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Discrete approximation of nonconvex hyperbolic optimal control problems with state constraints

by

I. Chryssoverghi, A. Bacopoulos, J. Coletsos and B. Kokkinis

Department of Mathematics,
National Technical University, Athens, Greece

Abstract: We consider an optimal control problem for systems defined by nonlinear hyperbolic partial differential equations with state constraints. Since no convexity assumptions are made on the data, we also consider the control problem in relaxed form. We discretize both the classical and the relaxed problems by using a finite element method in space and a finite difference scheme in time, the controls being approximated by piecewise constant ones. We develop the existence theory and the necessary conditions for optimality, for the continuous and the discrete problems. Finally, we study the behaviour in the limit of discrete optimality, admissibility and extremality properties.

Keywords: Optimal control, nonlinear hyperbolic systems, discretization, existence theory, minimum principle, relaxed controls.

1. Introduction

Optimal control problems without any convexity assumptions on the data have no classical solutions in general. In order to prove the existence of optimal controls, the convexity of some extended velocity set is usually assumed, which is clearly unrealistic when nonlinear systems are involved. To overcome this difficulty, one has to relax, or convexify, the problem in some manner, and then work on the relaxed problem. As a result, relaxation theory has been extensively used, not only to prove existence theorems and derive necessary conditions for optimality, see Warga (1972), Ekeland (1972), Chryssoverghi (1986), Fattorini (1994), Fattorini (1997), Roubicek (1997), but also to develop approximation schemes, see Roubicek (1991), Chryssoverghi et al. (1993), and optimization methods, see Warga (1977), Teo et al. (1984), Chryssoverghi et al. (1997).

Here, we consider an optimal distributed control problem for systems defined by nonlinear hyperbolic partial differential equations with several equality and inequality constraints (for hyperbolic systems, see also Bittner, 1975, Sloss et al., 1995, Sadek et al., 1996). Under reasonable assumptions, the existence of optimal controls and the necessary conditions for optimality are established for the relaxed problem. We then discrctize the classical (resp. relaxed) problem by using the Galerkin finite element method in space and the semi-implicit finite difference scheme in time for approximating the state equations, while the controls are approximated by piecewise constant classical (resp. relaxed) ones with respect to an independent partition of the space-time domain (see Cullum, 1971, and Casas, 1996, for discretizations of classical problems). This independent control discretization corresponds to the use of controls of simple and flexible form for numerical and/or engineering reasons. The discretization of both the classical and the relaxed continuous problems is motivated by the fact that in practice classical (resp. relaxed) optimization methods are usually applied to the classical (resp. relaxed) problem after some discretization. We then prove the existence of optimal controls for both the discrete classical and the discrete relaxed problems, and derive a piecewise minimum principle of optimality for the discrete relaxed problem. Finally, we study the behaviour in the limit of the above approximations. More precisely, we prove that, under appropriate assumptions, accumulation points of sequences of optimal discrete classical controls are optimal for the continuous relaxed problem, and that accumulation points of sequences of optimal (resp. admissible extremal) discrete relaxed controls are optimal (resp. admissible extremal) for the continuous relaxed problem.

2. The continuous optimal control problems

Let Ω be a bounded domain in R^d with Lipschitz boundary Γ, and let $I := (0, T)$, $0 < T < \infty$. Consider the following nonlinear hyperbolic state equations

$$\partial^2 y/\partial t^2 + A(t)y = f(x, t, y(x, t), w(x, t)), \quad \text{in } Q := \Omega \times I, \tag{1}$$

$$y(x, t) = 0, \quad \text{in } \Sigma := \Gamma \times I, \tag{2}$$

$$y(x, 0) = y^0(x), \quad \text{in } \Omega, \tag{3}$$

$$(\partial y/\partial t)(x, 0) = y^1(x), \quad \text{in } \Omega, \tag{4}$$

where $A(t)$ is the second order differential operator

$$A(t)y := -\sum_{i,j=1}^{d} (\partial/\partial x_i)[a_{ij}(x, t)(\partial y/\partial x_j)].$$

The constraints on the control w are

$$w(x, t) \in U, \quad \text{in } Q,$$

where U is a compact subset of $\mathbb{R}^{d'}$. The constraints on the state and the control variables y, w are

$$J_m(w) := \int_Q g_m(x, t, y(x,t), w(x,t))\,dx\,dt = 0, \quad 1 \leq m \leq p,$$

$$J_m(w) := \int_Q g_m(x, t, y(x,t), w(x,t))\,dx\,dt \leq 0, \quad p < m \leq q,$$

and the cost functional is

$$J_0(w) := \int_Q g_0(x, t, y(x,t), w(x,t))\,dx\,dt,$$

where $y := y_w$ is the solution of (1-4) for the control w. The optimal control problem is to minimize $J_0(w)$ subject to the above constraints.

Since such problems have no classical solutions in general, without additional convexity assumptions on the data, it is standard (see Warga, 1972) to work on the so-called relaxed form of the problem, which we define below.

Define the set of classical controls

$$W := \{w : (x,t) \to w(x,t) \mid w \text{ measurable from } \bar{Q} \text{ to } U\},$$

and the set of relaxed controls

$$R := \{r : (x,t) \to r(x,t) \mid r \text{ weakly measurable from } \bar{Q} \text{ to } M_1(U)\},$$

where the set $M_1(U)$ of probability measures on U is a subset of the space $M(U) \equiv C(U)^*$ of Radon measures on U, and has here the relative weak star topology. We have

$$R \subset L^\infty_w(\bar{Q}, M(U)) \equiv L^1(\bar{Q}, C(U))^* \subset B(\bar{Q}, U)^*,$$

where $L^\infty_w(\bar{Q}, M(U))$ is the set of (equivalence classes of) functions from $\bar{Q}$ to $M(U)$ which are measurable w.r.t. a weak norm topology on $M(U)$ (which coincides on $M_1(U)$ with the relative weak star topology) and essentially bounded w.r.t. the strong dual norm on $M(U)$, and $B(\bar{Q}, U)$ is the space of Caratheodory functions on $\bar{Q} \times U$ in the sense of Warga (1972). The subset R is endowed with relative weak star topology. The sets $M_1(U)$ and R are convex and, with their respective topology, metrizable and compact. For $\varphi \in B(\bar{Q}, U)$, and $r \in span(R)$, we use the notation

$$\varphi(x, t, r(x,t)) := \int_U \varphi(x, t, u)\, r(x,t)(du).$$

Note that this expression is linear in r. A sequence $\{r_k\}$ converges to r in R if

$$\lim_{k\to\infty} \int_Q \varphi(x, t, r_k(x,t))\,dx\,dt = \int_Q \varphi(x, t, r(x,t))\,dx\,dt,$$

for every $\phi \in B(\overline{Q}, U)$, or equivalently for every $\phi \in C(\overline{Q} \times U)$. In addition, we identify every classical control $w(\cdot,\cdot)$ with its associated Dirac relaxed control $\delta_{w(\cdot,\cdot)}$. Thus, we have $W \subset R$, and it is proved in Warga (1972) that W is dense in R.

