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Temporally sparse controls for infinite horizon semilinear parabolic equations with norm constraints*†

by

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Abstract: In this paper, infinite horizon optimal control problems subject to semilinear parabolic equations are investigated. A finite number of only time-dependent controls intervening at disjoint positions in the space domain are considered. The controls are subject to integral constraints and a term is included in the cost functional that promotes control sparsity. The existence of optimal controls is proven, first and second order optimality conditions are derived, and the approximation by finite horizon control problems is addressed.

Keywords: semilinear parabolic equation, sparse optimal control, infinite horizon problems, first and second order optimality conditions

1. Introduction

In this paper, we study the following control problem

$$(\mathrm{P})\begin{cases} \text{Minimize} \quad J(u) := F(u) + \alpha\, j(u) \\ u \in \mathcal{U}_{ad} \end{cases}$$

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where $\alpha > 0$,

$$F(u) = \int_0^\infty \int_\Omega |y_u(x,t) - y_d(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t,$$

$$j : L^1(0,\infty)^m \longrightarrow \mathbb{R}, \quad j(u) = \sum_{i=1}^m \int_0^\infty |u_i(t)|\,\mathrm{d}t,$$

$$\mathcal{U}_{ad} = \{u \in \mathcal{U} : \|u_i\|_{L^p(0,\infty)} \le \gamma_i,\ 1 \le i \le m\},$$

with $y_d \in L^2(Q)$, $\mathcal{U} = L^p(0,\infty)^m \cap L^1(0,\infty)^m$, $m \in \mathbb{N}$, $2 \le p \le \infty$, and $0 < \gamma_i < \infty$ for $1 \le i \le m$. Here y_u denotes the solution of the equation:

$$\left\{\begin{array}{l} \partial_t y_u + Ay_u + f(x,t,y_u) = g(x,t) + Bu \ \text{ in } Q = \Omega \times (0,\infty), \\ \partial_{n_A} y_u = 0 \text{ on } \Sigma = \Gamma \times (0,\infty), \quad y_u(0) = y_0 \text{ in } \Omega, \end{array}\right. \tag{1.1}$$

where $Bu = \sum_{i=1}^m u_i(t)\psi_i(x)$ for some functions $\{\psi_i\}_{i=1}^m \subset L^\infty(\Omega)$ with $\mathrm{supp}(\psi_i) \cap \mathrm{supp}(\psi_j) = \emptyset$ for $i \ne j$, and A is the linear elliptic operator

$$Ay = -\sum_{i,j=1}^n \partial_{x_j}[a_{ij}(x,t)\,\partial_{x_i} y] + a_0(x,t)y.$$

Assumptions on the coefficients of A and the functions f, g, and y_0 will be given in the next section.

We observe that $\mathcal{U}$ is continuously embedded in $L^2(0,\infty)^m$, which follows by interpolation between the spaces $L^1(0,\infty)^m$ and $L^p(0,\infty)^m$. As a consequence, we also have that $Bu \in L^2(Q) \cap L^p(0,\infty; L^\infty(\Omega))$.

By studying (P) we continue our investigations of nonlinear pde-constrained optimal control problems over infinite time horizons. Such problems have received little attention, although they arise quite naturally, for example, in the context of optimal stabilization, or in the case of modeling with finite horizons, where the length of the horizon is ambiguous and choosing an infinite horizon would be a safe way out. For infinite horizon optimal control problems. related to ordinary differential equations, we cite the monograph by Carlson, Haurie and Leizarowitz (1991), and selected papers by Aseer, Krastanov and Veliov (2017), Basco, Cannarsa and Frankowska (2018), and Kalkin (1974), where the latter might well be one of the earliest publications on the subject. In our work we treated infinite horizon problems related to semilinear parabolic equations in, e.g., Casas and Kunisch (2023a), and to the Navier Stokes equations in Casas and Kunisch (2024a). The specificity of problem (P), which is not considered in our previous work, consists in the fact that (P) does not contain a quadratic space-time cost of the control, but rather the sparsifying term in time only, and that the explicit control constraint is of energy type in time, rather than pointwise in time as it was in our previous work. This necessitates a different treatment of the optimality conditions, especially the second order necessary and sufficient optimality conditions. The control action enters (1.1) by means of finitely many time dependent controls u_i. The spatial distribution of the controllers is fixed and described by ψ_i. We shall utilize a feasibility assumption guaranteeing that there exists at least one control vector u satisfying the constraints and rendering the corresponding cost J finite. Such an assumption is justified by stabilizability results, which guarantee that for properly chosen m and ψ_i, there exists a control in feedback form for which the associated state decays exponentially, see, e.g., Azmi, Kunisch and Rodrigues (2021). The sparsifying term j in the cost then guarantees that optimal controls will individually shut off an remain shut off from times T_i^* on.

The paper is organised as follows. Section 2 presents properties of the state equation needed for the analysis that follows. In the following Section 3 we prove the existence of a solution for (P) and establish the first order necessary conditions. Second order necessary and sufficient optimality conditions for (P) are presented in Section 4, where special attention is paid to the fact that the gap between these two conditions is small. Finally, Section 5 is devoted to the approximation of the infinite horizon control problem by finite horizon control problems.

2. Analysis of the state equation

In this section, we are concerned with the existence, uniqueness, and regularity of the solution of (1.1). To this end, we make the following assumptions:

Assumption 2.1 *We assume* $1 \le n \le 3$, $y_0 \in L^\infty(\Omega)$ *and* $g \in L^r(0,\infty;L^s(\Omega)) \cap L^2(Q)$ *with* $\frac{1}{r} + \frac{n}{2s} < 1$.

The function $f : Q \times \mathbb{R} \longrightarrow \mathbb{R}$ *is measurable with respect to* $(x,t) \in Q$ *and of class* C^2 *with respect to* $y \in \mathbb{R}$ *and satisfies the following hypotheses*

$$f(x,t,0) = 0, \tag{2.1}$$

$$\exists \delta_f \in [0,1) : \frac{\partial f}{\partial y}(x,t,0) \ge -\delta_f a_0(x,t), \tag{2.2}$$

$$\exists M_f > 0 : f(x,t,y)y \ge 0 \text{ and } \frac{\partial f}{\partial y}(x,t,y) \ge 0 \quad \forall |y| > M_f, \tag{2.3}$$

$$\forall M > 0\ \exists C_{f,M} > 0 : \Big|\frac{\partial^j f}{\partial y^j}(x,t,y)\Big| \le C_{f,M} \quad \forall |y| \le M,\ j = 1,2, \tag{2.4}$$

$$\forall \rho > 0 \text{ and } \forall M > 0 \;\; \exists \varepsilon > 0 \text{ such that}$$
$$\Big|\frac{\partial^2 f}{\partial y^2}(x,t,y_2) - \frac{\partial^2 f}{\partial y^2}(x,t,y_1)\Big| \le \rho \;\; \forall |y_i| \le M, i = 1,2, \text{ if } |y_2 - y_1| \le \varepsilon, \tag{2.5}$$

for almost all $(x,t) \in Q$.

The coefficients of A *satisfy* $a_{i,j}, a_0 \in L^\infty(Q)$ *and*

$$\exists \Lambda > 0 : \sum_{i,j=1}^n a_{i,j}(x,t)\xi_i\xi_j \ge \Lambda|\xi|^2 \;\; \forall \xi \in \mathbb{R}^n \text{ and for a.a. } (x,t) \in Q, \tag{2.6}$$
$$a_0(x,t) \ge 0 \text{ for a.a. } (x,t) \in Q, \text{ and } a_0 \not\equiv 0. \tag{2.7}$$

We mention that (2.3) and (2.4) imply that

$$\frac{\partial f}{\partial y}(x,t,y) \ge -C_{f,M_f} \;\; \forall y \in \mathbb{R} \text{ and for a.a. } (x,t) \in Q. \tag{2.8}$$

Hereafter, we will follow the standard notation

$$W(0,T) = \{y \in L^2(0,T;H^1(\Omega)) : \partial_t y \in L^2(0,T;H^1(\Omega)^*)\} \text{ for } 0 < T \le \infty,$$

where $H^1(\Omega)^*$ denotes the conjugate of $H^1(\Omega)$. A function y is called solution of (1.1) if $y \in W(0,T) \cap L^\infty(Q_T)$ for all $0 < T < \infty$ and it satisfies the following equation in the variational sense

$$\left\{\begin{array}{l} \partial_t y + Ay + f(x,t,y) = g(x,t) + Bu \;\text{ in } Q_T = \Omega \times (0,T), \\ \partial_{n_A} y = 0 \text{ on } \Sigma_T = \Gamma \times (0,T), \quad y(0) = y_0 \text{ in } \Omega, \end{array}\right. \tag{2.9}$$

We know that (2.9) admits a unique solution y_u in $W(0,T) \cap L^\infty(Q_T)$ for every $u \in L^p(0,\infty)$ and all $T < \infty$. Moreover, if $u \in \mathcal{U}$ and $y_u \in L^2(Q)$, then the regularity $y_u \in W(0,\infty) \cap L^\infty(Q)$ holds. Further the following estimates are satisfied:

$$\|y_u\|_Q \le K_1\Big(\|y_u\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)} + \|g + Bu\|_{L^2(Q)}\Big), \tag{2.10}$$
$$\begin{aligned} \|y_u\|_{L^\infty(Q)} \le K_2\Big(&\|y_u\|_{L^2(Q)} + \|y_0\|_{L^\infty(\Omega)} + \|g + Bu\|_{L^2(Q)} \\ &+ \|g\|_{L^r(0,\infty,L^s(\Omega))} + \|Bu\|_{L^p(0,\infty;L^\infty(\Omega))} + M_f\Big), \end{aligned} \tag{2.11}$$

where

$$\|y_u\|_Q = \Big(\|y_u\|^2_{L^2(0,\infty;H^1(\Omega)} + \|y_u\|^2_{L^\infty(0,\infty;L^2(\Omega))}\Big)^{\frac{1}{2}};$$

see Casas and Kunisch (2024b) for the proof. Using (2.1), (2.4), and the mean value theorem we infer that

$$|f(x,t,y_u(x,t))| = \Big|\frac{\partial f}{\partial y}(x,t,\theta(x,t)y_u(x,t))\Big||y_u(x,t)| \le C_{f,M}|y_u(x,t)|,$$

where $M = \|y_u\|_{L^\infty(Q)}$ and $0 \le \theta(x,t) \le 1$. Since $y_u \in L^2(Q)\cap L^\infty(Q)$, the above inequality implies that $f(\cdot,\cdot,y_u) \in L^2(Q)\cap L^\infty(Q)$ and the following estimates hold

$$\|f(\cdot,\cdot,y_u)\|_{L^\infty(Q)} \le C_{f,M}M \quad\text{and}\quad \|f(\cdot,\cdot,y_u)\|_{L^2(Q)} \le C_{f,M}\|y_u\|_{L^2(Q)}. \tag{2.12}$$

Using these properties we deduce from the equation (1.1) that $y_u \in W(0,\infty)\cap L^\infty(Q)$. Moreover, from (2.10) and (2.12) we get

$$\|y_u\|_{W(0,\infty)} \le K_3\Big((1+C_{f,M})\|y_u\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)} + \|g+Bu\|_{L^2(Q)}\Big). \tag{2.13}$$

We define the set

$$\mathcal{A} = \{u \in L^2(0,\infty)^m : y_u \in L^2(Q)\}.$$

Now we introduce the mapping $G : \mathcal{A} \longrightarrow W(0,\infty)\cap L^\infty(Q)$ define by $G(u) = y_u$. The following theorem was proven in Casas and Kunisch (2023a, Theorems 2.2 and 3.1).

THEOREM 2.2 *Let us assume that $\mathcal{A}$ is not empty. Then, $\mathcal{A}$ is an open subset of $L^2(0,\infty)^m$ and the mapping G is of class C^2. Moreover, given $u \in \mathcal{A}$ and $v, v_1, v_2 \in L^2(0,\infty)^m$ we have that $z_v = G'(u)v$ and $z_{v_1,v_2} = G''(u)(v_1,v_2)$ are the unique solutions of the equations*

$$\left\{\begin{array}{l} \partial_t z + Az + \frac{\partial f}{\partial y}(x,t,y_u)z = Bv \ \text{ in } Q,\\ \partial_{n_A} z = 0 \text{ on } \Sigma, \quad z(0) = 0 \text{ in } \Omega, \end{array}\right. \tag{2.14}$$

$$\left\{\begin{array}{l} \partial_t z + Az + \frac{\partial f}{\partial y}(x,t,y_u)z = -\frac{\partial^2 f}{\partial y^2}(x,t,y_u)z_{v_1}z_{v_2} \ \text{ in } Q,\\ \partial_{n_A} z = 0 \text{ on } \Sigma, \quad z(0) = 0 \text{ in } \Omega. \end{array}\right. \tag{2.15}$$

3. Existence of a solution and first order optimality conditions for (P)

To address the existence of a solution of (P) first we observe that there exists a unique solution y_u of (1.1) for every $u \in \mathcal{U}_{ad}$. However, it could happen that the solution y_u does not belong to $L^2(Q)$ and, consequently, $J(u) = \infty$. In the sequel, we say that u is a feasible control for (P) if $u \in \mathcal{U}_{ad}$ and the associated state y_u belongs to $L^2(Q)$, or, equivalently, $u \in \mathcal{U}_{ad}\cap\mathcal{A}$. We point out that $\mathcal{U}_{ad}\cap\mathcal{A}$ is not necessarily convex. Hence, regarding the existence of a solution for (P) we have the following result.

