhu et al. (2019).Then, in 2020, a review was published, concerning the different types of hesitant fuzzy numbers, by Babakordi (2020b).

Lalotra and Singh (2020) considered fuzzy knowledge measure in the form of a dual measure of fuzzy entropy. They proposed an axiomatic framework to describe a hesitant fuzzy knowledge measure and investigated the duality property of hesitant fuzzy entropy and HF-knowledge measure.

Recently, the problem of dual hesitant fuzzy preference relations has been reviewed and a new class of decision making techniques has been proposed, based on specially built optimization models by Meng, Tang and Pedrycz (2021).

Corona virus made the prediction of various issues, like those related to the economy, highly doubtful, therefore, increased degree of doubt should be taken into consideration when modeling problems or solving them. Therefore, in Babakordi and Taghi-Nezhad (2021) the authors proposed and solved the hesitant fuzzy equations and investigated their applications in determining the equilibrium point of the market.

Modeling of many problems in various fields of science and practical applications leads to the linear equation systems. If the inputs are constituted by the hesitant fuzzy numbers, then hesitant fuzzy linear equation systems are dealt with.

Since there is no research in the field of hesitant fuzzy vector norm and hesitant fuzzy systems, in this paper; L_∞-norm and L_1-norm of hesitant fuzzy vectors are introduced and the solution of hesitant fuzzy system (HFS) is investigated using these tools.

The structure of the paper is as follows: In Section 2, first, the necessary basic definitions are presented, then hesitant fuzzy system is introduced and is solved using the appropriate techniques. In Section 3, the proposed method is applied to solve some numerical examples and the achieved results verify the effectiveness of the proposed method. Finally, conclusion and future work directions are presented in Section 4.

2. Hesitant fuzzy system

In this section we define some necessary tools and describe the required notation, and then the hesitant fuzzy system is introduced and solved.

Definition 2.1 *(see Torra and Narukawa, 2009, and Xia and Xu, 2011) Let X be a fixed set; an HFS on X is defined in terms of a function that when applied to X returns a finite subset of $[0,1]$. For better understanding, Xia and Xu (2011) expressed the HFS, here denoted A, in the following manner:*

$$A = \{< x, h_A(x) > | x \in X\}$$

where $h_A(x)$is a finite set of some values from $[0,1]$, corresponding to the possible membership degrees of the element $x \in X$ with respect to the HFSA. $h_A(x)$ is called a hesitant fuzzy element (HFE), for convenience, $h_A(x)$ is being referred to as h in the following.

Let $h,\ h_1,\ h_2$ be three HFEs and λ a real number, some of the operations, defined in Torra and Narukawa (2009), Torra (2010), and Xia and Xu (2011) are as follows:

$$\begin{aligned}
&h^c = \cup_{\gamma\in h}\{1-\gamma\}\\
&h^\lambda = \cup_{\gamma\in h}\left\{\gamma^\lambda\right\}\\
&\lambda\, h = \cup_{\gamma\in h}\left\{1-(1-\gamma)^\lambda\right\}\\
&h_1 \cup h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2} max\{\gamma_1,\gamma_2\}\\
&h_1 \cap h_2 = \cap_{\gamma_1\in h_1,\gamma_2\in h_2} min\{\gamma_1,\gamma_2\}\\
&h_1 + h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2}\{\gamma_1+\gamma_2-\gamma_1\gamma_2\}\\
&h_1 \times h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2}\{\gamma_1\gamma_2\}\,.
\end{aligned} \tag{1}$$

Definition 2.2 *Each set of real numbers such that all of its elements are zero is called hesitant fuzzy zero and the zero corresponding to the hesitant fuzzy element his denoted by 0_h= $\{0,0,..,0\}$, where the number of zeros of $0_h$is equal to the number of elements in h.*

Lemma 2.1 *The following always holds:*

1. $0_h + h = h,$
2. $0_h \times h = 0_h$

Proof The above results directly from (1).

Definition 2.3 *Assume* $h_1 = \{h_1^1, h_1^2, \ldots, h_1^m\}$ *and* $h_2 = \{h_2^1, h_2^2, \ldots, h_2^n\}$ *are hesitant fuzzy elements. We have:*

$$h_1 \leq h_2 \text{ iff } \quad \forall h_1^i \epsilon h_1,\, \exists h_2^j \epsilon h_2 \,:\, h_1^i \leq h_2^j\,, \;\; i = 1, \ldots, m, \;\; j = 1, \ldots, n.$$

Definition 2.4 *Assume* $h_1 = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ *and* $h_2 = \{\gamma'_1,\, \gamma'_2, \ldots,\, \gamma'_n\}$*. It is said that* $\mathrm{h_1 \approx h_2}$ *when*

$$\exists \varepsilon > 0 : |\gamma_1 - \gamma'_1| + |\gamma_2 - \gamma'_2| + \ldots + |\gamma_n - \gamma'_n| < \varepsilon$$

and in some cases there may be length $\{h_1\} \neq$ *length* $\{h_2\}$*; then, to operate correctly, we should extend the shorter element until both of them have the same length when we compare them. The extension of the shorter element is possible by adding different values, which may yield different results, and this might be reasonable, because the decision maker's risk preference can directly influence the final decision. In most cases, there is* $l(h_M(x_i)) \neq l\,(h_N\,(x_i))$ *and for simplicity consider* $l\,(x_i) = \max\{l\,(h_M\,(x_i))\,,\, l\,(h_N\,(x_i))\}$ *for each* x_i *in* X*. To proceed with a correct operation, the shorter one must be extended until both of them have the same length. The simplest way to extend the shorter one is to add the same value several times in it. Actually, the shorter one can be extended by adding any value to it. However, the value is highly dependent on the decision maker's risk preferences. The anticipation of an optimist may be related to the desirable outcome, meaning that the maximum value might be added, while the expectation of a pessimist may be related to the unfavorable outcome, and hence the minimum value might be added. For instance, consider* $h_M\,(x_i) = \{0.1,\, 0.2,\, 0.3\}\,,\; h_N\,(x_i) = \{0.4, 0.5\}$*, and* $l\,(h_M\,(x_i)) > l\,(h_N\,(x_i))$*. To perform an appropriate operation,* $h_N\,(x_i)$ *must be extended to, say,* $h_N\,(x_i) = \{0.4,\, 0.4,\, 0.5\}$*, i.e. until it has the same length as* $h_M\,(x_i)$*. Thus, the optimist may extend it as* $h_N\,(x_i) = \{0.4,\, 0.5,\, 0.5\}$*, while the pessimist may extend it as* $h_N\,(x_i) = \{0.4,\, 0.4,\, 0.5\}$*. It is worthwhile to note that it is reasonable that the shorter one be extended by adding different values, because the final decision is directly dependent on the decision maker's risk preferences (for more information, one can refer to Xu and Xia, 2011). And it is, of course, said that* $h_1 = h_2$ *when* $\gamma_1 = \gamma'_1,\, \gamma_2 = \gamma'_2, \ldots,\, \gamma_n = \gamma'_n$.