We denote by $(\cdot,\cdot)$ and $|\cdot|$ the inner product and norm in $L^2(D)$, by $((\cdot,\cdot))$ and $\|\cdot\|$ the usual inner product and the norm in the Sobolev space $V := H^1_0(D)$ and by $<\cdot,\cdot>$ the duality bracket between V and its dual V^*. Define the family of bilinear forms on V

$$a(t,v,w) := \sum_{i,j=1}^{d} \int_D a_{ij}(x,t)(\partial v/\partial x_j)(\partial w/\partial x_i)\,dx.$$

In order to define the relaxed as well as the classical solutions of our problem, we shall first interpret the state equation in the following weak and relaxed form:

$$<y_{tt},v> + a(t,y,v) = \int_D f(x,t,y(x,t),r(x,t))v(x)\,dx,$$

for every $v \in V$, a.e. in I, (5)

$$y(x,0) = y^0(x), \quad \text{in } D, \tag{6}$$

$$y_t(x,0) = y^1(x), \quad \text{in } \Omega, \tag{7}$$

where the derivatives are taken in the sense of distributions (cf. Lions, 1969, p. 115). Note that, accordingly to our notation, the relaxed control r appears here in mean-value form. Defining the functionals

$$J_m(r) := \int_Q g_m(x,t,y(x,t),r(x,t))\,dx\,dt, \quad 0 \le m \le q \tag{8}$$

the continuous relaxed optimal control problem (CRP) is to minimize $J_0(r)$ subject to the constraints $r \in R$, $J_m(r) = 0$, $1 \le m \le p$, and $J_m(r) \le 0$, $p < m \le q$, where $y = y_r$ is the (unique) solution of (2,5,6,7). The continuous classical problem (CCP) is the problem CRP with additional constraint $r \in W$.

We suppose that the operators $A(t)$ satisfy the following conditions

$$\sum_{i,j=1}^{d} a_{ij}(x,t)z_i z_j \ge \alpha \sum_{i=1}^{d} z_i^2, \quad (x,t) \in \overline{Q}, \; z_i \in R, \; 1 \le i \le d, \text{ with } \alpha > 0,$$

$$a_{ij} \in C^1(I, L^\infty(\Omega)), \; i,j = 1,\dots,d$$

$$a_{ij} = a_{ji}, \; i,j = 1,\dots,d,$$

which imply that

$$|a(t,v,w)| \le \alpha_1 \|v\|\,\|w\|, \quad t \in I, \; v,w \in V,$$

$$a(t,v,v) \ge \alpha_2 \|v\|^2, \quad t \in \bar{J}, \quad v \in V,$$

for some $\alpha_1 \ge 0$, $\alpha_2 > 0$, and that $a(t,v,w)$ is symmetric.

We suppose also that the function f is defined on $\bar{Q} \times R \times U$, measurable for fixed y, u, continuous for fixed x, t, and satisfies

$$|f(x,t,y,u)| \le F(x,t) + \beta|y|, \quad \text{for every } x, t, y, u$$

with $F \in L^2(Q)$, $\beta \ge 0$,

$$|f(x,t,y_1,u) - f(x,t,y_2,u)| \le L|y_1 - y_2|, \quad \text{for every } x, t, y_1, y_2, u.$$

Then, for every $r \in R$, $y^0 \in V$ and $y^1 \in L^2(D)$, it can be proved that equations (2,5,6,7) have a unique solution $y = y_r$, such that $y \in L^\infty(J,V)$, $y' \in L^\infty(J,L^2(D))$ and $y'' \in L^2(J,V^*)$ (see Lions, 1972, Chap. 4, for the uniqueness in the linear case, and proof of Lemma 2.1 below, with fixed control). It follows that y is essentially equal to a function in $C(\bar{J}, L^2(D))$, that y' is essentially equal to a function in $C(\bar{J}, V^*)$, and thus the initial conditions (6,7) make sense.

LEMMA 2.1 *The mapping* $r \to y_r$, *from* R *to* $L^2(Q)$, *is continuous.*

Proof. Let $r \in R$ be a fixed relaxed control. Let $\{r_k\}$ be any sequence converging to r in R, and set $y_k := y_{r_k}$. Note that, since the bilinear form $a(t,\cdot,\cdot)$ is defined on V only, we cannot directly replace v by $y(t) \in L^2(D)$ in (5). To overcome this difficulty, we shall use the following approximation. Since V is separable, there exists a sequence $\{v_i\}_{i=1}^\infty$ such that the elements $v_1, \dots, v_n$ are linearly independent for every n, and the set $span(\{v_i\}_{i=1}^\infty)$ is dense in V. For every k and n, define the approximate solution y_{nk}

$$y_{nk}(x,t) := \sum_{j=1}^{n} \xi_{jnk}(t) v_j(x),$$

which satisfies

$$(y''_{nk}(t), v_i) + a(t, y_{nk}(t), v_i) = (f(t, y_{nk}, r_k), v_i), \quad i = 1, \dots, n, \quad \text{a.e. in } I, \tag{9}$$

$$y_{nk}(0) = y_{0n} = \sum_{j=1}^{n} \xi^0_{jn} v_j, \tag{10}$$

$$y'_{nk}(0) = y_{1n} = \sum_{j=1}^{n} \xi^1_{jn} v_j, \tag{11}$$

where $y_{0n} \to y^0$ (resp. $y_{1n} \to y^1$) in V (resp. $L^2(D)$) strongly. For example, we can choose y_{0n} (resp. y_{1n}) to be the projection of y^0 (resp. y^1) onto the space $span(\{v_1, \ldots, v_n\})$ in V (resp. $L^2(\Omega)$). By our assumptions, for each n, k, equations (9,10,11) reduce to a system of linear ordinary differential equations in the ζ_{jnk}, which has a unique solution, with ζ_{jnk} and ζ'_{jnk} absolutely continuous on I (see Warga, 1972, Ch. II). Hence, in particular, $y_{nk}, y'_{nk} \in C(I, V)$. From equation (9), with v_i replaced by y'_{nk}, we obtain