Theorem 3.1 *If there exists a feasible control u_0 for* (P)*, then it has at least one solution.*

Proof Let $\{u_k\}_{k=1}^\infty$ be a minimizing sequence of (P) formed by feasible controls with associated states $\{y_k\}_{k=1}^\infty$. Since $J(u_k) \to \inf\,(\mathrm{P}) \le J(u_0) < \infty$, we deduce that $\{u_k\}_{k=1}^\infty$ and $\{y_k\}_{k=1}^\infty$ are bounded in $\mathcal{U}$ and $L^2(Q)$, respectively. From (2.11) and (2.13) we get that $\{y_k\}_{k=1}^\infty$ is bounded in $W(0,\infty) \cap L^\infty(Q)$. Moreover, the continuous embedding $\mathcal{U} \subset L^2(0,\infty)$ implies that $\{u_k\}_{k=1}^\infty$ is bounded in $L^2(0,\infty)^m$. Therefore, taking a subsequence, we obtain that $(u_k, y_k) \stackrel{*}{\rightharpoonup} (\bar u, \bar y)$ in $L^2(0,\infty)^m \cap L^p(0,\infty)^m \times W(0,\infty) \cap L^\infty(Q)$. This implies that $Bu_k \stackrel{*}{\rightharpoonup} B\bar u$ in $L^2(Q) \cap L^p(0,\infty; L^\infty(\Omega))$. Using these properties one can pass to the limit in the state equation (1.1) and deduce that $\bar y$ is the state associated with $\bar u$; see Casas and Kunisch (2023, Theorem 2.1) for details. Moreover, since $u_k \stackrel{*}{\rightharpoonup} \bar u$ in $L^p(0,\infty)^m$ we deduce that $\bar u$ satisfies the control constraints. It remains to prove that $\bar u \in L^1(0,\infty)^m$ and $J(\bar u) = \inf\,(\mathrm{P})$. To this end, we proceed as follows. For every $T < \infty$, using the compact embedding $W(0,T) \subset L^2(Q_T)$ and the weak convergence $u_k \rightharpoonup \bar u$ in $L^1(0,T)^m$ we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_0^T\int_\Omega |\bar y - y_d|^2 \,\mathrm{d}x\,\mathrm{d}t + \alpha \sum_{j=1}^m \int_0^T |\bar u_j| \,\mathrm{d}t \\
&\le \liminf_{k\to\infty} \Big(\frac{1}{2}\int_{Q_T} |y_k - y_d|^2 \,\mathrm{d}x\,\mathrm{d}t + \alpha \sum_{j=1}^m \int_0^T |u_{k,j}| \,\mathrm{d}t\Big) \le \liminf_{k\to\infty} J(u_k) = \inf\,(\mathrm{P}).
\end{aligned}
$$

Taking the supremum as $T \to \infty$ we infer that $\bar u \in L^1(0,\infty)^m$ and $J(\bar u) \le \inf\,(\mathrm{P})$, which concludes the proof. ■

Next we derive the first order optimality conditions satisfied by a local minimizer $\bar u$ of (P). If nothing is specifically said, $\bar u$ is called a local minimizer of (P) if $J(\bar u) < \infty$ and there exists $\varepsilon > 0$ such that $J(\bar u) \le J(u)$ for every $u \in \mathcal{U}_{ad}$ such that $\|u - \bar u\|_{\mathcal{U}} \le \varepsilon$. By interpolation we have that $\mathcal{U}$ is continuously embedded in $L^q(0,\infty)^m$ for every $q \in [1,p]$. Therefore, if $\bar u$ is a local minimizer of (P), then it is also a local minimizer in the $L^q(0,\infty)^m$ sense.

To write the optimality conditions, satisfied by a local minimizer, we need to analyze separately the functions F and j, defining the cost functional J. Regarding the functional F, we make the following assumption on y_d:

Assumption 3.2 $y_d \in L^2(Q) \cap L^{\hat r}(0,\infty; L^{\hat s}(\Omega))$ *with* $\frac{1}{\hat r} + \frac{n}{2\hat s} < 1$.

As a straightforward consequence of this assumption and Theorem 2.2 we get that $F : \mathcal{A} \longrightarrow \mathbb{R}$ is of class C^2 and for all $v, v_1, v_2 \in L^2(0,\infty)^m$ we have the following expressions

$$F'(u)v = \int_Q \varphi_u Bv \,\mathrm{d}x\,\mathrm{d}t = \sum_{i=1}^m \int_0^\infty \phi_{u,i}(t)v_i(t)\,\mathrm{d}t, \tag{3.1}$$

$$F''(u)(v_1,v_2) = \int_Q \Big(1 - \frac{\partial^2 f}{\partial y^2}(x,t,y_u)\varphi_u\Big) z_{v_1} z_{v_2}\,\mathrm{d}x\,\mathrm{d}t, \tag{3.2}$$

where $\phi_{u,i}(t) = \int_\Omega \varphi_u(t)\psi_i\,\mathrm{d}x$ and $\varphi_u \in W(0,\infty)\cap L^\infty(Q)$ is the adjoint state associated with u satisfying

$$\begin{cases} -\partial_t\varphi + A^*\varphi + \frac{\partial f}{\partial y}(x,t,y_u)\varphi = y_u - y_d & \text{in } Q,\\ \partial_{n_{A^*}}\varphi = 0 \text{ on } \Sigma, \quad \lim\limits_{t\to\infty}\|\varphi(t)\|_{L^2(\Omega)} = 0 \text{ in } \Omega. \end{cases} \tag{3.3}$$

For the proof of (3.1) and (3.2) the reader is referred to Casas and Kunisch (2023a, Theorem 2.3 and Corollary 3.1). The existence and uniqueness of a solution of (3.3) was established in Casas and Kunisch (2023a, Theorem A.4). We observe that the identity $\lim_{t\to\infty}\|\varphi_u(t)\|_{L^2(\Omega)} = 0$ is a consequence of the fact that $\varphi_u \in W(0,\infty)$; see Casas and Kunisch (2022, Theorem 2.4).

The functional $j : L^1(0,\infty)^m \longrightarrow \mathbb{R}$ can be written in the form $j(u) = \sum_{i=1}^m j_0(u_i)$, where $j_0 : L^1(0,\infty) \longrightarrow \mathbb{R}$ is defined by $j_0(w) = \|w\|_{L^1(0,\infty)}$. It is clear that j_0 is Lipschitz continuous and convex. Hence, the convex subdifferential $\partial j_0(w) \neq \emptyset$ and the directional derivatives $j_0'(w;v)$ exist for all $w,v \in L^1(0,\infty)$. The following properties hold:

$$\lambda \in \partial j_0(w) \text{ iff } \lambda(t) \begin{cases} = +1 & \text{if } w(t) > 0,\\ = -1 & \text{if } w(t) < 0,\\ \in [-1,+1] & \text{if } w(t) = 0, \end{cases} \tag{3.4}$$

$$j_0'(w;v) = \int_{I_w^+} v(t)\,\mathrm{d}t - \int_{I_w^-} v(t)\,\mathrm{d}t + \int_{I_w^0} |v(t)|\,\mathrm{d}t, \tag{3.5}$$

where $I_w^+ = \{t \in (0,\infty) : w(t) > 0\}$, $I_w^- = \{t \in (0,\infty) : w(t) < 0\}$, and $I_w^0 = \{t \in (0,\infty) : w(t) = 0\}$.

We define the sets

$$\mathcal{U}_{ad}^i = \{v \in L^p(0,\infty)\cap L^1(0,\infty) : \|v\|_{L^p(0,\infty)} \le \gamma_i\},\ 1 \le i \le m$$

and observe that $\mathcal{U}_{ad} = \prod_{i=1}^m \mathcal{U}_{ad}^i$. We have the following necessary optimality conditions.

THEOREM 3.3 *If $\bar u$ is a local minimizer of* (P)*, then there exist $\bar\lambda_i \in \partial j_0(\bar u_i)$ for $i = 1,\dots,m$, satisfying*

$$\int_0^\infty (\bar\phi_i(t) + \alpha\bar\lambda_i(t))(u(t) - \bar u_i(t))\,\mathrm{d}t \ge 0 \quad \forall u \in \mathcal{U}_{ad}^i, \tag{3.6}$$

where $\bar\phi_i(t) = \int_\Omega \bar\varphi(t)\psi_i\,\mathrm{d}x$ and $\bar\varphi$ is the adjoint state associated with $\bar u$.

Proof Using the convexity of $\mathcal{U}_{ad}$ and j we get for every $u \in \mathcal{U}_{ad}$

$$0 \le \lim_{\rho \to 0} \frac{J(\bar u + \rho(u - \bar u)) - J(\bar u)}{\rho} \le F'(\bar u)(u - \bar u) + \alpha[j(u) - j(\bar u)]. \tag{3.7}$$

Let us fix $i \in \{1, \dots, m\}$ and take $u \in \mathcal{U}_{ad}$ with $u_k = \bar u_k$ if $k \neq i$ and $u_i = v \in \mathcal{U}^i_{ad}$. Then, using (3.1) and the definition of $\bar\phi_i$, we infer from the above inequality that

$$\int_0^\infty \bar\phi_i(t)(v(t) - \bar u_i(t))\,\mathrm{d}t + \alpha[j_0(v) - j_0(\bar u_i)] \ge 0 \quad \forall v \in \mathcal{U}^i_{ad}.$$

Denoting by $I_{\mathcal{U}^i_{ad}} : L^1(0,\infty) \cap L^p(0,\infty) \longrightarrow [0,\infty]$ the indicator function of the convex set $\mathcal{U}^i_{ad}$ we obtain that $\bar u_i$ is the minimizer of the convex function

$$\mathcal{J}(v) = \int_0^\infty \bar\phi_i(t) v(t)\,\mathrm{d}t + \alpha j_0(v) + I_{\mathcal{U}^i_{ad}}(v).$$

This implies the existence of $\bar\lambda_i \in \partial j_0(\bar u_i)$ such that

$$0 \in \bar\phi_i + \alpha\bar\lambda_i + \partial I_{\mathcal{U}^i_{ad}}(\bar u_i),$$

which is equivalent to (3.6). ■

Corollary 3.4 *Let $\bar u = \{\bar u_i\}_{i=1}^m$ and $\{(\bar\lambda_i, \bar\phi_i)\}_{i=1}^m$ be as in Theorem 3.3. Then, $\{(\bar\lambda_i, \bar\phi_i)\}_{i=1}^m$ are continuous functions in $[0,\infty)$ and the following relations hold for all $t \in [0,\infty)$ and all $i = 1, \dots, m$*

$$\text{If } \|\bar u_i\|_{L^p(0,\infty)} < \gamma_i, \text{ then } \bar\phi_i(t) + \alpha\bar\lambda_i(t) = 0, \tag{3.8}$$

$$\bar\lambda_i(t) = \mathrm{Proj}_{[-1,+1]}\Big(-\frac{1}{\alpha}\bar\phi_i(t)\Big). \tag{3.9}$$

If $p \in [2,\infty)$ and $\bar\phi_i + \alpha\bar\lambda_i \equiv 0$, then for almost every $t \in [0,\infty)$ we have

$$\begin{aligned} &\text{if } \bar u_i(t) > 0, \text{ then } \bar\phi_i(t) = -\alpha,\\ &\text{if } \bar u_i(t) < 0, \text{ then } \bar\phi_i(t) = +\alpha,\\ &\text{if } |\bar\phi_i(t)| < \alpha, \text{ then } \bar u_i(t) = 0. \end{aligned} \tag{3.10}$$

If $p \in [2,\infty)$ and $\bar\phi_i + \alpha\bar\lambda_i \not\equiv 0$, then $\bar u$ is a continuous function and we have for all $t \in [0,\infty)$

$$\begin{aligned} &\bar u_i(t) > 0 \quad \text{iff} \quad \bar\phi_i(t) < -\alpha,\\ &\bar u_i(t) < 0 \quad \text{iff} \quad \bar\phi_i(t) > +\alpha,\\ &\bar u_i(t) = 0 \quad \text{iff} \quad |\bar\phi_i(t)| \le \alpha. \end{aligned} \tag{3.11}$$

If $p = \infty$, then we have for almost all $t \in [0,\infty)$

$$\begin{aligned} &\text{if } \bar\phi_i(t) < -\alpha \quad \text{then} \quad \bar u_i(t) = +\gamma_i,\\ &\text{if } \bar\phi_i(t) > +\alpha \quad \text{then} \quad \bar u_i(t) = -\gamma_i,\\ &\text{if } |\bar\phi_i(t)| < \alpha \quad \text{then} \quad \bar u_i(t) = 0. \end{aligned} \tag{3.12}$$

Finally, the multipliers $\{\bar\lambda_i\}_{i=1}^m$ satisfying (3.6) *are unique.*

Proof Since $\bar\varphi \in W(0,\infty) \subset C([0,\infty); L^2(\Omega))$, we deduce from the definition of $\bar\phi_i$ its continuity in $[0,\infty)$. The continuity of $\bar\lambda_i$ is a consequence of the identity (3.9), which will be proven below.

We start with assuming that $\|\bar u_i\|_{L^p(0,\infty)} < \gamma_i$ and set $\varepsilon_i = \gamma_i - \|\bar u_i\|_{L^p(0,\infty)}$. Given $T < \infty$, for every $\hat v \in L^p(0,T)$ with $\|\hat v\|_{L^p(0,T)} < \varepsilon_i$ we put

$$v(t) = \begin{cases} \hat v(t) + \bar u_i(t) & \text{if } t \in (0,T),\\ \bar u_i(t) & \text{if } t \ge T. \end{cases}$$

Then, it is evident that $v \in \mathcal{U}_{ad}^i$ and, consequently, we deduce from (3.6) that

$$\int_0^T (\bar\phi_i + \alpha\bar\lambda_i)\hat v \,\mathrm{d}t = \int_0^\infty (\bar\phi_i + \alpha\bar\lambda_i)(v - \bar u_i)\,\mathrm{d}t \ge 0 \ \text{ for every } \ \|\hat v\|_{L^p(0,T)} < \varepsilon_i.$$

This implies that $\bar\phi_i + \alpha\bar\lambda_i = 0$ in $(0,T)$. Since T was arbitrarily large, (3.8) follows.