Example 2.1 Consider two hesitant fuzzy elements, $h_A = \{0.8h_1 + h_2 - 0.48h_1h_2,\ 0.8h_1 + 0.5h_2 - 0.4h_1h_2\}$ and $h_B = \{0.9\}$. In order to possibly have $h_A \approx h_B$, first, h_A and h_B must have the same length. Therefore, h_B is extended as $h_B = \{0.9,\, 0.9\}$. Then, to check the relation we are after, h_1 and h_2 must be determined such that $|0.8h_1 + h_2 - 0.48h_1h_2 - 0.9| + |0.8h_1 + 0.5h_2 - 0.4h_1h_{2} - 0.9|$ takes its minimum possible value. If there is no h_1 and h_2 to satisfy this condition, then $h_A \approx h_B$ never happens. For more information on this subject one can refer to Xu and Xia (2011).

Definition 2.5 *A matrix is called hesitant fuzzy set matrix if at least one element of the matrix is a hesitant fuzzy set.*

Definition 2.6 *A matrix is called hesitant fuzzy element matrix if at least one element of the matrix is a hesitant fuzzy element.*

Definition 2.7 *The zero hesitant fuzzy vector, corresponding to hesitant fuzzy element vector* $H = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$ *is denoted by* O_H *and is defined as* $O_H = \begin{pmatrix} 0_{h_1} \\ 0_{h_2} \\ \vdots \\ 0_{h_n} \end{pmatrix}$.

Definition 2.8 *Consider the function* $\|.\|$, *whose input is a hesitant fuzzy element vector and output is a hesitant fuzzy element. This function is called hesitant fuzzy vector norm if for* H *and* H' *being hesitant fuzzy element vectors and* α *being a positive real number, the following properties hold:*

1. $\|H\| \geq 0,$
2. $\|H\| = 0 \Longleftrightarrow H = 0,$
3. $\|\alpha H\| = |\alpha| \, \|H\| , \ \alpha > 0,$
4. $\|H + H'\| \leq \|H\| + \|H'\| .$

Remark 2.1 *Vector norm for hesitant fuzzy element vector consists in choosing a representative hesitant fuzzy element for that vector. When a representative is chosen, the calculation is continued with this representative instead of dealing with the entire hesitant fuzzy vector.*

Definition 2.9 *Consider the* $n \times 1$ *hesitant fuzzy element vector*

$$H = \begin{pmatrix} h_1 = \left\{\gamma_{11},\, \gamma_{12}, \ldots ,\, \gamma_{1n_1}\right\} \\ h_2 = \left\{\gamma_{21},\, \gamma_{22}, \ldots ,\, \gamma_{2n_2}\right\} \\ \vdots \\ h_n = \left\{\gamma_{n1},\, \gamma_{n2}, \ldots ,\, \gamma_{nn_n}\right\} \end{pmatrix},$$

Its L_∞ *norm is denoted by* $\|H\|_\infty$ *and is defined as follows:*

$$\|H\|_\infty = \left\{\max\left\{\gamma_{11}, \gamma_{12}, \ldots, \gamma_{1n_1}\right\},\right. \\ \left.\max\left\{\gamma_{21}, \gamma_{22}, \ldots, \gamma_{2n_2}\right\}, \ldots, \max\left\{\gamma_{n1}, \gamma_{n2}, \ldots, \gamma_{nn_n}\right\}\right\}. \tag{2}$$

Theorem 2.1 *The* L_∞ *norm (2.9) is a vector norm.*

Proof Assume that:

$$\|H\|_\infty = \{\max\{\gamma_{11}, \gamma_{12}, \ldots, \gamma_{1n_1}\}, \\ \max\left\{\gamma_{21}, \gamma_{22}, \ldots, \gamma_{2n_2}\right\}, \ldots, \max\left\{\gamma_{n1}, \gamma_{n2}, \ldots, \gamma_{nn_n}\right\}\} \\ = \left\{\gamma_{1i_1},\, \gamma_{2i_2}, \ldots ,\, \gamma_{ni_n}\right\}. \tag{3}$$

1. Since $\gamma_{1i_1},\, \gamma_{2i_2}, \ldots ,\, \gamma_{ni_n} \in [0,1]$, then $\|H\|_\infty \geq 0$.

2. Assume that $\|H\|_{\infty} = 0_{\|H\|_{\infty}}$, then:

$$\left\{\gamma_{1i_1},\, \gamma_{2i_2}, \ldots,\, \gamma_{ni_n}\right\} = \{0, 0, \ldots, 0\} \underset{\text{Definition 2.4}}{\Longrightarrow} \forall\, 1 \leq j \leq n \;,\; \gamma_{ji_j} = 0 \quad (4)$$

Since $\forall\, 1 \leq j \leq n,\ \gamma_{j1},\, \gamma_{j2}, \ldots,\, \gamma_{jn_j} \in [0,1]$, then it can be deduced from (4) that $h_j = 0_{h_j}$. As a result, $H = O_H$ (the proof of the reverse case is similar to this proof and is omitted here).

3. Consider the real number $\alpha > 0$ and the hesitant fuzzy vector:

$$\alpha H = \begin{pmatrix} \alpha h_1 = \alpha \left\{\gamma_{11},\, \gamma_{12}, \ldots,\, \gamma_{1n_1}\right\} \\ \vdots \\ \alpha h_n = \alpha \left\{\gamma_{n1},\, \gamma_{n2}, \ldots,\, \gamma_{nn_n}\right\} \end{pmatrix}$$

$$= \begin{pmatrix} \left\{(1-(1-\gamma_{11})^{\alpha}), \ldots, (1-(1-\gamma_{1n_1})^{\alpha})\right\} \\ \vdots \\ \left\{(1-(1-\gamma_{n1})^{\alpha}), \ldots, \left(1-(1-\ \gamma_{nn_n})^{\alpha}\right)\right\} \end{pmatrix}$$

It can be obtained from (2.9) that:

$$\begin{aligned} \|\alpha H\|_{\infty} = \{&\max\left\{(1-(1-\gamma_{11})^{\alpha}), \ldots, 1-(1-\gamma_{1n_1})^{\alpha}\right\}, \ldots, \\ &\max\left\{(1-(1-\gamma_{n1})^{\alpha}), \ldots, \left(1-\left(1-\ \gamma_{nn_n}\right)^{\alpha}\right)\right\}\right\}. \end{aligned} \qquad (5)$$

On the other hand, from assumption (2), we deduce that

$$\forall 1 \leq k \leq n \max\left\{\gamma_{k1}, \gamma_{k2}, \ldots, \gamma_{kn_k}\right\} = \gamma_{ki_k},$$

therefore, for each $\alpha > 0$ we have:

$$\forall 1 \leq k \leq n,\, 1 \leq j \leq n_k \,(1-(1-\gamma_{ki_k})^{\alpha}) \geq (1-(1-\gamma_{kj})^{\alpha}).$$

By substituting in (2), it is obtained that:

$$\|\alpha H\|_{\infty} = \left\{(1-(1-\gamma_{1i_1})^{\alpha}), \ldots, (1-(1-\gamma_{ni_n})^{\alpha})\right\}. \qquad (6)$$

Then, consider $\alpha > 0$ and $\|H\|_{\infty}$ from (2). Using (1), we get the following result:

$$\begin{aligned} &\alpha \|H\|_{\infty} = \\ &\alpha \left\{\gamma_{1i_1}, \gamma_{2i_2}, \ldots, \gamma_{ni_n}\right\} = \left\{(1-(1-\gamma_{1i_1})^{\alpha}), \ldots, (1-(1-\gamma_{ni_n})^{\alpha})\right\} \end{aligned} \qquad (7)$$

It can be concluded from (6) and (2) that $\|\alpha H\|_{\infty} = \alpha \|H\|_{\infty}$.