$$(d/dt)[|y'_{nk}(t)|^2 + a(t, y_{nk}(t), y_{nk}(t))] - a_t(t, y_{nk}(t), y_{nk}(t))$$

$$= 2(f(t, y_{nk}(t), r_k(t)), y'_{nk}(t)),$$

where $a_t(t, v, w)$ is defined by replacing the coefficients a_{ij} by $\partial a_{ij}/\partial t$ in $a(t, v, w)$. Integrating on $[0, t]$, we get

$$|y'_{nk}(t)|^2 + a(t, y_{nk}(t), y_{nk}(t)) = |y_{1n}|^2 + a(0, y_{0n}, y_{0n})$$

$$+ \int_0^t a_t(s, y_{nk}(s), y_{nk}(s))ds + 2\int_0^t (f(s, y_{nk}(s), r_k(s)), y'_{nk}(s))ds.$$

By our assumptions, it follows easily that

$$|y'_{nk}(t)|^2 + \|y_{nk}(t)\|^2 \le c\,[|y_{1n}|^2 + \|y_{0n}\|^2 + \int_0^T |F(s)|^2 ds]$$

$$+ c' \int_0^t [|y'_{nk}(s)|^2 + \|y_{nk}(s)\|^2]\, ds.$$

Hence, by Gronwall's inequality

$$|y'_{nk}(t)|^2 + \|y_{nk}(t)\|^2 \le c, \quad t \in I,$$

which shows, in particular, that the double sequence $\{y_{nk}\}$ (resp. $\{y'_{nk}\}$) is bounded in $L^2(I, V)$ (resp. $L^2(Q)$). Now, let $\{y_{n_\mu k_\mu}\}_{\mu=1}^{\infty}$ be any subsequence such that $n_\mu \to \infty$ and $k_\mu \to \infty$. By the Alaoglu-Bourbaki theorem, there exists a subsequence $\{y_{n_\mu k_\mu}\}$ (same notation) such that

$$y_{n_\mu k_\mu} \to z \text{ in } L^2(I, V) \text{ weakly},$$

$$y'_{n_\mu k_\mu} \to z \text{ in } L^2(Q) \text{ weakly}.$$

We have, for every $v \in V$ and $\varphi \in C_0^\infty(I)$

$$\int_0^T (y'_{n_\mu k_\mu}, v)\varphi(t)\, dt = - \int_0^T (y_{n_\mu k_\mu}, v)\varphi'(t)\, dt,$$

and passing to the limit we see that $z = z'$ (Lemma 1.1 in Temam, 1977, p. 250). By the Aubin compactness theorem (see Temam, 1977, p. 271), we can suppose also that

$$y_{n_\mu k_\mu} \to z \text{ in } L^2(Q) \text{ strongly}.$$

We can then pass to the limit (in μ) in the approximate equations (9,10,11) in weak integrated form (see Lions, 1972), using here Proposition 2.1, from Chryssoverghi (1986), for the term containing *f*, to show that z = Y_r · Since the limit Y_r is unique, it follows by contradiction that Y_nk ---+ Y_n in L^2(Q) strongly, as *n*, *k*—+ oo. Note that the above proof shows, in particular, that for each *k*, i.e. keeping the control rk fixed, we have Y_nk -+ Yk := Y_{r_k} in L^2(Q) strongly, as n - + oo. These convergences imply that, for given s, there exist Nk, for each *k*, such that

IIYk - $Y_n$klIL2(Q) ::; s/2, for every *n* 2 Nk,

and *N* such that

IIY$_n$k -YrlIL2(Q)::; s/2, for every *n* 2 N and k 2 *N*.

Hence, for each *k* 2 N, by choosing some *n* 2 max(N, Nk), we get

IIYk - YrlIL2(Q)::; c.

Therefore Yk ---+ Y_r in L^2(Q) strongly, which proves the lemma, since R is metrizable.

■

In order to prove the existence of an optimal relaxed control, we suppose in addition that the functions 9_m, 0 ::; *m* ::; *q*, are measurable for fixed y, *u*, continuous for fixed *x*, *t*, and satisfy

l9m(x, *t*, *Y*, u)I ::; $G_m(x,t)$ + rmlYl^2, for every *x*, *t*, *Y*, *u*,

with G_m E L^1(Q), r_m 2 0, 0::; m::; *q*.

LEMMA 2.2 *The functionals* J_m, 0::; m::; *q*, *are continuous on R*.

Proof. Follows from Lemma 2.1 and Proposition 2.1. of Chryssoverghi (1986).

■

THEOREM 2.1 *If there exists an admissible control,* i. e. *a control satisfying the constraints, then there exists an optimal control for the GRP.*

Proof. By Lemma 2.2, the set RA C R of admissible controls is closed, hence compact, and the functional J_0 is continuous. The theorem follows.

■

In order to derive necessary conditions for optimality, we suppose in s3,ddition that f and g'_{my}, 0 ::; m::; *q*, exist, are measurable for fixed *y*, u, continuous for fixed x, *t*, and satisfy

lg $_y(x,t,y,$ u)I::; $Gl_m(x,t)$ + 11mlYI, for every *x*, *t*, *y*, *u*,

with Glm E L^2(Q), rl_m 2 0, 0::; m::; *q*.

Since *f* is Lipschitzian, we have also

lf (x, t, y, u)I ::; *L*, for every *x*, *t*, *y*, *u*.

LEMMA 2.3 For $\mu \geq 0$, $r \in R$, $\tilde{r}_i \in R$, $i = 0, \dots, \mu$, $\theta = (\theta_0, \dots, \theta_\mu)$, with $\theta_i > 0$, $i = 0, \dots, \mu$, and $\sum_{i=0}^{\mu} \theta_i \leq 1$, set $r_\theta := r + \sum_{i=0}^{\mu} \theta_i(\tilde{r}_i - r)$, $y := y_r$, $y_\theta := y_{r_\theta}$, $\Delta y_\theta := y_\theta - y$. Then

$$||\Delta y_\theta||_{L^2(Q)} \leq c||\theta||_\infty,$$

where $||\theta||_\infty = \max_i |\theta_i|$.