The relations (3.10) are an immediate consequence of (3.4). Let us prove (3.11). Since $\bar\phi_i + \alpha\bar\lambda_i \not\equiv 0$, the equality $\|\bar u_i\|_{L^p(0,\infty)} = \gamma_i$ follows from (3.8). Then, there exists $T_0 < \infty$ such that $\bar\phi_i + \alpha\bar\lambda_i \not\equiv 0$ in $[0,T_0]$ and $\|\bar u_i\|_{L^p(0,T_0)} > 0$ by (3.6). For every $T > T_0$ we put $\gamma_{i,T} = \|\bar u_i\|_{L^p(0,T)} > 0$. For all $v \in L^p(0,T)$ such that $\|v\|_{L^p(0,T)} \le \gamma_{i,T}$ we define

$$\hat v(t) = \begin{cases} v(t) & \text{if } t \in (0,T),\\ \bar u_i(t) & \text{if } t \ge T. \end{cases}$$

We observe that $\hat v \in L^p(0,\infty) \cap L^1(0,\infty)$ and $\|\hat v\|_{L^p(0,\infty)} \le \|\bar u_i\|_{L^p(0,\infty)} = \gamma_i$, and thus $\hat v \in \mathcal{U}_{ad}^i$. Then, for $\bar\eta_i = \bar\phi_i + \alpha\bar\lambda_i$, (3.6) leads to

$$\int_0^T \bar\eta_i(t)(v(t) - \bar u_i(t))\,\mathrm{d}t \ge 0.$$

This yields

$$-\int_0^T \bar\eta_i v\,\mathrm{d}t \le -\int_0^T \bar\eta_i \bar u_i\,\mathrm{d}t \le \|\bar\eta_i\|_{L^{p'}(0,T)}\|\bar u_i\|_{L^p(0,T)} = \gamma_{i,T}\|\bar\eta_i\|_{L^{p'}(0,T)}.$$

Taking the supremum over all elements $v \in L^p(0,T)$ such that $\|v\|_{L^p(0,T)} \le \gamma_{i,T}$, we obtain

$$\gamma_{i,T}\|\bar\eta_i\|_{L^{p'}(0,T)} \le -\int_0^T \bar\eta_i(t)\bar u_i(t)\,\mathrm{d}t \le \gamma_{i,T}\|\bar\eta_i\|_{L^{p'}(0,T)}.$$

This implies the representation formula

$$\bar u_i(t) = -\gamma_{i,T}\frac{|\bar\phi_i(t)+\alpha\bar\lambda_i(t)|^{p'-2}(\bar\phi_i(t)+\alpha\bar\lambda_i(t))}{\|\bar\phi_i+\alpha\bar\lambda_i\|^{p'-1}_{L^{p'}(0,T)}} \quad \text{a.e. in } (0,T). \tag{3.13}$$

Since $T > T_0$ is arbitrary, from the above formula and (3.4), the relations (3.11) follow. Moreover, if $p \in [2,\infty)$ then (3.9) is deduced from (3.8) and (3.11) by simple computations. The continuity of $\bar\lambda_i$ is the consequence of the continuity of $\bar\phi_i$ and (3.9). Finally, the continuity of $\bar u_i$ is the consequence of (3.13) and the continuity of $\bar\lambda_i$ and $\bar\phi_i$.

For the proof of (3.12) and the associated representation formula (3.9) the reader is referred to Casas (2012). The uniqueness of the multipliers $\{\bar\lambda_i\}_{i=1}^m$ is an immediate consequence of (3.9). ■

Corollary 3.5 *Let $\bar u \in \mathcal{U}_{ad}$ satisfy the optimality conditions* (3.6)*, then there exist time instances $\{T_i^*\}_{i=1}^m \subset (0,\infty)$ such that $\bar u_i(t) = \bar\phi_i(t) + \alpha\bar\lambda_i(t) = 0$ for every $t > T_i^*$, $1 \le i \le m$.*

Proof Since $\lim_{t\to\infty}\|\bar\varphi(t)\|_{L^2(\Omega)} = 0$, we deduce that

$$\lim_{t\to\infty}|\bar\phi_i(t)| \le \lim_{t\to\infty}\|\bar\varphi(t)\|_{L^2(\Omega)}\|\psi_i\|_{L^2(\Omega)} = 0, \quad 1 \le i \le m.$$

This leads to the existence of $T_i^* < \infty$ such that $|\bar\phi_i(t)| < \alpha$ for every $t > T_i^*$. Then, the equality $\bar u_i(t) = 0$ follows from (3.10)–(3.12). Finally, it is enough to use the representation formula (3.13) to deduce that $\bar\phi_i(t) + \alpha\bar\lambda_i(t) = 0$ for every $t > T_i^*$ as well. ■

Corollary 3.6 *Let $\bar u \in \mathcal{U}_{ad}$ satisfy the optimality conditions* (3.6) *and assume that $\bar\phi_i + \alpha\bar\lambda_i \not\equiv 0$ for some $1 \le i \le m$. Then, the following representation formula for $\bar u_i$ holds:*

$$\bar u_i(t) = -\gamma_i\frac{|\bar\phi_i(t)+\alpha\bar\lambda_i(t)|^{p'-2}(\bar\phi_i(t)+\alpha\bar\lambda_i(t))}{\|\bar\phi_i+\alpha\bar\lambda_i\|^{p'-1}_{L^{p'}(0,\infty)}} \quad \text{for a.e. } t \in [0,\infty). \tag{3.14}$$

Proof We use (3.13) for $T > T_i^*$. As a consequence of Corollary 3.4 we have that $\gamma_{i,T} = \gamma_i$ and $\|\bar\phi_i+\alpha\bar\lambda_i\|_{L^{p'}(0,T)} = \|\bar\phi_i+\alpha\bar\lambda_i\|_{L^{p'}(0,\infty)}$. Hence, (3.13) implies (3.14). ■

4. Second order optimality conditions for (P)

In this section we address the second order optimality conditions for problem (P). We will distinguish the cases $2 \le p < \infty$ and $p = \infty$.

4.1. Case I: $2 \le p < \infty$.

We associate the following Lagrangian functions to the control problem (P)

$$\mathcal{L}, \mathcal{F} : \mathcal{A} \times \mathbb{R}^m \longrightarrow \mathbb{R}$$
$$\mathcal{F}(u,\mu) = F(u) + \frac{1}{p}\sum_{i=1}^m \frac{\mu_i}{\gamma_i^p}\|u_i\|^p_{L^p(0,\infty)} \ \text{ and } \ \mathcal{L}(u,\mu) = \mathcal{F}(u,\mu) + \alpha j(u).$$

According to (3.1) and (3.4) we have the following directional derivative

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial u}(u,\mu;v) &= \frac{\partial \mathcal{F}}{\partial u}(u,\mu)v + \alpha j'(u;v)\\ &= \sum_{i=1}^m \Big(\int_0^\infty [\phi_i(t) + \frac{\mu_i}{\gamma_i^p}|u_i(t)|^{p-2}u_i(t)]v_i(t)\,\mathrm{d}t + \alpha j_0'(u_i;v_i)\Big),\end{aligned} \tag{4.1}$$

where $\phi_i(t) = \int_\Omega \varphi_u(t)\psi_i\,\mathrm{d}x$. The second derivative of $\mathcal{F}$ with respect to u is given by the expression

$$\begin{aligned}\frac{\partial^2 \mathcal{F}}{\partial u^2}(u,\mu)(v_1,v_2) &= \int_Q \Big(1 - \frac{\partial^2 f}{\partial y^2}(x,t,y_u)\varphi_u\Big) z_{v_1} z_{v_2}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + (p-1)\sum_{i=1}^m \frac{\mu_i}{\gamma_i^p}\int_0^\infty |u_i(t)|^{p-2}v_1(t)v_2(t)\,\mathrm{d}t.\end{aligned} \tag{4.2}$$

Let $\bar u \in \mathcal{U}_{ad}$ be a control satisfying the first order optimality conditions (3.6). Associated with $\bar u$ we define the Lagrange multipliers

$$\bar\mu = \{\bar\mu_i\}_{i=1}^m \ \text{ with } \ \bar\mu_i = \gamma_i\|\bar\phi_i + \alpha\bar\lambda_i\|_{L^{p'}(0,\infty)}. \tag{4.3}$$

Since $\bar\phi_i, \bar\lambda_i \in L^\infty(0,\infty)$ and $\bar\phi_i(t) + \alpha\bar\lambda_i(t) = 0$ for $t > T_i^*$, we have that $\bar\phi_i + \alpha\bar\lambda_i \in L^q(0,\infty)$ for all $q \in [1,\infty]$ and every $1 \le i \le m$. The choice of $\bar\mu$ is justified by the following proposition.

PROPOSITION 4.1 *Let $\bar u$ satisfy the first order optimality conditions* (3.6) *and* *let $\bar\mu$ be defined by* (4.3)*. Then, the following properties hold:*

i) $\bar\mu_i \geq 0$ *and* $\bar\mu_i(\|\bar u_i\|_{L^p(0,\infty)} - \gamma_i) = 0$ *for* $i = 1,\ldots,m$,

ii) $\dfrac{\partial\mathcal{L}}{\partial u}(\bar u,\bar\mu;v) \geq 0 \quad \forall v \in \mathcal{U}$,

iii) $\dfrac{\partial\mathcal{L}}{\partial u}(\bar u,\bar\mu;v) = 0$ *iff* $|v_i(t)| = \bar\lambda_i(t)v_i(t)$ *a.e. in* $I^0_{\bar u_i}$ *for* $i = 1,\ldots,m$.

Proof The statement *i)* is an immediate consequence of the definition (4.3) of $\bar\mu_i$ and (3.8). Let us prove *ii)*. If $\bar\mu_i = 0$, then the definition (4.3) along with (3.4) and (3.5) imply

$$\begin{aligned}&\int_0^\infty [\bar\phi_i(t) + \frac{\bar\mu_i}{\gamma_i^p}|\bar u_i(t)|^{p-2}\bar u_i(t)]v_i(t)\,\mathrm{d}t + \alpha j_0'(\bar u_i;v_i)\\ &= -\alpha\int_0^\infty \bar\lambda_i(t)v_i(t)\,\mathrm{d}t + \alpha j_0'(\bar u_i;v_i) = \alpha\int_{I^0_{\bar u_i}}[|v_i(t)| - \bar\lambda_i(t)v_i(t)]\,\mathrm{d}t \geq 0.\end{aligned} \tag{4.4}$$

Now we consider the case where $\bar\mu_i > 0$. Using the representation formula (3.14) we get

$$|\bar u_i(t)|^{p-2}\bar u_i(t) = -\gamma_i^{p-1}\frac{\bar\phi_i(t) + \alpha\bar\lambda_i(t)}{\|\bar\phi_i + \alpha\bar\lambda\|_{L^{p'}(0,\infty)}}.$$

By combining this with the definition of $\bar\mu_i$ we obtain

$$\frac{\bar\mu_i}{\gamma_i^p}|\bar u_i(t)|^{p-2}\bar u_i(t) = -(\bar\phi_i(t) + \alpha\bar\lambda_i(t)).$$

Therefore, the same expressions as in (4.4) apply in this situation. Finally, (4.4) yields *iii)*. ∎

Remark 4.2 *We observe that it was established in the above proof that*

$$\int_0^\infty [\bar\phi_i(t) + \frac{\bar\mu_i}{\gamma_i^p}|\bar u_i(t)|^{p-2}\bar u_i(t)]v_i(t)\,\mathrm{d}t + \alpha j_0'(\bar u_i;v_i) \geq 0 \quad \forall v \in \mathcal{U} \tag{4.5}$$

and for all $i = 1,\ldots,m$.

Now we address the necessary second order conditions. To this end, we define the cone of critical directions as follows

$$C_{\bar u} = \Big\{v \in \mathcal{U} : J'(\bar u;v) = 0 \text{ and } \int_0^\infty |\bar u_i(t)|^{p-2}\bar u_i(t)v_i(t)\,\mathrm{d}t \leq 0 \ \ \forall i \in S_0\Big\},$$

where $S_0 = \{i \in \{1,\ldots,m\} : \|\bar u_i\|_{L^p(0,\infty)} = \gamma_i\}$. We also define $S_0^+ = \{i \in S_0 : \bar\mu_i > 0\}$. We have the following property on $C_{\bar u}$.

Proposition 4.3 *If $v \in C_{\bar u}$ then we have*

i) $\displaystyle\int_0^\infty |\bar u_i(t)|^{p-2}\bar u_i(t)v_i(t)\,\mathrm{d}t = 0 \ \forall i \in S_0^+$,

ii) $\displaystyle\frac{\partial \mathcal{L}}{\partial u}(\bar u, \bar\mu; v) = 0.$

Proof Using Proposition (4.1), the definition of $C_{\bar u}$, and (4.1) we obtain that

$$0 \le \frac{\partial \mathcal{L}}{\partial u}(\bar u, \bar\mu; v) = J'(\bar u; v) + \sum_{i=1}^m \bar\mu_i \int_0^\infty |\bar u_i(t)|^{p-2}\bar u_i(t)v_i(t)\,\mathrm{d}t$$
$$= \sum_{i\in S_0^+} \bar\mu_i \int_0^\infty |\bar u_i(t)|^{p-2}\bar u_i(t)v_i(t)\,\mathrm{d}t \le 0,$$

This implies the statement of the proposition. ■

Theorem 4.4 *If $\bar u$ is a local minimizer of* (P)*, then* $\frac{\partial^2 \mathcal{F}}{\partial u^2}(\bar u; \bar\mu)v^2 \ge 0 \ \forall v \in C_{\bar u}$.

Proof First we take an element $v \in C_{\bar u} \cap L^\infty(0,\infty)$, satisfying the following property

$$\exists \delta > 0 \text{ such that } v_i(t) = 0 \text{ if } 0 < |\bar u_i(t)| < \delta \text{ for } 1 \le i \le m. \tag{4.6}$$

We will get rid of these assumptions later. Let us denote

$$E_0 = \Big\{ i \in S_0 : \int_0^\infty |\bar u_i(t)|^{p-2}\bar u_i(t)v_i(t)\,\mathrm{d}t = 0 \Big\}.$$

If $i \notin E_0$, we define the mappings $h_i : \mathbb{R} \longrightarrow L^p(0,\infty) \cap L^1(0,\infty)$ and $\sigma_i : \mathbb{R} \longrightarrow \mathbb{R}$ by $h_i(\rho) = \bar u_i + \rho v_i$ and $\sigma_i(\rho) = \|h_i(\rho)\|^p_{L^p(0,\infty)}$. If $i \notin S_0$, then $\sigma_i(0) < \gamma_i^p$ and, consequently, there exists $\varepsilon_i > 0$ such that $\sigma_i(\rho) < \gamma_i^p$ for every $|\rho| < \varepsilon_i$.