4. Consider the summation of $n \times 1$ hesitant fuzzy element vectors

$$H = \begin{pmatrix} h_1 = \{\gamma_{11}, \gamma_{12}, \ldots, \gamma_{1n_1}\} \\ \vdots \\ \vdots \\ h_n = \{\gamma_{n1}, \gamma_{n2}, \ldots, \gamma_{nn_n}\} \end{pmatrix}$$

and

$$H' = \begin{pmatrix} h'_1 = \{\gamma'_{11}, \gamma'_{12}, \ldots, \gamma'_{1n'_1}\} \\ \vdots \\ \vdots \\ h'_n = \{\gamma'_{n1}, \gamma'_{n2}, \ldots, \gamma'_{nn'_n}\} \end{pmatrix}$$

as follows:

$$\forall 1 \leq k \leq n,\ 1 \leq j_k \leq n_k,\ 1 \leq j'_k \leq n'_k$$

$$H + H^{'} = \begin{pmatrix} \cup_{\gamma_{1j_1} \in h_1, \gamma'_{1j_1} \in h'_1} \left\{ \gamma_{1j_1} + \gamma'_{1j'_1} - \gamma_{1j_1}\gamma'_{1j'_1} \right\} \\ \vdots \\ \cup_{\gamma_{nj_n} \in h_n, \gamma'_{nj'_n} \in h'_n} \left\{ \gamma_{nj_n} + \gamma'_{nj'_n} - \gamma_{nj_n}\gamma'_{nj'_n} \right\} \end{pmatrix}.$$

By calculating the L_∞ norm of the above vector using (2.9), it can be obtained that:

$$\begin{aligned} \|H + H'\|_\infty = &\left\{ max \left\{ \cup_{\gamma_{1j_1} \in h_1, \gamma'_{1j'_1} \in h'_1} \left\{ \gamma_{1j_1} + \gamma'_{1j'_1} - \gamma_{1j_1}\gamma'_{1j'_1} \right\} \right\}, \ldots, \right. \\ &\left. \max \left\{ \cup_{\gamma_{nj_n} \in h_n, \gamma'_{nj'_n} \in h'_n} \left\{ \gamma_{nj_n} + \gamma'_{nj'_n} - \gamma_{nj_n}\gamma'_{nj'_n} \right\} \right\} \right\} \\ = &\left\{ \gamma_{1k_1} + \gamma'_{1k'_1} - \gamma_{1k_1}\gamma'_{1k'_1}, \ldots, \gamma_{nk_n} + \gamma'_{nk'_n} - \gamma_{nk}\gamma'_{nk'_n} \right\} \end{aligned} \tag{8}$$

On the other hand, consider that:

$$\begin{aligned} \|H'\|_\infty = &\left\{ \max\left\{ \gamma'_{11}, \gamma'_{12}, \ldots, \gamma'_{1n'_1} \right\}, \max\left\{ \gamma'_{21}, \gamma'_{22}, \ldots, \gamma'_{2n'_2} \right\}, \ldots, \right. \\ &\left. \max\left\{ \gamma'_{n1}, \gamma'_{n2}, \ldots, \gamma'_{nn'_n} \right\} \right\} = \left\{ \gamma'_{1i'_1}, \gamma'_{2i'_2}, \ldots, \gamma'_{ni'_n} \right\}. \end{aligned} \tag{9}$$

According to (2) and (2) and using (1), we get:

$$\begin{aligned} \|H\|_\infty + \|H'\|_\infty &= \left\{ \gamma_{1i_1}, \ldots, \gamma_{ni_n} \right\} + \left\{ \gamma'_{1i'_1}, \ldots, \gamma'_{ni'_n} \right\} \\ &= \left\{ \gamma_{1i_1} + \gamma'_{1i'_1} - \gamma_{1i_1}\gamma'_{1i'_1}, \ldots, \gamma_{ni_n} + \gamma'_{ni'_n} - \gamma_{ni_n}\gamma'_{ni'_n} \right\}. \end{aligned} \tag{10}$$

Now, (8) and (2) result in $\forall \gamma \epsilon \, \|H + H'\|_\infty$; as we have $\gamma \epsilon \|H\|_\infty + \|H'\|_\infty$, and therefore, using Definition 2.3, the following can be obtained:

$$\|H + H'\|_\infty \leq \|H\|_\infty + \|H'\|_\infty.$$

Thus, since the four properties in Definition 2.8 hold, the L_∞ norm (2.9) is a vector norm.

EXAMPLE 2.2 Consider $H = \begin{pmatrix} \{0.1, 0.2\} \\ \{0.3, 0.4\} \end{pmatrix}$ and $H' = \begin{pmatrix} \{0.7, 0.8\} \\ \{0.5, 0.7\} \end{pmatrix}$. It can be obtained from Definition 2.9 that:

$$\|H\|_\infty = \{0.2, 0.4\}, \ \|H'\|_\infty = \{0.8, 0.7\}.$$

On the other hand, $\forall \alpha > 0$, and from the product, defined in (1), it can be deduced that:

$$\alpha H = \alpha \begin{pmatrix} \{0.1, 0.2\} \\ \{0.3, 0.4\} \end{pmatrix} = \begin{pmatrix} \alpha \{0.1, 0.2\} \\ \alpha \{0.3, 0.4\} \end{pmatrix} =$$
$$\begin{pmatrix} \{(1-(1-0.1)^\alpha), (1-(1-0.2)^\alpha)\} \\ \{(1-(1-0.3)^\alpha), (1-(1-0.4)^\alpha)\} \end{pmatrix} = \begin{pmatrix} \{(1-0.9^\alpha), (1-0.8^\alpha)\} \\ \{(1-0.7^\alpha), (1-0.6^\alpha)\} \end{pmatrix}$$

$$\begin{aligned} \|\alpha H\|_\infty &= \{\{(1-0.9^\alpha), (1-0.8^\alpha)\}, \{(1-0.7^\alpha), (1-0.6^\alpha)\}\} \\ &= \{(1-0.8^\alpha), (1-0.6^\alpha)\}, \end{aligned}$$

$$\alpha \|H\|_\infty = \alpha \{0.2, 0.4\} = \{(1-0.8^\alpha), (1-0.6^\alpha)\}.$$

It can be seen that $\|\alpha H\|_\infty$ and $\alpha \|H\|_\infty$ are equal on the basis of Definition 2.4.