Proof. Let $y_{n\theta}$ (resp. y_n) be the solution of equations (9,10,11), where y_{nk} is replaced by $y_{n\theta}$ (resp. y_n) and r_k by r_θ (resp. r). Setting $\Delta y_{n\theta} := y_{n\theta} - y_n$, we have

$$\begin{aligned}
&(\Delta y''_{n\theta}(t), \Delta y'_{n\theta}(t)) + a(t, \Delta y_{n\theta}(t), \Delta y'_{n\theta}(t)) \\
&= (f(t, y_{n\theta}, r_\theta) - f(t, y_n, r), \Delta y'_{n\theta}(t)) \\
&= (f(t, y_{n\theta}, r_\theta) - f(t, y_n, r_\theta), \Delta y'_{n\theta}(t)) \\
&\quad + \sum_{i=0}^{\mu} \theta_i (f(t, y_n, \tilde{r}_i - r), \Delta y'_{n\theta}(t)).
\end{aligned}$$

Since f is Lipschitzian, we deduce, similarly to the proof of Lemma 2.1, c denoting various constants, that

$$\begin{aligned}
&|\Delta y'_{n\theta}(t)|^2 + ||\Delta y_{n\theta}(t)||^2 \\
&\leq c\int_0^t [|\Delta y'_{n\theta}(s)|^2 + ||\Delta y_{n\theta}(s)||^2] ds + c||\theta||_\infty^2 \sum_{i=0}^{\mu} \int_0^T |f(y_n, \tilde{r}_i - r)|^2 ds,
\end{aligned}$$

hence, using Gronwall's inequality

$$\begin{aligned}
&||\Delta y_{n\theta}||^2_{L^2(Q)} \leq c||\Delta y_{n\theta}||^2_{L^\infty(I, L^2(\Omega))} \leq c||\Delta y_{n\theta}||^2_{L^\infty(I,V)} \\
&\leq c||\theta||_\infty^2 \sum_{i=0}^{\mu} \int_0^T |f(y_n, \tilde{r}_i - r)|^2 ds \\
&\leq c||\theta||_\infty^2 (\mu + 1)[||F||_{L^2(Q)} + \beta||y_n||_{L^2(Q)}]^2.
\end{aligned}$$

By Lemma 2.1, with fixed control, $y_n \to y$, $y_{n\theta} \to y_\theta$ and $\Delta y_{n\theta} \to \Delta y_\theta$, in $L^2(Q)$. Therefore

$$||\Delta y_\theta||_{L^2(Q)} \leq c||\theta||_\infty.$$

LEMMA 2.4 Dropping the index m, with the notation of Lemma 2.3, we have

$$\begin{aligned}
&J[r + \sum_{i=0}^{\mu} \theta_i(\tilde{r}_i - r)] - J(r) \\
&= \sum_{i=0}^{\mu} \theta_i \int_Q H(x, t, y(x,t), \phi(x,t), \tilde{r}_i(x,t) - r(x,t)) dx\, dt + o(||\theta||_\infty),
\end{aligned}$$

where, for each function g, the general Hamiltonian H is defined by

$$H(x, t, y, <p, u) := </Jf(x, t, y, u) + g(x, t, y, u), \quad (12)$$

and the general adjoint state $<p := <J$ satisfies

$$<J' + A(t)<p = f(y, r)<p + g(y, r), \quad in\ Q, \quad (13)$$

$$</J(x, t) = 0, \quad in\ E, \quad (14)$$

$$</J(x, T) = <J(x, T) = 0, \quad inn, \quad (15)$$

where $y := Y_r \cdot$ In particular, for $\mu = 0$, r, TE R, the directional derivative of J is

$$DJ(r, T - r) := \lim_{e\_,0+} [J(r + 0(T - r)) - J(r)]/0$$

$$= \int H(x, t, y(x, t), <j(x, t), T(x, t) - r(x, t))dx\, dt.$$

Proof. Under our assumptions on g and g, for given r ER, it can be proved (similarly to Warga (1972), II.6.7, II.6.8 and VIII.2.2, using essentially the mean-value theorem and the Lebesque dominated convergence theorem) that the functional 1¥ with

$$w(y) := \int g(x, t, Y, r)dx\, dt,$$

defined on $L^2(Q)$, has Frechet derivative W', with

$$w'(y)Ay = \int g(x, t, Y, r)Aydx\, dt.$$

By Lemma 2.3, we have

$$J(re) - J(r) = \int [g(ye, re) - g(y, re) + g(y, re) - g(y, r)]dx\, dt$$

$$= \int_Q g(y, r)Ayedx\, dt + \sum_0 0i \int_Q g(y, Ti - r)Ayedx\, dt$$

$$+ o(l\, 101loo) + \sum_{i=0} ei \int_Q g(y, Ti - r)dx\, dt$$

$$= \int_Q g(y, r)Ayedx\, dt + \sum_{i=0}^{\mu} ei \int_Q g(y, Ti - r)dx\, dt + o(ll0lloo)-$$

Since $Aye(0) = Ay(O) = </J(T) = <J(T) = 0$, integrating by parts, we obtain from the state equation

$$- \int_0^T (t:..y, </J')dt + \int_0 a(t, Aye, </J)dt$$

$$= \int_Q f_y(y,r)\Delta y\,\phi\,dx\,dt + \sum_{i=0} \theta_i \int_Q f(y, r_i' - r)\,\phi\,dx\,dt + o(\|\theta\|_\infty),$$

and from the adjoint equation

$$-\int_0^T (\phi_t, \Delta y)\,dt + \int_0^T a(t, \phi, \Delta y)\,dt$$

$$= \int_Q f_y(y, r)\,\phi\,\Delta y\,dx\,dt + \int_Q g_y(y, r)\,\Delta y\,dx\,dt,$$

and the lemma follows. ■

The following theorem states necessary conditions for optimality.

THEOREM 2.2 *If $r \in R$ is optimal for either the CRP or the CCP, then r is* extremal, i.e. *there exist multipliers $\lambda_m \in \mathbb{R}$, $0 \le m \le q$, with $\lambda_0 \ge 0$, $\lambda_m \ge 0$, $p < m \le q$, and $\sum_{m=0}^{q} |\lambda_m| \neq 0$, such that*

$$H(x, t, y(x, t), \phi(x, t), r(x, t))$$
$$= \min_{u \in U} H(x, t, y(x, t), \phi(x, t), u), \quad \text{a.e. in } Q, \tag{16}$$

and

$$\lambda_m J_m(r) = 0, \;\; p < m \le q, \tag{17}$$

where H and ϕ are defined by (12) and (13,14,15), with g replaced by $\sum_{m=0}^{q} \lambda_m g_m$.