If $i \in S_0 \setminus E_0$, then $\sigma_i(0) = \gamma_i^p$ and

$$\sigma_i'(0) = p \int_0^T |\bar u_i(t)|^{p-2}\bar u_i(t)v_i(t)\,\mathrm{d}t < 0.$$

Again, this implies the existence of $\varepsilon_i > 0$ such that $\sigma_i(\rho) < \gamma_i^p$ for all $\rho \in (0, \varepsilon_i)$. In both cases we have that $h_i(\rho) \in \mathcal{U}^i_{ad}$ for every $\rho \in (0, \varepsilon_i)$. In all cases, we assume that $\varepsilon_i \le \frac{\delta}{2\|v_i\|_{L^\infty(0,\infty)}}$.

If $i \in E_0$, then there exists $\varepsilon_i > 0$ such that

$$\|\bar{u}_i + \rho v_i\|_{L^p(0,\infty)} \geq \|\bar{u}_i\|_{L^p(0,\infty)} - \rho\|v_i\|_{L^p(0,\infty)} \geq \frac{\gamma_i}{2} \quad \text{if} \quad |\rho| < \varepsilon_i.$$

We define $h_i : (-\varepsilon_i, \varepsilon_i) \longrightarrow L^p(0,\infty) \cap L^1(0,\infty)$ by $h_i(\rho) = \gamma_i \dfrac{\bar{u}_i + \rho v_i}{\|\bar{u}_i + \rho v_i\|_{L^p(0,\infty)}}$. This choice also implies that $h_i(\rho) \in \mathcal{U}_{ad}^i$. For $i \in E_0$ we define

$$0 < \varepsilon_i \leq \frac{1}{2} \min\Big\{\frac{\gamma_i}{\|v_i\|_{L^p(0,\infty)}}, \frac{\delta}{\|v_i\|_{L^\infty(0,\infty)}}\Big\}.$$

For $0 < \varepsilon \leq \min\{\varepsilon_i : 1 \leq i \leq m\}$ the mapping $h : [0,\varepsilon) \longrightarrow \mathcal{U}_{ad}$, given by $h(\rho) = (h_i(\rho))_{i=1}^m$, is well defined and of class C^2. We observe that $h_i(0) = \bar{u}_i$ and $h'(0) = v_i$ for every $i = 1, \ldots, m$ and, as a consequence, we get that $h(0) = \bar{u}$ and $h'(0) = v$. Associated with this function we set $w : [0, \varepsilon) \longrightarrow \mathbb{R}$ by $w(\rho) = J(h(\rho))$. From Propositions 4.3-*ii* and 4.1-*iii*, using (4.6) and the choice of ε it follows that $|h_i(\rho)(t)| = \bar{\lambda}_i(t)h_i(\rho)(t)$ for almost all $t \in (0,\infty)$. Hence, we have

$$w(\rho) = F(h(\rho)) + \alpha \sum_{i=1}^m \int_0^\infty \bar{\lambda}_i h_i(\rho)\,\mathrm{d}t.$$

Therefore, w is of class C^2 and satisfies $w(0) = J(\bar{u})$ and

$$w'(0) = F'(\bar{u})h'(0) + \alpha \sum_{i=1}^m \int_0^\infty \bar{\lambda}_i h_i'(0)\,\mathrm{d}t = J'(\bar{u}; v) = 0.$$

Since $\bar{u}$ is a local minimizer of (P), then 0 is a local minimizer of w, hence $w''(0) \geq 0$. Let us compute this derivative. First we observe that

$$w'(\rho) = F'(h(\rho))h'(\rho) + \alpha \sum_{i=1}^m \int_0^\infty \bar{\lambda}_i h_i'(\rho)\,\mathrm{d}t.$$

By derivating this expression, we get

$$\begin{aligned} w''(0) &= F''(\bar{u})v^2 + F'(\bar{u})h''(0) + \alpha \sum_{i=1}^m \int_0^\infty \bar{\lambda}_i h_i''(0)\,\mathrm{d}t \\ &= F''(\bar{u})v^2 + \sum_{i=1}^m \int_0^\infty (\bar{\phi}_i + \alpha\bar{\lambda}_i)h_i''(0)\,\mathrm{d}t \\ &= F''(\bar{u})v^2 + \sum_{i \in S_0^+} \int_0^\infty (\bar{\phi}_i + \alpha\bar{\lambda}_i)h_i''(0)\,\mathrm{d}t, \end{aligned}$$

where we have used the fact that $\gamma_i\|\bar\phi_i+\alpha\bar\lambda\|_{L^p(0,\infty)}=\bar\mu_i=0$ if $i\notin S_0^+$. Let us compute $h_i''(0)$ for $i\in S_0^+\subset E_0$. The first derivative is given by

$$h_i'(\rho)=\frac{\gamma_i}{\|\bar u_i+\rho v_i\|_{L^p(0,\infty)}}v_i-\frac{\gamma_i\int_0^\infty|\bar u_i+\rho v_i|^{p-2}(\bar u_i+\rho v_i)v_i\,\mathrm{d}t}{\|\bar u_i+\rho v_i\|^{p+1}_{L^p(0,\infty)}}(\bar u_i+\rho v_i).$$

Now we use Proposition 4.3-i to deduce

$$h_i''(0)=(p-1)\frac{\int_0^\infty|\bar u_i|^{p-2}v_i^2\,\mathrm{d}t}{\gamma_i^p}\bar u_i.$$

Upon inserting this expression in the obtained formula for $w''(0)$ we infer with Hölder inequality, (4.3), (3.2), and (4.2) that

$$\begin{aligned}
0\le w''(0)&=F''(\bar u)v^2+(p-1)\sum_{i\in S_0^+}\frac{\int_0^\infty|\bar u_i|^{p-2}v_i^2\,\mathrm{d}t}{\gamma_i^p}\int_0^\infty(\bar\phi_i+\alpha\bar\lambda_i)\bar u_i\,\mathrm{d}t\\
&\le F''(\bar u)v^2+(p-1)\sum_{i\in S_0^+}\frac{\|\bar\phi_i+\alpha\bar\lambda_i\|_{L^{p'}(0,\infty)}}{\gamma_i^p}\|\bar u_i\|_{L^p(0,\infty)}\int_0^\infty|\bar u_i|^{p-2}v_i^2\,\mathrm{d}t\\
&\le F''(\bar u)v^2+(p-1)\sum_{i\in S_0^+}\frac{\bar\mu_i}{\gamma_i^p}\int_0^\infty|\bar u_i|^{p-2}v_i^2\,\mathrm{d}t\\
&=F''(\bar u)v^2+(p-1)\sum_{i=1}^m\frac{\bar\mu_i}{\gamma_i^p}\int_0^\infty|\bar u_i|^{p-2}v_i^2\,\mathrm{d}t=\frac{\partial^2\mathcal F}{\partial u^2}(\bar u,\bar\mu)v^2.
\end{aligned}$$

To conclude the proof we prove that all elements $v\in C_{\bar u}$ can be approximated in the norm of $\mathcal U$ by a sequence $\{v_k\}_{k=1}^\infty\subset C_{\bar u}\cap L^\infty(0,\infty)$, satisfying (4.6) for appropriate $\delta_k>0$. By doing this we obtain that

$$\frac{\partial^2\mathcal F}{\partial u^2}(\bar u,\bar\mu)v^2=\lim_{k\to\infty}\frac{\partial^2\mathcal F}{\partial u^2}(\bar u,\bar\mu)v_k^2\ge 0.$$

Let us construct such a sequence $\{v_k\}_{k=1}^\infty$. Given $v\in C_{\bar u}$, for every $i=1,\dots,m$ and every integer $k\ge1$ we introduce the functions

$$\hat v_{i,k}=\begin{cases}0 & \text{if } 0<|\bar u_i(t)|<\frac1k,\\ v_i(t) & \text{otherwise.}\end{cases}$$

Now, we set $\hat S_0=\{i\in S_0:\int_0^\infty|\bar u_i|^{p-2}\bar u_i v_i\,\mathrm{d}t=0\}$. For every $i\in\hat S_0$ we put

$$\begin{aligned}
\theta_{i,k}&=\int_0^\infty|\bar u_i|^{p-2}\bar u_i\,\mathrm{Proj}_{[-k,+k]}(\bar u_i)\,\mathrm{d}t \text{ and } \varepsilon_{i,k}\\
&=\frac{1}{\theta_{i,k}}\int_0^\infty|\bar u_i|^{p-2}\bar u_i\frac{\hat v_{i,k}}{1+\frac1k|\hat v_{i,k}|}\,\mathrm{d}t.
\end{aligned}$$

Finally, we define

$$v_{i,k}(t) = \begin{cases} \frac{\hat{v}_{i,k}(t)}{1+\frac{1}{k}|\hat{v}_{i,k}(t)|} - \varepsilon_{i,k}\operatorname{Proj}_{[-k,+k]}(\bar{u}_i(t)) & \text{if } i \in \hat{S}_0, \\ \frac{\hat{v}_{i,k}(t)}{1+\frac{1}{k}|\hat{v}_{i,k}(t)|} & \text{otherwise.} \end{cases}$$

It is obvious that $\theta_{i,k} \to \|\bar{u}_i\|^p_{L^p(0,\infty)} = \gamma_i^p$ for all $i \in S_0$. We also have that $\frac{\hat{v}_{i,k}(t)}{1+\frac{1}{k}|\hat{v}_{i,k}(t)|} \to v_i(t)$ pointwise in $(0,\infty)$ and the sequence is dominated by $v_i \in L^p(0,\infty) \cap L^1(0,\infty)$. Hence, $\frac{\hat{v}_{i,k}}{1+\frac{1}{k}|\hat{v}_{i,k}|} \to v_i$ strongly in $L^p(0,\infty)\cap L^1(0,\infty)$. As a consequence, we get

$$\varepsilon_{i,k} \to \frac{1}{\gamma_i^p}\int_0^\infty |\bar{u}_i|^{p-2}\bar{u}_i v_i \,\mathrm{d}t = 0 \quad \forall i \in \hat{S}_0.$$

Setting $v_k = (v_{i,k})_{k=1}^m$, we deduce from these facts that $v_k \to v$ strongly in $\mathcal{U}$ and by the choice of $\theta_{i,k}$ and $\varepsilon_{i,k}$ we find for k large enough

$$\int_0^\infty |\bar{u}_i|^{p-2}\bar{u}_i v_{i,k}\,\mathrm{d}t \begin{cases} = 0 & \forall i \in \hat{S}_0, \\ < 0 & \forall i \in S_0 \setminus \hat{S}_0. \end{cases}$$

Further, by construction it is obvious that $\{v_k\}_{k=1}^\infty \subset L^\infty(0,\infty)^m$. It remains to prove that $J'(\bar{u}; v_k) = 0$, so as to conclude that $\{v_k\}_{k=1}^\infty \subset C_{\bar{u}}$. To this end we first observe that since $v \in C_{\bar{u}}$, Propositions 4.3-*ii* and 4.1-*iii* imply that $|v_i(t)| = \bar{\lambda}_i(t)v_i(t)$ for almost all $t \in (0,\infty)$ and all $i = 1,\ldots,m$. By construction, it is immediate to check that the property is satisfied by the functions $v_{i,k}$. Using again Proposition 4.1-*iii* we infer that $\frac{\partial \mathcal{L}}{\partial u}(\bar{u},\bar{\mu}; v_k) = 0$ for every k. This leads to

$$0 = \frac{\partial \mathcal{L}}{\partial u}(\bar{u},\bar{\mu}; v_k) = J'(\bar{u}; v_k) + \sum_{i=1}^m \frac{\bar{\mu}_i}{\gamma_i^p}\int_0^\infty |\bar{u}_i(t)|^{p-2}\bar{u}_i(t)v_{i,k}(t)\,\mathrm{d}t = J'(\bar{u}; v_k),$$

where we have used that the above integral vanishes if $i \in \hat{S}_0$ and $\bar{\mu}_i = 0$ if $i \notin \hat{S}_0$. ∎

Now we address the second order sufficient optimality conditions. We limit this study to the case of $p = 2$. In infinite dimensional optimization, it is well known that we cannot consider the same cone $C_{\bar{u}}$ for the second order necessary and sufficient conditions. In general, an extended cone is necessary to deal with the sufficient conditions; see, for instance, Casas and Mateos (2020), Casas and Tröltzsch (2015), Dunn (1998), or Maurer and Zowe (1979). Given a control $\bar{u} \in \mathcal{U}_{ad}$, satisfying the first order optimality conditions (3.6), we define for every $\tau > 0$ the extended cone

$$C_{\bar{u}}^\tau = \Big\{ v \in \mathcal{U} : J'(\bar{u}; v) \le \tau \|z_v\|_{L^2(Q)}$$

and

$$\int_0^\infty \bar u_i v_i \,\mathrm{d}t \left\{ \begin{array}{l} \le 0 \ \forall i \in S_0 \\ \ge -\tau \|z_v\|_{L^2(Q)} \ \forall i \in S_0^+ \end{array} \right\},$$

where $z_v = G'(\bar u)v$ is the solution of (2.14) with u replaced by $\bar u$. Using Proposition 4.1-*ii* we get for every $v \in C^\tau_{\bar u}$

$$0 \le \frac{\partial \mathcal{L}}{\partial u}(\bar u, \bar\mu; v) = J'(\bar u; v) + \sum_{i \in S_0^+} \frac{\bar\mu_i}{\gamma_i^2} \int_0^\infty \bar u_i(t) v_i(t)\,\mathrm{d}t \le J'(\bar u; v).$$

Thus, for every small $\tau > 0$ and all $v \in C^\tau_{\bar u}$ the terms $J'(\bar u; v)$ and $\int_0^\infty \bar u_i v_i \,\mathrm{d}t$ for $i \in S_0^+$ are not necessarily zero, but they are small. Taking into account Proposition 4.3-*i*, we observe that $C_{\bar u} \subset C^\tau_{\bar u}$ for all $\tau > 0$ and $C^\tau_{\bar u}$ is a small extension of $C_{\bar u}$ if τ is small.