Also, using the definition of the L_∞ norm and (1), it can be obtained that:

$$H + H' = \begin{pmatrix} \{0.1, 0.2\} \\ \{0.3, 0.4\} \end{pmatrix} + \begin{pmatrix} \{0.7, 0.8\} \\ \{0.5, 0.7\} \end{pmatrix} = \begin{pmatrix} \{0.1, 0.2\} + \{0.7, 0.8\} \\ \{0.3, 0.4\} + \{0.5, 0.7\} \end{pmatrix}$$
$$= \begin{pmatrix} \{0.1+0.7-0.07, 0.1+0.8-0.08, 0.2+0.7-0.14, 0.2+0.8-0.16\} \\ \{0.3+0.5-0.15, 0.3+0.7-0.21, 0.4+0.5-0.2, 0.4+0.7-0.28\} \end{pmatrix}$$
$$= \begin{pmatrix} \{0.8-0.07, 0.9-0.08, 0.0.9-0.14, 1-0.16\} \\ \{0.8-0.15, 1-0.21, 0.9-0.2, 1.1-0.28\} \end{pmatrix}$$

$$\begin{aligned} \left\| H + H' \right\|_\infty = \{ &\max \{0.8-0.07, 0.9-0.08, 0.0.9-0.14, 1-0.16\}, \\ &\max \{0.8-0.15, 1-0.21, 0.9-0.2, 1.1-0.28\}\} = \\ &\{(1-0.16), (1.1-0.28)\} \end{aligned}$$

$$\begin{aligned} &\|H\|_\infty + \|H'\|_\infty = \{0.2, 0.4\} + \{0.8, 0.7\} \\ &= \{0.2+0.8-0.16, \ 0.2+0.7-0.14, 0.4+0.8-0.32, 0.4+0.7-0.28\} \\ &= \{1-0.16, \ 0.9-0.14, 1.2-0.32, 1.1-0.28\}. \end{aligned}$$

Therefore, from Definition 2.3, we have:

$$\|H + H'\|_\infty \leq \|H\|_\infty + \|H'\|_\infty.$$

Definition 2.10 *Consider the* $n\times 1$ *hesitant fuzzy element vector* $H = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$*, the* L_1 *norm of* H*, denoted by* $\|H\|_1$*, can be defined as follows:*

$$\|H\|_1 = \sum_{i=1}^{n} h_i \,. \tag{11}$$

Example 2.3 Consider $H = \begin{pmatrix} \{0.6, 0.5\} \\ \{0.3, 0.1\} \end{pmatrix}$. It can be obtained from (1), (11) that:

$$\|H\|_1 = \{0.6, 0.5\} + \{0.3, 0.1\} = \{0.72, 0.64, 0.65, 0.55\}\,.$$

Theorem 2.2 *The* L_1 *norm (11) is a vector norm.*

Proof The theorem is being proven for $H = \begin{pmatrix} \{\gamma_1,\, \gamma_2\} \\ \{\gamma'_1,\, \gamma'_2\} \end{pmatrix}$. The proof for other cases is the same.

In this case, it can be obtained from (1), (11) that:

$$\|H\|_1 = \{\gamma_1 + \gamma'_1 - \gamma_1\gamma'_1, \gamma_1 + \gamma'_2 - \gamma_1\gamma'_2, \gamma_2 + \gamma'_1 - \gamma_2\gamma'_1, \gamma_2 + \gamma'_2 - \gamma_2\gamma'_2\} \tag{12}$$

1. Since $\gamma_1,\, \gamma_2\gamma'_1,\, \gamma'_2 \in [0,1]$, therefore $\forall\, 1 \le i\,,\, j \le 2,\ \gamma_i + \gamma'_j \ge \gamma_i\gamma'_j$, so it can be concluded from (12) that $\|H\|_1 \ge 0$.

2. Assume $\|H\|_1 = 0_{\|H\|_1}$, then:

$$\begin{aligned}&\{\gamma_1 + \gamma'_1 - \gamma_1\gamma'_1,\, \gamma_1 + \gamma'_2 - \gamma_1\gamma'_2\,,\, \gamma_2 + \gamma'_1 - \gamma_2\gamma'_1,\, \gamma_2 + \gamma'_2 - \gamma_2\gamma'_2\} \\ &\quad = \{0, 0, 0, 0\}\end{aligned} \tag{13}$$

Therefore:
$\gamma_1 + \gamma'_1 = \gamma_1\gamma'_1$
$\gamma_1 + \gamma'_2 = \gamma_1\gamma'_2$
$\gamma_2 + \gamma'_1 = \gamma_2\gamma'_1$
$\gamma_2 + \gamma'_2 = \gamma_2\gamma'_2$.

It can be seen that $\gamma_1,\, \gamma_2,\, \gamma'_1,\, \gamma'_2 \in [0,1]$, thus $\gamma_1 = \gamma_2 = \gamma'_1 = \gamma'_2 = 0$, and, as a result, it can be obtained that $H = O_H$. Now, assuming $H = O_H$ and the corresponding substitution in (12), it can be obtained that $\|H\|_1 = 0_{\|H\|_1}$.

3. Consider the real positive number α and the valued hesitant fuzzy element vector H, i.e. the vector whose all elements are hesitant fuzzy elements. It can be obtained from (1) that:

$$\alpha H = \begin{pmatrix} \{1 - (1-\gamma_1)^{\alpha},\, 1 - (1-\gamma_2)^{\alpha}\} \\ \{1 - (1-\gamma'_1)^{\alpha},\, 1 - (1-\gamma'_2)^{\alpha}\} \end{pmatrix}$$

The use of (12) and (1) yields the following:

$$\begin{aligned}\|\alpha H\|_1 = \big\{&1-(1-\gamma_1)^\alpha + 1-(1-{\gamma'}_1)^\alpha - (1-(1-\gamma_1)^\alpha)\left(1-(1-{\gamma'}_1)^\alpha\right),\\ &1-(1-\gamma_1)^\alpha + 1-(1-{\gamma'}_2)^\alpha - (1-(1-\gamma_1)^\alpha)\left(1-(1-{\gamma'}_2)^\alpha\right),\\ &1-(1-\gamma_2)^\alpha + 1-(1-{\gamma'}_1)^\alpha - (1-(1-\gamma_2)^\alpha)\left(1-(1-{\gamma'}_1)^\alpha\right),\\ &1-(1-\gamma_2)^\alpha + 1-(1-{\gamma'}_2)^\alpha - (1-(1-\gamma_2)^\alpha)\left(1-(1-{\gamma'}_2)^\alpha\right)\big\}\\ = \big\{&1-(1-\gamma_1)^\alpha(1-{\gamma'}_1)^\alpha,\ 1-(1-\gamma_1)^\alpha(1-{\gamma'}_2)^\alpha,\\ &1-(1-\gamma_2)^\alpha(1-{\gamma'}_1)^\alpha, 1-(1-\gamma_2)^\alpha(1-{\gamma'}_2)^\alpha\big\} = \alpha\,\|H\|_1\end{aligned}$$