Proof. Let

$$S := \{\theta \in \mathbb{R}^{n+1} \mid \sum_{m=0}^{n} \theta_m \le 1\}.$$

By Lemma 2.4, the functions

$$\theta \to J_m(r_\theta), \quad 0 \le m \le q,$$

from S to $\mathbb{R}$, have a derivative at 0 (see Warga, 1972), and by Lemma 2.2, the functionals J_m are continuous on R. If r is optimal for the CRP or the CCP, then, by Theorem V.3.2 in Warga (1972), p. 310, there exist multipliers as above such that

$$\sum_{m=0}^{q} \lambda_m D J_m(r, r' - r) \ge 0, \quad \text{for every } r' \in R,$$

which is equivalent to (16) (see Warga, 1972, p. 360), and such that (17) holds. ■

We have also the following continuity result, whose proof is similar to that of Lemma 2.1.

LEMMA 2.5 *The mapping $r \to \phi_r$, from R to $L^2(Q)$, is continuous.*

3. The discrete optimal control problems

We shall now discretize our continuous optimal control problems. For simplicity, we suppose that the domain D is a polyhedron.

For every integer n, let $\{S_i^n\}_{i=1}^{M(n)}$ be an admissible regular triangulation (see Temam, 1977, p. 73) of $\bar{\Omega}$ into closed d-simplices, $\{I_j^n\}_{j=0}^{N(n)-1}$ a subdivision of the interval I into $N(n)$ intervals $I_j^n := [t_j, t_{j+1})$ of equal length $\Delta t = T/N$, and $\{E_k^n\}_{k=1}^{P(n)}$ a partition of $\bar{Q}$ into $P(n)$ Borel subsets with $\max_k[\operatorname{diam}(E_k^n)] \to 0$, as $n \to \infty$. Set $Q_{ij}^n := S_i^n \times I_j^n$. For example, the E_k^n may be unions of some of the Q_{ij}^n. Let $V^n \subset V$ be the subspace of functions which are continuous on $\bar{\Omega}$ and affine on each S_i^n. Let $R^n \subset R$ be the set of piecewise constant relaxed controls

$$R^n := \{r^n \in R \mid r^n(x,t) := r_k^n \in M_1(U), \quad \text{on } E_k^n,\ k = 1, \dots, P\},$$

and $W^n := R^n \cap W$ the set of piecewise constant classical controls.

For given values v^j, $j = 0, \dots, N$, in a vector space, we define the following functions, a.e. on I

$$v_-(t) := v^j, \quad t \in \mathring{I}_j^n, \quad j = 0, \dots, N-1,$$

$$v_+(t) := v^{j+1}, \quad t \in \mathring{I}_j^n, \quad j = 0, \dots, N-1,$$

$v(t) :=$ the function which is affine on each I_j^n such that

$$v(t_j) = v^j, \quad j = 0, \dots, N.$$

For a given discrete relaxed control $r^n := (r_1^n, \dots, r_k^n, \dots, r_P^n)$, the corresponding discrete state $y^n := (y^0, \dots, y^j, \dots, y^N)$ is given by

$$y^{j+1} - y^j = \Delta t\, z^{j+1}, \quad j = 0, \dots, N-1, \tag{18}$$

$$(1/\Delta t)(z^{j+1} - z^j, v) + a(t_{j+1}, y^{j+1}, v) = (f^j, v),$$
$$\text{for every } v \in V^n,\ j = 0, \dots, N-1, \tag{19}$$

$$y^0, z^0 \text{ given}, \quad y^j, z^j \in V^n, \quad j = 0, \dots, N, \tag{20}$$

$$f^j(x) := (1/\Delta t) \int_{I_j} f(x, t, y^j(x), r^n(x,t))\,dt, \quad j = 0, \dots, N-1. \tag{21}$$

Choosing a basis $\{v_\mu\}$ in V^n and writing

$$y^{j+1} = \sum_\mu c_\mu^{j+1} v_\mu, \quad z^{j+1} = \sum_\mu d_\mu^{j+1} v_\mu,$$

we reduce equations (18,19) to a linear system of the form

c = /j_tc' + b (22)

$$(D + \Delta t^2 E)c' = b', \tag{23}$$

where D and E are the corresponding mass and stiffness matrices, respectively. This system is regular since (23) has a positive definite matrix. Defining the discrete functionals

J:;,(rn) := lgm(x,t,y'.':_(x,t),rn(x,t))dxdt, 0::; m::; q,

the discrete constraints are rn E Rn, either of the two following constraints

$$|J_m^n(r^n)| \leq \varepsilon_m^n, \ 1 \leq m \leq p, \tag{24}$$

$$J_m^n(r^n) = \varepsilon_m^n, \ 1 \leq m \leq p, \tag{25}$$

and

$$J_m^n(r^n) \leq \varepsilon_m^n, \ \varepsilon_m^n \geq 0, \ p < m \leq q, \tag{26}$$

where the c, 1 ::; *m* ::; *q* are given numbers. The discrete relaxed problem DRPn (resp. DRP'n) is to minimize J"0(rn) subject to the above constraints, case (24) (res. (25)). The discrete classical problem DCPn is the problem DRPn with the additional constraint rn E wn.

THEOREM 3.1 *If there exists an admissible control for any of the above discrete problems, then there exists an optimal control for this problem.*

Proof. We first remark that the convergence in Rn = [M₁ (U)j^P implies the convergence in R, hence in each set Rij of restrictions r lQ , for r E R. The continuity of the mapping rn → yn is proved by induction on *j*, using system (22,23), and Proposition 2.1, from Chryssoverghi (1986). The same proposition shows that the functionals J;:, are continuous. The existence of an optimal control for the DRPn and the DRP'n follows then from the compactness of the set R_ of admissible discrete relaxed controls. The existence for the DCPn is easily proved, using here the ordinary topology of wn.

■

LEMMA 3.1 *Dropping the index* m, *define the general discrete relaxed adjoint state* <J' = (<P0, ... , <Pj, ... , <PN) *by*

<Pj+l - rpj = fj_tijJj, j = N - 1, ... , 0, (27)

(1/fj_t)(ijJj+i - 1/Jj, v) + a(t:J, <Pj, v) = (fJ<P:J+i, v) + (g";, v),

$$\text{for every } v \in V^n,\ j = N-1,\dots,0, \tag{28}$$

$$\phi_N^n = \psi_N^n = 0,\ \phi_j^n, \psi_j^n \in V^n,\ j = 0,\dots,N,$$

$$f_j^n(x) = (1/\Delta t)\int_{I_j} f(x, t, y_j^n, r^n)dt,\ j = 0,\dots,N-1, \tag{29}$$

$$g_j^n(x) = (1/\Delta t)\int_{I_j} g(x, t, y_j^n, r^n)dt,\ j = 0,\dots,N-1. \tag{30}$$

The directional derivative of the functional J^n on R^n is given by

$$DJ^n(r^n, \tilde{r}^n - r^n) = \int_Q H(x, t, y_-^n, \phi_+^n, \tilde{r}^n - r^n)dx\,dt,$$

where H is defined by (12).