Theorem 4.5 *Let $\bar u \in \mathcal{U}_{ad} \cap \mathcal{A}$ satisfy the first order optimality conditions* (3.6) *and the following second order condition:*

$$\exists \delta > 0 \text{ and } \exists \tau > 0 : \frac{\partial^2 \mathcal{F}}{\partial u^2}(\bar u, \bar\mu) v^2 \ge \delta \|z_v\|^2_{L^2(Q)} \ \forall v \in C^\tau_{\bar u}, \tag{4.7}$$

where $z_v = G'(\bar u)v$. Then, there exist $\varepsilon > 0$ and $\kappa > 0$ such that

$$J(\bar u) + \frac{\kappa}{2} \|y_u - \bar y\|^2_{L^2(Q)} \le J(u) \ \forall u \in \mathcal{U}_{ad} : \|y_u - \bar y\|_{L^2(Q)} + \|y_u - \bar y\|_{L^\infty(Q)} \le \varepsilon. \tag{4.8}$$

Before proving this theorem we establish two auxiliary lemmas.

Lemma 4.6 *Assume that $\bar u \in \mathcal{U}_{ad} \cap \mathcal{A}$. Then, there exist $\varepsilon_1 > 0$ and $\bar M > 0$ such that for every $u \in \mathcal{U}_{ad}$ with $\|y_u - \bar y\|_{L^\infty(Q)} \le \varepsilon_1$ we have that $u \in \mathcal{A}$ and $\|y_u\|_{W(0,\infty)} \le \bar M$. Moreover, the following inequalities hold*

$$\|y_u - (\bar y + z_{u-\bar u})\|_{L^2(Q)} \le K_1 \|y_u - \bar y\|_{L^\infty(Q)} \|y_u - \bar y\|_{L^2(Q)}, \tag{4.9}$$

$$\|y_u - \bar y\|_{L^2(Q)} \le 2 \|z_{u-\bar u}\|_{L^2(Q)}, \tag{4.10}$$

$$\|z_{u-\bar u}\|_{L^2(Q)} \le \frac{3}{2} \|y_u - \bar y\|_{L^2(Q)}, \tag{4.11}$$

$$\|z_{u,v} - z_v\|_{L^2(Q)} \le K_2 \|y_u - \bar y\|_{L^\infty(Q)} \|z_v\|_{L^2(Q)} \quad \forall v \in L^2(0,\infty)^m, \tag{4.12}$$

$$\|z_{u,v}\|_{L^2(Q)} \le 2 \|z_v\|_{L^2(Q)} \quad \forall v \in L^2(0,\infty)^m, \tag{4.13}$$

where $z_{u,v} = G'(u)v$, $z_v = G'(\bar u)v$, and $z_{u-\bar u} = G'(\bar u)(u - \bar u)$.

Proof Let u satisfy the assumptions of the lemma and set $w = y_u - \bar y$. By subtracting the equations satisfied by y_u and $\bar y$ and performing a Taylor expansion of f around $\bar y$ we obtain

$$\begin{cases}\partial_t w + Aw + \frac{\partial f}{\partial y}(x,t,\bar y)w = B(u-\bar u) - \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x,t,y_\theta)(y_u-\bar y)^2 \ \text{ in } Q,\\ \partial_{n_A} w = 0 \text{ on } \Sigma, \quad w(0) = 0 \text{ in } \Omega,\end{cases} \tag{4.14}$$

where $0 \le \theta(x,t) \le 1$ and $y_\theta = \bar y + \theta(y_u - \bar y)$. Since $\|y_\theta\|_{L^\infty(Q)} \le M = \|\bar y\|_{L^\infty(Q)} + \varepsilon_1$, we infer from assumption (2.4) and Casas and Kunisch (2023a, Theorem A.3)

$$\|w\|_{W(0,T)} \le C\Big(C_\gamma + C_{f,M}\varepsilon_1\|y_u - \bar y\|_{L^2(Q_T)}\Big) \ \forall T < \infty,$$

where C_γ depends on the parameters $\{\gamma_i\}_{i=1}^m$. This implies that ε_1 can be chosen small enough such that $\|w\|_{W(0,T)} \le \hat M$ for every $T < \infty$ and some constant $\hat M$. Hence, the inequality $\|y_u\|_{W(0,\infty)} \le \bar M = \hat M + \|\bar y\|_{W(0,\infty)}$ holds and, consequently, we have that $u \in \mathcal{A}$.

Now we set $\hat w = y_u - (\bar y + z_{u-\bar u}) = w - z_{u-\bar u}$. Upon subtracting the equations satisfied by w and $z_{u-\bar u}$ it follows that

$$\begin{cases}\partial_t \hat w + A\hat w + \frac{\partial f}{\partial y}(x,t,\bar y)\hat w = -\frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x,t,y_\theta)(y_u-\bar y)^2 \ \text{ in } Q,\\ \partial_{n_A} \hat w = 0 \text{ on } \Sigma, \quad \hat w(0) = 0 \text{ in } \Omega.\end{cases}$$

Arguing as we did for w we deduce (4.9). Now, we redefine $\varepsilon_1 = \min\{\varepsilon_1, \frac{1}{2K_1}\}$. Then, using (4.9), we infer

$$\begin{aligned}\|y_u - \bar y\|_{L^2(Q)} &\le \|y_u - (\bar y + z_{u-\bar u})\|_{L^2(Q)} + \|z_{u-\bar u}\|_{L^2(Q)}\\ &\le \frac{1}{2}\|y_u - \bar y\|_{L^2(Q)} + \|z_{u-\bar u}\|_{L^2(Q)},\end{aligned}$$

which implies (4.10). Inequality (4.11) follows in a similar way:

$$\|z_{u-\bar u}\|_{L^2(Q)} \le \|y_u - (\bar y + z_{u-\bar u})\|_{L^2(Q)} + \|y_u - \bar y\|_{L^2(Q)} \le \frac{3}{2}\|y_u - \bar y\|_{L^2(Q)}.$$

Now, we prove (4.12). Setting $z = z_{u,v} - z_v$, by subtracting the equations satisfied by $z_{u,v}$ and z_v, and using the mean value theorem we arrive at

$$\begin{cases}\partial_t z + Az + \frac{\partial f}{\partial y}(x,t,y_u)z = -\frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x,t,\hat y_{\hat\theta})(y_u-\bar y)z_v \ \text{ in } Q,\\ \partial_{n_A} z = 0 \text{ on } \Sigma, \quad z(0) = 0 \text{ in } \Omega.\end{cases}$$

Then, (4.12) is the consequence of Casas and Kunisch (2023a, Theorem A.3) applied to the above equation and the assumptions on y_u.

Finally, we redefine again $\varepsilon_1 = \min\{\varepsilon_1, \frac{1}{K_2}\}$. Then (4.13) is an immediate consequence of (4.12). ■

Lemma 4.7 *Let $\bar u \in \mathcal{U} \cap \mathcal{A}$ and ε_1 be as in Lemma 4.6. Given $\rho > 0$, there exists $\varepsilon_\rho \in (0, \varepsilon_1]$ such that*

$$F(u) - F(\bar u) \geq F'(\bar u)(u - \bar u) + \frac{1}{2}F''(\bar u)(u - \bar u)^2 - \frac{\rho}{2}\|z_{u-\bar u}\|^2_{L^2(Q)} \qquad (4.15)$$

for every $u \in \mathcal{U}$ such that $\|y_u - \bar y\|_{L^\infty(Q)} \leq \varepsilon_\rho$, where $z_{u-\bar u} = G'(\bar u)(u - \bar u)$.

Proof As in the proof of Lemma 4.6, we set $w = y_u - \bar y$. Then, using the adjoint state equation (3.3) associated with $\bar u$ and (4.14), we get

$$\begin{aligned}
&F(u) - F(\bar u) = \int_Q (\bar y - y_d)(y_u - \bar y)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_Q (\bar y - y_u)^2\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_Q \Big(- \partial_t \bar\varphi + A^*\bar\varphi + \frac{\partial f}{\partial y}(x,t,y_u)\bar\varphi\Big)(y_u - \bar y)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_Q (\bar y - y_u)^2\,\mathrm{d}x\,\mathrm{d}t\\
&\int_Q \bar\varphi\Big(\partial_t w + Aw + \frac{\partial f}{\partial y}(x,t,\bar y)w\Big)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_Q (\bar y - y_u)^2\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_Q \bar\varphi B(u - \bar u)\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_Q \frac{\partial^2 f}{\partial y^2}(x,t,y_\theta)\bar\varphi(y_u - \bar y)^2\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_Q (\bar y - y_u)^2\,\mathrm{d}x\mathrm{d}t\\
&= F'(\bar u)(u - \bar u) + \frac{1}{2}\int_Q \Big[1 - \frac{\partial^2 f}{\partial y^2}(x,t,y_\theta)\bar\varphi\Big](y_u - \bar y)^2\,\mathrm{d}x\,\mathrm{d}t. \qquad (4.16)
\end{aligned}$$

From here and (3.2) we deduce

$$\begin{aligned}
&F(u) - F(\bar u) = F'(\bar u)(u - \bar u) + \frac{1}{2}F''(\bar u)(u - \bar u)^2\\
&-\frac{1}{2}\Big(\int_Q \Big[1 - \frac{\partial^2 f}{\partial y^2}(x,t,\bar y)\bar\varphi\Big] z^2_{u-\bar u}\,\mathrm{d}x\,\mathrm{d}t - \int_Q \Big[1 - \frac{\partial^2 f}{\partial y^2}(x,t,y_\theta)\bar\varphi\Big](y_u - \bar y)^2\,\mathrm{d}x\,\mathrm{d}t\Big).
\end{aligned}$$

To prove (4.15) we have to estimate the difference of the last two integrals. To this end we proceed as follows:

$$\begin{aligned}
&\Big|\int_Q \Big[1 - \frac{\partial^2 f}{\partial y^2}(x,t,\bar y)\bar\varphi\Big] z^2_{u-\bar u}\,\mathrm{d}x\,\mathrm{d}t - \int_Q \Big[1 - \frac{\partial^2 f}{\partial y^2}(x,t,y_\theta)\bar\varphi\Big](y_u - \bar y)^2\,\mathrm{d}x\,\mathrm{d}t\Big|\\
&\leq \int_Q \Big| z^2_{u-\bar u} - (y_u - \bar y)^2\Big|\,\mathrm{d}x\,\mathrm{d}t + \int_Q \Big|\frac{\partial^2 f}{\partial y^2}(x,t,y_\theta) - \frac{\partial^2 f}{\partial y^2}(x,t,\bar y)\Big| |\bar\varphi| z^2_{u-\bar u}\,\mathrm{d}x\,\mathrm{d}t\\
&+ \int_Q \Big|\frac{\partial^2 f}{\partial y^2}(x,t,y_\theta)\bar\varphi\Big| |z^2_{u-\bar u} - (y_u - \bar y)^2|\,\mathrm{d}x\,\mathrm{d}t = I_1 + I_2 + I_3.
\end{aligned}$$

For the first term we have with (4.9) and (4.10)

$$\begin{aligned}|I_1| &\le \|y_u-(\bar y+z_{u-\bar u})\|_{L^2(Q)}(\|z_{u-\bar u}\|_{L^2(Q)}+\|y_u-\bar y\|_{L^2(Q)})\\ &\le 6K_1\|y_u-\bar y\|_{L^\infty(Q)}\|z_{u-\bar u}\|^2_{L^2(Q)}\le\frac{\rho}{3}\|z_{u-\bar u}\|^2_{L^2(Q)}\end{aligned}$$

if $\|y_u-\bar y\|_{L^\infty(Q)}\le\varepsilon_{\rho,1}=\min\{\varepsilon_1,\frac{\rho}{18K_1}\}$.

To estimate I_2 we use (2.5) and the fact that $\|y_\theta-\bar y\|_{L^\infty(Q)}\le\|y_u-\bar y\|_{L^\infty(Q)}\le\varepsilon_1$. Hence, we deduce the existence of $\varepsilon_{\rho,2}\in(0,\varepsilon_1]$ such that for $\|y_u-\bar y\|_{L^\infty(Q)}\le\varepsilon_{\rho,2}$ we have

$$|I_2|\le\Big\|\frac{\partial^2 f}{\partial y^2}(x,t,y_\theta)-\frac{\partial^2 f}{\partial y^2}(x,t,\bar y)\Big\|_{L^\infty(Q)}\|\bar\varphi\|_{L^\infty(Q)}\|z_{u-\bar u}\|^2_{L^2(Q)}\le\frac{\rho}{3}\|z_{u-\bar u}\|^2_{L^2(Q)}.$$

The term I_3 is estimated in almost the same way as I_1

$$\begin{aligned}|I_3| &\le C_{f,M}\|\bar\varphi\|_{L^\infty(Q)}\int_Q\Big|z^2_{u-\bar u}-(y_u-\bar y)^2\Big|\,\mathrm{d}x\,\mathrm{d}t\\ &\le 6C_{f,M}\|\bar\varphi\|_{L^\infty(Q)}K_1\|y_u-\bar y\|_{L^\infty(Q)}\|z_{u-\bar u}\|^2_{L^2(Q)}\le\frac{\rho}{3}\end{aligned}$$

if $\|y_u-\bar y\|_{L^\infty(Q)}\le\varepsilon_{\rho,3}=\min\{\varepsilon_1,\frac{\rho}{6C_{f,M}\|\bar\varphi\|_{L^\infty(Q)}K_1}\}$. Then, it is enough to take $\varepsilon_\rho=\min\{\varepsilon_{\rho,1},\varepsilon_{\rho,2},\varepsilon_{\rho,3}\}$ to deduce (4.15). ∎

Proof of Theorem 4.5. Let ε_1 be the number given in Lemma 4.6. From (4.16) and (4.10) we infer

$$\begin{aligned}F(u)-F(\bar u) &\ge F'(\bar u)(u-\bar u)-\frac12(1+C_{f,M}\|\bar\varphi\|_{L^\infty(Q)})\|y_u-\bar y\|^2_{L^2(Q)}\\ &\ge F'(\bar u)(u-\bar u)-(1+C_{f,M}\|\bar\varphi\|_{L^\infty(Q)})\|z_{u-\bar u}\|^2_{L^2(Q)}\\ &= F'(\bar u)(u-\bar u)-K_3\|z_{u-\bar u}\|^2_{L^2(Q)}.\end{aligned}\tag{4.17}$$

Let $i_0\in S_0^+$ satisfy $\frac{\bar\mu_{i_0}}{\gamma^2_{i_0}}=\min_{i\in S_0^+}\frac{\bar\mu_i}{\gamma_i^2}$. We put $\nu=\min\{1,\frac{\bar\mu_{i_0}}{\gamma^2_{i_0}}\}$ and $\tau_0=\nu\tau$.