4. The following can be obtained by using (11) and (1) for the matrices $H = \begin{pmatrix} \{\gamma_1,\, \gamma_2\} \\ \{{\gamma'}_1,\, {\gamma'}_2\} \end{pmatrix}$ and $H' = \begin{pmatrix} \{\delta_1,\, \delta_2\} \\ \{{\delta'}_1,\, {\delta'}_2\} \end{pmatrix}$:

$$\|H\|_1 = \{\gamma_1 + {\gamma'}_1 - \gamma_1{\gamma'}_1,\ \gamma_1 + {\gamma'}_2 - \gamma_1{\gamma'}_2,\ \gamma_2 + {\gamma'}_1 - \gamma_2{\gamma'}_1,\ \gamma_2 + {\gamma'}_2 - \gamma_2{\gamma'}_2\}$$

$$\|H'\|_1 = \{\delta_1 + {\delta'}_1 - \delta_1{\delta'}_1,\ \delta_1 + {\delta'}_2 - \delta_1{\delta'}_2,\ \delta_2 + {\delta'}_1 - \delta_2{\delta'}_1,\ \delta_2 + {\delta'}_2 - \delta_2{\delta'}_2\}$$

$$\begin{aligned}&\|H\|_1 + \|H'\|_1 =\\ &\{\gamma_1 + {\gamma'}_1 - \gamma_1{\gamma'}_1, \gamma_1 + {\gamma'}_2 - \gamma_1{\gamma'}_2, \gamma_2 + {\gamma'}_1 - \gamma_2{\gamma'}_1, \gamma_2 + {\gamma'}_2 - \gamma_2{\gamma'}_2\}\\ &+\{\delta_1 + {\delta'}_1 - \delta_1{\delta'}_1,\ \delta_1 + {\delta'}_2 - \delta_1{\delta'}_2,\ \delta_2 + {\delta'}_1 - \delta_2{\delta'}_1,\ \delta_2 + {\delta'}_2 - \delta_2{\delta'}_2\}\end{aligned} \tag{14}$$

$$H + H' = \begin{pmatrix} \{\gamma_1 + {\gamma'}_1 - \gamma_1{\gamma'}_1, \gamma_1 + {\gamma'}_2 - \gamma_1{\gamma'}_2, \gamma_2 + {\gamma'}_1 - \gamma_2{\gamma'}_1, \gamma_2 + {\gamma'}_2 - \gamma_2{\gamma'}_2\} \\ \{\delta_1 + {\delta'}_1 - \delta_1{\delta'}_1,\ \delta_1 + {\delta'}_2 - \delta_1{\delta'}_2, \delta_2 + {\delta'}_1 - \delta_2{\delta'}_1, \delta_2 + {\delta'}_2 - \delta_2{\delta'}_2\} \end{pmatrix}$$

Hence, it can be obtained from (11) that:

$$\begin{aligned}&\|H + H'\|_1 =\\ &\quad\{\gamma_1 + {\gamma'}_1 - \gamma_1{\gamma'}_1,\ \gamma_1 + {\gamma'}_2 - \gamma_1{\gamma'}_2,\ \gamma_2 + {\gamma'}_1 - \gamma_2{\gamma'}_1,\ \gamma_2 + {\gamma'}_2 - \gamma_2{\gamma'}_2\}\\ &\quad+\{\delta_1 + {\delta'}_1 - \delta_1{\delta'}_1,\ \delta_1 + {\delta'}_2 - \delta_1{\delta'}_2,\ \delta_2 + {\delta'}_1 - \delta_2{\delta'}_1,\ \delta_2 + {\delta'}_2 - \delta_2{\delta'}_2\}\end{aligned}$$

It can be deduced from the above equality, (14) and Definition 2.3 that:

$$\|H + H'\|_1 \le \|H\|_1 + \|H'\|_1\,.$$

Therefore, the proof is complete. □

Example 2.4 Assume $H = \begin{pmatrix} \{0.2, 0.1\} \\ \{0.4\} \end{pmatrix}$ and $H' = \begin{pmatrix} \{0.5\} \\ \{0.3, 0.6\} \end{pmatrix}$. Using (11), we get:

$$\|H\|_1 = \{0.2 + 0.4 - 0.2 \times 0.4,\ 0.1 + 0.4 - 0.1 \times 0.4\}$$

$$\|H'\|_1 = \{0.5 + 0.3 - 0.5 \times 0.3, 0.5 + 0.6 - 0.5 \times 0.6\}$$

On the other hand, $\forall \alpha > 0$:

$$\begin{aligned}
&\alpha \|H\|_1 = \alpha \{0.2 + 0.4 - 0.2 \times 0.4,\ 0.1 + 0.4 - 0.1 \times 0.4\} \\
&= \{1 - (1 - (0.2 + 0.4 - 0.2 \times 0.4))^{\alpha}, 1 - (1 - (0.1 + 0.4 - 0.1 \times 0.4))^{\alpha}\} \\
&= \{1 - (1 - 0.2)^{\alpha} (1 - 0.4)^{\alpha}, 1 - (1 - 0.1)^{\alpha} (1 - 0.4)^{\alpha}\}
\end{aligned}$$

$$\alpha H = \alpha \begin{pmatrix} \{0.2, 0.1\} \\ \{0.4\} \end{pmatrix} = \begin{pmatrix} \alpha \{0.2, 0.1\} \\ \alpha \{0.4\} \end{pmatrix} =$$

$$\begin{pmatrix} \{1 - (1 - 0.2)^{\alpha}, 1 - (1 - 0.1)^{\alpha}\} \\ \{1 - (1 - 0.4)^{\alpha}\} \end{pmatrix}$$

$$\begin{aligned}
&\|\alpha H\|_1 = \\
&\{1 - (1 - 0.2)^{\alpha} + 1 - (1 - 0.4)^{\alpha} - (1 - (1 - 0.2)^{\alpha}) (1 - (1 - 0.4)^{\alpha}), \\
&1 - (1 - 0.1)^{\alpha} + 1 - (1 - 0.4)^{\alpha} - (1 - (1 - 0.1)^{\alpha}) (1 - (1 - 0.4)^{\alpha})\} \\
&= \{1 - (1 - 0.2)^{\alpha} (1 - 0.4)^{\alpha},\ 1 - (1 - 0.1)^{\alpha} (1 - 0.4)^{\alpha}\} \\
&= \alpha \|H\|_1 .
\end{aligned}$$

It can be seen that the third property of the previous theorem is satisfied ($\|\alpha H\|_1 = \alpha \|H\|_1$).

At the same time:

$$\begin{aligned}
&\{0.2 + 0.4 - 0.2 \times 0.4,\ 0.1 + 0.4 - 0.1 \times 0.4\} + \\
&\{0.5 + 0.3 - 0.5 \times 0.3, 0.5 + 0.6 - 0.5 \times 0.6\} \\
&= \{0.2 + 0.4 + 0.3 + 0.5 - 0.2 \times 0.4 - 0.5 \times 0.3 - 0.2 \times 0.5 - 0.2 \times 0.3 \\
&-0.2 \times 0.5 \times 0.3 - 0.4 \times 0.5 - 0.4 \times 0.3 + 0.4 \times 0.5 \times 0.3 + 0.2 \times 0.4 \times 0.5 \\
&+0.2 \times 0.4 \times 0.3 - 0.2 \times 0.4 \times 0.3 \times 0.5, 0.2 + 0.4 + 0.5 + 0.6 - 0.2 \times 0.4 \\
&-0.5 \times 0.6 - 0.2 \times 0.5 - 0.2 \times 0.6 + 0.2 \times 0.5 \times 0.6 - 0.4 \times 0.5 - 0.4 \times 0.6 \\
&+0.4 \times 0.5 \times 0.6 + 0.2 \times 0.4 \times 0.5 + 0.2 \times 0.4 \times 0.6 - 0.2 \times 0.4 \times 0.5 \times 0.6, 0.1 \\
&+0.4 + 0.5 + 0.3 - 0.1 \times 0.4 - 0.5 \times 0.3 - 0.1 \times 0.5 - 0.1 \times 0.3 \\
&+0.1 \times 0.5 \times 0.3 - 0.4 \times 0.5 - 0.4 \times 0.3 + 0.4 \times 0.5 \times 0.3 + 0.1 \times 0.4 \times 0.5 \\
&+0.1 \times 0.4 \times 0.3 - 0.1 \times 0.4 \times 0.5 \times 0.3, 0.1 + 0.4 + 0.5 \\
&+0.6 - 0.1 \times 0.4 - 0.5 \times 0.6 - 0.1 \times 0.5 - 0.1 \times 0.6 \\
&+0.1 \times 0.5 \times 0.6 - 0.4 \times 0.5 - 0.4 \times 0.6 + 0.4 \times 0.5 \times 0.6 + 0.1 \times 0.4 \times 0.5 \\
&+0.1 \times 04 \times 0.6 - 0.1 \times 0.4 \times 0.5 \times 0.6\},
\end{aligned}$$