Proof. (Outline.) The proof parallels that of the continuous case (Lemmas 2.3 and 2.4). The discrete analogue of Lemma 2.3 is first proved similarly to Lemma 4.2 below. The analogue of Lemma 2.4 is then proved by interpreting the discrete estimates and the discrete equations in terms of y_-^n, y_+^n, y, etc... and using the discrete integration by parts formula

$$\int_0^T ((v)', w_+)dt = -\int_0^T (v_-, (w)')dt + (v_N, w_N) - (v_0, w_0), \tag{31}$$

which is directly verified.

■

THEOREM 3.2 *If r^n is optimal for the DRP^n, then r^n is* extremal, i.e. *there exist multipliers $\lambda_m^n \in \mathbb{R}$, $0 \le m \le q$, with $\lambda_0^n \ge 0$, $\lambda_m^n \ge 0$, $p < m \le q$, and $\sum_{m=0}^{q} |\lambda_m^n| = 1$, such that*

$$\int_{I_{B_k^n}} H(x,t,y_-^n,\phi_+^n,r_k)dxdt =$$

$$\min_{u \in U}\int_{I_{B_k^n}} H(x,t,y_-^n,\phi_+^n,u)dxdt,\quad k = 1,\dots,P, \tag{32}$$

and

$$\lambda_m^n [J_m^n(r^n) - \varepsilon_m^n] = 0,\ p < m \le q, \tag{33}$$

where H, y^n, q^n are given by (12), (18), (19,20,21) and (21-30) with g replaced by $\sum \lambda g^n$.

Proof. Follows from Theorem V.2.3 in Warga (1972), p. 303, noting that the inequality

$$\sum_{m=0}^{q} \lambda_m^n D\Phi_m^n(r^n, \bar{r}^n - r^n) \geq 0, \quad \text{for every } \bar{r}^n \in R^n,$$

or

$$\sum_{k=1}^{p} \int_{B_k^n} H(x, t, y_k^n, z_k^n, \bar{r}_k - r_k^n)\,dx\,dt \geq 0, \quad \text{for every } \bar{r}^n \in R^n,$$

is equivalent to (32), since the $\bar{r}_k$ are independent. Relations (33) are the transversality conditions.

■

4. Convergence

In this section we study the behaviour of the discrete problems in the limit, as $n \to \infty$. We first state the following control approximation lemma which is proved in Chryssoverghi et al. (1993); see also Roubicek (1991).

LEMMA 4.1 *For every $r \in R$, there exists a sequence $\{u^n \in W^n\}$ such that $u^n \to r$ in R.*

LEMMA 4.2 *(Stability) For every $r^n \in R^n$, if $\|y_0\|$ and $|z_0|$ remain bounded and Δt is sufficiently small, then*

$$\|y_j^n\| \leq c, \quad j = 0, \ldots, N, \tag{34}$$

$$|z_j^n| \leq c, \quad j = 0, \ldots, N, \tag{35}$$

$$\sum_{j=0}^{N-1} \|y_{j+1}^n - y_j^n\|^2 \leq c, \tag{36}$$

$$\sum_{j=0}^{N-1} |z_{j+1}^n - z_j^n|^2 \leq c. \tag{37}$$

Proof. It is easily proved (see Chryssoverghi et al., 1993) that

$$|f_j^n| \leq F_j^n + \beta|y_j^n|,$$

where

$$\bar{F}_j^n := \left((1/\Delta t) \int_{\Omega \times I_j^n} F^2\,dx\,dt \right)^{1/2}.$$

We have, since y'/+1 - Y'/ = 6.tzj+1

a(t'J+1,Y'J+1,Y"/+1) - a(t'J,Y"/,Y"/)
= 6.ta(t'J+ 1, Y"/+1, z"/+1) + 6.ta(t'J+1 , z"/+ 1, Y"/)+
a(t'J+ 1 , Y'J , Y'/) - a(t'],Y"/,Y"/)
= 26.ta(t'J+i, y'/+1, z7+1) - 6.ta(t'J+ 1, Y"/+1, z7+1) + 6.ta(t'J+1 , z7+ 1, Y"/)
+a(t'J+ 1 , Y'/ , Y'/) - a(t'],Y'J,Y"/)
= 26.ta(t'J+1, Y"/+1, z.'J+1) - 6.t^2 a(t'J+1, z7+ 1, z"/+1)+
a(t'J+ 1, Y"/, Y"/) - a(t'J, Y"/, Y'J).

The discrete state equation yields

(zj+1 - z"/, z"/+1) + 6.ta(t'J+ 1, Y'J+ 1, z'J+1) = 6.t(f?, z'J+1).

Hence, by the assumptions and the mean value theorem

lz"/+11 ^2 - lz"/1 ^2 + lz"/+1 - z"/1 ^2 + a(t'J+ 1, Y"/+1, Y"/+1)
-a(t"/, Y"/, Y"/) + 6.t^2 a (t1J+1, z"/+ 1, z"/+1)
S ia(t'J+1, Y"/, Y"/) - a(t'], Y"/, Y"/)1 + 26.tl(f]', Z"/+1)1
S c ' 6.tliY"/11 ^2 + c6.t((FT) ^2 + IIY"/11 ^2 + lz"/1 ^2 + izf+1 - zfl ^2).

Assuming that cb..t S 1/2 (where c does not depend on the triangulation of 0), and summing over *j*, we obtain

$$|z_{i+1}^n|^2 + (1/2)\sum_{j=0}^{i} |z_{j+1}^n - z_j^n|^2 + \alpha_2 ||y_{i+1}^n||^2 + \alpha_2 \sum_{j=0}^{i} ||y_{j+1}^n - y_j^n||^2$$

S lzil ^2 + a1ilYill ^2 + ci!Flli2(Q) + c" 6.t $\sum_{j=0}^{i}$ (IIYfll ^2 + izfl ^2),

and the lemma follows from the discrete Gronwall inequality (see Lees, 1960). ■

From now on, we choose (for example) y0 to be the projection of y^0 onto V^n in V and Zo the projection of y^1 onto V^n in $L^2(0)$. This implies here that Yo-+ y^0 in V strongly and that Zo-+ y^1 in $L^2(0)$ strongly.