We take $\varepsilon\in(0,\varepsilon_1]$ such that $\frac{2\tau_0}{3\varepsilon}-K_3\ge\frac{\delta}{4}$, where δ satisfies (4.7). We also assume that $\varepsilon\le\varepsilon_\rho$ and $\rho=\frac{\delta}{2}$, where ε_ρ was introduced in Lemma 4.7.

Given an element $u\in\mathcal{U}_{ad}$ such that $\|y_u-\bar y\|_{L^2(Q)}+\|y_u-\bar y\|_{L^\infty(Q)}\le\varepsilon$, we distinguish two cases to prove (4.8).

Case I: $J'(\bar u)(u-\bar u)>\tau_0\|z_{u-\bar u}\|_{L^2(Q)}$.

Using the convexity of j, (4.17),(4.10), and the assumption $\|y_u-\bar y\|_{L^2(Q)}\le\varepsilon$ we obtain

$$\begin{aligned}J(u)-J(\bar u) &\ge J'(\bar u;u-\bar u)-K_3\|z_{u-\bar u}\|^2_{L^2(Q)}\ge\tau_0\|z_{u-\bar u}\|_{L^2(Q)}-K_3\|z_{u-\bar u}\|^2_{L^2(Q)}\\ &\ge\Big(\frac{2\tau_0}{3\|y_u-\bar y\|_{L^2(Q)}}-K_3\Big)\|z_{u-\bar u}\|^2_{L^2(Q)}\ge\frac{\delta}{4}\|z_{u-\bar u}\|^2_{L^2(Q)}.\end{aligned}$$

Case II: $J'(\bar u)(u-\bar u) \le \tau_0 \|z_{u-\bar u}\|_{L^2(Q)}$.

Under this assumption we have that $u - \bar u \in C^\tau_{\bar u}$. Indeed, since $\tau_0 \le \tau$, we have that $J'(\bar u)(u-\bar u) \le \tau \|z_{u-\bar u}\|_{L^2(Q)}$. Moreover, since $u \in \mathcal{U}_{ad}$, we have for every $i \in S_0$

$$\int_0^\infty \bar u_i(t)(u_i(t) - \bar u_i(t))\,\mathrm{d}t \le \|\bar u_i\|_{L^2(0,\infty)}\|u_i\|_{L^2(0,\infty)} - \|\bar u_i\|^2_{L^2(0,\infty)}$$
$$= \gamma_i\Big(\|u_i\|_{L^2(0,\infty)} - \gamma_i\Big) \le 0.$$

Now we check the last condition to prove that $u - \bar u$ belongs to the critical cone $C^\tau_{\bar u}$. From Proposition 4.1-*ii* we get

$$0 \le \frac{\partial \mathcal{L}}{\partial u}(\bar u, \bar \mu; u - \bar u) = J'(\bar u; u-\bar u) + \sum_{i \in S_0^+} \frac{\bar\mu_i}{\gamma_i^2}\int_0^\infty \bar u_i(t)(u_i(t) - \bar u_i(t))\,\mathrm{d}t,$$

which implies with the definition of ν

$$\int_0^\infty \bar u_i(t)(u_i(t) - \bar u_i(t))\,\mathrm{d}t \ge -\frac{\gamma_i^2}{\bar\mu_i}J'(\bar u; u-\bar u) \ge -\frac{\tau_0}{\nu}\|z_{u-\bar u}\|_{L^2(Q)} = -\tau\|z_{u-\bar u}\|_{L^2(Q)}$$

for every $i \in S_0^+$. Thus, $u - \bar u \in C^\tau_{\bar u}$ holds. Using the fact that $\|u_i\|_{L^2(0,\infty)} - \|\bar u_i\|_{L^2(0,\infty)} \le 0$ for all $i \in S_0^+$, the convexity of j, Proposition 4.1-*ii*, (4.7), and (4.15) with $\rho = \frac{\delta}{2}$, we obtain

$$J(u) - J(\bar u) \ge \mathcal{L}(u, \bar\mu) - \mathcal{L}(\bar u, \bar \mu) \ge \frac{\partial \mathcal{L}}{\partial u}(\bar u, \bar\mu; u - \bar u)$$
$$+ \frac{1}{2}\mathcal{F}''(\bar u)(u-\bar u)^2 - \frac{\delta}{4}\|z_{u-\bar u}\|^2_{L^2(Q)} \ge \frac{\delta}{4}\|z_{u-\bar u}\|^2_{L^2(Q)}.$$

Finally, using (4.10) we infer for both cases that

$$J(u) - J(\bar u) \ge \frac{\delta}{4}\|z_{u-\bar u}\|^2_{L^2(Q)} \ge \frac{\delta}{16}\|y_u - \bar y\|^2_{L^2(Q)}. \qquad \square$$

Remark 4.8 *The inequality* (4.15) *is a key issue in the proof of Theorem 4.5. The way in which* (4.15) *is proven here is different from the usual procedure; see Casas and Tröltzsch (2016) or Casas, Mateos and Rösch (2019). Here, the main difficulty in following the approach of Casas and Tröltzsch (2016) or Casas, Mateos and Rösch (2019) is that we cannot perform a Taylor expansion of $F(u)$ around $\bar u$ for arbitrary $u \in \mathcal{A}$, since it is not known whether $u_\theta = \bar u + \theta(u - \bar u)$ is an element of $\mathcal{A}$. Despite the fact that $y_u, \bar y \in L^\infty(Q)$, we do not know if y_{u_θ} belongs to $L^\infty(Q)$.*

Corollary 4.9 *Under the assumptions of Theorem 4.5 there exist* $\hat\varepsilon > 0$ *and* $\delta > 0$ *such that*

$$J(\bar u) + \frac{\kappa}{2}\|y_u - \bar y\|^2_{L^2(Q)} \le J(u)\ \forall u \in \mathcal{U}_{ad} \cap B_{\hat\varepsilon}(\bar u),$$

where $B_{\hat\varepsilon}(\bar u) \subset \mathcal{A} \subset L^2(0,\infty)^m$ *is the ball centered at* $\bar u$ *and radius* $\hat\varepsilon$*.*

This corollary is an immediate consequence of Theorem 4.5 and the continuity of the mapping $G : \mathcal{A} \longrightarrow W(0,\infty) \cap L^\infty(Q)$.

4.2. Case II: $p = \infty$

In this case, the control constraints are linear, consequently, the second order analysis is simpler. We start by establishing the second order necessary conditions for optimality. Assuming that $\bar u \in \mathcal{U}_{ad}$ satisfies the first order optimality conditions (3.6), we define the associated cone of critical directions as follows:

$$C_{\bar u} = \{v \in \mathcal{U} : J'(\bar u; v) = 0 \text{ and } v \text{ satisfies (4.18)}\}$$

with

$$v_i(t)\begin{cases} \ge 0 & \text{if } \bar u_i(t) = -\gamma_i, \\ \le 0 & \text{if } \bar u_i(t) = +\gamma_i, \end{cases}, 1 \le i \le m. \tag{4.18}$$

The proof of the following lemma can be found in Casas, Herzog and Wachsmuth (2017).

Lemma 4.10 *The following properties hold:*

i) $J'(\bar u; v) \ge 0$ *for every* v *satisfying the sign conditions* (4.18)*.*

ii) For every $v \in C_{\bar u}$ *we have*

$$\int_0^T (\bar\phi_i(t) + \alpha\bar\lambda_i(t))v_i(t)\,\mathrm{d}t = 0 \quad \textit{and} \quad j_0'(\bar u_i, v_i) = \int_0^T \bar\lambda_i(t) v_i(t)\,\mathrm{d}t.$$

iii) $C_{\bar u}$ *is a closed, convex cone in* $\mathcal{U}$*.*

Since $u - \bar u$ satisfies the sign conditions (4.18) for every $u \in \mathcal{U}_{ad}$, the statement *i)* implies that $J'(\bar u; u - \bar u) \ge 0$ for all $u \in \mathcal{U}_{ad}$.

Now, we formulate the second order necessary optimality conditions.

Theorem 4.11 *Let* $\bar u \in \mathcal{U}_{ad} \cap \mathcal{A}$ *be a local minimizer of* (P)*, then* $F''(\bar u)v^2 \ge 0$ *for every* $v \in C_{\bar u}$*.*

PROOF Given $v \in C_{\bar u}$ we define for every integer $k \geq 1$

$$v_{i,k}(t) = \begin{cases} 0 & \text{if } \gamma_i - \frac{1}{k} < |\bar u_i(t)| < \gamma_i, \\ \text{Proj}_{[-k,+k]}(v_i(t)) & \text{otherwise,} \end{cases} \quad 1 \leq i \leq m.$$

It is immediate that $v_k \to v$ strongly in $\mathcal{U}$ and there exists $\rho_k > 0$ such that $\bar u + \rho v_k \in \mathcal{A}$ for every $\rho \in (0, \rho_k)$. Following the steps of the proof of Casas, Herzog and Wachsmuth (2012, Theorem 3.7), we obtain that $F''(\bar u)v_k^2 \geq 0$. Then, we pass to the limit as $k \to \infty$ and get the desired result. ■

As we did for $p = 2$, we need to extend the critical cone $C_{\bar u}$ to formulate the second order sufficient optimality conditions. For every $\tau > 0$ we define

$$C^\tau_{\bar u} = \{v \in \mathcal{U} : J'(\bar u; v) \leq \tau \|z_v\|_{L^2(Q)} \text{ and } v \text{ satisfies (4.18)}\}.$$

Then we have the following result.

THEOREM 4.12 *Let $\bar u \in \mathcal{U}_{ad} \cap \mathcal{A}$ satisfy the first order optimality conditions* (3.6) *and the following second order condition:*

$$\exists \delta > 0 \text{ and } \exists \tau > 0 : \frac{\partial^2 F}{\partial u^2}(\bar u, \bar\mu)v^2 \geq \delta \|z_v\|^2_{L^2(Q)} \ \forall v \in C^\tau_{\bar u}, \tag{4.19}$$

where $z_v = G'(\bar u)v$. Then, there exist $\varepsilon > 0$ and $\kappa > 0$ such that

$$J(\bar u) + \frac{\kappa}{2}\|y_u - \bar y\|^2_{L^2(Q)} \leq J(u) \ \forall u \in \mathcal{U}_{ad} : \|y_u - \bar y\|_{L^2(Q)} + \|y_u - \bar y\|_{L^\infty(Q)} \leq \varepsilon. \tag{4.20}$$

The proof of this theorem follows the same steps as the one of Theorem 4.5, with some simplifications. Given $u \in \mathcal{U}_{ad} \cap \mathcal{A}$, we know that $u - \bar u$ satisfies the sign conditions (4.18). Hence, $u - \bar u \in C^\tau_{\bar u}$ holds if and only if $J'(\bar u; u - \bar u) \leq \tau \|z_{u-\bar u}\|_{L^2(Q)}$. Then, we distinguish two cases as in the proof of Theorem 4.5, but with $\tau_0 = \tau$. Then, the proof is the same, with just replacing $\mathcal{L}$ by J and $\mathcal{F}$ by F.

5. Approximation by finite horizon problems

In this section we consider the approximation of (P) by finite horizon optimal control problems and provide error estimates for these approximations. For every $0 < T < \infty$ we consider the control problem

$$(\mathrm{P}_T) \quad \min_{u \in \mathcal{U}_{T,ad}} J_T(u),$$

where $\mathcal{U}_{T,ad} = \{u \in L^p(0,T)^m : \|u_i\|_{L^p(0,T)} \le \gamma_i\}$,

$$J_T(u) = F_T(u) + \alpha j_T(u) = \frac{1}{2}\int_{Q_T} (y_{T,u} - y_d)^2 \, dx \, dt + \alpha \sum_{i=1}^m \int_0^T |u_i(t)| \, dt,$$

and $y_{T,u}$ is the solution of

$$\left\{ \begin{array}{l} \dfrac{\partial y}{\partial t} + Ay + ay + f(x,t,y) = g + Bu \text{ in } Q_T = \Omega \times (0,T), \\ \partial_{n_A} y = 0 \text{ on } \Sigma_T = \Gamma \times (0,T), \ y(0) = y_0 \text{ in } \Omega. \end{array} \right. \tag{5.1}$$

For every control $u \in L^2(0,T)^m$ with associated state $y_{T,u}$ and adjoint state $\varphi_{T,u}$ we define extensions to $(0,\infty)$ and Q, denoted by $\hat{u}$, $\hat{y}_{T,u}$, and $\hat{\varphi}_{T,u}$, by setting $(\hat{u}, \hat{\varphi}_{T,u})(x,t) = (0,0)$ if $t > T$ and $\hat{y}_{T,u}$ is the solution of (1.1) for $u = \hat{u}_T$. It is obvious that if $u \in \mathcal{U}_{T,ad}$, then $\hat{u} \in \mathcal{U}_{ad}$ holds. Given a local minimizer u_T of (P_T), we denote by y_T and φ_T its associated state and adjoint state, respectively. Arguing as in the proof of Theorem 3.3, we obtain that u_T satisfies the following optimality conditions

$$\int_0^T (\phi_{T,i}(t) + \alpha\lambda_{T,i}(t))(u(t) - u_{T,i}(t)) \, dt \ge 0 \quad \forall u \in \mathcal{U}^i_{T,ad}, \tag{5.2}$$

where $\phi_{T,i}(t) = \int_\Omega \varphi_T(t)\psi_i \, dx$ and $\lambda_{T,i} \in \partial j_{T,0}(u_{T,i})$, $1 \le i \le m$. Hereafter, $j_{T,0} : L^1(0,T) \longrightarrow \mathbb{R}$ denotes the mapping $j_{T,0}(u) = \|u\|_{L^1(0,T)}$.