$$H + H^{'} = \begin{pmatrix} \{0.2 + 0.5 - 0.2 \times 0.5, 0.1 + 0.5 - 0.1 \times 0.5\} \\ \{0.4 + 0.3 - 0.4 \times 0.3, 0.4 + 0.6 - 0.4 \times 0.6\} \end{pmatrix}$$

$$\begin{aligned}
&\|H+H'\|_1 = 0.2+0.4+0.3+0.5-0.2\times0.4-0.5\times0.3\\
&-0.2\times0.5-0.2\times0.3-0.2\times0.5\times0.3\\
&-0.4\times0.5-0.4\times0.3+0.4\times0.5\times0.3+0.2\times0.4\times0.5+0.2\times0.4\times0.3\\
&-0.2\times0.4\times0.3\times0.5, 0.2+0.4+0.5+0.6-0.2\times0.4-0.5\times0.6\\
&-0.2\times0.5-0.2\times0.6+0.2\times0.5\times0.6-0.4\times0.5\\
&-0.4\times0.6+0.4\times0.5\times0.6+0.2\times0.4\times0.5+0.2\times0.4\times0.6\\
&-0.2\times0.4\times0.5\times0.6, 0.1+0.4+0.5+0.3-0.1\times0.4-0.5\times0.3\\
&-0.1\times0.5-0.1\times0.3+0.1\times0.5\times0.3-0.4\times0.5-0.4\times0.3\\
&+0.4\times0.5\times0.3+0.1\times0.4\times0.5+0.1\times0.4\times0.3\\
&-0.1\times0.4\times0.5\times0.3, 0.1+0.4+0.5+0.6-0.1\times0.4-0.5\times0.6\\
&-0.1\times0.5-0.1\times0.6+0.1\times0.5\times0.6-0.4\times0.5-0.4\times0.6\\
&+0.4\times0.5\times0.6+0.1\times0.4\times0.5\\
&+0.1\times0.4\times0.6-0.1\times0.4\times0.5\times0.6
\end{aligned}$$

As a result, it can be deduced from Definition 2.3 that:

$$\|H+H'\|_1 \le \|H\|_1 + \|H'\|_1 .$$

DEFINITION 2.11 *The vector system*

$$\begin{pmatrix} <c_{11},a_{11}> \cdots <c_{1n},a_{1n}> \\ <c_{21},a_{21}> \cdots <c_{2n},a_{2n}> \\ \vdots \\ <c_{n1},a_{n1}> \cdots <c_{nn},a_{nn}> \end{pmatrix} \begin{pmatrix} <x_1,h_1> \\ <x_2,h_2> \\ \vdots \\ <x_n,h_n> \end{pmatrix} = \begin{pmatrix} <d_1,b_1> \\ <d_2,b_2> \\ \vdots \\ <d_n,b_n> \end{pmatrix} \quad (15)$$

is called hesitant fuzzy vector system if for each $1 \le i,\ j \le n$, c_{ij} *and* d_j *being known real numbers,* x_j *are unknown real numbers,* a_{ij} *and* b_j *are known hesitant fuzzy elements and* h_j *are unknown hesitant fuzzy elements. By assuming* $A = (<c_{ij}, a_{ij}>)$, $B = \left(<d_j, b_j>\right)$ *and* $X = \left(<x_j, h_j>\right)$ *the system is briefly denoted* $AX = B$.

Solution of the hesitant fuzzy system

In order to calculate X, first the following crisp system is solved:

$$\begin{pmatrix} c_{11}\cdots c_{1n} \\ c_{21}\cdots c_{2n} \\ \vdots \\ c_{n1}\cdots c_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}. \quad (16)$$

By solving the above system, $x_1x_2,\ldots,x_n$ are obtained.

Now, consider the following hesitant fuzzy element system:

$$\begin{pmatrix} a_{11} \cdots a_{1n} \\ \vdots \\ a_{n1} \cdots a_{nn} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}. \tag{17}$$

There is:

$$\begin{pmatrix} a_{11} \times h_1 + \ \cdots + \ a_{1n} \times h_n \\ \vdots \\ a_{n1} \times h_1 + \ \cdots + \ a_{nn} \times h_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}. \tag{18}$$

For each $1 \leq i, \, j \leq n$, $a_{ij} \times h_1 + \ \cdots + \ a_{ij} \times h_n$ can be calculated using (1). Also, for each $1 \leq i \leq n$, there is $h'_i = a_{ij} \times h_1 + \ \cdots + a_{ij} \times h_n$. By substituting h'_i in (18), it can be obtained that:

$$\begin{pmatrix} h'_1 \\ \vdots \\ h'_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

By assuming $H^{'} = \begin{pmatrix} h'_1 \\ \vdots \\ h'_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ and considering $\|H'\|_\infty = \|B\|_\infty$, it can be found that:

$$\{\max\{h'_1\}, \ldots, \max\{h'_n\}\} = \{\max\{b_1\}, \ldots, max\{b_n\}\}. \tag{19}$$

Finally, for each $1 \leq i \leq n$ and assuming $b'_i = max\{b_i\}$ and $h''_i = max\{h'_i\}$, upon solving the following minimization problem, the solution of hesitant fuzzy element system (17) can be calculated in the following manner:

$$\min\, z = \sum_{i=1}^{n} \left| h_i^{''} - b_i^{'} \right| \; st: \; 0 \leq h_i \leq 1, \, z \geq 0. \tag{20}$$

Theorem 2.3 *If the real-valued matrix*

$$\begin{pmatrix} c_{11} \cdots c_{1n} \\ c_{21} \cdots c_{2n} \\ \vdots \\ c_{n1} \cdots c_{nn} \end{pmatrix}$$

is nonsingular and the linear programming problem (20) has a feasible solution, then the hesitant fuzzy system (15) has a hesitant fuzzy solution.

3. Numerical examples

In this section, some examples are solved to illustrate the utility of the proposed method.