LEMMA 4.3 (Convergence) If r^n -+ r in R, then the corresponding discrete states y::_,y7+,y converge to Yr in $L^2(Q)$ strongly, as n-+ oo.

Proof. Since IIYol I and lzol are bounded, inequalities (34), (35) show that y::_,y7+,y are bounded in $L^2(J, V)$ and that z"::-,z7+,z are bounded in $L^2(Q)$. By inequalities (36), (37), we have

Y+ - y':_-+ 0, in $L^2(J, V)$ and in $L^2(Q)$ strongly,

z+ - z':_-+ 0, in $L^2(Q)$ strongly.

It follows easily that

Y+ - y -+ 0, in L^2 *(Q)* strongly,

z+ - z -+ 0, in L^2 *(Q)* strongly.

Therefore there exist subsequences y ,Y+,YA converging to some y in L^2 *(I, V)* weakly, hence in L^2 *(Q)* weakly, and subsequences z , z+, zA converging to some *z* in L^2 *(Q)* weakly. By the Aubin compactness theorem (Temam, 1977, p. 271), there exists a subsequence YA converging to the same y in L^2 *(Q)* strongly, hence y, Y+ converge to y in L^2 *(Q)* strongly, for the corresponding subsequences.

In order to pass to the limit in the discrete equations, let *v* E Cg"(D.), E C^1 (I), with (T) = 0, w := v, v^n (resp. ^{n}) the continuous piecewise affine interpolant of *v* (resp.), with respect to the partition of IΓ (resp. *I*), and w^n := v^n ^{n}. The discrete state equation (19), with v = w'J$_{+ i}$, yields by summation

T *T* *T*
l ((z)', w+)dt + *l* a(t'+, Y+, w7+)dt = *l* (f(t, y7:_, r^n), w7+)dt,

hence, by the discrete integration by parts formula (31)

T *T*
-*l* (z7:_ (w)')dt + *l* a(t7+, Y+, w+)dt =

T
l (f(t, y7:_ r^n), w7+)dt + (z*()*, v^n) (0).

Moreover, since (YA)' = z+, then, by integrating by parts, we get

$$-\int_0^T (y^n_\wedge, v)\xi' dt = \int_0^T (z^n_+, v)\xi dt + (y^n_0, v)\xi(0).$$

Now, we have clearly

z, z+-+ z in L^2 *(Q)* weakly,
(wA)' -+ w' in L^{00} *(Q)* strongly,
f+ -+ *t* in L^{00} *(I)* strongly,
Y+-+ y in L^2 *(I, V)* weakly,
W+ -+ *w* in L^{00} *(I*, V) strongly,
Y, Y+, YA -+ y in L^2 *(Q)* strongly,
r^n -+ r in R,
W+-+ w in L^2 *(Q)* strongly,
z'[j -+ y^1 in L^2 *(0)* strongly,
Yo -+ y^0 in V strongly,
v^n -+ v in C(IT) strongly.

We can thus pass to the limit in the above two equations, using Proposition 2.1, from Chryssoverghi (1986), for the term containing f, to obtain

$$-l^{T} (z, v) \{ dt + l^{T} a(t, Y, v) e dt = l^{T} (J(t, Y, r), v) e dt + (y^1, v) e(o),$$

and

$$-l^{T} (y, v) \{ dt = f_o^{T} (z, v) e dt + (y^0, v) e(o),$$

which hold for every v E V, since C(D) is dense in V. Choosing first e E C (I), these equations yield $y' = z$ and equation (5) (see Temam, 1977, p. 250). Finally, integrating by parts, we obtain the initial conditions (6), (7). Therefore, $y = Y_r$, and the convergences of the lemma hold for the original sequences.

■

LEMMA 4.4 *If* r^n - r *in* R, *then* $J^n_m(r^n)$ - $J_m(r)$, $0 \leq m \leq q$, *as* n - oo.

Proof. We have

$$J^n_m(r^n) = \int g_m (x, t, y^n_{r^n}, r^n) dx\, dt,$$

and the lemma follows from Lemma 4.3. and Proposition 2.1 from Chryssoverghi (1986).

■

First we study the behaviour in the limit of optimal discrete controls. We assume that the CRP is feasible. Consider the DRP^n (resp. DCP^n). In the presence of state constraints, we assume that the sequences {c }, in (24) and (26), converge to zero as n - oo and satisfy

$$|J^n_m(r^n)| \leq c, \quad 1 \leq m \leq p,$$

$$J^n_m(\overline{r}^n) \leq \varepsilon^n_m, \; \varepsilon^n_m \geq 0, \; p < m \leq q,$$

for every n, where $\{r^n$ E $R^n\}$ (resp. { E w^n}) is a sequence converging to some optimal control f of the CRP. Such sequences exist since, by Lemma 4.1, there exists a sequence { E w^n} such that - f, for some optimal f, and by Lemma 4.4

$$\lim_{n \to \infty} J^n_m(\ ) = J_m(r) = 0, \quad 1 \leq m \leq p,$$

$$\lim_{n \to \infty} J^n_m(\overline{r}^n) = J_m(\overline{r}) \leq 0, \; p < m \leq q.$$

In particular, the DRP^n (resp. DCP^n) is thus feasible for every n. We have the following, rather theoretical, result.

THEOREM 4.1 *For each n, let r^n be optimal for the DRP' (resp. DCP''). Then the sequence $\{r^n\}$ has accumulation points, and every such point is optimal for the GRP.*

Proof. Since R is compact, let $\{r^n\}$ (same notation) be a subsequence converging to some r E R. Since r^n is optimal and r^n admissible for the DRP^n (resp. DCP^n), we have

$$Jf;(r^n) \leq Jf;(r^n),$$

$$IJ;;;,(r^n)I \leq s;:,, \quad 1:::; m:::; p,$$

$$J;;;\_(r^n) \leq s;:,, \quad p < m :Sq,$$

Taking the limit and using Lemma 4.4, we see that r is optimal for the CRP. If there are no state constraints, by using any sequence $\{r^n$ E $w^n\}$ converging to some optimal control of the CRP, we also obtain in the limit that r is optimal for the CRP.