As a consequence of (5.2), Corollary 3.4 is also satisfied, with $(0,\infty)$ and $(\bar{u}, \bar{\lambda}, \bar{\phi})$ replaced by $(0,T)$ and (u_T, λ_T, ϕ_T).

The next two theorems establish the convergence of the approximating problems (P_T) to (P) as $T \to \infty$.

Theorem 5.1 *For every $T > 0$ the control problem (P_T) has at least one solution u_T. If (P) has a feasible control u_0, then the extensions $\{\hat{u}_T\}_{T>0}$ of any family of solutions are bounded in $L^p(0,\infty)^m$. Every weak limit $\bar{u}$ in $L^p(0,\infty)^m$ of a sequence $\{\hat{u}_{T_k}\}_{k=1}^\infty$ with $T_k \to \infty$ as $k \to \infty$ is a solution of (P). Moreover, the weak convergence $\hat{u}_{T_k} \rightharpoonup \bar{u}$ in $L^q(0,\infty)^m$ for $q \in (1,p]$ and the strong convergence $\hat{y}_{T_k} \to \bar{y}$ in $L^2(Q) \cap L^\infty(Q)$ hold.*

Before proving this theorem, we establish the following lemma.

Lemma 5.2 *Let $\bar{u} \in \mathcal{U} \cap \mathcal{A}$ satisfy $\bar{u}(t) = 0$ for $t \ge T^*$ with $T^* \in (0,\infty)$ and denote by $\bar{y} \in W(0,\infty) \cap L^\infty(Q)$ its associated state. Then, for every $T \in [T^*,\infty)$ there exists $\varepsilon > 0$ such that for all $\phi \in L^\infty(\Omega)$ with $\|\phi\|_{L^\infty(\Omega)} < \varepsilon$ the problem*

$$\left\{ \begin{array}{l} \partial_t y + Ay + f(x,t,y) = g(x,t) \quad in\ Q^T = \Omega \times (T,\infty), \\ \partial_{n_A} y = 0 \ on\ \Sigma^T = \Gamma \times (T,\infty), \quad y(T) = \bar{y}(T) + \phi \ in\ \Omega \end{array} \right. \tag{5.3}$$

has a unique solution $y \in W(T,\infty)\cap L^\infty(Q^T)$. *Moreover, there exists a constant* C *independent of* ϕ *such that*

$$\|y\|_{W(T,\infty)} + \|y\|_{L^\infty(Q^T)} \le C\Big(\|y\|_{L^2(Q^T)} + \|\bar y(T)+\phi\|_{L^\infty(\Omega)} + \|g\|_{L^2(Q^T)} + \|g\|_{L^r(T,\infty,L^s(\Omega))} + M_f\Big). \tag{5.4}$$

Proof We are going to deduce this result by applying the implicit function theorem. First, we define the space

$$Y = \{y \in W(T,\infty)\cap L^\infty(Q^T) : \partial_t y + Ay \in L^r(T,\infty;L^s(\Omega))\cap L^2(Q^T)\}$$

and the function $\mathcal{G} : Y \times L^\infty(\Omega) \longrightarrow [L^r(T,\infty;L^s(\Omega))\cap L^2(Q^T)]\times L^\infty(\Omega)$ by

$$\mathcal{G}(y,\phi) = \Big(\partial_t y + Ay + f(\cdot,\cdot,y) - g, y(T) - (\phi + \bar y(T))\Big).$$

Endowed with the graph norm, Y is a Banach space and $\mathcal{G}$ is of class C^1. For every $z\in Y$ we have

$$\frac{\partial\mathcal{G}}{\partial y}(y,\phi)z = (\partial_t z + Az + \frac{\partial f}{\partial y}(\cdot,\cdot,y)z, z(T)).$$

Obviously, we have that $\mathcal{G}(\bar y,0)=0$ and $\frac{\partial\mathcal{G}}{\partial y}(\bar y,0) : Y \longrightarrow [L^r(T,\infty;L^s(\Omega))\cap L^2(Q^T)]\times L^\infty(\Omega)$ is a continuous linear mapping. To prove that it is an isomorphism we have to check that the equation

$$\begin{cases} \partial_t z + Az + \frac{\partial f}{\partial y}(x,t,y)z = h \ \text{ in } Q^T,\\ \partial_{n_A} z = 0 \text{ on } \Sigma^T, \quad z(T) = z_T \text{ in } \Omega \end{cases}$$

has a unique solution $z\in Y$ for every $(h,z_T)\in[L^r(T,\infty;L^s(\Omega))\cap L^2(Q^T)]\times L^\infty(\Omega)$. This follows from Casas and Kunisch (2023a, Theorem A.3) and Casas and Kunisch (2024b). Then, the statement of the lemma follows from the implicit function theorem. ■

Proof of Theorem 5.1. Since $\mathcal{U}_{T,ad}$ is not empty, the existence of solution for (P_T) is a classical result. Actually, one can easily adapt the existence proof of solution for (P) to (P_T). Let y^0 be the solution of (1.1) corresponding to u_0. By definition of feasible control we have that $J(u_0)<\infty$. Using the optimality of u_T we obtain

$$J_T(u_T) \le J_T(u_0) \le J(u_0) \ \forall T>0.$$

This proves the boundedness of $\{\hat u_T\}_{T>0}$ in $L^1(0,\infty)^m$ and the existence of a constant K such that $\|y_T\|_{L^2(Q_T)} \le K$ for every T. Moreover, from the fact that

$\{\hat{u}_T\}_{T>0} \subset \mathcal{U}_{ad}$ we deduce the boundedness of $\{\hat{u}_T\}_{T>0}$ in $L^p(0,\infty)^m$. By interpolation between the spaces $L^1(0,\infty)^m$ and $L^p(0,\infty)^m$ we infer that $\{\hat{u}_T\}_{T>0}$ is a bounded sequence in $L^q(0,\infty)^m$ for every $q \in [1,p]$. Let $\{(\hat{u}_{T_k}, y_{T_k}\chi_{(0,T_k)})\}_{k=1}^\infty$ be a sequence with $T_k \to \infty$ as $k \to \infty$ converging weakly to $(\bar{u}, \bar{y})$ in $L^p(0,\infty)^m \times L^2(Q)$. This implies the weak convergence of $\{\hat{u}_{T_k}\}_{k=1}^\infty$ in $L^q(0,\infty)^m$ for every $q \in (1,p]$. Since $\{\hat{u}_{T_k}\}_{k=1}^\infty \subset \mathcal{U}_{ad}$ and $\mathcal{U}_{ad}$ is closed in $L^p(0,\infty)^m$ and convex, we infer that $\bar{u} \in \mathcal{U}_{ad}$. Moreover, we can apply Casas and Kunisch (2023a, Theorem A1) to the equation (5.1) and deduce the existence of a constant M_1 such that for all $k \geq 1$

$$\begin{aligned}\|y_{T_k}\|_{L^2(0,T_k;H^1(\Omega))} + \|y_{T_k}\|_{L^\infty(Q_{T_k})} \leq M_1 = C\Big(&\|g + B\hat{u}_{T_k}\|_{L^2(Q)}\\ &+ \|g\|_{L^r(0,\infty;L^s(\Omega))} + \|B\hat{u}_{T_k}\|_{L^p(0,\infty;L^\infty(\Omega))} + \|y_0\|_{L^\infty(\Omega)} + K + M_f\Big).\end{aligned}$$

From this estimate and (2.12) we get the existence of a constant M_2 such that

$$\|f(\cdot,\cdot,y_{T_k})\|_{L^2(Q_{T_k})} + \|f(\cdot,\cdot,y_{T_k})\|_{L^\infty(Q_{T_k})} \leq M_2 \quad \forall k \geq 1.$$

The two above estimates and (5.1) imply that

$$\|y_{T_k}\|_{W(0,T_k)} + \|y_{T_k}\|_{L^\infty(Q_{T_k})} \leq M_3 \quad \forall k \geq 1 \tag{5.5}$$

for a new constant M_3. Using the convergence of $y_{T_k} \rightharpoonup \bar{y}$ in $L^2(Q_T)$ for every $T < \infty$, the compactness of the embedding $W(0,\hat{T}) \subset L^2(Q_T)$, and the above estimate, it is straightforward to pass to the limit in the equation

$$\begin{cases} \dfrac{\partial y_{T_k}}{\partial t} + Ay_{T_k} + ay_{T_k} + f(x,t,y_{T_k}) = g + Bu_{T_k} \text{ in } Q_T,\\ \partial_{n_A} y = 0 \text{ on } \Sigma_T,\ y_{T_k}(0) = y_0 \text{ in } \Omega \end{cases} \tag{5.6}$$

for each $T_k \geq T$, and to deduce that $\bar{y}$ is the solution of (5.1), associated to $\bar{u}$ for arbitrary $0 < T < \infty$. This proves that $\bar{y}$ is the solution of (1.1), corresponding to $\bar{u}$ and $\|\bar{y}\|_{W(0,\infty)} + \|\bar{y}\|_{L^\infty(Q)} \leq M_3$. This implies that $\bar{u} \in \mathcal{A}$. Let us prove that $\bar{u}$ is a solution of (P). Using the convergence $u_{T_k} \rightharpoonup \bar{u}$ in $L^1(Q_T)^m$ for every $T < \infty$, we get for every feasible control u of (P)

$$\begin{aligned}J_T(\bar{u}) &\leq \liminf_{k\to\infty} J_T(u_{T_k}) \leq \liminf_{k\to\infty} J_{T_k}(u_{T_k})\\ &\leq \limsup_{k\to\infty} J_{T_k}(u_{T_k}) \leq \limsup_{k\to\infty} J_{T_k}(u) = J(u).\end{aligned}$$

Hence, the inequality $J(\bar{u}) = \sup_{T\to\infty} J_T(\bar{u}) \leq J(u)$ holds, which proves that $\bar{u}$ is a solution of (P). Moreover, replacing u by $\bar{u}$ in the above inequalities we infer

$$\lim_{k\to\infty}\Big(\frac{1}{2}\int_{Q_{T_k}} (y_{T_k} - y_d)^2 \,\mathrm{d}x\,\mathrm{d}t + \alpha j_{T_k}(u_{T_k})\Big) = \frac{1}{2}\int_Q (\bar{y} - y_d)^2 \,\mathrm{d}x\,\mathrm{d}t + \alpha j(\bar{u}).$$

This is equivalent to the identity

$$\lim_{k\to\infty}\Big(\frac{1}{2}\int_Q (y_{T_k}-y_d)^2\chi_{0,T_k})\,\mathrm{d}x\,\mathrm{d}t+\alpha j(\hat u_{T_k})\Big)=\frac{1}{2}\int_Q(\bar y-y_d)^2\,\mathrm{d}x\,\mathrm{d}t+\alpha j(\bar u).$$

Once more, using the convergence $u_{T_k}\rightharpoonup \bar u$ in $L^1(Q_T)^m$ for every $T<\infty$ we obtain

$$j_T(\bar u)\le \liminf_{k\to\infty} j_T(u_{T_k})\le \liminf_{k\to\infty} j(\hat u_{T_k}).$$

Taking the supremum in T we deduce $j(\bar u)\le \liminf_{k\to\infty} j(\hat u_{T_k})$. This convergence along with the weak convergence $y_{T_k}\chi_{(0,T_k)}\rightharpoonup \bar y$ in $L^2(Q)$ and the above equality yield the strong convergence $\lim_{k\to\infty}\|y_{T_k}-\bar y\|_{L^2(Q_{T_k})}=0$; see Casas and Kunisch (2023b, Lemma 5.2). It remains to prove that $\hat y_{T_k}\to\bar y$ in $L^2(Q)\cap L^\infty(Q)$. The proof of this convergence is split in several steps.

Step I.- $\lim_{k\to\infty}\|\hat y_{T_k}-\bar y\|_{L^\infty(Q_T)}=0$ *for every* $T<\infty$. Let us set $w_k=\hat y_{T_k}-\bar y$. Then, we have for every $T_k\ge T$

$$\left\{\begin{array}{l}\dfrac{\partial w_k}{\partial t}+Aw_k+\dfrac{\partial f}{\partial y}(x,t,y_{\theta_k})w_k=B(u_{T_k}-\bar u)\text{ in }Q_T,\\ \partial_{n_A}w_k=0\text{ on }\Sigma_T,\ \ w_k(0)=0\text{ in }\Omega,\end{array}\right. \tag{5.7}$$

where $y_{\theta_k}=\bar y+\theta_k(\hat y_{T_k}-\bar y)$ and $\theta_k:Q\longrightarrow[0,1]$ is a measurable function. Since $\{w_k\}_{k=1}^\infty$ is bounded in $L^2(Q_T)\cap L^\infty(Q_T)$, we get with (2.4) that $B(u_{T_k}-\bar u)-\frac{\partial f}{\partial y}(x,t,y_{\theta_k})w_k$ is bounded in $L^p(0,T;L^\infty(\Omega))$. Then, we deduce from Disser, ter Elst and Rehberg (2017) the boundedness of $\{w_k\}_{k=1}^\infty$ in $C^{0,\beta}(\bar Q_T)$ for some $\beta\in(0,1)$. Using the compactness of the embedding $C^{0,\beta}(\bar Q_T)\subset C(\bar Q_T)$ along with the strong convergence $y_{T_k}\to\bar y$ in $L^2(Q_T)$, we infer the strong convergence $w_k\to 0$ in $C(\bar Q_T)$ and $\hat y_{T_k}\to\bar y$ in $L^\infty(Q_T)$ as $k\to\infty$ for every $T<\infty$.