Example 3.1 Assume

$$A = \begin{pmatrix} <2, \{0.7, 0.8\}> & <1, \{0.9, 0.6\}> \\ <-3, \{0.5, 0.7\}> & <6, \{0.6, 0.9\}> \end{pmatrix},$$

$$X = \begin{pmatrix} < x_1, h_1 > \\ < x_2, h_2 > \end{pmatrix}$$

and

$$B = \begin{pmatrix} <5, \{0.1, 0.4\}> \\ < 0, \{0.3, 0.1\} > \end{pmatrix}.$$

First, by solving the crisp system

$$\begin{pmatrix} 2 & 1 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

it can be established that

$$x_1 = 2,\ x_2 = 1.$$

Now, consider the following hesitant fuzzy element vector system:

$$\begin{pmatrix} \{0.7, 0.8\} & \{0.9, 0.6\} \\ \{0.5, 0.7\} & \{0.6, 0.9\} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \frac{\{0.1, 0.4\}}{\{0.3, 0.1\}} \end{pmatrix}.$$

There is:

$$\begin{pmatrix} \{0.7, 0.8\}\, h_1 + \{0.9, 0.6\}\, h_2 \\ \{0.5, 0.7\}\, h_1 + \{0.6, 0.9\}\, h_2 \end{pmatrix} = \begin{pmatrix} \frac{\{0.1, 0.4\}}{\{0.3, 0.1\}} \end{pmatrix}.$$

It can be obtained from (1) that:

$$\left(\begin{array}{c} \left\{ \begin{array}{c} 0.7\, h_1 + 0.9\, h_2 - 0.63\, h_1 h_2, \\ 0.7\, h_1 + 0.6\, h_2 - 0.42\, h_1 h_2, \\ 0.8\, h_1 + 0.9\, h_2 - 0.72\, h_1 h_2, \\ 0.8\, h_1 + 0.6\, h_2 - 0.48\, h_1 h_2 \end{array} \right\} \\ \left\{ \begin{array}{c} 0.5\, h_1 + 0.6\, h_2 - 0.3\, h_1 h_2, \\ 0.5\, h_1 + 0.9\, h_2 - 0.45\, h_1 h_2, \\ 0.7\, h_1 + 0.6\, h_2 - 0.42\, h_1 h_2, \\ 0.7\, h_1 + 0.9\, h_2 - 0.63\, h_1 h_2 \end{array} \right\} \end{array} \right) = \begin{pmatrix} \frac{\{0.1, 0.4\}}{\{0.3, 0.1\}} \end{pmatrix}.$$

By making the L_∞ norms of the above two vectors equal to each other,

$$\left\| \left(\begin{Bmatrix} 0.7\,h_1 +\,0.9\,h_2 - 0.63\,h_1h_2, \\ 0.7\,h_1 +\,0.6\,h_2 - 0.42\,h_1h_2, \\ 0.8\,h_1 +\,0.9\,h_2 - 0.72\,h_1h_2, \\ 0.8\,h_1 +\,0.6\,h_2 - 0.48\,h_1h_2 \end{Bmatrix} \atop \begin{Bmatrix} 0.5\,h_1 +\,0.6\,h_2 - 0.3\,h_1h_2, \\ 0.5\,h_1 +\,0.9\,h_2 - 0.45\,h_1h_2, \\ 0.7\,h_1 +\,0.6\,h_2 - 0.42\,h_1h_2, \\ 0.7\,h_1 +\,0.9\,h_2 - 0.63\,h_1h_2 \end{Bmatrix} \right) \right\|_\infty = \left\| \left(\frac{\{0.1, 0.4\}}{\{0.3, 0.1\}} \right) \right\|_\infty,$$

one obtains:

$$\{\,0.8\,h_1 +\,0.9\,h_2 - 0.72\,h_1h_2,\;\; 0.7\,h_1 +\,0.9\,h_2 - 0.63\,h_1h_2\} \;= \{0.4, 0.3\}\,.$$

Finally, the following minimization problem is formulated in order to calculate h_1 and h_2:

$$\min\;\; z = |0.8\,h_1 +\,0.9\,h_2 - 0.72\,h_1h_2 - 0.4| + |0.7\,h_1 +\,0.9\,h_2 - 0.63\,h_1h_2 - 0.3|$$

$$0 \le h_1,\, h_2 \le 1,\, z \ge 0.$$

The result is found to be:

$$h_1 = 0.5\,, h_2 = 0,\; z = 0.05.$$

Therefore

$$X = \begin{pmatrix} <2, 0.1> \\ <1, 0> \end{pmatrix}.$$

Figure 1 illustrates the nature of the obtained results for this example.

Example 3.2 Consider the following hesitant fuzzy system:

$$\begin{pmatrix} <1, \{0.7, 0.8, 0.2\}> & <1, \{0.3, 0.2, 0.4, 0.6\}> \\ <2, \{0.5, 0.8, 0.4, 0.1\}> & <1, \{0.1, 0.2, 0.5\}> \end{pmatrix} \begin{pmatrix} < x_1, h_1> \\ < x_2, h_2> \end{pmatrix}$$
$$= \left(\frac{<5, \{0.6, 0.4, 0.9\}>}{<7, \{0.5, 0.7, 0.8, 0.3\}>} \right).$$

First, by solving the crisp system

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{5}{7} \end{pmatrix}$$

it is obtained that:

$$x_1 = 2,\; x_2 = 3.$$

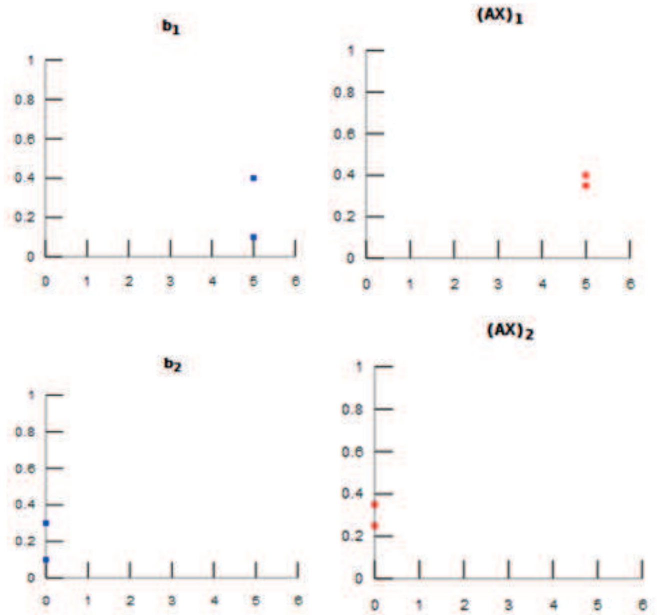

Figure 1. Comparison between AX and B (Example 3.1)

Now, consider the hesitant fuzzy element system:

$$\begin{pmatrix} \{0.7, 0.8, 0.2\} & \{0.3, 0.2, 0.4, 0.6\} \\ \{0.5, 0.8, 0.4, 0.1\} & \{0.1, 0.2, 0.5\} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \{0.6, 0.4, 0.9\} \\ \{0.5, 0.7, 0.8, 0.3\} \end{pmatrix}.$$

In order to solve the above system, the following corresponding linear programming problem is solved:

$$\min z = |0.8\, h_1 + 0.6\, h_2 - 0.48\, h_1 h_2 - 0.9| + |0.8\, h_1 + 0.5\, h_2 - 0.4\, h_1 h_2 - 0.8|$$
$$0 \leq h_1,\, h_2 \leq 1,\, z \geq 0.$$

The result is:

$$h_1 = 0.75\,,\; h_2 = 1,\; z = 0.06.$$

Therefore, the final solution of the system is as follows:

$$X = \begin{pmatrix} < 2, 0.75 > \\ < 3, 1 > \end{pmatrix}$$

The results from this example are illustrated in Fig. 2.