■

In the presence of inequality state constraints only, under appropriate assumptions, we can take s;;:, = 0 in the discrete state constraints, following the approach of Casas (1996). Given *8* E R, define the parametrized continuous problems CRP_6 and *CCP0*, with state constraints Jm ≤ *8* 1 ≤ m ≤ *q* and discrete problems DRP6 and DCP6, with constraints J;;,:::; 8, *1:::;* m:::; *q* and suppose that

(i) 8 := inf{81 the *CRP0* is feasible}< 0,

(ii) the CRP0 is stable to the left, i.e. $\lim_{0 \to 0-}$ (min *CRP0)* = min CRP0.

Note that since W is dense in R, we have

$$8 = \min_{rER} \{ \max_{1:Sm:Sq} Jm(r)\} = \inf_{wEW} \{ \max_{1:Sm:Sq} Jm(w)\}.$$

Condition (i) implies that the CRP_6 is feasible for every *8* 2: 8 and not feasible for every *8* < 8. It follows from condition (i) and Lemmas 4.1 and 4.4 that the DRP0 and the DCP0 are feasible for n sufficiently large. If 8 < 0, condition (ii) implies that the CRP_0 is stable, since it is always stable to the right, if *8:::;* 0.

For *8:::;* 8 < 0, it can be shown that

$$\min\ CRP0:::; \inf\ CCP0:::; \min\ CRP0:::; \inf\ CCP_0.$$

Now suppose that 8 < 0 and that the CCP_0 is stable, i.e.

$$\lim_{0 \to 0}(\inf\ CCP0) = \inf\ CCPo.$$

Then min CRP0 = inf CCP0 (Casas, 1996), hence condition (ii) holds. Note that conditions (i) and (ii) also imply that

$$\min\ CCP0 = \inf\ CCP0.$$

Since the function 0 - min CRP0, defined on [8, +oo), is non increasing, it is continuous (i.e. the CRP_0 is stable) for every 0 2 8, except at most a countable number of them. This assures us in some manner that the CRPo is almost always table.

Under conditions (i) and (ii), Theorem 4.1, with c = 0, remains valid. To see this, let $\{r^{nk}\}$ be a subsequence of $\{r^n\}$ such that r^{nk} - r E R, where r is feasible for the CRP_0 by Lemma 4.4. Now let 8 < 8 < 0 be given, and let r0 be optimal for the CRP0. There exists a sequence {r8 E w ⁿ} (Lemma 4.1) such that r8 - r0, where r8 is feasible for the DRP0 (resp. DCP0), for n 2 No (Lemma 4.4). We have

$J('^{k}(r^{nk})$:S $Jf:k(r^{k})$, for nk 2 No,

and taking the limit, as k - oo

Jo(r) :S Jo(ro)-

Finally,

$$Jo(r) :S \lim_{6-+0-} Jo(r0) = \lim_{6-+0-} (\min CRP0) = \min CRPo,$$

which shows that r is optimal for the CRPo =CRP.

Next, we examine the behaviour in the limit of extremal discrete controls. This is motivated by the fact that numerical optimization methods often compute approximations of extremal controls. We shall need the following lemma whose proof is similar to that of Lemma 4.3.

LEMMA 4.5 *If r^n - r in R, then the corresponding discrete adjoint states* 1>7!, <//t, 1>!. *converge to* 1>r *in* $L^2(Q)$ *strongly, as n -* oo.

Now consider the DRP'n. Sequences {c }, in (25) and (26), converging to zero and such that the DRP'n is feasible for every n can be constructed here as follows. Let be any solution of the unconstrained problem

c'' , - r/$t'' { [J;:; (r'')]² + j : ; max(0, J'; (r''))]² } ,

and set

= J;;:,( ), 1 :S m :Sp,

E'= max(O, J;;:,( )), p < m :Sq.

Let f be an admissible control of the CRP and {pⁿ} a sequence converging to f. We have

$$\lim_{n\to\infty} J^n_m(\tilde{r}^n) = J_m(\tilde{r}) = 0,\ 1 \le m \le p,$$

$$\lim_{n\to\infty} J_m^n(r^n) = J_m(r) \le 0, \; p < m \le q,$$

which show that $\varepsilon_m^n \to 0$, hence $E_m^n \to 0$, $1 \le m \le q$. Now choose $\varepsilon_m^n = E_m^n$, $1 \le m \le p$, and ε_m^n such that $\varepsilon_m^n \ge E_m^n$ and $\varepsilon_m^n \to 0$, $p < m \le q$. Then clearly the DRP'^n is feasible for every n. We suppose that the ε_m^n, $1 \le m \le q$, are chosen as above.

THEOREM 4.2 *For each n, let r^n be admissible and extremal for the DRP'^n. Then the sequence $\{r^n\}$ has accumulation points, and every such point is admissible and extremal for the GRP.*

Proof. Since R is compact and $\sum_m |\lambda_m^n| = 1$, we can suppose that $r^n \to r \in R$ and $\lambda_m^n \to \lambda_m$, as $n \to \infty$, with $\sum_m |\lambda_m| = 1$, $\lambda_0 \ge 0$, $\lambda_m \ge 0$, $p < m \le q$. By Lemmas 4.1, 4.3, 4.5 and Proposition 2.1, in Chryssoverghi (1986), we have, for given $\bar r \in R$ and $\bar r^n \to \bar r$

$$\sum_{m=0}^{q} \lambda_m DJ_m(r, \bar r - r) = \lim_{n\to\infty} \sum_{m=0}^{q} \lambda_m^n DJ_m^n(r^n, \bar r^n - r^n) \ge 0,$$

$$\lambda_m J_m(r) = \lim_{n\to\infty} \lambda_m^n [J_m^n(r^n) - \varepsilon_m^n] = 0, \; p < m \le q,$$

$$J_m(r) = \lim_{n\to\infty} [J_m^n(r^n) - \varepsilon_m^n] = 0, \; 1 \le m \le p,$$

$$J_m(r) = \lim_{n\to\infty} [J_m^n(r^n) - \varepsilon_m^n] \le 0, \; p < m \le q,$$

which show that r is extremal and admissible for the CRP.

■

5. Final comments

In order to obtain in the limit the strong relaxed minimum principle, we have used the relaxed discrete problems, since (i) the strong discrete classical minimum principle does not hold in general (see Canon et al., 1970), and (ii) weak discrete classical necessary conditions obviously cannot yield the strong conditions in the limit.

Optimization methods, when applied to the discrete relaxed problem, usually compute discrete Gamkrelidze controls. These controls, which are convex combinations of Dirac relaxed controls, can then easily be approximated by piecewise constant classical controls by a simple procedure (see Warga, 1972, and Chryssoverghi et al., 1997).

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