Step II.- There exist $T^*<\infty$ *and* $k^*\ge 1$ *such that* $\hat u_{T_k}(t)=0$ *for all* $k\ge k^*$ *and almost all* $t>T^*$. Indeed, we have that the adjoint states $\hat\varphi_{T_k}$ satisfy the adjoint state equations

$$\left\{\begin{array}{l}-\partial_t\hat\varphi_{T_k}+A^*\hat\varphi_{T_k}+\frac{\partial f}{\partial y}(x,t,\hat y_{T_k}\chi_{(0,T_k)})\hat\varphi_{T_k}=(\hat y_{T_k}-y_d)\chi_{(0,T_k)}\text{ in }Q,\\ \partial_{n_{A^*}}\hat\varphi_{T_k}=0\text{ on }\Sigma,\quad \displaystyle\lim_{t\to\infty}\|\hat\varphi_{T_k}(t)\|_{L^2(\Omega)}=0\text{ in }\Omega.\end{array}\right. \tag{5.8}$$

Given $\varepsilon>0$, the convergence $\lim_{k\to\infty}\|y_{T_k}-\bar y\|_{L^2(Q_{T_k})}=0$ and the fact that $\bar y-y_d\in L^2(Q)$ imply the existence of k_ε and T_ε such that for $k>k_\varepsilon$ and $T_k>T_\varepsilon$

$$\|(\hat y_{T_k}-y_d)\chi_{(0,T_k)}\|_{L^2(T_\varepsilon,\infty;L^2(\Omega))}\le\|\hat y_{T_k}-\bar y\|_{L^2(Q_{T_k})}+\|\bar y-y_d\|_{L^2(T_\varepsilon,\infty;L^2(\Omega))}<\varepsilon.$$

Using this, we deduce from (5.8) and Casas and Kunisch (2023a, Theorem A.4) that $\|\hat\varphi_{T_k}\|_{L^\infty(T_\varepsilon,\infty;L^2(\Omega))}\le C\varepsilon$ for every $k\ge k_\varepsilon$. For every $1\le i\le m$, this yields

$$|\phi_{T_k,i}(t)| \le \|\hat\varphi_{T_k}(t)\|_{L^2(\Omega)}\|\psi_i\|_{L^2(\Omega)} \le C\|\psi_i\|_{L^2(\Omega)}\varepsilon \ \forall k \ge k_\varepsilon \text{ and for a.a. } t \ge T_\varepsilon.$$

Upon selecting $\varepsilon > 0$ such that $C\|\psi_i\|_{L^2(\Omega)}\varepsilon < \alpha$ and applying Corollary 3.4 we infer that $\hat u_{T_k}(t) = 0$ for every $k \ge k_\varepsilon$ and almost all $t > T_\varepsilon$.

Step III.- $\exists k_0 \ge 1$ such that $\{\hat y_{T_k}\}_{k\ge k_0} \subset W(0,\infty)\cap L^\infty(Q)$ and the convergence $\lim_{k\to\infty}\Big(\|\hat y_{T_k} - \bar y\|_{L^2(Q)} + \|\hat y_{T_k} - \bar y\|_{L^\infty(Q)} = 0\Big)$ *holds.* Without loss of generality we can assume that $T^* > T_i^*$ for $1 \le i \le m$, where $\{T_i^*\}_{i=1}^m$ are given in Corollary 3.5. Thus, we have that $B(u_{T_k} - \bar u)(t) = 0$ $\forall k \ge k^*$ and for almost all $t \ge T^*$.

We take $T > T^*$. Since $w_k \to 0$ in $C(\bar Q_T)$, we have that $w_k(T) \to 0$ in $C(\bar\Omega)$. Then, applying Lemma 5.2 we infer the existence of k_0 such that $\{\hat y_{T_k}\}_{k\ge k_0} \subset W(T,\infty)\cap L^\infty(Q^T)$ and it is uniformly bounded in this space. Combining this with (5.5) we infer that $\{\hat y_{T_k}\}_{k\ge k_0} \subset W(0,\infty)\cap L^\infty(Q)$. Moreover, by applying Casas and Kunisch (2023a, Theorem A.3) along with Casas and Kunisch (2024b) to the equation

$$\begin{cases} \dfrac{\partial w_k}{\partial t} + Aw_k + aw_k + \dfrac{\partial f}{\partial y}(x,t,y_{\theta_k})w_k = 0 \text{ in } Q^T,\\ \partial_{n_A} w_k = 0 \text{ on } \Sigma^T, \end{cases}$$

we obtain

$$\|w_k\|_{W(T,\infty)} + \|w_k\|_{L^\infty(Q^T)} \le C\|w_k(T)\|_{L^\infty(\Omega)} \to 0 \ \text{ as } \ k\to\infty.$$

Combining this with *Step I* we get the desired convergence. □

Now we address a kind of converse theorem for strong local minimizers. We say that $\bar u$ is a strong local minimizer of (P) is there exists $\varepsilon > 0$ such that

$$J(\bar u) \le J(u)\ \forall u \in \mathcal U_{ad}\cap\mathcal A \text{ satisfying } \|y_u - \bar y\|_{L^2(Q)} + \|y_u - \bar y\|_{L^\infty(Q)} \le \varepsilon. \quad (5.9)$$

If the above inequality is strict for $u \ne \bar u$, then we say that $\bar u$ is a strict strong local minimizer.

Theorem 5.3 *Let $\bar u$ be a strict strong local minimizer of* (P). *Then, there exist $T_0 \in (0,\infty)$ and a family $\{u_T\}_{T>T_0}$ of strong local minimizers to* (P_T) *such that the weak convergence $\hat u_T \rightharpoonup \bar u$ in $L^q(0,\infty)^m$ for all $q \in (1,p]$ and the strong convergence $\hat y_T \to \bar y$ in $L^2(Q)\cap L^\infty(Q)$ hold as $T\to\infty$.*

Proof Let $\bar u$ satisfy (5.9). We consider the control problems

$$(\mathrm{P}_\varepsilon) \quad \min_{u\in\mathcal U_{ad}^\varepsilon} J(u) \quad \text{and} \quad (\mathrm{P}_{T,\varepsilon}) \quad \min_{u\in\mathcal U_{T,ad}^\varepsilon} J_T(u),$$

where

$$\mathcal{U}^{\varepsilon}_{ad} = \{u \in \mathcal{U}_{ad} \cap \mathcal{A} : \|y_u - \bar{y}\|_{L^2(Q)} + \|y_u - \bar{y}\|_{L^\infty(Q)} \le \varepsilon\},$$
$$\mathcal{U}^{\varepsilon}_{T,ad} = \{u \in \mathcal{U}_{T,ad} : \|y_u - \bar{y}\|_{L^2(Q_T)} + \|y_u - \bar{y}\|_{L^\infty(Q_T)} \le \varepsilon\}.$$

Obviously $\bar{u}$ is the unique solution of (P_ε). Regarding the problem ($\mathrm{P}_{T,\varepsilon}$), we first observe that $\bar{u}_{|(0,T)} \in \mathcal{U}^{\varepsilon}_{T,ad}$. Moreover, it is easy to check that if $\{u_k\}_{k=1}^\infty \subset \mathcal{U}^{\varepsilon}_{T,ad}$ and $u_k \rightharpoonup u$ in $L^p(0,T)^m$, then $y_{u_k} \stackrel{*}{\rightharpoonup} y_u$ in $L^\infty(Q_T)$. Hence, $\mathcal{U}^{\varepsilon}_{T,ad}$ is nonempty, bounded, and sequentially weakly closed in $L^p(0,T)^m$. Then, for every T the existence of a solution u_T of ($\mathrm{P}_{T,\varepsilon}$) can be proven as usual by taking a minimizing sequence. Now, arguing as in the proof of Theorem 5.1 and using the uniqueness of the solution of (P_ε), we deduce the convergence $\hat{u}_T \rightharpoonup \bar{u}$ in $L^p(0,\infty)^m$ as $T \to \infty$ and $\hat{y}_T \to \bar{y}$ in $L^2(Q) \cap L^\infty(Q)$. This implies the existence of T_0 such that $\|\hat{y}_T - \bar{y}\|_{L^2(Q_T)} + \|\hat{y}_T - \bar{y}\|_{L^\infty(Q_T)} < \varepsilon$ for all $T > T_0$. Hence, u_T is also a strong local minimizer of (P_T) for $T > T_0$. Indeed, let us set $\varepsilon_T = \|\hat{y}_T - \bar{y}\|_{L^2(Q_T)} + \|\hat{y}_T - \bar{y}\|_{L^\infty(Q_T)}$. Then, for every $u \in \mathcal{U}_{T,ad}$ with $\|y_T - y_{T,u}\|_{L^2(Q_T)} + \|y_T - y_{T,u}\|_{L^\infty(Q_T)} \le \varepsilon - \varepsilon_T$ we have

$$\|y_{T,u} - \bar{y}\|_{L^2(Q_T)} + \|y_{T,u} - \bar{y}\|_{L^\infty(Q_T)} \le \|y_T - y_{T,u}\|_{L^2(Q_T)} + \|y_T - y_{T,u}\|_{L^\infty(Q_T)} + \|y_T - \bar{y}\|_{L^2(Q_T)} + \|y_T - \bar{y}\|_{L^\infty(Q_T)} < \varepsilon.$$

Since u_T is a minimizer of ($\mathrm{P}_{T,\varepsilon}$) and u is a feasible control for ($\mathrm{P}_{T,\varepsilon}$), the inequality $J_T(u_T) \le J_T(u)$ follows. ∎

In the previous theorem we proved the existence of strong local minimizers $\{u_T\}_{T>T_0}$ of problems (P_T) weakly converging to $\bar{u}$, assuming that $\bar{u}$ is a strict strong local minimizer of (P). Moreover, strong convergence of the associated states $\hat{y}_T \to \bar{y}$ in $L^2(Q) \cap L^\infty(Q)$ was established. In addition, the inequality $J_T(u_T) \le J_T(\bar{u})$ holds for every $T > T_0$. In the next theorem we provide an estimate for the difference of the corresponding states.

THEOREM 5.4 *Suppose that $p = 2$ or $p = \infty$ and that $\bar{u}$ is a strong local minimizer of* (P) *satisfying the second order sufficient optimality condition. We assume that $\frac{\partial f}{\partial y}(x,t,y) \ge 0$ holds for all $y \in \mathbb{R}$ and almost all $(x,t) \in Q$. Let $\{u_T\}_{T>T_0}$ be a sequence of local minimizers of problems* (P_T) *such that $\hat{u}_T \rightharpoonup \bar{u}$ in $L^q(0,\infty)^m$ $\forall q \in (1,p]$, $\hat{y}_T \to \bar{y}$ in $L^2(Q) \cap L^\infty(Q)$, and $J_T(\bar{u}) \le J_T(u_T)$. Then, there exist $T^* \in [T_0,\infty)$ and a constant C such that for every $T \ge T^*$*

$$\|\hat{y}_T - \bar{y}\|_{L^2(Q)} \le C\Big(\|y_T(T)\|_{L^2(\Omega)} + \|y_d\|_{L^2(T,\infty;L^2(\Omega))} + \|g\|_{L^2(T,\infty;L^2(\Omega))}\Big). \quad (5.10)$$

PROOF We use the inequalities (4.8) or (4.20). For this purpose, we take $T^* \in [T_0,\infty)$ such that $\|\hat{y}_T - \bar{y}\|_{L^2(Q)} + \|\hat{y}_T - \bar{y}\|_{L^\infty(Q)} < \varepsilon$ for all $T \ge T^*$. Then, we have

$$\frac{\kappa}{2}\|\hat y_T-\bar y\|^2_{L^2(Q)} \le J(\hat u_T)-J(\bar u) \le J_T(u_T)-J_T(\bar u)$$
$$+\frac{1}{2}\int_T^\infty \|\hat y_T(t)-y_d(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \le \frac{1}{2}\int_T^\infty \|\hat y_T(t)-y_d(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t,$$

which leads to

$$\|\hat y_T-\bar y\|_{L^2(Q)} \le \frac{1}{\sqrt{\kappa}}\|\hat y_T-y_d\|_{L^2(T,\infty;L^2(\Omega))}. \tag{5.11}$$

To prove (5.10) we observe that $\hat y_T$ satisfies the equation

$$\begin{cases} \dfrac{\partial \hat y_T}{\partial t}+A\hat y_T+f(x,t,\hat y_T)=g \text{ in } \Omega\times(T,\infty),\\ \partial_{n_A}\hat y_T=0 \text{ on } \Gamma\times(T,\infty),\ \hat y_T(T)=y_T(T) \text{ in } \Omega. \end{cases}$$

Testing this equation with $\hat y_T$, and using the fact that $f(x,t,\hat y_T)\hat y_T \ge 0$ due to the monotonicity of f with respect to y and (2.1), it follows that

$$\frac{1}{2}\|\hat y_T(t)\|^2_{L^2(\Omega)}+\int_T^\infty \langle A\hat y_T, y_T\rangle\,\mathrm{d}t \le \frac{1}{2}\|y_T(T)\|^2_{L^2(\Omega)}+\int_T^\infty\int_\Omega g\hat y_T\,\mathrm{d}x\,\mathrm{d}t.$$

From this inequality we infer

$$\|\hat y_T\|_{L^2(T,\infty;L^2(\Omega))} \le C'\Big(\|y_T(T)\|_{L^2(\Omega)}+\|g\|_{L^2(T,\infty;L^2(\Omega))}\Big).$$

This inequality and (5.9) imply (5.8). ■

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