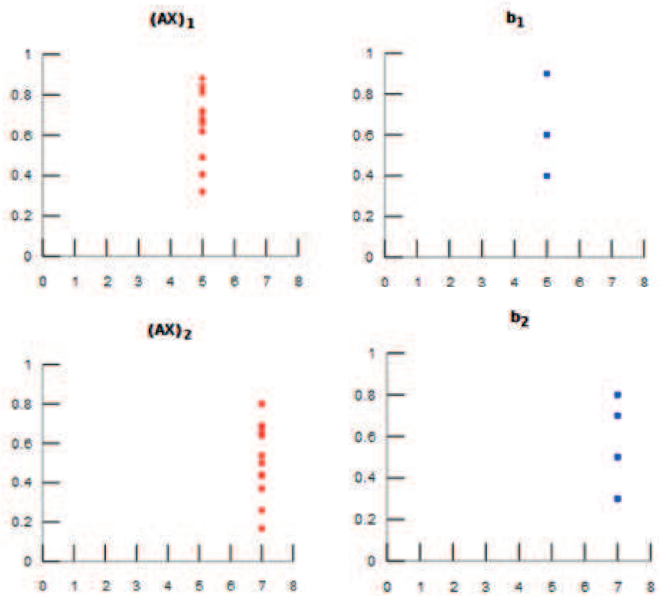

Figure 2. Comparison between AX and B (Example 3.2)

Example 3.3 Consider the following hesitant fuzzy system:

$$\begin{pmatrix} <1,\{0.7,0.8,0.3\}> & <1,\{0.2,0.4\}> & <1,\{0.6,0.5,0.7\}> \\ <2,\{0.4,0.1\}> & <-3,\{0.5,0.6\}> & <4,\{0.2,0.3,0.9\}> \\ <3,\{0.1,0.8\}> & <4,\{0.4,0.3\}> & <5,\{0.9,0.5\}> \end{pmatrix} \times$$
$$\times \begin{pmatrix} <x_1,h_1> \\ <x_2,h_2> \\ <x_3,h_3> \end{pmatrix}$$
$$= \begin{pmatrix} <9,\{0.6,0.4,0.7\}> \\ <13,\{0.4,0.2,0.6\}> \\ <40,\{0.9,0.3,0.5,0.7\}> \end{pmatrix}$$

First, the following crisp system is solved:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 13 \\ 40 \end{pmatrix}.$$

The solution is found to be:

$$x_1 = 1,\ x_2 = 3,\ x_3 = 5.$$

Now, the hesitant fuzzy element system is considered:

$$\begin{pmatrix} \{0.7, 0.8, 0.3\} & \{0.2, 0.4\} & \{0.6, 0.5, 0.7\} \\ \{0.4, 0.1\} & \{0.5, 0.6\} & \{0.2, 0.3, 0.9\} \\ \{0.1, 0.8\} & \{0.4, 0.3\} & \{0.9, 0.5\} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} \{0.6, 0.4, 0.7\} \\ \{0.4, 0.2, 0.6\} \\ \{0.9, 0.3, 0.5, 0.7\} \end{pmatrix}.$$

The following corresponding linear programming problem must be solved in order to find the solution of the above system:

$$\begin{aligned} &\min z = \\ &= |0.8h_1 + 0.4\,h_2 - 0.32h_1h_2 + 0.7h_3 - 0.56h_1h_3 - 0.28h_2h_3 + 0.224h_1h_2h_3 - 0.7| \\ &+ |0.4h_1 + 0.6h_2 - 0.24h_1h_2 + 0.9h_3 - 0.36h_1h_3 - 0.54h_2h_3 + 0.216h_1h_2h_3 - 0.6| \\ &+ |0.8h_1 + 0.4h_2 - 0.32h_1h_2 + 0.9h_3 - 0.72h_1h_3 - 0.36h_2h_3 + 0.288h_1h_2h_3 - 0.9| \\ &\qquad st : 0 \leq h_1,\ h_2h_3 \leq 1,\ z \geq 0. \end{aligned}$$

It can be obtained that:

$$h_1 = 0.67,\ h_2 = 0.06, h_3 = 0.48,\ z = 0.16.$$

As a result, the final solution of the system is established as:

$$\mathrm{X} = \begin{pmatrix} <1, 0.67> \\ <3, 0.06> \\ <5, 0.48> \end{pmatrix}$$

The results for this example are illustrated in Fig. 3.

4. Conclusion

In this paper, L_∞ and L_1 norms for hesitant fuzzy sets and hesitant fuzzy zero are defined and then used to solve the hesitant fuzzy vector system. The advantage of the proposed method is that the hesitant fuzzy solution of a hesitant fuzzy system can be established with lower complexity and computational effort and only through solving a linear minimization problem in a very short time. In the future research, it is planned to solve the dual hesitant fuzzy matrix system and Silvester hesitant fuzzy system.

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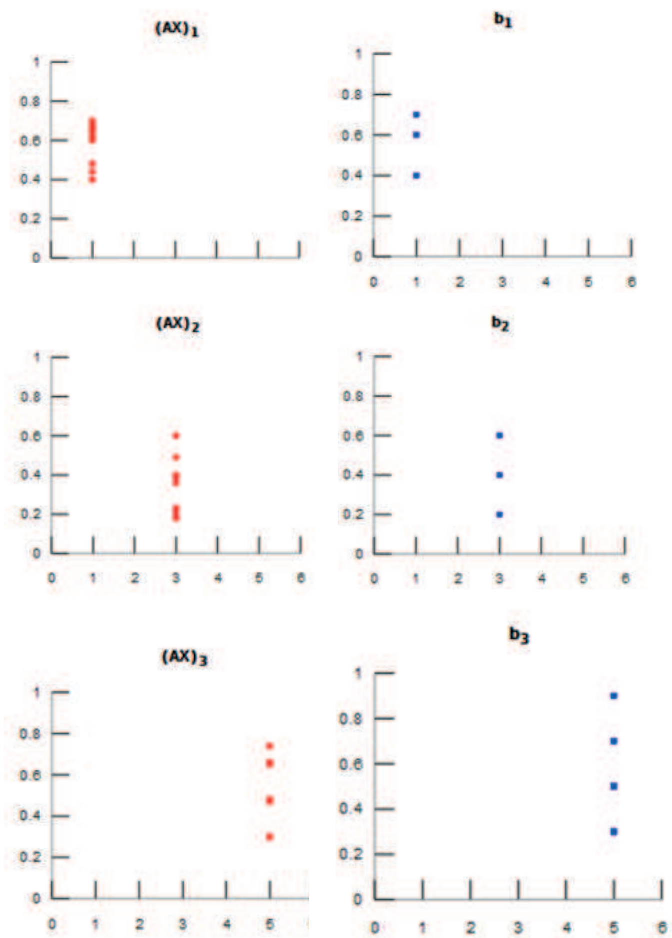

Figure 3. Comparison between AX and B (Example 3.3)

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